

Integration using substitution

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1 Indefinite integral

U-substitution is the first techniques we will learn, let's first look at an example question:

Evaluate

$$\int 2xe^{-x^2} dx$$

To solve this integral: Let $u = -x^2$, then $du = -2xdx$, let's first substitute u back to the integral:

$$\int 2xe^u dx$$

At first glance it seems that we make this integral more complicated as we introduce 2 variables in, but if we examine the integral carefully, we notice that we already have $2xdx$ present in the integrand, we just need a minus sign. So if we add the minus sign like this, we can substitute du in and evaluate the integral:

$$\int -e^u(-2xdx) = \int -e^u du = -e^u + C$$

All that is left is to do is substitute $u = -x^2$ back and we get the final result:

$$\int 2xe^{-x^2} dx = -e^{-x^2} + C$$

If we differentiate our results, we will arrive at the integrand, meaning that our process is correct.

This is essentially U-substitution, our thought process can be summarized as follow:

1. Let something equals to u
2. Calculate $du = \text{some expression} \cdot dx$
3. Manipulate the integral so we found the some expression $\cdot dx$
4. Substitute du and evaluate the integral
5. Replace u with x and finish the integral

Let's look at another example:

$$\int 2x \sin(x^2) dx$$

We can complete this integral with u-substitution, let $u = x^2$, $du = 2x dx$, the integral turns to:

$$\int \sin(u) du = -\cos(u) + C$$

Substitute $u = x^2$ back to the integral:

$$\int 2x \sin(x^2) dx = -\cos(x^2) + C$$

Another example would be:

$$\int \frac{x}{x^2 + 8} dx$$

This looks tricky, but we do see a x^2 term and x term, and the derivative of x^2 is $2x$. Let $u = x^2 + 8$, $du = 2x dx$:

$$\int \frac{1}{2u} du = \frac{1}{2} \ln |u| + C = \frac{1}{2} \ln |x^2 + 8| + C$$

2 Definite Integral

Definite integral can be treated the same as indefinite integral, however we need to account for the upper and lower bound.

Let's take a look at an example:

$$\int_0^2 \frac{x}{x^2 + 8} dx$$

In our previous example, we let $u(x) = x^2 + 8$, we introduce a new function to simplify the integrand, so we also need to change the upper and lower bound of integration to match the newly created integrand.

Since $u(0) = 8$ and $u(2) = 12$, we substitute this into the integration bound:

$$\int_0^2 \frac{x}{x^2 + 8} dx = \int_8^{12} \frac{1}{2u} du = \frac{1}{2} (\ln 12 - \ln 8) = \ln \frac{\sqrt{6}}{2}$$

Thought process for definite integral would be:

1. Let something equals to u
2. Calculate $du = \text{some expression} \cdot dx$
3. Replace the upper and lower bound with $u(a)$ and $u(b)$, where a and b are old integration bounds
4. Manipulate the integral so we found the some expression $\cdot dx$
5. Substitute du and evaluate the integral