Improper Integral

Polaris

2024/12/14

Welcome to this guide on Evaluating Improper Integral of the FiveHive Calculus BC course. This article will guide you through the following:

1. What is an improper integral?
2. How to define the convergence and divergence of the integral?
3. How to calculate a convergent improper integral?

1 Definition

We define improper integral as:

1. In the integration bound $[a, b]$, there is a infinite discontinuity
2. The integral has a upper/lower bound of ∞ or $-\infty$

2 Calculate by Definition

2.1 Infinite Discontinuity in Integration Bound

Let's first take a look at how to calculate them:

$$\int_0^1 \frac{1}{x} \mathrm{d}x \tag{1}$$

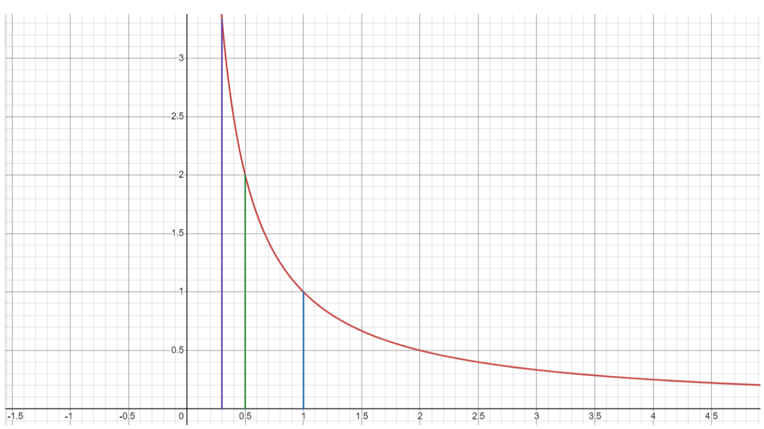

Figure 1: Graph of $\frac{1}{x}$

Since we know that $f(x) = \frac{1}{x}$ is not defined at $x = 0$, we need to use some clever trick to calculate this integral.

Let's rewrite the lower bound as a:

$$\int_a^1 \frac{1}{x} \mathrm{d}x$$

where $a > 0$.

It is obvious that this integral is not equal to the original integral we want to find, since the lower bound is not equal, but if we take the limit as $a \to 0$, this integral will approach the integral we want to find, then we can apply the fundamental theorem of calculus and find the result:

$$\begin{aligned}\lim_{a \to 0^-} \int_a^1 \frac{1}{x} \mathrm{d}x &= \lim_{a \to 0^-} \ln a - \ln 1 \\ &= \infty - 0 \\ &= \infty\end{aligned}$$

Hence we can see that this integral diverges (the limit goes to infinity).

Let's take a look at where the integral converges:

$$\int_0^1 \frac{1}{\sqrt{x}} \mathrm{d}x$$

We can apply the same trick we used:

$$\begin{aligned}\lim_{a \to 0^-} \int_a^1 \frac{1}{\sqrt{x}} \mathrm{d}x &= \lim_{a \to 0^-} -2\sqrt{a} + 2\sqrt{1} \\ &= 0 + 2 \\ &= 2\end{aligned}$$

Which means this integral converges (the limit has a finite value)

2.2 Infinite Integration Bound

We will use some example to illustrate the idea of integrating on a infinte bound:

$$\int_1^\infty \frac{1}{x} \mathrm{d}x$$

Let's first replace the upper bound to b, and note when b become very large, the integral will get very close to the original integral:

$$\begin{aligned}\int_1^\infty \frac{1}{x} \mathrm{d}x &= \lim_{b \to \infty} \int_1^b \frac{1}{x} \mathrm{d}x \\ &= \lim_{b \to \infty} \ln b - \ln 1 \\ &= \infty - 0 \\ &= \infty\end{aligned}$$

Which means this integral diverges, we can see that there is no difference between this improper integral and the one listed above, it is both taking a limit, the same also apply for negative infinity, the lower bound will approach negative infinity.

Let's see a integral that converges:

$$\int_1^\infty \frac{1}{x^2} \mathrm{d}x$$

$$\begin{aligned}\int_1^\infty \frac{1}{x^2} \mathrm{d}x &= \lim_{b \to \infty} \int_0^b \frac{1}{x^2} \mathrm{d}x \\ &= \lim_{b \to \infty} -\frac{1}{b} + 1 \\ &= 1\end{aligned}$$

2.3 Both Infinite Discontinuity and Infinite Integration Bound

There are other improper integral that can be seen as a combination of both case 1 and case 2, to determine their convergence, we need to consider things separately. Let $\int_a^b f(x)\mathrm{d}x$ be an impropoer integral, the integral converges *only if* $\int_a^c f(x)\mathrm{d}x$ and $\int_c^b f(x)\mathrm{d}x$ both converges.

This gives a method to calculate some other improper integrals:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\mathrm{d}x$$

We can rewrite the integral as:

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{1+x^2}\mathrm{d}x &= \int_{-\infty}^{0} \frac{1}{1+x^2}\mathrm{d}x + \int_{0}^{\infty} \frac{1}{1+x^2}\mathrm{d}x \\
&= \lim_{b\to-\infty} \int_b^0 \frac{1}{1+x^2}\mathrm{d}x + \lim_{a\to\infty} \int_0^a \frac{1}{1+x^2}\mathrm{d}x \\
&= \arctan 0 - \lim_{b\to-\infty} \arctan b + \arctan 0 - \lim_{a\to\infty} \arctan a \\
&= -\frac{-\pi}{2} + \frac{\pi}{2} \\
&= \pi
\end{aligned}$$

Which means this integral converges to π.

$$\int_0^{\infty} \frac{1}{x^2}\mathrm{d}x$$

It is convenient to split this integral into two parts and analyze them separately:

$$\begin{aligned}
\int_0^{\infty} \frac{1}{x^2}\mathrm{d}x &= \int_0^1 \frac{1}{x^2}\mathrm{d}x + \int_1^{\infty} \frac{1}{x^2}\mathrm{d}x \\
&= \lim_{b\to0^+} \int_b^1 \frac{1}{x^2}\mathrm{d}x + \lim_{a\to\infty} \int_1^a \frac{1}{x^2}\mathrm{d}x \\
&= -\frac{1}{1} + \lim_{b\to0^+} \frac{1}{b} + 1 \\
&= \infty
\end{aligned}$$

There is a quicker way to do this:

$$\begin{aligned}
\int_0^1 \frac{1}{x^2}\mathrm{d}x &= -\frac{1}{1} + \lim_{b\to0^+} \frac{1}{b} \\
&= -1 + \infty \\
&= \infty
\end{aligned}$$

This integral diverges, which by the theorem introduce earlier, the entire integral diverges.

It is *extremely important* to check the infinite discontinuity within the integration bound, consider the following example:

$$\int_{-1}^{1} \frac{1}{x^2}dx$$

If one ignore that at $x = 0$, the function has an infinte discontinuity and directly apply the fundamental theorem of calculus, one will get the incorrect result of -2:

$$\begin{aligned}\int_{-1}^{1} \frac{1}{x^2} dx &= -\frac{1}{x}\Big|_{-1}^{1} \\ &= -1 - 1 \\ &= -2\end{aligned}$$

The correct approach is to recognize there is an infinite discontinuity at $x = 0$, thus

$$\begin{aligned}\int_{-1}^{1} \frac{1}{x^2} dx &= \int_{-1}^{0} \frac{1}{x^2} dx + \int_{0}^{1} \frac{1}{x^2} dx \\ &= \lim_{a \to 0^-} \int_{-1}^{a} \frac{1}{x^2} dx + \lim_{b \to 0^+} \int_{b}^{1} \frac{1}{x^2} dx \\ &= \lim_{a \to 0^-} -\frac{1}{a} - 1 + (-1) - \lim_{b \to 0^+} -\frac{1}{b} \\ &= \infty\end{aligned}$$

Do not assume that 2 infinities can cancel each other, in other words $\infty - \infty \neq 0$, this expression is undefined.

3 Practice

1. $\int_1^\infty \frac{1}{x^4} dx$

   A. This integral diverges

   B. 1

   C. $\frac{1}{3}$

   D. $\frac{1}{9}$

2. $\int_0^1 \ln x dx$

   A. This integral diverges

   B. $-e$

   C. 1

   D. -1

3. $\int_0^2 \frac{1}{(1-x)^2} dx$

   A. This integral diverges

   B. -2

   C. 2

   D. 1

4. $\int_0^1 \frac{1}{\sqrt{x}}$

   A. This integral diverges

   B. $\frac{1}{2}$

   C. -2

   D. 2

5. $\int_0^\infty e^{-x} dx$

   A. This integral diverges

   B. 1

   C. -1

   D. e

1.

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x^4} dx &= \lim_{b\to\infty} \frac{1}{x^4} dx \\
&= \lim_{b\to\infty} \left(-\frac{1}{3x^3}\Big|_1^b \right) \\
&= -\lim_{b\to\infty} \frac{1}{3b^3} + \frac{1}{3} \\
&= \frac{1}{3}
\end{aligned}$$

The answer is C.

2. Let $u = \ln x$, $dv = dx$, thus $du = \dfrac{1}{x}$, $v = x$.

$$\begin{aligned}
\int \ln x dx &= x \ln x - \int x \frac{1}{x} dx \\
&= x \ln x - \int dx \\
&= x \ln x - x + C
\end{aligned}$$

Thus

$$\begin{aligned}
\int_0^1 \ln x dx &= \lim_{a\to 0^-} \int_a^1 \ln x dx \\
&= \lim_{a\to 0^-} (x \ln x - x)\Big|_a^1 \\
&= 0 - 1 - \lim_{a\to 0^-} (a \ln x - a) \\
&= -1 - \lim_{a\to 0^-} a \ln a \\
&= -1 - \lim_{a\to 0} \frac{\ln a}{\frac{1}{a}} \\
&= -1 - \lim_{a\to 0} \frac{\frac{1}{a}}{-\frac{1}{a^2}} \\
&= -1 - \lim_{a\to 0} a \\
&= -1
\end{aligned}$$

The answer is D.

3.

$$\begin{aligned}
\int_0^2 \frac{1}{(1-x)^2}dx &= \int_0^1 \frac{1}{(1-x)^2}dx + \int_1^2 \frac{1}{(1-x)^2}dx \\
&= \lim_{b\to 1^-}\int_0^b \frac{1}{(1-x)^2}dx + \lim_{a\to 1^+}\int_a^2 \frac{1}{(1-x)^2}dx \\
&= \lim_{b\to 1^-} \frac{1}{1-x}\bigg|_0^b + \lim_{a\to 1^+} \frac{1}{1-x}\bigg|_a^2 \\
&= \infty
\end{aligned}$$

The integral diverges, the answer is A.

If one did not notice the infinite discontinuity at $x = 1$, the incorrect result of -2 will be obtained.

4.

$$\begin{aligned}
\int_0^1 \frac{1}{\sqrt{x}}dx &= \lim_{a\to 0^+}\int_a^1 x^{-1/2}dx \\
&= \lim_{a\to 0^+} 2x^{1/2}\bigg|_a^1 \\
&= \lim_{a\to 0^+}(2\sqrt{1} - 2\sqrt{a}) \\
&= 2
\end{aligned}$$

The answer is D.

5.

$$\begin{aligned}
\int_0^\infty e^{-x}dx &= \lim_{b\to\infty}\int_0^b e^{-x}dx \\
&= \lim_{b\to\infty}\left(-e^{-x}\right)\bigg|_0^b \\
&= \lim_{b\to\infty}\left(-e^{-b}\right) - (-e^0) \\
&= 0 + 1 = 1
\end{aligned}$$

The answer is B