

# Partial Fraction

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Consider this integral:

$$\int \frac{x+1}{x^2-5x+6} dx$$

This integral looks scary, but notice that we can factor the denominator:  $x^2-5x+6 = (x-2)(x-3)$

Let's first take a look at fraction addition:

$$\frac{A}{B} + \frac{C}{D} = \frac{AD+BC}{BD}$$

Since the integrand is a fraction, and we successfully write the denominator as a product, we should be able to split the fraction into a sum of two fractions.

Assume we have split the fraction like this:

$$\frac{x+1}{(x-2)(x-3)} = \frac{A}{(x-2)} + \frac{B}{(x-3)}$$

Here  $A$  and  $B$  are different constants, let's try to combine them together:

$$\frac{A(x-3) + B(x-2)}{(x-2)(x-3)} = \frac{Ax - 3A + Bx - 2B}{(x-2)(x-3)} = \frac{(A+B)x + (-3A - 2B)}{(x-2)(x-3)}$$

Notice that we reformed the denominator into  $\text{constant} \cdot x + \text{constant}$ , which does indeed look like  $x+1$  in the original integrand, thus we have this relation:

$$\begin{cases} A+B=1 \\ -3A-2B=1 \end{cases}$$

This linear set of equations can be easily solved, which gives the solution of  $A = -3$  and  $B = 4$ . Thus we turned this integrand into a fraction of sum:

$$\int \frac{x+1}{x^2-5x+6} dx = \int \left( -\frac{3}{(x-2)} + \frac{4}{(x-3)} \right) dx$$

Then we can evaluate the integral easily:

$$\int \left( -\frac{3}{(x-2)} + \frac{4}{(x-3)} \right) dx = -3 \ln|x-2| + 4 \ln|x-3| + C$$

In general, partial fraction follow this process:

Consider a integrand where the denominator can be factored, first split the fraction into a sum of fractions:

$$\frac{Ax + B}{(x + C)(x + D)} = \frac{E}{(x + C)} + \frac{F}{(x + D)} = \frac{(E + F)x + (DE + CF)}{(x + C)(x + D)}$$

Where  $A, B, C, D, E, F$  are all constants, thus:

$$\begin{cases} E + F = A \\ DE + CF = B \end{cases}$$

Then solve for  $E$  and  $F$  to complete the fraction split.

Here is another example:

$$\int \frac{2x - 1}{x^2 - 4x + 3} dx = \int \frac{x + 1}{(x - 3)(x - 1)} dx$$

Split the fraction into a sum of fraction:

$$\frac{A}{x - 3} + \frac{B}{x - 1} = \frac{A(x - 1) + B(x - 3)}{(x - 3)(x - 1)} = \frac{(A + B)x + (-A - 3B)}{(x - 3)(x - 1)}$$

Solve for  $A$  and  $B$ :

$$\begin{cases} A + B = 1 \\ -A - 3B = 1 \end{cases}$$

Thus  $A = 2$  and  $B = -1$ , and the integral turns to

$$\int \left( \frac{2}{x - 3} - \frac{1}{x - 1} \right) dx = 2 \ln |x - 3| - \ln |x - 1| + C$$