

# Representing Functions as Power Series

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## 1 Functions and Power Series

As one see in the Taylor Series section, a function (like trigonometric function) can be represented as a infinite series, for example

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

This hints us that perhaps all functions can be written as a infinite series.

### 1.1 Example Question

1. Represent  $f(x) = \int e^{-x^2} dx$  as a power series

This function cannot be obtained easily, since directly integrating  $e^{-x^2}$  requires techniques that is not taught in Calc BC, however, recall that

$$e^{-x^2} = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{2n}}{n!} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \dots$$

We can integrate this term by term, and obtain this function

$$f(x) = \int e^{-x^2} dx = \int \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{2n}}{n!} dx$$

Notice that  $(-1)^{n+1}$  and  $n!$  are all constant, therefore we only need to worry about the  $x$  term in the series.

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n!} \int x^{2n} dx = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{2n+1}}{(2n+1)n!}$$

Just like functions can be represented as a power series, a power series can also be turned into a function.

2. Represent  $f(x) = \sum_{n=0}^{\infty} 3x^n$ , where  $|x| < 1$

In the section on Taylor series, we derived that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

Therefore

$$f(x) = \sum_{n=0}^{\infty} 3x^n = \frac{3}{1-x}$$

## 2 Practice Problems

1. Represent  $\arctan x$  as a power series

- A.  $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$
- B.  $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$
- C.  $\sum_{n=0}^{\infty} \frac{x^{2n}}{2n!}$
- D.  $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n!}$

2. Represent  $f(x) = \frac{12}{3+x}$  as a power series

- A.  $\sum_{n=0}^{\infty} \frac{(-1)^{n+1} 4}{3^n} x^{2n}$
- B.  $\sum_{n=0}^{\infty} \frac{4}{3^n} x^{2n}$
- C.  $\sum_{n=0}^{\infty} \frac{4}{3^n} x^n$
- D.  $\sum_{n=0}^{\infty} \frac{(-1)^n 4}{3^n} x^n$

3. Represent  $f(x) = 2^x$  as a power series

- A.  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$
- B.  $\sum_{n=0}^{\infty} \frac{(\ln 2)^n}{n!} x^n$
- C.  $\sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!}$
- D.  $\sum_{n=0}^{\infty} \frac{(\ln 2)^{n+1}}{(n+1)!} x^{n+1}$

### 3 Solution

1. Notice that

$$\int \frac{1}{1+x^2} dx = \arctan x + C$$

Therefore we can apply term by term integration and find the power series of  $\arctan x$ , We developed that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

Therefore

$$\sum_{n=0}^{\infty} (-1)^{n+1} x^n = \frac{1}{1+x}$$

Changing  $x$  to  $x^2$ , the series turns to

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^{n+1} x^{2n}$$

Therefore the power series one desired is

$$\arctan x = \int \frac{1}{1+x^2} dx = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{2n+1}}{(2n+1)!}$$

The answer is A.

2. Divide both the denominator and numerator by 3

$$\frac{12}{3+x} = \frac{4}{1 - \left(-\frac{x}{3}\right)}$$

Which looks very similar to the Taylor series of  $\sum_{n=0}^{\infty} a_0 x^n = \frac{a_0}{1-x}$ , therefore we can construct the power series

$$\frac{12}{3+x} = \sum_{n=0}^{\infty} 4 \left(-\frac{x}{3}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n 4}{3^n} x^n$$

The answer is D.

3. First, recall the Taylor series of  $e^x$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Notice that

$$a^x = e^{x \ln a}$$

Therefore we can turn the Taylor series of  $a^x$  into something related to  $e^x$

$$a^x = \sum_{n=0}^{\infty} \frac{(\ln a)^n}{n!} x^n$$

When  $a = 2$ , this series turns to the Taylor Series of  $2^x$

$$2^x = \sum_{n=0}^{\infty} \frac{(\ln 2)^n}{n!} x^n$$

The answer is B.