Suggested exercises for stochastic GW background lectures

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Abstract

Some suggested exercises accompanying the lectures on searches for stochastic gravitational-wave backgrounds.

Contents

1. *Practical application of Bayes' theorem*

   Suppose on your last visit to the doctor's office you took a test for some rare disease. This type of disease occurs in only 1 out of 10,000 people, as determined by a random sample of the population. The test that you took is rather effective in that it can correctly identify the presence of the disease 95% of the time, but it gives false positives 1% of the time.

   Suppose the test came up positive. What is the probability that you have the disease?

2. *Comparing frequentist and Bayesian analyses for a constant signal in white noise*

   Consider a constant amplitude signal $a = \text{const}$ in N samples of white Gaussian noise with fixed known variance σ^2. The likelihood functions describing the noise-only and signal+noise models are

$$p(d|\mathcal{M}_0) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^N \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{N} d_i^2\right] \tag{1}$$

   and

$$p(d|a,\mathcal{M}_1) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^N \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{N} (d_i - a)^2\right] \tag{2}$$

   respectively. Asumme for the Bayesian analyses that the amplitude a of the signal is described by prior distribution

$$p(a|\mathcal{M}_1) = \frac{1}{a_{\max}} \tag{3}$$

   This exercise asks you to perform both frequentist and Bayesian analyses for the above signal and noise models, calculating various analytic expressions (see below) and doing numerical calculation for simulated data.

   For the analytic calculations:

   (a) Show that the ML estimator of a is given by

$$\hat{a} \equiv a_{\mathrm{ML}}(d) = \frac{1}{N}\sum_{i=1}^{N} d_i \equiv \bar{d} \tag{4}$$

   (b) Derive the following identity

$$\sum_{i=1}^{N}(d_i - a)^2 = \sum_i d_i^2 - N\hat{a}^2 + N(a-\hat{a})^2 = N\left(\mathrm{Var}[d] + (a-\hat{a})^2\right) \tag{5}$$

   expressing the argument of the exponential in terms of $\hat{a}$.

   (c) Show that the likelihood function $p(d|a,\mathcal{M}_1)$ can be rewritten in terms of $\hat{a}$ as:

$$p(d|a,\mathcal{M}_1) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^N \exp\left[-\frac{\mathrm{Var}[d]}{2\sigma_{\hat{a}}^2}\right] \exp\left[-\frac{(a-\hat{a})^2}{2\sigma_{\hat{a}}^2}\right] \tag{6}$$

   (d) Show that the evidence is given by

$$p(d|\mathcal{M}_1) = \frac{\exp\left[-\frac{\mathrm{Var}[d]}{2\sigma_{\hat{a}}^2}\right]\left[\mathrm{erf}\left(\frac{a_{\max}-\hat{a}}{\sqrt{2}\sigma_{\hat{a}}}\right) + \mathrm{erf}\left(\frac{\hat{a}}{\sqrt{2}\sigma_{\hat{a}}}\right)\right]}{2a_{\max}\left(\sqrt{2\pi}\sigma\right)^{N-1}\sqrt{N}} \tag{7}$$

   (e) Show that the posterior distribution for a is given by

$$p(a|d,\mathcal{M}_1) = \frac{1}{\sqrt{2\pi}\sigma_{\hat{a}}}\exp\left[-\frac{(a-\hat{a})^2}{2\sigma_{\hat{a}}^2}\right] 2\left[\mathrm{erf}\left(\frac{a_{\max}-\hat{a}}{\sqrt{2}\sigma_{\hat{a}}}\right) + \mathrm{erf}\left(\frac{\hat{a}}{\sqrt{2}\sigma_{\hat{a}}}\right)\right]^{-1} \tag{8}$$

(f) Show that the Bayes factor is

$$\begin{aligned}\mathcal{B}_{10}(d) &= \exp\left[\frac{\hat{a}^2}{2\sigma_{\hat{a}}^2}\right]\left(\frac{\sqrt{2\pi}\sigma_{\hat{a}}}{a_{\max}}\right)\frac{1}{2}\left[\operatorname{erf}\left(\frac{a_{\max}-\hat{a}}{\sqrt{2}\sigma_{\hat{a}}}\right)+\operatorname{erf}\left(\frac{\hat{a}}{\sqrt{2}\sigma_{\hat{a}}}\right)\right]\\ &\simeq \exp\left[\frac{\hat{a}^2}{2\sigma_{\hat{a}}^2}\right]\left(\frac{\sqrt{2\pi}\sigma_{\hat{a}}}{a_{\max}}\right)\end{aligned} \tag{9}$$

where the last approximate equality assumes that the likelihood is peaked relative to the prior.

(g) Show that the maximum-likelihood ratio statistic is

$$\Lambda_{\rm ML}(d) = \exp\left(\frac{\hat{a}^2}{2\sigma_{\hat{a}}^2}\right) \tag{10}$$

(h) Show that frequentist test statistic $\Lambda(d)$ constructed from $\Lambda_{\rm ML}(d)$ is

$$\Lambda(d) \equiv 2\ln\Lambda_{\rm ML}(d) = \frac{\hat{a}^2}{\sigma_{\hat{a}}^2} = \left(\frac{\sqrt{N}\bar{d}}{\sigma}\right)^2 \equiv \rho^2 \tag{11}$$

with sampling distributions

$$p(\Lambda|\mathcal{M}_0) = \frac{1}{\sqrt{2\pi\Lambda}}e^{-\Lambda/2} \tag{12}$$

$$p(\Lambda|a,\mathcal{M}_1) = \frac{1}{\sqrt{2\pi\Lambda}}\frac{1}{2}\left[e^{-\frac{1}{2}(\sqrt{\Lambda}-\sqrt{\lambda})^2}+e^{-\frac{1}{2}(\sqrt{\Lambda}+\sqrt{\lambda})^2}\right] \tag{13}$$

where

$$\lambda = \langle\rho\rangle^2 = \frac{Na^2}{\sigma^2} \tag{14}$$

For the numerical simulations, take e.g.,

$$N = 100\,,\quad \sigma = 1\,,\quad 0 \le a \le a_{\max} = 1\,,\quad a_0 = 0.1 = \text{true value}\,, \tag{15}$$

but these values can be changed accordingly to consider stronger (or weaker) injections, etc. Calculate the following quantities:

(a) the value of the threshold Λ_* corresponding to a false alarm probability $\alpha = 0.1$.

(b) the observed value $\Lambda_{\rm obs}$ of the test statistic $\Lambda(d)$ for simulated data, and its corresponding p value.

(c) the frequentist detection probability curve $\gamma(a) \equiv 1-\beta(a)$, and the value $a^{90\%,\rm DP}$ of a needed for 90% detection probability.

(d) the frequentist 95% confidence interval $[\hat{a}-\sigma_{\hat{a}}, \hat{a}+\sigma_{\hat{a}}]$.

(e) the frequentist 90% confidence level upper limit $a^{90\%,\rm UL}$.

(f) the Bayesian 95% credible interval centered on the mode of $p(a|d,\mathcal{M}_1)$.

(g) the Bayesian 90% credible upper limit $a^{90\%,\rm UL}$.

(h) the Bayes factor $\mathcal{B}_{10}(d)$, $2\ln\mathcal{B}_{10}(d)$, and the Laplace approximation of that quantity.

3. *Rate estimate of stellar-mass binary black hole mergers:*

Estimate the total rate (number of events per time) of stellar-mass binary black hole mergers throughout the universe by multiplying LIGO's local rate estimate $R_0 \sim 10$ - $200\ \text{Gpc}^{-3}\,\text{yr}^{-1}$ by the comoving volume out to some large redshift, e.g., $z = 10$. (For this calculation you can ignore any dependence of the rate density with redshift.) You should find a merger rate of $\sim$1 per minute to a few per hour.

Hint: You will need to do numerically evaluate the following integral for proper distance today as a function of source redshift:

$$d_0(z) = \frac{c}{H_0}\int_0^z \frac{\mathrm{d}z'}{E(z')}\,, \qquad E(z) \equiv \sqrt{\Omega_{\mathrm{m}}(1+z)^3 + \Omega_\Lambda}\,, \tag{16}$$

with

$$\Omega_{\mathrm{m}} = 0.31\,, \qquad \Omega_\Lambda = 0.69\,, \qquad H_0 = 68\ \mathrm{km\,s^{-1}\,Mpc^{-1}}\,. \tag{17}$$

Doing that integral, you should find what's shown in Figure 1, which you can then evaluate at $z = 10$ to convert R_0 (number of events per comoving volume per time) to total rate (number of events per time) for sources out to redshift $z = 10$.

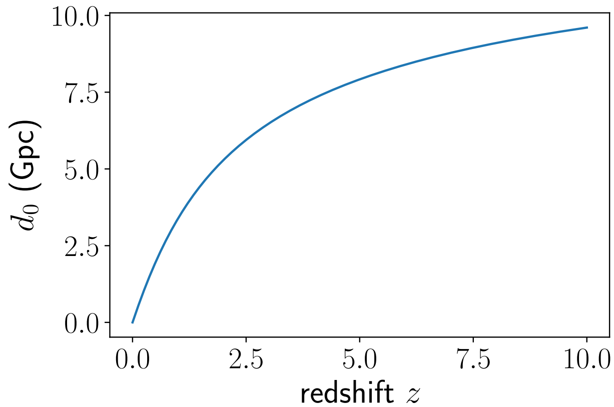

Figure 1:

4. *Relating $S_h(f)$ and $\Omega_{\mathrm{gw}}(f)$:*
   Derive the relationship

$$S_h(f) = \frac{3H_0^2}{2\pi^2}\frac{\Omega_{\mathrm{gw}}(f)}{f^3} \tag{18}$$

   between the strain power spectral density $S_h(f)$ and the dimensionless fractional energy density spectrum $\Omega_{\mathrm{gw}}(f)$. (*Hint*: You will need to use the various definitions of these quantities and also

$$\rho_{\mathrm{gw}} = \frac{c^2}{32\pi G}\langle \dot{h}_{ab}(t,\vec{x})\dot{h}^{ab}(t,\vec{x})\rangle\,, \tag{19}$$

   which expresses the energy-density in gravitational-waves to the metric perturbations $h_{ab}(t,\vec{x})$.)

5. *Cosmology and the "Phinney formula" for astrophysical backgrounds:*
   (a) Using the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left(\frac{\Omega_{\mathrm{m}}}{a^3} + \Omega_\Lambda\right) \tag{20}$$

   for a spatially-flat FRW spacetime with matter and cosmological constant, and the relationship

$$1 + z = \frac{1}{a(t)}\,, \qquad a(t_0) \equiv 1 \quad (t_0 \equiv \text{today})\,, \tag{21}$$

   between redshift z and scale factor $a(t)$, derive

$$\frac{\mathrm{d}t}{\mathrm{d}z} = -\frac{1}{(1+z)H_0E(z)}\,, \qquad E(z) = \sqrt{\Omega_{\mathrm{m}}(1+z)^3 + \Omega_\Lambda}\,. \tag{22}$$

(b) Using this result for $\mathrm{d}t/\mathrm{d}z$, show that

$$\Omega_{\rm gw}(f) = \frac{f}{\rho_{\rm c} H_0} \int_0^\infty \mathrm{d}z\, R(z)\, \frac{1}{(1+z)E(z)} \left(\frac{\mathrm{d}E_{\rm gw}}{\mathrm{d}f_{\rm s}}\right)\bigg|_{f_{\rm s}=f(1+z)} \tag{23}$$

in terms of the rate density $R(z)$ as measured in the source frame (number of events per comoving volume per time interval in the source frame). (*Hint*: The expression for $\mathrm{d}t/\mathrm{d}z$ from part (a) will allow you to go from the "Phinney formula" for $\Omega_{\rm gw}(f)$ written in terms of the number density $n(z)$,

$$\Omega_{\rm gw}(f) = \frac{1}{\rho_c} \int_0^\infty \mathrm{d}z\, n(z)\, \frac{1}{1+z} \left(f_{\rm s}\, \frac{\mathrm{d}E_{\rm gw}}{\mathrm{d}f_{\rm s}}\right)\bigg|_{f_{\rm s}=f(1+z)}, \tag{24}$$

to one in terms of the rate density $R(z)$, where $n(z)\,\mathrm{d}z = R(z)\,|\mathrm{d}t|_{t=t(z)}$. Note: Both of the above expressions for $\Omega_{\rm gw}(f)$ assume that there is only one type of source, described by some set of average source parameters. If there is more than one type of source, one must sum the contributions of each source to $\Omega_{\rm gw}(f)$.)