1 Integral Test for convergence

If f is a **positive**, **continous** and **decreasing** for $x \geq m$, where $m \geq 1$

and the n-th term expression $a_n = f(x)$, then:

$$\sum_{n=1}^{\infty} a_n \text{ and } \int_1^{\infty} f(x)\mathrm{d}x \tag{1.1}$$

both converge or diverge.

1.1 Example questions

1. Evaluate this series:

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1}$$

First, check for the three criteria: positive, coutinous, decreasing

(a) Positive: obviously $f(x) = \frac{x}{x^2+1}$ is positive in $[1, \infty)$

(b) Continous: the function is continous for all real number

(c) Decreasing: $f'(x) = -\frac{x^2-1}{(x^2+1)^2}$, and $f'(x) < 0$ when $x > 1$

Now evaluate the indefinite integral:

$$\begin{aligned}
\int_1^{\infty} \frac{x}{x^2+1}\mathrm{d}x &= \lim_{b\to\infty} \int_1^{b} \frac{x}{x^2+1}\mathrm{d}x \\
&= \lim_{b\to\infty} \frac{1}{2} \int_1^{b} \frac{1}{x^2+1}\mathrm{d}(x^2+1) \\
&= \frac{1}{2} \lim_{b\to\infty} \left(\ln\left|b^2+1\right| - \ln\left|1^2+1\right|\right) \\
&= \infty
\end{aligned}$$

The integral diverges, meaning that the series also diverges

2. Evaluate this series:

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

Let

$$a_n = f(x) = \frac{1}{x^2+1}$$

Checking if the method work is omitted, but it does work Then we can construct and solve this improper integral:

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x^2+1}\mathrm{d}x &= \lim_{b\to\infty} \int_1^{b} \frac{1}{x^2+1}\mathrm{d}x \\
&= \lim_{b\to\infty} \arctan b - \arctan 1 \\
&= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}
\end{aligned}$$

Meaning the series converge, **but the series doesn't necessarily converge to** $\pi/4$