BenchQC:
QUBO models for bin packing

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22nd August 2023

1 Bin packing with conflicts – IP model

Consider the following bin packing IP model. (Notation adapted from OptWare document, p. 13.)

$$
\begin{aligned}
\min \quad & \sum_{b \in B o \in O} b \cdot x_{ob} && && \text{(1a)}\\
\text{s.t.} \quad & \sum_{b \in B} x_{ob} = 1 && \forall o \in O && \text{(1b)}\\
& \sum_{o \in O} w_o \cdot x_{ob} \leq c && \forall b \in B && \text{(1c)}\\
& x_{ob} + x_{o'b} \leq 1 && \forall (o, o') \in I, b \in B && \text{(1d)}\\
& x_{ob} - x_{o'b} = 0 && \forall (o, o') \in K, b \in B && \text{(1e)}\\
& x_{ob} \in \{0, 1\} && \forall b \in B o \in O &&
\end{aligned}
$$

Without loss of generality, Constraints (1e) can be dropped by combining connected objects into a single object.

2 Bin packing with conflicts – QUBO model

Define the QUBO cost function as

$$H(x, y) = A \cdot H_A(x, y) + B \cdot H_B(x)$$

The cost term H_B models the costs in eq. (1a):

$$H_B(x) = \sum_{b \in B o \in O} b \cdot x_{ob}\,.$$

The penalty term H_A encodes feasibility. For every constraint in model (1), a penalty term for its violation is introduced. Constraints (1b):

$$\sum_{o \in O} \left(\sum_{b \in B} x_{ob} - 1 \right)^2 \ .$$

Constraints (1d):

$$\sum_{b \in B} \sum_{o, o' \in I} \left(x_{ob} \cdot x_{o'b} \right) \ .$$

For Constraints (1c), auxiliary variables y are necessary. For $b \in B$, the *slack* of inequality (1c) is

$$S_b \coloneqq c - \sum_{o \in O} w_o \cdot x_{ob} \geq 0\,.$$

Unary (one-hot) encoding of slack. We introduce $y_{sb} \in \{0,1\},\ b \in B,\ s \in \{0,1,\dots,c\}$, where $y_{sb} = 1 \Leftrightarrow S_b = s$. Thus,

$$S_b = \sum_{s=0}^{c} s \cdot y_{sb} \quad \forall b \in B \ .$$

This is modeled by introducing two additional penalty terms into H_A. First, for each $b \in B$, only a single y_{sb} is equal to one:

$$\sum_{b \in B} \left(1 - \sum_{s=1}^{c} y_{sb} \right)^2 \ .$$

Second, for each $b \in B$, negative slack is penalized:

$$\sum_{b \in B} \left(\sum_{s=0}^{c} s \cdot y_{sb} - \sum_{o \in O} w_o \cdot x_{ob} \right)^2 \ .$$

In total, we need $|B| \cdot (c+1)$ additional variables.

Binary (logarithmic) encoding of slack. Here, we introduce a different slack-encoding with $|B| \cdot \lceil \log_2(c+1) \rceil$ additional variables. We introduce $y_{kb} \in \{0,1\},\ b \in B,\ k \in \{0,1,\dots,M\}$, where

$$M := \lceil \log_2(c+1) \rceil \ .$$

Then, the slack has a binary representation

$$S_b = \sum_{k=0}^{M-2} 2^k \cdot y_{kb} + (c - 2^{M-1} + 1) \cdot y_{M-1,b} \ .$$

We add a penalty to H_A if the slack does not have such a binary representation, i.e. Constraint (1c) is violated:

$$\sum_{b \in B} \left(\sum_{k=0}^{M-2} 2^k \cdot y_{kb} + (c - 2^{M-1} + 1) \cdot y_{M-1,b} - \sum_{o \in O} w_o \cdot x_{ob} \right)^2 \ .$$

References