

Integration by parts

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1 Indefinite Integral

The formula for integration by parts is simple:

$$\int u dv = uv - \int v du$$

First let's start with an example question:

$$\int x e^x dx$$

At first glance we don't see the dv structure, but notice that $e^x dx = d(e^x)$:

$$\int x e^x dx = \int x d(e^x)$$

Now we completed the udv structure and we can use integration by parts:

$$\int x d(e^x) = x e^x - \int e^x dx = x e^x - e^x + C$$

The key of integration by parts is turn some expression $\cdot dx$ into $d(\text{some expression})$, thus finding the expression becomes a skill, here are some expression to consider:

1. e^x
2. $\sin x$ and $\cos x$
3. x^n

Let's take a look at another another example:

$$\int x \cos x dx$$

Let $u = x$ and $dv = d(\sin x) = \cos x dx$, we can apply integration by parts:

$$\int x d(\sin x) = x \sin x - \int \sin x dx = x \sin x + \cos x + C$$

Sometimes we need to apply integration by parts more than once, for example:

$$\int e^x \sin x dx$$

Let $u = \sin x$, $dv = d(e^x) = e^x dx$, by integration by parts:

$$\int e^x \sin x dx = \int \sin x d(e^x) = e^x \sin x - \int e^x \cos x dx$$

Here we arrived at a new integral of $\int e^x \cos x dx$, which again can be evaluated by integration by parts:

Let $u = \cos x$, $dv = d(e^x) = e^x dx$, by integration by parts:

$$\int e^x \cos x dx = \int \cos x d(e^x) = e^x \cos x + \int e^x \sin x dx$$

Here we see a problem, it seems that we need to evaluate our original integral to get an expression for our original integral, but this can be easily bypassed, notice that

$$\int e^x \sin x dx = e^x \sin x - e^x \cos x - \int e^x \sin x dx$$

We can treat our original integral as an unknown value and solve this equation, thus:

$$2 \int e^x \sin x dx = e^x \sin x - e^x \cos x + C$$

It is not hard to see that

$$\int e^x \sin x dx = \frac{1}{2} e^x (\sin x - \cos x) + C$$

Integration by parts can be used to calculate the indefinite integral of inverse trig function:

$$\int \arcsin x dx$$

We immediately see a udv structure, let $u = \arcsin x$ and $dv = dx$:

$$\int \arcsin x dx = x \arcsin x - \int x d(\arcsin x) = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} dx$$

The last integral can be evaluated with a u -substitution, thus we arrive at our final example:

$$\int \arcsin x dx = x \arcsin x + \sqrt{1-x^2} + C$$

2 Definite Integral

For definite integral, the formula for integration by parts turn to:

$$\int_a^b u dv = uv \Big|_a^b - \int_a^b v du$$

An example would be:

$$\int_0^{\frac{1}{2}} \arcsin x dx$$

Previously we derived that

$$\begin{aligned} \int_0^{\frac{1}{2}} \arcsin x dx &= x \arcsin x \Big|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}} dx \\ &= \frac{1}{2} \arcsin \frac{1}{2} + \sqrt{1-x^2} \Big|_0^{\frac{1}{2}} \\ &= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1 \end{aligned}$$