Alternating Series Test for Convergence

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Previously, we are all working with sequence with only positive number, now we will deal with sequence with both positive and negative number.

1 Alternating Series

We define alternating series as a series where the sign of terms switch from positive to negative, or negative to positive. ex.

$$\sum_{n=0}^{\infty}(-1)^n a_n \text{ and } \sum_{n=1}^{\infty}\cos(n\pi)a_n$$

1.1 Convergence Test

Let $a_n > 0$, the alternating series:

$$\sum_{n=1}^{\infty}(-1)^n a_n \text{ and } \sum_{n=1}^{\infty}(-1)^{n+1}a_n$$

converge if the following two requirements are met:

1. $\lim_{n\to\infty} a_n = 0$
2. $a_{n+1} \le a_n$ (a_n is decreasing)

Do not use this test to test for divergence

1.2 Example Questions

1. Determine if the series converge or diverge:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$$

Check the two requirements:

(a) $\lim_{n\to\infty} \frac{1}{n} = 0$

(b) $a_{n+1} \le a_n$ (this function is decreasing in $[1,\infty)$)

Both requirements fit, meaning that this series **converge**, although this series look like the harmoic series.

2 Practice Question

Determine if the following series converge using the Alternating Series Convergence Test:

1.

$$\sum_{n=1}^{\infty}(-1)^n\frac{n^2}{n^2+5}$$

A. Converge

B. Cannot be determined by Alternating Series Convergence Test

2.

$$\sum_{n=2}^{\infty}\frac{\cos(n\pi)}{\sqrt{n}}$$

A. Converge

B. Cannot be determined by Alternating Series Convergence Test

3.

$$\sum_{n=0}^{\infty}(-1)^{n-1}\frac{n}{3^{n-1}}$$

A. Converge

B. Cannot be determined by Alternating Series Convergence Test

4.

$$\sum_{n=0}^{\infty}(-1)^{n-1}\frac{1}{\ln(n-1)}$$

A. Converge

B. Cannot be determined by Alternating Series Convergence Test

5.

$$\sum_{n=0}^{\infty}(-1)^{n-1}\frac{n}{\ln(n-1)}$$

A. Converge

B. Cannot be determined by Alternating Series Convergence Test

3 Solutions

1.

$$\lim_{n\to\infty}\frac{n^2}{n^2+5}=\lim_{n\to\infty}\frac{2n}{2n}=1$$

Which means one cannot determine the convergence of this series. Although this series diverge because $\lim\limits_{n\to\infty}(-1)^n\dfrac{n^2}{n^2+5}$ DNE

The answer is B.

2. First, recognize that $\cos(n\pi)=(-1)^n$, thus the original series become

$$\sum_{n=2}^{\infty}(-1)^n\frac{1}{\sqrt{n}}$$

Apply the test to this series:

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

and $\dfrac{1}{\sqrt{n}}$ is decreasing, thus the series is convergent, the answer is A.

3. Apply the test to this limit:

$$\lim_{n\to\infty} \frac{n}{3^{n-1}} = \lim_{n\to\infty} \frac{1}{\ln 3 \cdot 3^{n-1}} = 0$$

and the series is decreasing, meaning the series converge, the answer is A.

4. Apply the test to the limit:

$$\lim_{n\to\infty} \frac{1}{\ln(n-1)} = 0$$

and the series is decreasing, meaning that the series converge, the answer is A.

5. Apply the test to the limit:

$$\lim_{n\to\infty} \frac{n}{\ln(n-1)} = \lim_{n\to\infty} \frac{1}{\frac{1}{n-1}} = \lim_{n\to\infty} n-1 = \infty$$

Which means this test is not applicable to this series, the convergece of the series cannot be determined by Alternating Series Convergence Test