

# Radius of Convergence

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## 1 Radius and Interval of Convergence of Power Series

A Power Series takes in a form of

$$\sum_{n=1}^{\infty} a_n(x-c)^n$$

Where  $c$  is the center of the series, since there is an extra  $x$ , for some  $x$  the series might diverge, for example, consider a series where  $a_n = 1$  and  $c = 0$ :

$$\sum_{n=1}^{\infty} x^n$$

If  $x \geq 1$ , this series will diverge to infinity, if  $0 < x < 1$ , this series will converge.

There are 3 possibilities of the convergence of a power series:

1. Converges only at the center (every power series converge at center)
2. Converges for some value within a finite distance to the center (we call this distance Radius of Convergence)
3. Converge for all values

### 1.1 Interval of Convergence

Interval of Convergence is a fancy word for all the values that make the series converge, it is an important parameter of an power series.

1. Within the radius of convergence, the series will converge **absolutely**
2. At the endpoint of the interval, the series could **converge absolutely, conditionally or diverge**.

To determine the radius of convergence, use the ratio test

If

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Where  $a$  is the coefficient in front of the power term. Then radius of convergence of the series is  $R = \frac{1}{\rho}$ , if  $\rho = \infty$ , then  $R = 0$ ; if  $\rho = 0$ ,  $R = \infty$

## 1.2 Example Questions

1. Find the radius of convergence of this power series:

$$\sum_{n=0}^{\infty} (x+3)^n$$

By ratio test:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{1} \right| = 1$$

Thus the radius of convergence is 1.

To find the interval of convergence for this series, notice that the center is at  $x = -3$ , and the radius of convergence is 1, it means that for all values with in  $-2$  and  $-4$ , the series will converge.

To test for the endpoints, substitute the value to the series and test if it converge, for example, to determine the convergence of this series at  $x = -2$ , substitute  $x = -2$  in

$$\sum_{n=0}^{\infty} (-1)^n$$

This is a geometric series with a common ratio of  $r = -1$ , which diverge, same logic applies to  $x = -4$ , therefore the interval of convergence for this series is  $(-4, -2)$

2. Find the radius of convergence of this power series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^{n+1}}{n^2}$$

By ratio test:

$$\lim_{n \rightarrow \infty} \left| \frac{n^2}{(n+1)^2} \right| = 1$$

Therefore radius of convergence is 1.

### 1.3 Practice Question

Determine the radius of convergence of the following power series.

1.  $\sum_{n=1}^{\infty} n!x^n$

A.  $\infty$

B. 0

C. 1

D. 10

2.  $\sum_{n=1}^{\infty} \frac{(x-1)^2}{2^n \cdot n}$

A.  $\frac{1}{2}$

B.  $\infty$

C. 2

D. 0

3.  $\sum_{n=1}^{\infty} \frac{(2x)^n}{n \cdot 3^n}$

A.  $\frac{3}{2}$

B.  $\frac{2}{3}$

C. 2

D. 3

### 1.4 Solution

1. By ratio test

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)!}{n!} \right| = \infty$$

Therefore the radius of convergence is 0, the answer is B.

2. By ratio test

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{2^n \cdot n}{2^{n+1} \cdot (n+1)} \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{2} \frac{n}{n+1} \right| = \frac{1}{2}$$

Therefore the radius of convergence of  $R = 2$ , the answer is C.

3. By ratio test

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{2^{n+1}}{(n+1) \cdot 3^{n+1}} \frac{n \cdot 3^n}{2^n} \right| = \frac{3}{2} \lim_{n \rightarrow \infty} \left| \frac{n}{n+1} \right| = \frac{2}{3}$$

Therefore the radius of convergence is  $\frac{3}{2}$ , the answer is A.