Integration using Subsitution

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1 Indefinite integral

U-subsutition is the first techniques we will learn, let's first look at an example question:

Evaluate

$$\int 2xe^{-x^2}\mathrm{d}x$$

To solve this integral: Let $u = -x^2$, then $\mathrm{d}u = -2x\mathrm{d}x$, let's first substitute u back to the integral:

$$\int 2xe^{u}\mathrm{d}x$$

At first glance it seems that we make this integral more complicated as we introduce 2 variables in, but if we examine the integral carefully, we notice that we already have $2x\mathrm{d}x$ present in the integrand, we just need a minus sign. So if we add the minus sign like this, we can substitute $\mathrm{d}u$ in and evaluate the integral:

$$\int -e^{u}(-2x\mathrm{d}x) = \int -e^{u}\mathrm{d}u = -e^{u} + C$$

All that is left is to do is substitute $u = -x^2$ back and we get the final result:

$$\int 2xe^{-x^2}\mathrm{d}x = -e^{-x^2} + C$$

If we differentiate our results, we will arrive at the integrand, meaning that our process is correct.

This is essentially U-substitution, our thought process can be summarized as follow:

1. Let something equals to u
2. Calculate $\mathrm{d}u = \text{some expression} \cdot \mathrm{d}x$
3. Manipulate the integral so we found the some expression $\cdot\, \mathrm{d}x$
4. Substitute $\mathrm{d}u$ and evaluate the integral
5. Replace u with x and finish the integral

Let's look at another example:

$$\int 2x \sin(x^2) \mathrm{d}x$$

We can complete this integral with u-substitution, let $u = x^2$, $\mathrm{d}u = 2x\mathrm{d}x$, thus $x = \sqrt{u}$ and $\mathrm{d}x = \frac{1}{2x}\mathrm{d}u$, the integral turns to:

$$\int 2\sqrt{u} \sin(u) \frac{1}{2\sqrt{u}} \mathrm{d}u = -\cos(u) + C$$

Substitute $u = x^2$ back to the integral:

$$\int 2x \sin(x^2) \mathrm{d}x = -\cos(x^2) + C$$

Another example would be:

$$\int \frac{x}{x^2+8} \mathrm{d}x$$

This looks tricky, but we do see a x^2 term and x term, and the derivative of x^2 is $2x$. Let $u = x^2 + 8$, $\mathrm{d}u = 2x\mathrm{d}x$, thus $x = \sqrt{u-8}$, $\mathrm{d}x = 1/2x\mathrm{d}u$:

$$\int \frac{\sqrt{u-8}}{u} \frac{1}{2\sqrt{u-8}} \mathrm{d}u \int \frac{1}{2u} \mathrm{d}u = \frac{1}{2} \ln|u| + C = \frac{1}{2} \ln\left|x^2+8\right| + C$$

2 Definite Integral

Definite integral can be treated the same as indefinite integral, however we need to account for the upper and lower bound.

Let's take a look at an example:

$$\int_0^2 \frac{x}{x^2+8} \mathrm{d}x$$

In our previous example, we let $u(x) = x^2 + 8$, we introduce a new function to simplify the integrand, so we also need to change the upper and lower bound of integration to match the newly created integrand.

Since $u(0) = 8$ and $u(2) = 12$, we substitute this into the integration bound:

$$\int_0^2 \frac{x}{x^2+8} \mathrm{d}x = \int_8^{12} \frac{1}{2u} \mathrm{d}u = \frac{1}{2}(\ln 12 - \ln 8) = \ln \frac{\sqrt{6}}{2}$$

Thought process for definite integral would be:

1. Let something equals to u
2. Calculate $\mathrm{d}u = \text{some expression} \cdot \mathrm{d}x$
3. Replace the upper and lower bound with $u(a)$ and $u(b)$, where a and b are old integration bounds
4. Manipulate the integral so we found the some expression $\cdot\, \mathrm{d}x$
5. Substitute $\mathrm{d}u$ and evaluate the integral