Integration by parts

assassin3552

2025/03/27

1 Proving Integration by Parts Formula

Start from the product rule of differentiation:

$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u'v + v'u$$

If we multiply both side by $\mathrm{d}x$ on both side and take the indefinite integral, we have:

$$\int \mathrm{d}(uv) = \int u'v\mathrm{d}x + \int v'u\mathrm{d}x$$

Evaluate the integral, we have

$$\int u\mathrm{d}v = uv - \int v\mathrm{d}u$$

Here we introduced a new variable that $v = v'\mathrm{d}x$ and $u = u'\mathrm{d}x$, you can see them as completely new variable and have no relation with the original u and v

2 Indefinite Integral

The formula for integration by parts is simple:

$$\int u\mathrm{d}v = uv - \int v\mathrm{d}u$$

First let's start with an example question:

$$\int xe^x\mathrm{d}x$$

At first glance we don't see the dv structure, but notice that $e^x\mathrm{d}x = \mathrm{d}(e^x)$:

$$\int xe^x\mathrm{d}x = \int x\mathrm{d}(e^x)$$

Now we completed the udv structure and we can use integration by parts:

$$\int x\mathrm{d}(e^x) = xe^x - \int e^x\mathrm{d}x = xe^x - e^x + C$$

The key of integration by parts is turn some expression $\cdot$ dx into d(some expression), thus finding the expression becomes a skill, here are some expression to consider:

1. e^x
2. $\sin x$ and $\cos x$
3. x^n

Let's take a look at another another example:

$$\int x\cos x\mathrm{d}x$$

Let $u = x$ and $\mathrm{d}v = \mathrm{d}(\sin x) = \cos x\mathrm{d}x$, we can apply integration by parts:

$$\int x\mathrm{d}(\sin x) = x\sin x \int \sin x\mathrm{d}x = x\sin x - \cos x + C$$

Sometimes we need to apply integration by parts more than once, for example:

$$\int e^x \sin x \mathrm{d}x$$

Let $u = \sin x$, $\mathrm{d}v = \mathrm{d}(e^x) = e^x \mathrm{d}x$, by integration by parts:

$$\int e^x \sin x \mathrm{d}x = \int \sin x \mathrm{d}(e^x) = e^x \sin x - \int e^x \cos x \mathrm{d}x$$

Here we arrived at a new integral of $\int e^x \cos x \mathrm{d}x$, which again can be evaluated by integration by parts:

Let $u = \sin x$, $\mathrm{d}v = \mathrm{d}(e^x) = e^x \mathrm{d}x$, by integration by parts:

$$\int e^x \cos x \mathrm{d}x = \int \cos x \mathrm{d}(e^x) = e^x \cos x + \int e^x \sin x \mathrm{d}x$$

Here we see a problem, it seems that we need to evaluate our original integral to get an expression for our original integral, but this can be easliy bypassed, notice that

$$\int e^x \sin x \mathrm{d}x = e^x \sin x - e^x \cos x - \int e^x \sin x \mathrm{d}x$$

We can treat our original integral as an unknown value and solve this equation, thus:

$$2 \int e^x \sin x \mathrm{d}x = e^x \sin x - e^x \cos x + C$$

It is not hard to see that

$$\int e^x \sin x \mathrm{d}x = \frac{1}{2} e^x (\sin x - \cos x) + C$$

Integration by parts can be used to calculate the indefinite integral of inverse trig function:

$$\int \arcsin x \mathrm{d}x$$

We immediately see a $u\mathrm{d}v$ structure, let $u = \arcsin x$ and $\mathrm{d}v = \mathrm{d}x$:

$$\int \arcsin x \mathrm{d}x = x \arcsin x - \int x \mathrm{d}(\arcsin x) = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} \mathrm{d}x$$

The last integral can be evaluate with a u-substitution, thus we arrive at our final example:

$$\int \arcsin x \mathrm{d}x = x \arcsin x + \sqrt{1-x^2} + C$$

3 Definite Integral

For definite integral, the formula for integration by parts turn to:

$$\int_a^b u\mathrm{d}v = uv\Big|_a^b - \int_a^b v\mathrm{d}u$$

An example would be:

$$\int_0^{\frac{1}{2}} \arcsin x \mathrm{d}x$$

Previously we derived that

$$\begin{aligned}\int_0^{\frac{1}{2}} \arcsin x \mathrm{d}x &= x \arcsin x\Big|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}}\mathrm{d}x \\ &= \frac{1}{2}\arcsin\frac{1}{2} + \sqrt{1-x^2}\Big|_0^{\frac{1}{2}} \\ &= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1\end{aligned}$$