Mathematical Modeling of a Ball-Beam System with Beam Pinned at Midpoint

1. System Description

We consider a ball-beam system where:

- The beam is of length $2L$ and is pinned at its midpoint.
- A servo motor attached at the pivot rotates the beam by an angle $\theta(t)$ with respect to the horizontal.
- A ball of mass m and radius r rolls without slipping along the beam.
- The ball's position along the beam (measured from the midpoint) is $x(t)x(t)$, positive to the right.

2. Assumptions

- Rolling without slipping.
- Beam is rigid, friction only sufficient to ensure rolling.
- Servo input directly sets $\theta(t)$ (servo dynamics neglected in basic model).
- Symmetry about the pivot, so ball can move in $[-L, L]$.

3. Kinematics of the Ball

Let $x(t)x(t)$ be the displacement along the beam.

Rolling without slipping:

$\dot{x}=r\dot{\phi}$

where $\phi(t)$ is the angular displacement of the ball.

4. Forces Along the Beam

Along the beam axis:

- Gravity component along the beam: $F_g = mg\sin(\theta)$
- Friction provides torque but no net work (rolling without slipping).

Using Newton's second law along the beam and including rotational inertia:

For a solid sphere:

$$I = (2/5)mr^2$$

Effective inertia:

$$m_{eff} = m + Ir^2 = m + (2/5)m = (7/5)m$$

Equation of motion along the beam:

$$m_{eff}\ddot{x} = mg\sin(\theta)$$

$$\ddot{x} = (m/m_{eff})g\sin(\theta)$$

$$\ddot{x} = 5/7g\sin(\theta)$$

5. Linearization for Small Angles

For small θ:

$$\sin(\theta) \approx \theta$$

$$\ddot{x} = 5/7g\,\theta$$

6. Including Servo Dynamics (Optional)

If servo dynamics are considered:

$$Jb\ddot{\theta} + b\theta\dot{\theta} = Kmu(t)$$

Couple this with:

$\ddot{x} = 5/7 g \sin(\theta)$

This yields a higher-order system where the control input is motor voltage $u(t)u(t)$.

7. Final Mathematical Model

Nonlinear form:

$\ddot{x} = 5/7 g \sin(\theta)$

Linearized form:

$\ddot{x} = 5/7 g \theta$

Remarks:

- The model is symmetric about the midpoint due to the pivot at the center.
- Domain of motion: $x \in [-L, L] x \backslash in [-L, L]$.
- For PID control, these models directly relate $\theta(t) \backslash theta(t)$ to the ball acceleration.

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