Partial Fraction

assassin3552

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Consider this integral:

$$\int \frac{x+1}{x^2-5x+6}\mathrm{d}x$$

This integral looks scary, but notice that we can factor the denominator: $x^2-5x+6=(x-2)(x-3)$

Let's first take at look at fraction addition:

$$\frac{A}{B}+\frac{C}{D}=\frac{AD+BC}{BD}$$

Since the integrand is a fraction, and we successfully write the denominator as a product, we should be able to split the fraction into a sum of two fraction.

Assume we have split the fraction like this:

$$\frac{x+1}{(x-2)(x-3)}=\frac{A}{(x-2)}+\frac{B}{(x-3)}$$

Here and A and B are different constant, let's try to combine them together:

$$\frac{A(x-3)+B(x-2)}{(x-2)(x-3)}=\frac{Ax-3A+Bx-2B}{(x-2)(x-3)}=\frac{(A+B)x+(-3A-2B)}{(x-2)(x-3)}$$

Notice that we reformed the denominator into constant $\cdot\, x+$ constant, which does indeed look like $x+1$ in the original integrand, thus we have this relation:

$$\begin{cases} A+B=1 \\ -3A-2B=1 \end{cases}$$

This linear set of equation can be easily solved, which gives the solution of $A=-3$ and $B=4$. Thus we turned this integrand into a fraction of sum:

$$\int \frac{x+1}{x^2-5x+6}\mathrm{d}x=\int\left(-\frac{3}{(x-2)}+\frac{4}{(x-3)}\right)\mathrm{d}x$$

Then we can evaluate the integral easily:

$$\int\left(-\frac{3}{(x-2)}+\frac{4}{(x-3)}\right)\mathrm{d}x=-3\ln|x-2|+4\ln|x-3|+C$$

In general, partial fraction follow this process:

Consider a integrand where the denominator can be factored, first split the fraction into a sum of fractions:

$$\frac{Ax+B}{(x+C)(x+D)} = \frac{E}{(x+C)} + \frac{F}{(x+D)} = \frac{(E+F)x+(DE+CF)}{(x+C)(x+D)}$$

Where A, B, C, D, E, F are all constants, thus:

$$\begin{cases} E+F=A \\ DE+CF=B \end{cases}$$

Then solve for E and F to complete the fraction split.

Here is another example:

$$\int \frac{2x-1}{x^2-4x+3}\mathrm{d}x = \int \frac{x+1}{(x-3)(x-1)}\mathrm{d}x$$

Split the fraction into a sum of fraction:

$$\frac{A}{x-3} + \frac{B}{x-1} = \frac{A(x-1)+B(x-3)}{(x-3)(x-1)} = \frac{(A+B)x+(-A-3B)}{(x-3)(x-1)}$$

Solve for A and B:

$$\begin{cases} A+B=1 \\ -A-3B=1 \end{cases}$$

Thus $A=2$ and $B=-1$, and the integral turns to

$$\int \left(\frac{2}{x-3} - \frac{1}{x-1}\right)\mathrm{d}x = 2\ln|x-3| - \ln|x-1| + C$$