Harmoic Series and p-Series

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A **p-Series** is a series that has the form of

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + ... + \frac{1}{n^p}$$

Where $p > 0$.

1 Convergence of p-series

Through integral test, one can determine the convergence of a p-series, first, check if the requirement of integral test is met:

1. Positive: $f(x) = \dfrac{1}{x^p}$ is positive in $(0, \infty)$
2. Continous: the function is continous for all real number expect $x = 0$
3. Decreasing: $f'(x) = -\dfrac{p}{x^{p+1}}$ is negative when $x > 0$

This means we can apply the integral test to this series, set up this integral:

$$\int_1^{\infty} \frac{1}{x^p} dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^p} dx = \lim_{b\to\infty} \frac{1}{-p+1} x^{-p+1} \Big|_1^b = \lim_{b\to\infty} \frac{1}{-p+1} b^{-p+1} - \frac{1}{-p+1}$$

We know that $\dfrac{1}{-p+1}$ is a real number that has a finite value, so we should examine this limit $\displaystyle\lim_{b\to\infty} \frac{1}{-p+1} b^{-p+1}$ to determine the convergence of the series.

$$\lim_{b\to\infty} \frac{1}{-p+1} b^{-p+1} = \frac{1}{-p+1} \lim_{b\to\infty} b^{-p+1}$$

This limit has a finite value when $-p + 1 < 0$, or $p > 1$, thus the series converge if $p > 1$, diverge if $0 < p \leq 1$.

When $p = 1$, the series is called **harmoic series**, which diverges.

2 Example Problem

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{k^3}}$$

The exponent can be written as $3/4 < 1$, which means it diverges

3 Practice Problems

Determine if the following series converge:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}}$$

A. Converge

B. Diverge

$$\sum_{n=1}^{\infty} \frac{1}{n^5}$$

A. Converge

B. Diverge

$$\sum_{n=1}^{\infty} \frac{1}{3^n}$$

A. Converge

B. Diverge

4 Solution

1. $\frac{1}{5} < 1$, meaning this series diverge, the answer is B.
2. $5 > 1$, meaning this series converge, the answer is A.
3. This is not a p-series, but a harmonic series, $\frac{1}{3} < 1$, meaning this series converge, the answer is A.