Absolute Convergence and Conditional Convergence

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2025/07/17

Welcome to this guide on Absolute Convergence and Conditional Convergence, this article will guide you through the following:

What is Absolute Convergence?

What is Conditional Convergence?

1 Absolute/Conditional Convergence

Let $\sum a_n$ be an infinte series, if

1. $\sum |a_n|$ and $\sum a_n$ both converges, then the series is **absolutely convergent**.
2. $\sum a_n$ converges but $\sum |a_n|$ diverge, then the series is **conditionally convergent**

1.1 Example Questions

1. Determine if the series converge absolutely, conditionally, or diverge

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$$

Apply the Alternating series test to this series

(a) $\lim_{n\to\infty} \frac{1}{\sqrt[3]{n}} = 0$

(b) The function is decreasing

Which means by Alternating Series Test, this series converge.

By p-series test, $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$ diverge, which means the original series **converge conditionally**.

2. Determine if the series converge absolutely, conditionally or diverge

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$$

Note that this series is **not** an alternating series. By direct comparison test, we have:

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Meaning the original series **converge absolutely**

2 Practice Problems

Determine if the following is absolute convergent or conditional convergent

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3^{n-1}}$

A. Absolutely convergent

B. Conditionally convergent

C. Divergent

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3} \frac{1}{2^n}$

A. Absolutely convergent

B. Conditionally convergent

C. Divergent

3. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{\ln(n+1)}$

A. Absolutely convergent

B. Conditionally convergent

C. Divergent

4. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^{n^2}}{n!}$

A. Absolutely convergent

B. Conditionally convergent

C. Divergent

5. $\displaystyle\sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{n \ln n}$

A. Absolutely convergent

B. Conditionally convergent

C. Divergent

3 Solution

1. First apply the alternating series test to test the convergence of the original series.

   1. $$\lim_{n\to\infty} \frac{n}{3^{n-1}} = \lim_{n\to\infty} \frac{1}{\ln 3 \cdot 3^{n-1}} = 0$$

      Here we applied L'Hopital's Rule

   2. Let $f(n) = \frac{n}{3^{n-1}}$, find the first order derivative of this function.

      $$f'(n) = \frac{1}{(3^{n-1})^2}\left(3^{n-1} - n\ln 3 \cdot 3^{n-1}\right) = \frac{1 - n\ln 3}{3^{n-1}}$$

      $f'(n) < 0$ when $n > \frac{1}{\ln 3}$, which means $a_{n+1} < a_n$

      Both condition satisfy, the series is convergent.

      Then check if $|\sum a_n|$ converge, apply the ratio test for convergence.

      $$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{n+1}{3^n}\frac{3^{n-1}}{n}\right| = \lim_{n\to\infty} \frac{1}{3}\frac{n+1}{n} = \frac{1}{3} < 1$$

      Which means this series converges as well, the series is absolutely convergent, the answer is A.

2. First apply the alternating series test to test the convergence of the original series.

   1. $$\lim_{n\to\infty} \frac{1}{2^n} = 0$$

   2. Let $f(n) = \frac{1}{2^n} = 2^{-n}$, find the first order derivative of this function.

      $$f'(n) = -\ln 2 \cdot 2^{-n}$$

      $f'(n) < 0$ for all real numbers, which means $a_{n+1} < a_n$

      Both condition satisfy, the series is convergent.

      Then check if $|\sum a_n|$ converge, apply the ratio test for convergence.

      $$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2^n}{2^{n+1}}\right| = \frac{1}{2} < 1$$

      Which means this series converges as well, the series is absolutely convergent, the answer is A.

3. First apply the alternating series test to test the convergence of the original series.

   1. $$\lim_{n\to\infty} \frac{1}{\ln(n+1)} = 0$$

   2. Let $f(n) = \frac{1}{\ln(n+1)} = (\ln(n+1))^{-1}$, find the first order derivative of this function.

      $$f'(n) = -\frac{(\ln(n+1))^{-2}}{n+1}$$

$f'(n) < 0$ for all real numbers, which means $a_{n+1} < a_n$

Both condition satisfy, the series is convergent.

Then check if $|\sum a_n|$ converge, here the ratio test cannot draw any conclusion. We turn our attention to comparison test.

Let $a_n = \frac{1}{n}$ and $b_n = \frac{1}{\ln n}$. When $n > 1$, $n > \ln n$, meaning $\frac{1}{\ln n} > \frac{1}{n}$, note a_n is a divergent harmonic series, meaning b_n is also divergent.

This means this series diverge, the series is conditionally convergent, the answer is B.

4. First apply the alternating series test to test the convergence of the original series.

$$\frac{2^{n^2}}{n!} = \frac{2^n \cdot 2^n \cdot 2^n ... \cdot 2^n}{1 \cdot 2 \cdot 3 ... \cdot n}$$

Where there are n 2^n multiplying each other, also $2^n > n$, thus each 2^n is greater that each of the natrual number in the denominator, as n increase, the fraction will only become larger, meaning the limit is not 0.

This means this series diverge, the answer is C

5. First apply the alternating series test to test the convergence of the original series.

   1.
$$\lim_{n\to\infty} \frac{1}{n \ln n} = 0$$

   2. Let $f(n) = \frac{1}{n \ln n} = (n \ln n)^{-1}$, find the first order derivative of this function.

$$f'(n) = -\frac{\ln n + 1}{(n \ln n)^2}$$

$f'(n) < 0$ for all real numbers, which means $a_{n+1} < a_n$

Both condition satisfy, the series is convergent.

Then check if $|\sum a_n|$ converge, applying the integral test would be the most convenient

$$\int_2^\infty \frac{1}{x \ln x} dx$$

Let $u = \ln x$, $du = \frac{1}{x} dx$, the upper bound turns into $\ln 2$ and ∞, thus

$$\int_2^\infty \frac{1}{x \ln x} dx = \int_{\ln 2}^\infty \frac{1}{u} du = \lim_{b\to\infty} (\ln b - \ln(\ln 2)) = \infty$$

This means this series diverge, the series is conditionally convergent, the answer is B.