Partial Fraction

assassin3552

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1 Partial Fraction technique

Welcome to this guide on integration using Partial Fraction of the FiveHive Calculus BC course. This article will guide you through how to preform a partial fraction integration technique.

Partial fraction is a technique that focus on spliting a fraction into a sum of multiple fractions, the example below demonstrate this technique well.

Consider this integral:

$$\int \frac{x+1}{x^2-5x+6}\mathrm{d}x$$

This integral looks scary, but notice that we can factor the denominator: $x^2-5x+6=(x-2)(x-3)$

Let's first take at look at fraction addition:

$$\frac{A}{B}+\frac{C}{D}=\frac{AD+BC}{BD}$$

Since the integrand is a fraction, and we successfully write the denominator as a product, we should be able to split the fraction into a sum of two fraction.

Assume we have split the fraction like this:

$$\frac{x+1}{(x-2)(x-3)}=\frac{A}{(x-2)}+\frac{B}{(x-3)}$$

Here and A and B are different constant, let's try to combine them together:

$$\frac{A(x-3)+B(x-2)}{(x-2)(x-3)}=\frac{Ax-3A+Bx-2B}{(x-2)(x-3)}=\frac{(A+B)x+(-3A-2B)}{(x-2)(x-3)}$$

Notice that the numerator is the same form as before: constant $\cdot\, x$ + constant, which does indeed look like $x+1$ in the original integrand, thus we have this relation:

$$\begin{cases} A+B=1 \\ -3A-2B=1 \end{cases}$$

This linear set of equation can be easily solved, which gives the solution of $A=-3$ and $B=4$. Thus we turned this integrand into a fraction of sum:

$$\int \frac{x+1}{x^2-5x+6}\mathrm{d}x=\int\left(-\frac{3}{(x-2)}+\frac{4}{(x-3)}\right)\mathrm{d}x$$

Then we can evaluate the integral easily:

$$\int\left(-\frac{3}{(x-2)}+\frac{4}{(x-3)}\right)\mathrm{d}x=-3\ln|x-2|+4\ln|x-3|+C$$

In general, partial fraction follow this process:

Consider a integrand where the denominator can be factored, first split the fraction into a sum of fractions:

$$\frac{Ax+B}{(x+C)(x+D)} = \frac{E}{(x+C)} + \frac{F}{(x+D)} = \frac{(E+F)x+(DE+CF)}{(x+C)(x+D)}$$

Where A, B, C, D, E, F are all constants, thus:

$$\begin{cases} E+F=A \\ DE+CF=B \end{cases}$$

Then solve for E and F to complete the fraction split.

Here is another example:

$$\int \frac{2x-1}{x^2-4x+3}\mathrm{d}x = \int \frac{2x-1}{(x-3)(x-1)}\mathrm{d}x$$

Split the fraction into a sum of fraction:

$$\frac{A}{x-3} + \frac{B}{x-1} = \frac{A(x-1)+B(x-3)}{(x-3)(x-1)} = \frac{(A+B)x+(-A-3B)}{(x-3)(x-1)}$$

Solve for A and B:

$$\begin{cases} A+B=2 \\ -A-3B=-1 \end{cases}$$

Thus $A=\frac{5}{2}$ and $B=-\frac{1}{2}$, and the integral turns to

$$\int \left(\frac{5}{2}\frac{1}{x-3} - \frac{1}{2}\frac{1}{x-1}\right)\mathrm{d}x = \frac{5\ln|x-3| - \ln|x-1|}{2} + C$$

2 Practice

Evaluate the following integrals

1. $\displaystyle\int \frac{x}{(x+1)(x+2)}dx$

   A. $\ln|x+1| - 2\ln|x+2| + C$

   B. $-\ln|x+1| + 2\ln|x+2| + C$

   C. $2\ln|x+1| - \ln|x+2| + C$

   D. $-2\ln|x+1| + \ln|x+2| + C$

2. $\displaystyle\int \frac{1}{x^2-1}dx$

   A. $\dfrac{1}{2}\ln\left|x^2-1\right| + C$

   B. $\dfrac{1}{2}\ln\left|\dfrac{x+1}{x-1}\right| + C$

   C. $\dfrac{1}{2}\ln\left|\dfrac{x-1}{x+1}\right| + C$

   D. $\arctan x + C$

3. $\displaystyle\int \frac{3x+1}{x^2-5x+6}dx$

   A. $7\ln|x-2| - 10\ln|x-3| + C$

   B. $-7\ln|x-2| + 10\ln|x-3| + C$

   C. $10\ln|x-2| - 7\ln|x-3| + C$

   D. $-10\ln|x-2| + 7\ln|x-3| + C$

4. $\displaystyle\int_0^1 \frac{3x+2}{x^2-3x-10}dx$

   A. $\dfrac{4(\ln 5 - \ln 4) + 17(\ln 2 - \ln 3)}{7}$

   B. $\dfrac{17(\ln 5 - \ln 4) + 17(\ln 2 - \ln 3)}{7}$

   C. $\dfrac{4(\ln 4 - \ln 5) + 17(\ln 3 - \ln 2)}{7}$

   D. $\dfrac{17(\ln 4 - \ln 5) + 4(\ln 3 - \ln 2)}{7}$

5. $\displaystyle\int_0^1 \frac{5x+4}{x^2+7x+6}dx$

   A. $\dfrac{26(\ln 7 - \ln 6) - \ln 2}{5}$

   B. $\dfrac{\ln 7 - \ln 6 - 26\ln 2}{5}$

   C. $\dfrac{-26(\ln 7 - \ln 6) + \ln 2}{5}$

D. $\dfrac{-\ln 7 + \ln 6 + 26\ln 2}{5}$

3 Solution

1.

$$\begin{aligned}\frac{x}{(x+1)(x+2)} &= \frac{A}{(x+1)} + \frac{B}{(x+2)}\\ &= \frac{A(x+1)+B(x+2)}{(x+1)(x+2)}\\ &= \frac{(A+B)x+(A+2B)}{(x+1)(x+2)}\end{aligned}$$

Thus

$$\begin{cases} A+B=1\\ A+2B=0\end{cases}$$

The solution is $A=-1,\ B=2$, substitute this back:

$$\begin{aligned}\int \frac{x}{(x+1)(x+2)}dx &= \int\left(-\frac{1}{(x+1)}+\frac{2}{x+2}\right)dx\\ &= -\int\frac{1}{x+1}dx + 2\int\frac{1}{x+2}dx\\ &= -\ln|x+1| + 2\ln|x+2| + C\end{aligned}$$

The answer is B.

2.

$$\begin{aligned}\frac{1}{x^2-1} &= \frac{1}{(x+1)(x-1)}\\ &= \frac{A}{(x-1)}+\frac{B}{(x+1)}\\ &= \frac{A(x+1)+B(x-1)}{(x+1)(x-1)}\\ &= \frac{(A+B)x+(A-B)}{(x+1)(x-1)}\end{aligned}$$

Thus

$$\begin{cases} A+B=0\\ A-B=1\end{cases}$$

The solution is $A = \frac{1}{2}$, $B = -\frac{1}{2}$, substitute this back:

$$\begin{aligned}
\int \frac{1}{(x+1)(x-1)} dx &= \int \left(\frac{1}{2(x-1)} - \frac{1}{2(x+1)} \right) dx \\
&= \frac{1}{2} \int \frac{1}{x-1} dx - \frac{1}{2} \int \frac{1}{x+1} dx \\
&= \frac{1}{2} \left(\ln|x-1| - \ln|x+1| \right) + C \\
&= \frac{1}{2} \ln \left| \frac{x-1}{x+1} \right| + C
\end{aligned}$$

The answer is C.

3.

$$\begin{aligned}
\frac{3x+1}{x^2-5x+6} &= \frac{3x+1}{(x-2)(x-3)} \\
&= \frac{A}{(x-2)} + \frac{B}{(x-3)} \\
&= \frac{A(x-3) + B(x-2)}{(x-2)(x-3)} \\
&= \frac{(A+B)x + (-3A-2B)}{(x-2)(x-3)}
\end{aligned}$$

Thus

$$\begin{cases} A + B = 3 \\ -3A - 2B = 1 \end{cases}$$

The solution is $A = -7$, $B = 10$, substitute this back:

$$\begin{aligned}
\int \frac{3x+1}{(x-2)(x-3)} dx &= \int \left(-\frac{7}{x-2} + \frac{10}{x-3} \right) dx \\
&= -7 \int \frac{1}{x-2} dx + 10 \int \frac{1}{x-3} dx \\
&= -7 \ln|x-2| + 10 \ln|x-3| + C
\end{aligned}$$

The answer is B.

4.

$$\begin{aligned}
\frac{3x+2}{x^2-3x-10} &= \frac{3x-2}{(x-5)(x+2)} \\
&= \frac{A}{(x-5)} + \frac{B}{(x+2)} \\
&= \frac{A(x+2) + B(x-5)}{(x+2)(x-5)} \\
&= \frac{(A+B)x + (2A-5B)}{(x+2)(x-5)}
\end{aligned}$$

Thus

$$\begin{cases} A+B=3 \\ 2A-5B=-2 \end{cases}$$

The solution is $A=\frac{17}{7}$, $B=\frac{4}{7}$, substitute this back:

$$\begin{aligned}
\int \frac{3x+2}{(x-5)(x+2)}dx &= \int \left(\frac{17}{7}\frac{1}{x-5}+\frac{4}{7}\frac{1}{x+2}\right)dx \\
&= \frac{17}{7}\int \frac{1}{x-5}dx+\frac{4}{7}\int \frac{1}{x+2}dx \\
&= \frac{17}{7}\ln|x-5|+\frac{4}{7}\ln|x+2|+C
\end{aligned}$$

Hence

$$\begin{aligned}
\int_0^1 \frac{3x+2}{x^2-3x-10}dx &= \frac{17}{7}\ln|x-5|+\frac{4}{7}\ln|x+2|\Big|_0^1 \\
&= \frac{17}{7}\ln|1-5|+\frac{4}{7}\ln|1+2|-\left(\frac{17}{7}\ln|0-5|+\frac{4}{7}\ln|0-2|\right) \\
&= \frac{17\ln 4+4\ln 3-17\ln 5+4\ln 2}{7} \\
&= \frac{17(\ln 4-\ln 5)+4(\ln 3-\ln 2)}{7}
\end{aligned}$$

The answer is D.

5.

$$\begin{aligned}
\frac{5x+4}{x^2+7x+6}dx &= \frac{5x+4}{(x+1)(x+6)} \\
&= \frac{A}{(x+1)}+\frac{B}{(x+6)} \\
&= \frac{A(x+6)+B(x+1)}{(x+1)(x+6)} \\
&= \frac{(A+B)x+(6A+B)}{(x+1)(x+6)}
\end{aligned}$$

Thus

$$\begin{cases} A+B=5 \\ 6A+B=4 \end{cases}$$

The solution is $A=-\frac{1}{5}$, $B=\frac{26}{5}$, substitute this back:

$$\begin{aligned}
\int \frac{5x-4}{(x+1)(x+6)}dx &= \int \left(-\frac{1}{5}\frac{1}{x+1}+\frac{26}{5}\frac{1}{x+6}\right)dx \\
&= -\frac{1}{5}\int \frac{1}{x+1}dx+\frac{26}{5}\int \frac{1}{x+6}dx \\
&= -\frac{1}{5}\ln|x+1|+\frac{26}{5}\ln|x+6|+C
\end{aligned}$$

Hence

$$\begin{aligned}
\int_0^1 \frac{5x+4}{x^2+7x+6}dx &= -\frac{1}{5}\ln|x+1| + \frac{26}{5}\ln|x+6|\Big|_0^1 \\
&= -\frac{1}{5}\ln|1+1| + \frac{26}{5}\ln|1+6| - \left(-\frac{1}{5}\ln|1+0| - \frac{26}{5}\ln|0+6|\right) \\
&= \frac{-\ln 2 + 26\ln 7 + \ln 1 + 26\ln 6}{5} \\
&= \frac{26(\ln 7 - \ln 6) - \ln 2}{5}
\end{aligned}$$

The answer is A.