Absolute Convergence and Conditional Convergence

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1 Alternating Series Error Bound

Consider an alternating series:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$$

This series converge, but how far is the first n-th (say 5) terms from the actual convergence value? We introduce the Alternating Series Error Bound Theorem:

$$\text{If an alternating series converge, then } \sigma = |S - S_n| \leq |a_{n+1}| \tag{1.1}$$

Where σ is the error.

1.1 Intuitive Understanding

This section will introduce an intuitive understanding (not rigorius proof) to the theorem introduced earlier. Consider this series:

$$\sum_{n=0}^{\infty}(-1)^n a_n = a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + ... = L$$

Let the sum of the first 4 terms of this series be S_n, thus

$$\sum_{n=0}^{\infty}(-1)^n a_n = S_n + a_4 - a_5 + ... = L$$

Since this series converge, $a_{n+1} < a_n$, which means that $|a_4| > |-a_5 + a_6 - ...|$, or $|a_4| > |\sigma - a_4|$thus error $\sigma = |a_4 - a_5 + a_6 - ...| < 2|a_4|$, meaning that

$$S_n - a_{n+1} \leq L \leq S_n + a_{n+1}$$

1.2 Example

1. Estimate the convergence value of this series (give an estimation on upper and lower bound):

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$$

   Let's first add up the first 5 term of the series:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} = \frac{47}{60}$$

By the introduced theorem, we have $\sigma \leq |a_6|$, or $\sigma \leq \left|\frac{1}{6}\right|$, meaning that:

$$\frac{47}{60} - \frac{1}{6} \leq \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} \leq \frac{47}{60} + \frac{1}{6}$$

$$\frac{37}{60} \leq \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} \leq \frac{19}{20}$$

This series actually converge to $\ln(2) \approx 0.693$, which is with in the error bound

2 Practice Problems

Estimate the error bound of the following series using the first 5 terms

1. $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^2}$

A. $-\frac{3019}{3600} \leq \sum_{n=1}^{\infty}(-1)^n\frac{1}{n^2} \leq -\frac{2731}{3600}$

B. $-\frac{3119}{3600} \leq \sum_{n=1}^{\infty}(-1)^n\frac{1}{n^2} \leq -\frac{973}{1200}$

C. This series diverge

2. $\sum_{n=2}^{\infty}(-1)^n\frac{n}{\ln n}$

A. $11.360 \leq \sum_{n=2}^{\infty}(-1)^n\frac{n}{\ln n} \leq 18.554$

B. $8.260 \leq sum_{n=2}^{\infty}(-1)^n\frac{n}{\ln n} \leq 14.957$

C. This series diverge

3. $\sum_{n=1}^{\infty}(-1)^n\frac{n}{e^n}$

A. $-0.222 \leq \sum_{n=1}^{\infty}(-1)^n\frac{n}{e^n} \leq -0.192$

B. $-0.207 \leq \sum_{n=1}^{\infty}(-1)^n\frac{n}{e^n} \leq -0.140$

C. This series diverge

3 Solution

1. First apply the convergence test for this series, the limit is 0 and the function is decreasing.

Find the sum of the first 5 terms

$$\sum_{n=1}^{5}(-1)^n\frac{1}{n^2} = -\frac{3019}{3600}$$

Thus the error turns to

$$-\frac{3019}{3600} - \frac{1}{36} \leq \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} \leq -\frac{3019}{3600} + \frac{1}{36}$$

Which is equivalent to

$$-\frac{3119}{3600} \leq \sum_{n=1}^{\infty}(-1)^n\frac{1}{n^2} \leq -\frac{973}{1200}$$

The answer is B.

This series converge to $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^2} = -\frac{\pi^2}{12}$, which is within the error bound. Finding the convergent value of this series can be neatly solved with the Basel Problem.

2. First apply the convergence test for this series,

$$\lim_{n\to\infty}\frac{n}{\ln n} = \lim_{n\to\infty}\frac{1}{1/n} = \infty$$

Here we applied the L'Hopital Rule, the limit DNE, meaning the series diverge.

The answer is C.

3. First apply the convergence test for this series, the limit is 0 and the function is decreasing.

Find the sum of the first 5 terms

$$\sum_{n=1}^{5}(-1)^n\frac{n}{e^n} = -0.207$$

Thus the error turns to

$$-0.207 - \frac{6}{e^6} \leq \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} \leq -0.207 + \frac{6}{e^6}$$

Which is equivalent to

$$-0.222 \leq \sum_{n=1}^{\infty}(-1)^n\frac{n}{e^n} \leq -0.192$$

The answer is A. This series actually converge to $-\frac{e}{(e+1)^2} \approx -0.197$, which is within the error bound.