1 Integral Test for convergence

If f is a **positive**, **continous** and **decreasing** for $x \geq m$, where $m \geq 1$, and the n-th term expression $a_n = f(x)$, then:

$$\sum_{n=1}^{\infty} a_n \text{ and } \int_1^{\infty} f(x)\mathrm{d}x \tag{1.1}$$

both converge or diverge.

1.1 Example questions

1. Evaluate this series:

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1}$$

   First, check for the three criteria: positive, coutinous, decreasing

   (a) Positive: obviously $f(x) = \frac{x}{x^2+1}$ is positive in $[1, \infty)$

   (b) Continous: the function is continous for all real number

   (c) Decreasing: $f'(x) = -\frac{x^2-1}{(x^2+1)^2}$, and $f'(x) < 0$ when $x > 1$

   Now evaluate the indefinite integral:

$$\begin{aligned}\int_1^{\infty} \frac{x}{x^2+1}\mathrm{d}x &= \lim_{b\to\infty} \int_1^{b} \frac{x}{x^2+1}\mathrm{d}x \\ &= \frac{1}{2}\lim_{b\to\infty} \left(\ln\left|b^2+1\right| - \ln\left|1^2+1\right|\right) \\ &= \infty\end{aligned}$$

   Here a u-substitution of $u = x^2 + 1$ and $du = 2x$ is performed. The integral diverges, meaning that the series also diverges

2. Evaluate this series:

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

   Let

$$a_n = f(x) = \frac{1}{x^2+1}$$

   Checking if the method work is omitted, but it does work Then we can construct and solve this improper integral:

$$\begin{aligned}\int_1^{\infty} \frac{1}{x^2+1}\mathrm{d}x &= \lim_{b\to\infty} \int_1^{b} \frac{1}{x^2+1}\mathrm{d}x \\ &= \lim_{b\to\infty} \arctan b - \arctan 1 \\ &= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}\end{aligned}$$

   Meaning the series converge, **but the series doesn't necessarily converge to** $\pi/4$

2 Practice

Determine whether the following series converge or diverge using the integral test.

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

A. Converge

B. Diverge

C. Insufficient information

$$\sum_{n=1}^{\infty} \sin n$$

A. Converge

B. Diverge

C. Insufficient information

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

A. Converge

B. Diverge

C. Insufficient information

$$\sum_{n=1}^{\infty} \frac{6n^2}{n^3 + 1}$$

A. Converge

B. Diverge

C. Insufficient information

$$\sum_{n=1}^{\infty} e^{-n}$$

A. Converge

B. Diverge

C. Insufficient information

3 Solution

1. First check if the series fit the requirement of integral test

Positive: obviously $f(x) = \dfrac{1}{x^2}$ is positive in $[1, \infty)$

Continous: the function is continous for all real number except at $x = 0$

Decreasing: $f'(x) = -\dfrac{2}{x^3}$, and $f'(x) < 0$ when $x > 1$

Thus meaning we can apply the integral test to test the convergence of this series:

$$\int_1^\infty \frac{1}{x^2}dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}dx = \lim_{b\to\infty} -\frac{1}{x}\Big|_1^b = -\lim_{b\to\infty}\frac{1}{b} + 1 = 1$$

Which means this integral converge, the answer is A.

2. First check if the series meet the requirement of integral test:

Positive: $f(x) = \sin x$ is positive in some interval, but not positive in $[1, \infty)$

Continous: the function is continous for all real number

Decreasing: $f'(x) = \cos x$, $f'(x) < 0$ for only certain x

This means the integral test cannot be applied to this test, the answer is C.

3. First check if the series meet the requirement of integral test:

Positive: $f(x) = \dfrac{1}{x \ln x}$ is positive in $[2, \infty)$

Continous: the function is continous in its domain

Decreasing: $f'(x) = \dfrac{\ln x + 1}{(x \ln x)^2}$, where $f'(x) < 0$ when $x > 1$.

This means we can apply the integral test to test the convergence of this series

$$\int_2^\infty \frac{1}{x\ln x}dx = \int_2^\infty \frac{1}{x}\frac{1}{\ln x}dx = \int_{\ln 2}^\infty \frac{1}{u}du = \infty$$

The integral diverge, thus this series diverge. The answer is B.

4. First check if the series meet the requirement of integral test:

Positive: $f(x) = \dfrac{6x^2}{x^3+1}$ is positive in $[1, \infty)$

Continous: the function is continous in its domain

Decreasing: $f'(x) = \dfrac{-6x(x^3-2)}{(x^3+1)^2}$, where $f'(x) < 0$ when $x > \sqrt[3]{2}$.

This means we can apply the integral test to test the convergence of this series.

Let $u = x^3 + 1$, $du = 3x^2$, thus

$$\int_1^\infty \frac{2}{u}du = 2\lim_{b\to\infty}\int_1^b \frac{1}{u}du = 2\lim_{b\to\infty} \ln b\Big|_1^b = 2\lim_{b\to\infty}\ln b = \infty$$

5. First check if the series meet the requirement of integral test:

Positive: $f(x) = e^{-x}$ is positive in $[1, \infty)$

Continous: the function is continous in its domain

Decreasing: $f'(x) = -e^{-x}$, where $f'(x) < 0$ when $x > 1$.

This means we can apply the integral test to test the convergence of this series.

Let $u = -x$, $du = -dx$, thus

$$\int_1^\infty e^{-x} dx = -\lim_{b\to\infty} \int_1^b e^u du = -\lim_{b\to\infty} e^u \Big|_1^b = -\lim_{b\to\infty} e^{-x} \Big|_1^b = -\lim_{b\to\infty} e^{-x} + 0 = \frac{1}{e}$$

The integral converge, this means the series also converge, the answer is A.