1 Lagrange Error Bound

Sometimes finding error is also an important part of calculations, Lagrange Error Bound provided a method to estimate the error.

Consider a function f and its nth order Taylor Polynomial, we define the error between the two as such:

$$R_n(x) = f(x) - P_n(x)$$

Where $P_n(x)$ is the Taylor Polynomial to nth degree, Lagrange proved that this error function can be written as:

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$$

Where z is a number within x and c. Practically, it is often impossible to find z, therefore there needs to be some ways to apprximate it.

Let M be a number that satisfy

$$\left|f^{(n+1)}(z)\right| \leq M$$

Thus the maximum value of the error is

$$R_n(x) \leq M\frac{\left|(x-c)^{n+1}\right|}{(n+1)!}$$

This offers a neat way to estimate the error.

In other words, M is the maximum value of the derivative within the interval of x and c.

Example 1:

Estimate the Lagrange error bound of e^x and its 3rd order Taylor polynomial centered at $x = 0$:

The Lagrange error bound can be expressed as such

$$R_3(x) = M\frac{|x-c|^4}{4!}$$

Note the interval of estimation is not directly given, but can be figure out anyway, the Taylor polynomial is centered at $x = 0$, and the apprximation ends at x, therefore the interval is $(0, x)$

Now one need to find M, the maximum value of the derivative within the interval $(0, x)$, the nth derivative of e^x is always e^x, and e^x is an increase function, which means the maximum value M occurs at the end of the interval.

$$\left|f^{n+1}(z)\right| \leq \left|f^{n+1}(x)\right| \leq |e^x| \leq M$$

The Lagrange error bound turns to

$$R_3(x) = \frac{e^x x^4}{4!}$$

To understand this error, it means that the apprximated difference between the real function and apprximated function at point x is $R_3(x)$

2 Practice Problems

1. Estimate the Lagrange error bound of $\sin x$ and its 3rd order Taylor polynomial centered at $x = 0$ over the interval of $(0, \pi)$

A. $\dfrac{\pi x^4}{4!}$

B. $\dfrac{x^4}{4!}$

C. $\dfrac{\pi x^5}{5!}$

D. $\dfrac{x^5}{5!}$

2. Estimate the Lagrange error bound of $\ln(x + 1)$ and its 3rd order Taylor polynomial centered at $x = 0$

A. $\dfrac{x^4}{4(x+1)^4}$

B. $\dfrac{x^4}{(x+1)^4}$

C. $\dfrac{x^3}{3(x+1)^3}$

D. $\dfrac{x^3}{(x+1)^3}$

3 Solution

1. The Lagrange error bound can be expressed as such

$$R_3(x) = M\frac{|x-c|^4}{4!}$$

Now one need to find M, the maximum value of the derivative within the interval $(0, \pi)$. The 4th derivative of $\sin x$ is $\sin x$.

$$\left|f^4(z)\right| \leq |\sin z| \leq 1 \leq M$$

The Lagrange error bound turns to

$$R_3(x) = \frac{x^4}{4!}$$

2. The Lagrange error bound can be expressed as such

$$R_3(x) = M\frac{|x-c|^4}{4!}$$

Now one need to find M, the maximum value of the derivative within the interval $(0, x)$. The 4th derivative of $\ln(x+1)$ is $-\dfrac{6}{(x+1)^4}$.

$$\left|f^4(z)\right| \leq \left|-\frac{6}{(x+1)^4}\right| \leq M$$

The Lagrange error bound turns to

$$R_3(x) = \frac{6}{(x+1)^4}\frac{x^4}{4!} = \frac{x^4}{4(x+1)^4}$$