Surprise

Polaris

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1 Stellar Interiors

The density profile of a hypothetical star given by:

$$\rho(r) = \rho_0 \left(1 - \alpha \left(\frac{r}{R}\right)^2\right)$$

where r is the distance of a random point from the center of the star, R is the radius of the star, and α is a constant.

1. What region of this star has the density ρ_0
2. What is the mean density of the star
3. Find $m(r)$ and total mass of the star
4. Find the change of pressure P with respect to r, $\dfrac{dP}{dr}$, in terms of G, M, R and r.
5. Give a possible range of α and explain your reasoning

2 Solutions

1. When $\rho(r) = \rho_0$, it is not hard to notice that $r = 0$ when this happens. Thus, the answer is the core of the star.

2. By the definition of average, one can write:

$$\begin{aligned}
\overline{\rho} &= \frac{1}{R-0}\int_0^R \rho(r)dr \\
&= \frac{1}{R}\int_0^R \rho_0\left(1-\alpha\left(\frac{r}{R}\right)^2\right)dr \\
&= \frac{1}{R}\int_0^R \left(\rho_0 - \frac{1}{R}\rho_0\alpha\left(\frac{r}{R}\right)^2\right)dr \\
&= \frac{1}{R}\int_0^R \rho_0 dr - \frac{1}{R}\int_0^R \rho_0\alpha\left(\frac{r}{R}\right)^2 dr \\
&= \frac{1}{R}\rho_0 R - \frac{1}{R}\frac{\rho_0\alpha}{R^2}\int_0^R r^2 dr \\
&= \rho_0 - \frac{\rho_0\alpha}{R^3}\frac{1}{3}R^3 \\
&= \frac{3-\alpha}{3}\rho_0
\end{aligned}$$

3. $m(r)$ can be given by integrating density with a volume element

$$\begin{aligned}
m(r) &= \int_0^r \rho(r)dV \\
&= \int_0^r \rho_0\left(1-\alpha\left(\frac{r}{R}\right)^2\right)4\pi r^2 dr \\
&= 4\pi\rho_0\int_0^r r^2\left(1-\alpha\left(\frac{r}{R}\right)^2\right)dr \\
&= 4\pi\rho_0\int_0^r r^2 dr - \frac{4\pi\alpha\rho_0}{R^2}\int_0^r r^4 dr \\
&= \frac{4}{3}\pi\rho_0 r^3 - \frac{4\pi\alpha\rho_0}{5R^2}r^5
\end{aligned}$$

Total mass is thus given by $M = m(R)$:

$$\begin{aligned}
M = m(R) &= \frac{4}{3}\pi\rho_0 R^3 - \frac{4\pi\alpha\rho_0}{5R^2}R^5 \\
&= \frac{4}{3}\pi R^3\rho_0 - \frac{4}{5}\alpha\pi R^3\rho_0 \\
&= 4\pi R^3\rho_0\left(\frac{1}{3} - \frac{1}{5}\alpha\right) \\
&= \frac{4}{15}\pi R^3\rho_0(5-3\alpha)
\end{aligned}$$

4. By hydrostatic equalibrium equation:

$$\frac{dP}{dr} = -\frac{Gm(r)\rho(r)}{r^2}$$

Subsitute both $\rho(r)$ and $m(r)$ in, one get:

$$\begin{aligned}
\frac{dP}{dr} &= -\frac{G}{r^2}\left(\frac{4}{3}\pi\rho_0 r^3 - \frac{4\pi\alpha\rho_0}{5R^2}r^5\right)\rho_0\left(1-\alpha\left(\frac{r}{R}\right)^2\right) \\
&= -4\pi G\rho_0^2 r\left(\frac{1}{3}-\frac{\alpha r^2}{5R^2}\right)\left(1-\alpha\left(\frac{r}{R}\right)^2\right) \\
&= -4\pi G\rho_0^2 r\left(\frac{5R^2}{15R^2}-\frac{3\alpha r^2}{15R^2}\right)\left(\frac{R^2-\alpha r^2}{R^2}\right) \\
&= -4\pi G\rho_0^2 r\left(\frac{5R^2-3\alpha r^2}{15R^2}\right)\left(\frac{R^2-\alpha r^2}{R^2}\right) \\
&= -4\pi G\rho_0^2 r\frac{5R^4-8\alpha R^2 r^2+3\alpha^2 r^4}{15R^4} \\
&= -\frac{4\pi G\rho_0^2 r}{15R^4}5R^4 + \frac{4\pi G\rho_0^2 r}{15R^4}8\alpha R^2 r^2 - \frac{4\pi G\rho_0^2 r}{15R^4}3\alpha^2 r^4 \\
&= -\frac{4\pi G\rho_0^2 r}{3} + \frac{32\pi G\rho_0^2 r^3}{15R^2}\alpha - \frac{4\pi G\rho_0^2 r^5}{5R^4}\alpha^2
\end{aligned}$$