n-th Term Test

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This article will guide you through how to preform a n-th Term Test to determine a convergence of a series.

1 n-th Term Test

$$\text{For an infinte series} \sum_{n=1}^{\infty} a_n \text{ , if } \lim_{x\to\infty} a_n \neq 0 \text{ or the limit does not exist, then} \sum_{n=1}^{\infty} a_n \text{ diverge}$$

This theorem is the *n-th term test*, it is important to point out that if the limit equals to 0, this test *cannot draw any conclusion.*

$$\text{If } \lim_{x\to\infty} a_n = 0, \text{ then} \sum_{n=1}^{\infty} a_n \text{ don't have to converge}$$

We will not go over the proof of this test, but we can intuitively understand this test, for a series to diverge, the n-th term of a sequence needs to be larger than 0, otherwise there is a chance of this series converging, hence we require that the limit of n-th term is not equal to 0.

2 Example Questions

State if the following series diverge or converge

1. $$\sum_{n=1}^{\infty} \left(\frac{n}{n+100}\right)$$

   For this series, if $\lim_{n\to\infty} \frac{n}{n+100} \neq 0$, the series diverges:

   $$\lim_{n\to\infty} \frac{n}{n+100} = \lim_{n\to\infty} \frac{1}{1} = 1$$

   Which means the original series diverge.

2. $$\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^2$$

   Examine this limit:

   $$\lim_{x\to\infty} \left(\frac{1}{n}\right)^2 = 0$$

   By the n-th term test, we have no way to prove that the series converge or diverge through this method(though the series converge, it is the Basel Problem)

3 Practice Problem

State if the following series diverge or converge.

1. $$\sum_{n=1}^{\infty} \frac{1}{n}$$

2. $$\sum_{n=1}^{\infty} \left(\frac{e^n}{\ln n} \right)$$

3. $$\sum_{n=1}^{\infty} \left(\sin \frac{1}{n} \right)$$

4. $$\sum_{n=1}^{\infty} \left(n \sin \left(\frac{1}{n} \right) \right)$$

5. $$\sum_{n=1}^{\infty} n \tan n$$

4 Answer Key

1. Apply the n-th term test for this series:
$$\lim_{n\to\infty} \frac{1}{n} = 0$$
Which means this test is inconclusive, we cannot draw any conclusion from this test.

2. Apply the n-th term test for this series
$$\begin{aligned} \lim_{x\to\infty} \frac{e^x}{\ln x} &= \lim_{n\to\infty} \frac{e^x}{1/x} \\ &= \lim_{x\to\infty} xe^x \\ &= \infty \end{aligned}$$
Which means this series diverge.

3. Apply the n-th term test for this series:
$$\lim_{x\to\infty} \sin \frac{1}{x}$$
We know that
$$-\frac{1}{x} \leq \sin \frac{1}{x} \leq \frac{1}{x}$$
By squeeze theorem, we have
$$\begin{aligned} \lim_{x\to\infty} -\frac{1}{x} \leq \lim_{x\to\infty} \sin \frac{1}{x} \leq \lim_{x\to\infty} \frac{1}{x} \\ 0 \leq \lim_{x\to\infty} \sin \frac{1}{x} \leq 0 \end{aligned}$$

Which gives

$$\lim_{x\to\infty} \sin\frac{1}{x} = 0$$

This means this test is inconclusive for this series.

4. Apply the n-th term test for this series:

$$\begin{aligned}\lim_{x\to\infty} x\sin\frac{1}{x} &= \frac{\sin\frac{1}{x}}{\frac{1}{x}} \\ &= 1\end{aligned}$$

Which means this test is inconclusive for this series.

5. Apply the n-th term test for this series:

$$\lim_{x\to\infty} x\tan x = \text{DNE}$$

Which means this series diverge.