Geometric Series

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1 Definition

Consider this sequence

$$3, 6, 12, 24, 48...$$

It is easy to see that there is a common factor of 2 between every term. Thus we can rewrite the sequence like this:

$$3 \cdot 2^0, 3 \cdot 2^1, 3 \cdot 2^2, 3 \cdot 2^3, 3 \cdot 2^4$$

In general, the n-th term expression can be written as (where n starts from 0):

$$a_n = 3 \cdot 2^n$$

If we take the sum of all terms, we arrive at what we called **geometric series**, the quotient of two neiboring terms of a geometric series is a constant (2 in the previous example).

With this knowledge, we can generalize the n-th term expression given before:

$$a_n = a_0 \cdot r^n$$

Where a_n is the n-th term, a_0 is the first term, r is the **common ratio**

2 Convergence of Geometric Series

Consider a geometric sequence of

$$\sum_{n=0}^{\infty} a_0 \cdot r^n$$

This series diverge if $|n| > 1$, converge if $0 < |n| < 1$. If the series converge, it converge to a value of

$$S = \sum_{n=0}^{\infty} a_0 \cdot r^n = \frac{a_0}{1-r}$$

2.1 Example questions

State if the following geometric series diverge or find the value it converge to:

1. $$\sum_{n=0}^{\infty} 1.1^n$$

   since $1.1 > 1$, this series diverge

2.
$$\sum_{n=1}^{\infty} \frac{3}{2^n}$$

since $\frac{1}{2} < 1$, this series converges, by equation (2.2):

$$\sum_{n=1}^{\infty} 3(\frac{1}{2})^n = \frac{\frac{3}{2}}{1-\frac{1}{2}} = 3$$

3.
$$\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$$

since $\frac{3}{2} > 1$, the series diverge

4.
$$\sum_{n=0}^{\infty} \left(-\frac{e}{\pi}\right)^n$$

since $0 < \left|\frac{-e}{\pi}\right| < 1$, the series converge, by equation (2.2):

$$\sum_{n=0}^{\infty} \left(-\frac{e}{\pi}\right)^n = \frac{1}{1-\frac{-e}{\pi}} = \frac{\pi}{e}$$