Long Division and Complete the Square

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This article will guide you through what is long division, what is complete the squares, and how to apply it to integration

1 Long Division

1.1 What is Long Division

Long division is also known as *polynomial division*, it is division design for dividing two polynomials, for example, if we want to find the result of $\frac{x^3+x^2-1}{x-1}$, we can set up this division

$$
\begin{array}{r|rrr}
\multicolumn{1}{r}{} & & & \\ \cline{2-4}
x-1) & x^3 & +x^2 & -1
\end{array}
$$

Which is just like division we learn in primary school. Now we want to find a quadratic term multiplied with $x-1$ produce something like x^3+x^2 (just like normal division), first try x^2:

$$
\begin{array}{r|rrr}
\multicolumn{1}{r}{} & & x^2 & \\ \cline{2-4}
x-1) & x^3 & +x^2 & -1
\end{array}
$$

Multiply $x^2(x-1) = x^3 - x^2$, we have:

$$
\begin{array}{r|rrr}
\multicolumn{1}{r}{} & & x^2 & \\ \cline{2-4}
x-1) & x^3 & +x^2 & -1 \\
 & -x^3 & +x^2 & \\ \cline{2-3}
 & & 2x^2 &
\end{array}
$$

Apply the same logic, try $2x$ and multiply it with $x-1$:

$$
\begin{array}{r|rrrr}
\multicolumn{1}{r}{} & & x^2 & +2x & \\ \cline{2-5}
x-1) & x^3 & +x^2 & & -1 \\
 & -x^3 & +x^2 & & \\ \cline{2-3}
 & & 2x^2 & & \\
 & & -2x^2 & +2x & \\ \cline{3-4}
 & & & 2x & -1
\end{array}
$$

Finally, we try 2 as a part of the quotient and multiply it with $x-1$:

$$
\begin{array}{r|rrrr}
 & & x^2 & +2x & +2 \\
\hline
x-1 & x^3 & +x^2 & & -1 \\
 & -x^3 & +x^2 & & \\
\hline
 & & 2x^2 & & \\
 & & -2x^2 & +2x & \\
\hline
 & & & 2x & -1 \\
 & & & -2x & +2 \\
\hline
 & & & & 1
\end{array}
$$

The remainder of 1 means there is a leftover term with 1 as its numerator, overall, we can write the following equation:

$$\frac{x^3+x^2-1}{x-1} = x^2+2x+2+\frac{1}{x-1}$$

You can verify this by combining the two fraction and see if it returns to the original fraction.

1.2 Application to Integration

Consider this integral:

$$\int \frac{x^3+x^2-1}{x-1}\mathrm{d}x$$

It is hard to preform a u-substitution or a trig substitution, that's where we try long division, we know that the integrant can be rewritten as such:

$$\int \frac{x^3+x^2-1}{x-1}\mathrm{d}x = \int \left(x^2+2x+2+\frac{1}{x-1}\right)\mathrm{d}x$$

We can split the integral and easily compute the result of this integral:

$$\int \frac{x^3+x^2-1}{x-1}\mathrm{d}x = \int \left(x^2+2x+2+\frac{1}{x-1}\right)\mathrm{d}x = \frac{1}{3}x^3+x^2+2x+\ln|x-1|+C$$

Let's take a look at another example:

$$\int \frac{5x^2+x-1}{x+1}\mathrm{d}x$$

First apply polynomial division:

$$
\begin{array}{r|rrr}
 & & 5x & -4 \\
\hline
x+1 & 5x^2 & +x & -1 \\
 & -5x^2 & -5x & \\
\hline
 & & -4x & -1 \\
 & & 4x & +4 \\
\hline
 & & & 3
\end{array}
$$

This means that we can re-write this integral as and easily compute the integral:

$$\int 5x-4+\frac{3}{x+1}\mathrm{d}x = \frac{5}{2}x^2-4x+3\ln|x+1|+C$$

2 Complete the Square

Recall when learning about the quadratic function, we learnt about a form of quadratic function called the standard form, which is:

$$y = (x-h)^2 + k$$

When we want to complete the square, we want to fit a quadratic equation into this form, for example:

$$\begin{aligned} x^2 + 6x + 10 &= x^2 + 6x + \left(\frac{6}{2}\right)^2 - \left(\frac{6}{2}\right)^2 + 10 \\ &= (x+3)^2 + 1 \end{aligned}$$

In general, consider a quadratic expression of $x^2 + bx + c$, in order to turn this into a form that looks like the standard form of quadratic function, we add $\left(\frac{b}{2}\right)^2$ to the equation:

$$\begin{aligned} x^2 + bx + c &= x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c \\ &= \left(x + \frac{b}{2}\right)^2 - \frac{b^2 - 4c}{4} \end{aligned}$$

Let's take a look at an example:

$$\int \frac{8}{x^2 + 6x + 10} \mathrm{d}x$$

We know that $x^2 + 6x + 10 = (x+3)^2 + 1$, thus:

$$\begin{aligned} \int \frac{8}{x^2 + 6x + 10} \mathrm{d}x &= 8 \int \frac{1}{(x+3)^2 + 1} \mathrm{d}x \\ &= 8 \int \frac{1}{u^2 + 1} \mathrm{d}u \\ &= 8 \arctan u + C = 8 \arctan (x+3) + C \end{aligned}$$

Here we preformed an u-substitution of $u = x + 3$ and $\mathrm{d}u = \mathrm{d}x$

Another example is:

$$\int \frac{2}{\sqrt{-x^2 + 10x - 24}} \mathrm{d}x$$

First complete the square for $-x^2 + 10x - 24$:

$$\begin{aligned} -x^2 + 10x + 24 &= -(x^2 - 10x + 24) \\ &= -(x^2 - 10x + 25 - 25 + 24) \\ &= -((x-5)^2 - 1) \\ &= 1 - (x-5)^2 \end{aligned}$$

Thus:

$$\begin{aligned}
\int \frac{2}{\sqrt{-x^2+10x-24}}\mathrm{d}x &= 2\int \frac{1}{\sqrt{1-(x-5)^2}}\mathrm{d}x \\
&= 2\int \frac{1}{\sqrt{1-u^2}}\mathrm{d}u \\
&= 2\arcsin u + C \\
&= 2\arcsin\left(x-5\right) + C
\end{aligned}$$

Here we preformed an u-substitution of $u = x - 5$ and $\mathrm{d}u = \mathrm{d}x$

3 Exercises

1. $\displaystyle\int \frac{x^3}{x+3}\mathrm{d}x$
2. $\displaystyle\int \frac{6x^3-7x^2+1}{2x-1}\mathrm{d}x$
3. $\displaystyle\int \frac{2x+3}{x^2+3x+10}\mathrm{d}x$
4. $\displaystyle\int \frac{1}{x^2-2x+5}\mathrm{d}x$
5. $\displaystyle\int \frac{3}{3x^2-5x+4}\mathrm{d}x$

Finish the exercises first and then check your answer

4 Solutions

1. First, preform long division

$$\begin{array}{r l}
 & \quad x^2 - 3x \quad + 9 \\ \cline{2-2}
x+3) & \quad x^3 \\
 & -x^3 - 3x^2 \\ \cline{2-2}
 & \qquad -3x^2 \\
 & \qquad\quad 3x^2 + 9x \\ \cline{2-2}
 & \qquad\qquad\qquad 9x \\
 & \qquad\qquad\quad -9x - 27 \\ \cline{2-2}
 & \qquad\qquad\qquad\quad -27
\end{array}$$

We can convert the integral into the following

$$\begin{aligned}
\int \frac{x^3}{x+3}\mathrm{d}x &= \int \left(x^2 - 3x + 9 - \frac{27}{x+3}\right)\mathrm{d}x \\
&= \frac{1}{3}x^3 - \frac{3}{2}x^2 + 9x - 27\ln|x+3| + C
\end{aligned}$$

2. First, preform a long division

$$
\begin{array}{r}
3x^2 - 2x - 1 \\
2x - 1 \overline{) \; 6x^3 - 7x^2 \qquad\quad + 1} \\
-\, 6x^3 + 3x^2 \qquad\qquad \\
\hline
-\, 4x^2 \qquad\qquad \\
4x^2 - 2x \quad\; \\
\hline
-\, 2x + 1 \\
2x - 1 \\
\hline
0
\end{array}
$$

We can now split the integral as such

$$
\begin{aligned}
\int \frac{6x^3 - 7x^2 + 1}{2x - 1} \mathrm{d}x &= \int \left(3x^2 - 2x - 1\right) \mathrm{d}x \\
&= x^3 - x^2 - x + C
\end{aligned}
$$

3. Surprisingly, this question does not involve any of the method taught in this article, it is a review of simple u-substitution.

   Let $u = x^2 + 3x + 10$, thus $\mathrm{d}u = (2x + 3)\mathrm{d}x$, we can turn the integral into

$$
\begin{aligned}
\int \frac{2x + 3}{x^2 + 3x + 10} \mathrm{d}x &= \int \frac{1}{u} \mathrm{d}u \\
&= \ln |u| + C = \ln \left| x^2 + 3x + 10 \right| + C
\end{aligned}
$$

4.

$$
\begin{aligned}
\int \frac{1}{x^2 - 2x + 5} \mathrm{d}x &= \int \frac{1}{(x-1)^2 + 4} \mathrm{d}x \\
&= \int \frac{1}{\left(\frac{x-1}{2}\right)^2 + 1} \mathrm{d}u \\
&= \int \frac{1}{u^2 + 1} \frac{\mathrm{d}u}{2} \\
&= \frac{1}{2} \arctan u + C \\
&= \frac{1}{2} \arctan \left(\frac{x-1}{2} \right) + C
\end{aligned}
$$

Here we preformed a u-substitution of $u = \frac{x-1}{2}$ and $\mathrm{d}u = \frac{1}{2}\mathrm{d}x$

5.

$$
\begin{aligned}
\int \frac{3}{3x^2-5x+4}\mathrm{d}x &= \int \frac{3}{3(x^2-\frac{5}{3}x+\frac{4}{3})}\mathrm{d}x \\
&= \int \frac{1}{x^2-\frac{5}{3}x+\left(\frac{5}{6}\right)^2-\left(\frac{5}{6}\right)^2+\frac{4}{3}}\mathrm{d}x \\
&= \int \frac{1}{\left(x+\frac{5}{6}\right)^2+\frac{23}{36}}\mathrm{d}x \\
&= \int \frac{1}{\frac{23}{36}\left(\left(\frac{36}{23}\left(x+\frac{5}{6}\right)^2\right)+1\right)}\mathrm{d}x \\
&= \frac{36}{23}\int \frac{1}{\left(\frac{6}{\sqrt{23}}\left(x+\frac{5}{6}\right)^2\right)+1}\mathrm{d}x \\
&= \frac{36}{23}\frac{\sqrt{23}}{6}\int \frac{1}{u^2+1}\mathrm{d}u \\
&= \frac{6\sqrt{23}}{23}\arctan u + C \\
&= \frac{6\sqrt{23}}{23}\arctan\left(\frac{6x+5}{\sqrt{23}}\right)
\end{aligned}
$$

Here we preform a u-substitution of $u = \dfrac{6}{\sqrt{23}}\left(x+\dfrac{5}{6}\right) = \dfrac{6x+5}{\sqrt{23}}$, $\mathrm{d}u = \dfrac{6}{\sqrt{23}}\mathrm{d}x$