Fundamental Theorem of Calculus

Polaris

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1 Fundamental Theorem of Calculus

If $f(x)$ is continous on the interval of $[a, b]$, then

$$\int_a^b f(x) = F(b) - F(a)$$

Where $\frac{d}{dx}F(x) = f(x)$. This is very powerful, as it links differentiation and integration, the two fundamental operation in calculus.

Let's take a look at an example question:

$$\int_0^{\frac{\pi}{2}} \cos x \mathrm{d}x$$

To evaluate this definite integral, we first need to find a function that has a derivative of $\cos x$, and the function is $\sin x$ (consult back to differentiation), thus by the Fundamental Theorem of Calculus:

$$\int_0^{\frac{\pi}{2}} \cos x \mathrm{d}x = \sin\left(\frac{\pi}{2}\right) - \sin(0) = 1$$

Where $\sin\left(\frac{\pi}{2}\right) = 1$.

2 Application of FTC

Let's start with a question, evaluate

$$\int_2^x (3t^2 - 2)\mathrm{d}t$$

At first glance it seems weird that the variable x is on the upper bound, but let's pretend that x is a number and do the integral (which is literally algerbra)

$$\int_2^x (3t^2 - 2)\mathrm{d}t = t^3 - 2t\Big|_2^x = x^3 - 2x - 8 + 4 = x^3 - 2x - 4$$

This question inspires us to recognize that when a variable is on the integration bound, it means the integral will be a function of the variable, or:

$$\int_a^x F(t)\mathrm{d}t = f(x) - f(a)$$

Where again $\frac{d}{dx}F(x) = f(x)$ and $f(a)$ is a number.

Another way to apply FTC is finding a value of a function at one point with an integral, let's look at a question:

$F(x)$ is an antiderivative of $f(x)$, if $F(5) = 10$, find an expression that equals to $F(11)$

In order to solve this problem, let's first use FTC:

$$\int_5^{11} f(x)\mathrm{d}x = F(11) - F(5)$$

Notice here $F(11)$ appears, so we can easily find an expression for $F(11)$:

$$F(11) = F(5) + \int_5^{11} f(x)\mathrm{d}x = 10 + \int_5^{11} f(x)\mathrm{d}x$$

Another way you will see it is in a table:

Consider a function $f(x)$ and its derivative $f'(x)$:

x	0	2	3	5
$f(x)$	-15	-8	4	7
$f'(x)$	12	8	3	-2

Find the value of $\int_0^5 f'(x)\mathrm{d}x$

We know that by FTC:

$$\int_0^5 f'(x)\mathrm{d}x = f(5) - f(0)$$

While $f(5) = 7$ and $f(0) = -15$, thus

$$\int_0^5 f'(x)\mathrm{d}x = f(5) - f(0) = 7 + 15 = 22$$

Another way to test about FTC is through graphs:

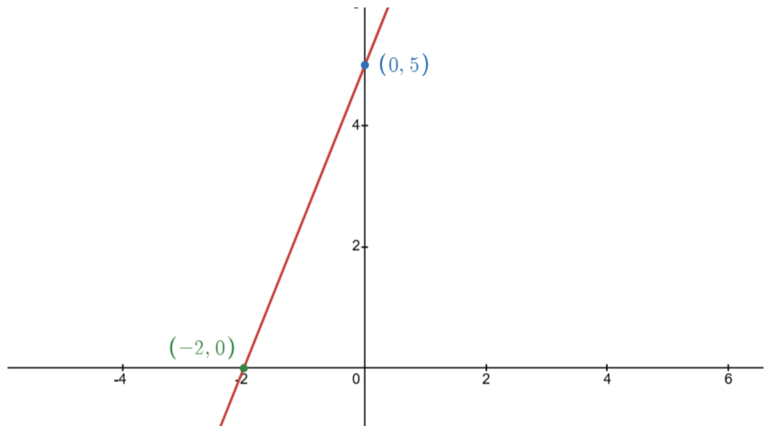

Figure 1: Graph of $f'(x)$

Consider a function $f(x)$ with a derivative of $f'(x)$. If this is the graph of $f'(x)$ and $f(-2) = 3$, find $f(0)$.

First, by FTC we have

$$\int_{-2}^{0} f'(x)\mathrm{d}x = f(0) - f(-2)$$

Thus

$$f(0) = f(-2) + \int_{-2}^{0} f'(x)\mathrm{d}x$$

By the geometric meaning of integrals, $\displaystyle\int_{-2}^{0} f'(x)\mathrm{d}x$ is the area under the curve of $f'(x)$, which is the triangle formed by the graph of $f'(x)$ and the coordinate axis.

This offers a way to calculate the integral, the area of the triangle is simply $S = \dfrac{1}{2} \cdot 2 \cdot 5 = 5$, thus the integral also equals to 5, meaning

$$f(0) = f(-2) + \int_{-2}^{0} f'(x)\mathrm{d}x = 3 + 5 = 8$$

The answer is $f(0) = 8$