Geometric Series

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1 Definition

Consider this sequence

$$3, 6, 12, 24, 48...$$

It is easy to see that there is a common factor of 2 between every term. Thus we can rewrite the sequence like this:

$$3 \cdot 2^0, 3 \cdot 2^1, 3 \cdot 2^2, 3 \cdot 2^3, 3 \cdot 2^4$$

In general, the n-th term expression can be written as (where n starts from 0):

$$a_n = 3 \cdot 2^n$$

If we take the sum of all terms, we arrive at what we called **geometric series**, the quotient of two neighboring terms of a geometric series is a constant (2 in the previous example).

With this knowledge, we can generalize the n-th term expression given before:

$$a_n = a_0 \cdot r^n$$

Where a_n is the n-th term, a_0 is the first term, r is the **common ratio**

2 Convergence of Geometric Series

Consider a geometric sequence of

$$\sum_{n=0}^{\infty} a_0 \cdot r^n$$

This series diverge if $|n| \geq 1$, converge if $0 < |n| < 1$. If the series converge, it converge to a value of

$$S = \sum_{n=0}^{\infty} a_0 \cdot r^n = \frac{a_0}{1 - r}$$

2.1 Proof

You don't need to know this for AP exam, this section is for the completeness of knowledge. Let S be the number the series converge to

$$\begin{aligned}
S &= a + ar + ar^2 + ... + ar^n + ... \\
rS &= ar + ar^2 + ar^3 + ... + ar^n + ... \\
rS &= S - a \\
(1-r)S &= a \\
S &= \frac{a}{1-r}
\end{aligned}$$

2.2 Example questions

State if the following geometric series diverge or find the value it converge to:

1. $$\sum_{n=0}^{\infty} 1.1^n$$

   since $1.1 > 1$, this series diverge

2. $$\sum_{n=1}^{\infty} \frac{3}{2^n}$$

   since $\frac{1}{2} < 1$, this series converges, by equation (2.2):

   $$\sum_{n=1}^{\infty} 3\left(\frac{1}{2}\right)^n = \frac{\frac{3}{2}}{1-\frac{1}{2}} = 3$$

3 Practice Problem

Determine if the following series converge, if it converge, find the value it converge to

1. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$

A. Converge to 3

B. Converge to $\dfrac{1}{3}$

C.Diverge

2. $\displaystyle\sum_{n=0}^{\infty} \left(-\frac{e}{\pi}\right)^n$

A. Converge to $\dfrac{\pi}{e}$

B. Converge to $-\dfrac{\pi}{e}$

C. Diverge

3. Consider this geometric series $4+3+\frac{9}{4}+\frac{27}{16}+...$, which value does this series converge to?

A. 16

B. $\frac{16}{3}$

C. diverge

4. A ball is dropped from a height of 10 m, everytime it bounces back to $\frac{4}{5}$ of its original height, what is the total distance (both up and down) the ball travelled when it stops?

A. 90 m

B. 50 m

C. This value is infinite

4 Solution

1. The common ratio $r=\frac{3}{2}>1$, therefore this series diverge. The answer is C

2. Since $0<\left|\frac{-e}{\pi}\right|<1$, the series converge to $\sum_{n=0}^{\infty}\left(-\frac{e}{\pi}\right)^n=\frac{1}{1-\frac{-e}{\pi}}=\frac{\pi}{e}$, the answer is A.

3. The general term of this series is $4\left(\frac{3}{4}\right)^n$, the common ratio $r=\frac{3}{4}<1$, which means the series converge to $S=\frac{4}{1-\frac{3}{4}}=16$, the answer is A.

4. Everytime, the ball will reach a height of $10\left(\frac{4}{5}\right)^n$, where n is the number of bounce. The total distance it bounce is thus

$$2\sum_{n=0}^{\infty}10\cdot\left(\frac{4}{5}\right)^n-10$$

Mulitpling the series by 2 is because the ball bounces both up and down, and subtract 10 because it began to travel through falling, therefore the total distance travelled is $2\frac{10}{1-\frac{4}{5}}-10=90$ m