By Planck Law, the energy the star emitted at the specific waveband per unit area is

$$F = \int_{5\cdot 10^{14}}^{6\cdot 10^{14}} \left(\frac{2h\nu^3}{c^2} \frac{1}{\exp\left\{\frac{h\nu}{k_B T}\right\} - 1} - 1 \right) d\nu = 7.99 \cdot 10^6 \text{ W/m}^2$$

The total energy that comes out of the star in the selected waveband is

$$L = 4\pi R^2 F = 4.86 \cdot 10^{25} \text{ W}$$

The photon flux arriving at the telescope is therefore

$$N = \frac{AQLt}{4h\pi\bar{\nu}d^2}$$

Where $\bar{\nu}$ is the average wavelength of the photon.

The flux due to the background light is

$$F_b = 6 \cdot 10^{-29} \text{W/m}^2 \cdot \text{Hz} \cdot \text{sr} \cdot 1 \cdot 10^{14} \text{Hz} \cdot \left(\frac{1}{206265} \right)^2 \text{sr} = 1.41 \cdot 10^{-25} \text{W/m}^2$$

The incomming photon over the integration period is therefore

$$N_b = \frac{F_b}{h\bar{\nu}} \pi r^2 Qt = 0.131 \text{ photon}$$

By the definition of SNR

$$\text{SNR} = \frac{N}{\sqrt{N_b}} = 10$$

Therefore, distance is related to SNR as such

$$\frac{QL}{4h\pi\bar{\nu}d^2} = 10\sqrt{N_b}$$

Or

$$d = \sqrt{\frac{AQLt}{40\sqrt{N_b}h\pi\bar{\nu}}} = 9.98 \cdot 10^{23} \text{ m} = 32 \text{ Mpc}$$

If the photon from the star is also accounted for the calulation of SNR, one have

$$\text{SNR} = \frac{N}{\sqrt{N + N_b}} = 10$$

This requires

$$N^2 - 100N - N_b = 0$$

Taking the positive solution, $N = 100.00131$ and the distance is

$$d' = \sqrt{\frac{AQLt}{40\sqrt{N_b}h\pi\bar{\nu}}} = 1.898 \cdot 10^{23} \text{ m} = 6.15 \text{ Mpc}$$

The magnitude and flux is related as

$$m = -2.5 \lg S + C$$

By introducing noise, the magnitude will deviate

$$m \pm \delta m = -2.5 \lg(S \pm N) + C$$

Therefore, the deviation in magnitude is

$$\delta m = \pm 2.5 \lg \left(1 + \frac{N}{S}\right) = \pm 2.5 \lg \left(1 + \frac{1}{\sqrt{S}}\right)$$

Therefore the signal needed to register a deviation of 0.02 will be

$$S = \left(\frac{1}{10^{0.4\delta m} - 1}\right)^2 = 2893 \text{ photons/s}$$

For a 15 magnitude star, the telescope can capture $S_{15} = F \cdot A = 7854$ photons/s. The time needed is therefore

$$t = \frac{S}{S_{15}} = 0.37 \text{ s}$$

By hydrostatic equilibrium:

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

Multiply $dV = 4\pi r^2 dr$ on both side

$$4\pi r^2 dP = -\frac{GM(r)\rho(r)}{r^2} 4\pi r^2 dr$$

Rearrange and integrate both side

$$\int \frac{4}{3}\pi r^3 dP = \int -\frac{GM(r)dm}{3r}$$

The LHS can be solved by integration by parts and the RHS is the total gravitaional potential energy.

$$VP - \int PdV = -\frac{1}{3}\Omega$$

VP should equal to 0 based on the boundary condition of zero volume in center and zero pressure at outer radius. Therefore

$$\int PdV = -\frac{1}{3}\Omega$$

The pressure of a gas is related to the average speed of particles

$$P = \frac{1}{3}\frac{N}{V}m\bar{v}^2$$

The internal energy density for ideal gas is the sum of all kinetic energy of particles

$$u = \frac{1}{2}\frac{N}{V}m\bar{v}^2$$

Compare the two, we arrive at

$$\frac{P}{u} = \frac{2}{3}$$

Rearrange and integrate both side with respect to volume

$$U = \frac{3}{2}\int PdV$$

For photon gas, consider the pressure integral

$$P = \frac{1}{3}\int_0^\infty n_p v p dp$$

For photon $v = c$ and $E = pc$, therefore the integral can be turned into

$$P = \frac{1}{3}\int_0^\infty n_p E dp$$

Internal energy density is

$$u = \int_0^\infty n_p E dp$$

Therefore, for photon gas, the internal energy is

$$U = 3\int PdV$$

Since the photon pressure is half of gas pressure, the gas pressure is 2/3 of total pressure and the photon pressure is 1/3 of total pressure.

$$U_{tot} = U_{\text{gas}} + U_{\text{photon}} = \frac{3}{2}\int \frac{2}{3}PdV + 3\frac{1}{3}\int PdV = 2\int PdV = -\frac{2}{3}\Omega$$

The total gravitaional potential energy is given by integration

$$\Omega = -\int_0^R \frac{GM(r)}{r}dm = \int_0^R \frac{GM(r)\rho(r)}{r}4\pi r^2 dr$$

The mass as a function of radius is

$$M(r) = \int_0^r 4\pi r^2 \rho(r)dr = 4\pi\rho_0 \int_0^r \left(r^2 - \frac{r^{\alpha+2}}{R^\alpha}\right)dr = 4\pi\rho_0\left(\frac{1}{3}r^3 - \frac{1}{\alpha+3}\frac{r^{\alpha+3}}{R^{\alpha+3}}\right)$$

The gravitaional energy integral turns into

$$\Omega = -16G\pi^2\rho_0^2 \int_0^R \left(r - \frac{r^{\alpha+1}}{R^\alpha}\right)\left(\frac{1}{3}r^3 - \frac{1}{\alpha+3}\frac{r^{\alpha+3}}{R^{\alpha+3}}\right)dr$$

This integral gives

$$\Omega = -16G\pi^2\rho_0^2 \frac{\alpha^2(2\alpha+11)}{15(\alpha+5)(\alpha+3)(2\alpha+5)}$$

Finally, solve ρ_0 in terms of M, notice that $M(R) = M$, this gives

$$\rho_0 = \frac{3M(\alpha+3)}{4\pi R^3\alpha}$$

The total gravitational potential energy of the star is

$$\Omega = -16G\pi^2 \frac{9M^2(\alpha+3)^2}{16\pi^2\alpha^2R^6}\frac{\alpha^2(2\alpha+11)R^5}{15(\alpha+5)(\alpha+3)(2\alpha+5)}$$

After simplification

$$\Omega = -\frac{3}{5}\frac{GM^2}{R}\frac{(2\alpha+11)(\alpha+3)}{(\alpha+5)(2\alpha+5)}$$

The internal energy of the star is therefore

$$U = \frac{2}{3}\Omega = \frac{2}{5}\frac{GM^2}{R}\frac{(2\alpha+11)(\alpha+3)}{(\alpha+5)(2\alpha+5)}$$