New Year Homework

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1 Star with changing luminosity

(a) The luminosity of the star as a function of time is

$$L(t) = L_0 \cdot 100^{t/100}$$

Thus

$$\begin{aligned} m - m_6 &= -2.5 \lg\left(\frac{F}{F_6}\right) = -2.5 \lg\left(\frac{L}{L_6}\right) \\ &= -2.5 \lg\left(100^{-t/100}\right) \\ 10^{-2t/100} &= 10^{0.4(m-m_6)} \\ t &= 50 \cdot 0.4(m - m_6) = 170\text{yr} \end{aligned}$$

(b) Factoring in interstellar extinction, the apparent magitude of this star is

$$m_0 = m - 0.05 \cdot 100 = 9.5\text{mag}$$

Compare this star with the Sun, where $m_s = -26.74$

$$\begin{aligned} m - m_s &= -2.5 \lg\left(\frac{F}{F_s}\right) \\ &= -2.5 \lg\left(\frac{L}{L_s}\frac{d_s^2}{d^2}\right) \\ &= -2.5 \lg\frac{L}{L_s} - 5 \lg\frac{d_s}{d} \\ \frac{L}{L_s} &= 10^{-0.4(m-m_s+5\lg(d_s/d))} \approx 1.36 \end{aligned}$$

Therefore the initial radius of the star in terms of solar radius is

$$\frac{R}{R_s} = \sqrt{\frac{L}{L_s}\frac{T_s^4}{T^4}} \approx 1.55$$

The orbital radius of Jupiter is 5.2 AU which is equal to $1117R_s$, therefore, the luminosity of the star will increase by a factor of

$$\frac{L_J}{L_0} = \frac{R_J^2}{R^2} = 5.19 \cdot 10^5$$

The time it takes is therefore

$$t = 50 \lg\frac{L_J}{L_0} = 286\text{yr}$$

2 Shadow's Movement

I don't know how to do this problem.

3 Tully-Fisher Relation

(a) Mass-to-light ratio is defined as $C_1 = M/L$, where M is the mass inside the disk with radius of R. Therefore

$$L = \frac{M}{C_1} = \frac{Rv^2}{C_1 G}$$

Making a furthur approximation that the galactic disk have even brightness everywhere, this means that $L/R^2 = C_2$, substitute this in, one arrive at the T-L relation

$$L = \frac{v^4}{G^2 C_1^2 C_2} \propto v_\infty^4$$

(b) The absolute magnitude of Milky way is $M_{mw} \approx -21$ mag, for Milky Way, $v_{\infty,mw} \approx 220$ km/s. From the graph, the redshift at the edge of the galaxy is around $7.7 \cdot 10^{-3}$, deduct $3 \cdot 10^{-3}$ for the redshift due to expasion of the universe, the rotational velocity of the edge of the galaxy is

$$v_\infty = cz = 1410 \text{ km/s}$$

By TF relation, the luminosity of the galaxy is

$$\frac{L}{L_{mw}} = \frac{v_\infty^4}{v_{\infty,mw}^4} = 1687$$

The absolute magnitude of the galaxy is

$$M_{gal} = M_{mw} - 2.5 \lg \left(\frac{L}{L_{mw}} \right) = -29$$

Distance can be estimated from Hubble law ($H_0 = 70$ km/s/Mpc)

$$d = \frac{v}{H_0} = 12.9 \text{ Mpc}$$

The apparent magnitude of the galaxy is therefore

$$m = M - 5 + 5 \lg d = 1.55$$

4 Colliding Star

(a) In the CM frame, the total energy of the system is

$$E = \frac{1}{2}\mu v^2 - \frac{GM_{tot}\mu}{r}$$

Where μ is the reduced mass. When $E = 0$, the system is just bounded. Therefore the farthest distance of binding is

$$r_s = \frac{4GM}{v^2}$$

(b) The mean free-path of the system is

$$l = \frac{1}{n\sigma}$$

Where $\sigma = \pi r_s^2$. The mean collision time is thus

$$t_{scale} = \frac{1}{n\sigma v}$$

(c) For Milky Way $L = 10^3$ ly, $R = 5 \cdot 10^4$ ly, $N = 2 \cdot 10^{11}$. The number density is therefore

$$n = \frac{N}{V} = \frac{N}{\pi R^2 L} = 3 \cdot 10^{-50} \text{ /m}^3$$

The velocity dispersion of the stars can be used to approximate the mean velocity between the stars. By virial theorem and dimensional analysis, the velocity dispersion is

$$v = \sqrt{\frac{2GM}{R}} = 3.36 \cdot 10^5 \text{m/s}$$

Substitute everything into the equation, $t_{scale} \approx 1.41 \cdot 10^{24}\text{s} \approx 4.46 \cdot 10^{16}\text{yr}$, which is longer than the age of universe.

(d) The number density of the three types of star is related as such

$$\frac{n_1}{n_2} = \frac{n_2}{n_3} = 0.1^{-2.35} = 223.8$$

The sum of all three densities should equal to the mean density. Thus

$$n = n_1 + n_2 + n_3 = n_1 + \frac{1}{223.8} n_1 + \frac{1}{223.8^2} n_1 = 1.004 n_1$$

This gives $n_1 = 2.99 \cdot 10^{-50}/\text{m}^3$, $n_2 = 1.33 \cdot 10^{-52}/\text{m}^3$ and $n_3 = 5.96 \cdot 10^{-56}/\text{m}^3$

The equivalent mean free path is therefore

$$l_{eq} = \frac{1}{n_1\sigma_1 + n_2\sigma_2 + n_3\sigma_3} = 3.25 \cdot 10^{31}\text{m}$$

The mean time between collision is

$$t_{scale} = \frac{l_{eq}}{v} \approx 9.67 \cdot 10^{25}\text{s} \approx 3.06 \cdot 10^{18}\text{yr}$$

Which is still longer than the age of the universe.

5 Energy recieved by Earth

(a) At a distance r, the energy recieved by Earth per unit area is

$$f = \frac{L}{4\pi r^2}$$

The energy recieved throughout the year is

$$F = \int_0^T \frac{L}{4\pi r^2} dt$$

By Kepler's Second Law, one have

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{h}{2}$$

Thus, by preforming an substituition on the integral above, one have

$$F = \frac{L}{4\pi}\int_0^{2\pi} \frac{1}{r^2} r^2 d\theta = \frac{L}{2h}$$

In an elliptical orbit, reduced angular momentum is

$$h = \frac{2\pi a^2\sqrt{1-e^2}}{T}$$

Which means the total energy recieved by unit area throughout an year is

$$F = \frac{L \cdot T}{4\pi a^2\sqrt{1-e^2}}$$

The total energy recieved by the land is

$$E = \frac{1}{4}\pi R_E^2 F = \frac{LTR_E^2}{16a^2\sqrt{1-e^2}} = 1.37 \cdot 10^{24}\text{J}$$

(b) The temperature change of water is related with the heat the water absorb

$$Q = Cm\Delta T = 5 \cdot 10^{26}\text{J}$$

Where $C = 4200$ J/kg·K, $m = \rho V = 1.4 \cdot 10^{21}$ kg

Assume the Earth rotated $k\pi$ rad by the time the ocean evaporate, the total energy recieved by the Earth is

$$E = k\frac{LTR_E^2}{4a^2\sqrt{1-e^2}} = Q$$

This gives $k = 91.3$, which translate to 45.6 rotations around the Sun, or 45.6 years for the entire ocean to evaporate.

(c) Compare this to the power of TNT

$$n = \frac{E}{\text{16kiloton of TNT}} = 2 \cdot 10^{10}$$

The energy Earth recieve from radiation every year is equivalent to 20 billion nuclear bomb dropped on Hiroshima.

6 Farmer and the Dog

The outer boundary of the places the dog can visit is an ellipse with $a = 12.5$ m and $c = 5$ m, the semi-minor axis of this ellipse is

$$b = \sqrt{a^2 - c^2} = 11.5\text{m}$$

The total area the dog can gaurd is the area of the ellipse

$$S = \pi ab = 451.6\text{m}^2$$

After placing the obstacle on the rope, the area the dog can gaurd is constrain to a circular sector. The opening angle of this sector can be found through cosine law.

$$\theta = \arccos\left(\frac{15^2 - 10^2 - 10^2}{2 \cdot 10 \cdot 10}\right) = 82.8^\circ$$

The new area the dog can gaurd is

$$S' = \pi r^2 \cdot \frac{\theta}{360^\circ} = 72.3\text{m}^2$$

The decrease in area is

$$\Delta S = S - S' = 379.3\text{m}^2$$

7 Superluminal jet

(a) Converting the proper motion into rad/s: $\mu = 1.23 \cdot 10^{-16}$ rad/s

The velocity of the jet is thus

$$v_{\text{app}} = \mu D = 2.38 \cdot 10^9 \text{m/s}$$

Therefore

$$\beta_{\text{app}} = \frac{v_{\text{app}}}{c} = 7.95$$

(b) Consider two photons that is emitted at two different time Δt_e, the first photon will reach the observer at a time of

$$t_1 = \frac{D}{c}$$

The second photon will reach the observer at

$$t_2 = \Delta t_e + \frac{D - v\Delta t_e \cos\theta}{c}$$

Thus the time difference of observing the photons is

$$\Delta t = t_2 - t_1 = \Delta t_e - \frac{v\Delta t_e \cos\theta}{c}$$

$$v_{\text{app}} = \frac{v\Delta t_e \sin\theta}{\Delta t} = \frac{v \sin\theta}{1 - v\cos\theta/c}$$

This is equivalent to

$$\beta_{\text{app}} = \frac{\beta \sin\theta}{1 - \beta\cos\theta}$$

(c) Take derivative of the obtained relation, one obtain

$$\frac{d}{d\theta}\beta_{\text{app}} = \frac{\beta}{(1 - \beta\cos\theta)^2}(\cos\theta - \beta)$$

Let this equal to 0, the maximum apparent velocity will happen when $\beta = \cos\theta$

For superluminal to happen, the following is true

$$\frac{\beta \sin\theta}{1 - \beta\cos\theta} > 1$$

which translate to

$$\beta > \frac{1}{\sin\theta + \cos\theta}$$

(d) Reversing the relationship found in (b), one have

$$\beta = \frac{\beta_{\text{app}}}{\beta_{\text{app}}\cos\theta + \sin\theta} < 1$$

Isolating β_{app}, one have

$$\beta_{\text{app}} < \frac{\sin\theta}{1 - \cos\theta}$$

Which can be translated to

$$\tan\frac{\theta}{2} > \frac{1}{\beta_{\text{app}}}$$

Therefore the maximum angle is

$$\theta_{\max} = 2\arctan\frac{1}{\beta_{app}} = 14^\circ$$

From the relation, $\theta \propto \beta^{-1}$, which means for smaller θ (closer to line of sight), superluminal motion appears more extreme.

8 Coplanar circular orbiting planets

(a) By cosine law, the distance between the two planets is depended on the orbital radius of each planet and the angle between the radius vector

$$r^2 = R_1^2 + R_2^2 - 2R_1R_2\cos(\theta)$$

Where $\theta = \omega_{\text{rel}}(t - t_0) = 2\pi(t - t_0)/T_{\text{rel}}$, where t_0 is the offset between the arbitrary moment and the moment that the planets are closest to each other.

Therefore

$$r(t) = \sqrt{R_1^2 + R_2^2 - 2R_1R_2\cos\left(\frac{2\pi(t - t_0)}{T_{\text{rel}}}\right)}$$

(b) From sinusodial regression

$$r^2 = 16.4829 + 16.0498\cos\left(\frac{2\pi}{T_{\text{rel}}} + 3.1239\right)$$

This gives

$$\begin{cases} R_1^2 + R_2^2 = 16.4829 \\ 2R_1R_2 = 16.0498 \end{cases}$$

This gives

$$\begin{cases} R_1 = 3.1813\text{AU} \\ R_2 = 2.5224\text{AU} \end{cases}$$

(c) The relative angular velocity is equal to

$$\omega_{\text{rel}} = \omega_2 - \omega_1 = \sqrt{\frac{GM}{R_2^3}} - \sqrt{\frac{GM}{R_1^3}}$$

Compare this to the data of Earth

$$\frac{\omega_{\text{rel}}}{\omega_E} = \sqrt{\frac{M}{M_s}}\sqrt{\frac{R_E^3}{R_2^3}} - \sqrt{\frac{M}{M_s}}\sqrt{\frac{R_E^3}{R_1^3}} = \frac{T_e}{T_{\text{rel}}}$$

Substitute the nessasary numbers, the ratio of mass is $M = 1.136M_s$