Ratio Test for Convergence

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Welcome to this guide on Ratio Test for Convergence, this article will guide you through how to apply the ratio test to test for convergence

1 Ratio Test for Convergence

Consider an infinite series:

$$\sum_{n=1}^{\infty} \text{ converge if } \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1 \tag{1.1}$$

$$\sum_{n=1}^{\infty} \text{ diverge if } \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1 \text{ or } \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty \tag{1.2}$$

Note that if the limit is 1, this test cannot draw any conclusion about the convergence and divergence of the series.

1.1 Example Questions

1. Determine the convergence of the series:

   $$\sum_{n=1}^{\infty} \frac{2^n}{n!}$$

   From equation (8.2), we can see that

   $$\begin{aligned}\lim_{n\to\infty}\left|\frac{2^{n+1}}{(n+1)!}\frac{n!}{2^n}\right| &= \lim_{n\to\infty}\left|\frac{2}{(n+1)}\right| \\ &= 0 < 1\end{aligned}$$

   Meaning this series **converge**.

2. Determine the convergence of this series:

   $$\sum_{n=1}^{\infty} \frac{n^2 2^{n+1}}{3^n}$$

   We use the ratio test:

   $$\begin{aligned}\lim_{n\to\infty}\frac{(n+1)^2 2^{n+2}}{3^{n+1}}\frac{3^n}{n^2 2^{n+1}} &= \lim_{n\to\infty}\frac{2}{3}\left(\frac{n+1}{n}\right)^2 \\ &= \frac{2}{3}\left(\lim_{n\to\infty}\frac{n+1}{n}\right)^2 \\ &= \frac{2}{3} < 1\end{aligned}$$

Meaning this series **converge**.

2 Practice Question

Determine if the following series converge or diverge using the ratio test:

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n-1)!}$

A. Converge

B. Diverge

C. Cannot be determined

2. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{10^n}$

A. Converge

B. Diverge

C. Cannot be determined

3. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$

A. Converge

B. Diverge

C. Cannot be determined

4. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n \cdot 2^n}$

A. Converge

B. Diverge

C. Cannot be determined

5. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{3^n}$

A. Converge

B. Diverge

C. Cannot be determined

3 Solutions

1. Apply the ratio Test

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n-1)!}{n!} \right| = \lim_{n\to\infty} \frac{1}{n} = 0$$

The series converge, the answer is A.

2. Apply the ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{10^{n+1}}\frac{10^n}{n!}\right| = \lim_{n\to\infty}\frac{n+1}{10} = \infty$$

The series diverge, the answer is B.

3.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}(n+1)!}{(n+1)^{n+1}}\frac{n^n}{2^n n!}\right| = \lim_{n\to\infty}\frac{2(n+1)n^n}{(n+1)^{n+1}} = 2\lim_{n\to\infty}\frac{n^n}{(n+1)^n} = 2\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n$$

Recall the definition of e

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

It is not hard to see that

$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \frac{1}{e}$$

Thus the limit is

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{2}{e} < 1$$

The series converge, the answer is A.

4.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{3^{n+1}}{(n+1)2^{n+1}}\frac{n2^n}{3^n} = \lim_{n\to\infty}\frac{3}{2}\frac{n}{n+1} = \frac{3}{2} > 1$$

The series diverge, the answer is B

5.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)^2}{3^{n+1}}\frac{3^n}{n^2} = \frac{1}{3}\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^2 = \frac{1}{3} < 1$$

The series converge, the answer is A.