3. LIGO BBH rate calculation:

$R_0 = 10 - 200 \text{ Gpc}^{-3} \text{ yr}^{-1}$ (local rate estimate)

$$r = R_0 \underbrace{\frac{4}{3}\pi d_0^3(z)}_{\text{comoving volume}}$$

where

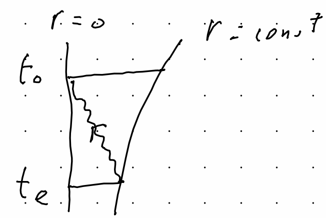
$d_0(z)$ = proper distance today to a source which emitted GWs at redshift $z$

FRW: $ds^2 = -c^2 dt^2 + a^2(t)\left[dr^2 + S_k^2(r)\, d\Omega^2\right]$

$\rightarrow$ $d_0(z) = a(t_0) \int_0^r dr'$

$= a(t_0) r$

$= r$ (taking $a(t_0) = 1$)




Radial photon: $ds^2 = 0 = -c^2 dt^2 + a^2(t)\, dr^2$

$\rightarrow$ $c\,dt = a(t)\, dr$

$$dr = \frac{c\,dt}{a(t)}$$

Thus,

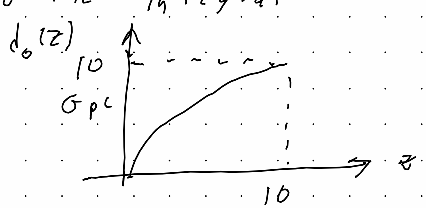
$$d_0(z) = \int_0^r dr' = \int_{t_e}^{t_0} \frac{c\,dt'}{a(t')} \qquad 1+z = \frac{1}{a(t)}$$

$$= \int_z^0 c\,(1+z') \left(\frac{dt'}{dz'}\right) dz'$$

<u>Now</u>: $\frac{dt}{dz} = \frac{-1}{(1+z)\,H_0\,E(z)}$, $E(z) = \sqrt{\Omega_m (1+z)^3 + \Omega_\Lambda}$

$$\rightarrow d_0(z) = \frac{c}{H_0} \int_z^0 \frac{-\cancel{(1+z')}\,dz'}{\cancel{(1+z')}\,E(z)} = \frac{c}{H_0} \int_0^z \frac{dz'}{\sqrt{\Omega_m (1+z')^3 + \Omega_\Lambda}}$$

Do the integral




$\rightarrow d_0(z=10) \approx 10\ \text{Gpc}$

$\rightarrow r = R_0 \frac{4}{3}\pi\,(10\ \text{Gpc})^3$

Now: L160 local rate estimate: $R_0 = 10 - 200 \ \mathrm{Gpc}^{-3}\,\mathrm{yr}^{-1}$

$R_0 = 10$:

$$r \simeq 10 \ \cancel{\mathrm{Gpc}^{-3}}\,\mathrm{yr}^{-1} \ \frac{4}{3}\cancel{\pi}\left(10 \ \cancel{\mathrm{Gpc}}\right)^3$$

$$\simeq 4 \times 10^4 \ \frac{1}{\mathrm{yr}} \underbrace{\left(\frac{1\,\mathrm{yr}}{\pi \times 10^7\,\cancel{s}}\right)\left(\frac{3600\,\cancel{s}}{\mathrm{hr}}\right)}_{\approx\ 10^{-4}\ \frac{1}{\mathrm{hr}}}$$

$$\approx \boxed{\frac{4 \ \mathrm{events}}{\mathrm{hr}}}$$

$R_0 = 200$: $20 \times$ larger

$$r \approx \frac{80 \ \mathrm{events}}{\mathrm{hr}} \approx \boxed{\frac{1 \ \mathrm{event}}{\mathrm{minute}}}$$

4.* Relationship between $S_h(f)$ and $\Omega_{gw}(f)$:

$$h_{ab}(t,\vec{x}) = \int_{-\infty}^{\infty} df \int d^2\Omega_{\hat{k}} \sum_A h_A(f,\hat{k})\, e^A_{ab}(\hat{k})\, e^{i2\pi f(t-\hat{k}\cdot\vec{x}/c)}$$

$$\langle h_A(f,\hat{k})\rangle = 0$$

$$\langle h_A(f,\hat{k})\, h^*_{A'}(f',\hat{k}')\rangle = \frac{1}{16\pi} S_h(f)\, \delta(f-f')\, \delta_{AA'}\, \delta^2(\hat{k},\hat{k}')$$

$$\Omega_{gw}(f) = \frac{1}{\rho_c}\frac{d\rho_{gw}}{d\ln f} = \frac{f}{\rho_c}\frac{d\rho_{gw}}{df}$$

$$\rho_{gw} = \frac{c^2}{32\pi G}\langle \dot{h}_{ab}(t,\vec{x})\, \dot{h}^{ab}(t,\vec{x})\rangle$$

$$= \frac{c^2}{32\pi G}\int_{-\infty}^{\infty} df \int_{-\infty}^{\infty} df' \int d^2\Omega_{\hat{k}} \int d^2\Omega_{\hat{k}'} \sum_A \sum_{A'}$$

$$\langle h_A(f,\hat{k})\, h^*_{A'}(f',\hat{k}')\rangle\, e^A_{ab}(\hat{k})\, e^{A'ab}(\hat{k}')$$

$$(i2\pi f)(-i2\pi f')\, e^{i2\pi f(t-\hat{k}\cdot\vec{x}/c)}\, e^{-i2\pi f'(t-\hat{k}'\cdot\vec{x}/c)}$$

$$= \frac{c^2}{32\pi G}\int_{-\infty}^{\infty} df \int d^2\Omega_{\hat{k}} \sum_A e^A_{ab}(\hat{k})\, e^{Aab}(\hat{k})\, 4\pi^2 f^2\, \frac{1}{16\pi} S_h(f)$$

Now: $e^{+}_{ab}(\hat{n}) = l_a l_b - m_a m_b$  $\{\hat{n}, \hat{l}, \hat{m}\}$ right-hand set of unit vectors

$e^{\times}_{ab}(\hat{n}) = l_a m_b + m_a l_b$

$$e^{+}_{ab}(\hat{n})\, e^{+ab}(\hat{n}) = (\hat{l}\cdot\hat{l})^2 - 2\cancelto{0}{(\hat{l}\cdot\hat{m})}^2 + (\hat{m}\cdot\hat{m})^2 = 2$$

$$e^{\times}_{ab}(\hat{n})\, e^{\times ab}(\hat{n}) = 2(\hat{l}\cdot\hat{l})(\hat{m}\cdot\hat{m}) + 2\cancelto{0}{(\hat{l}\cdot\hat{m})}^2 = 2$$

thus,

$$\rho_{gw} = \frac{c^2}{32\pi G}\, \cancel{4}\,\cancel{\pi}\cdot\cancel{4}\, \frac{1}{\cancel{16\pi}} \int d^2\Omega_{\hat{n}} \int_{-\infty}^{\infty} df\, f^2 S_h(f)$$

$$= \frac{c^2}{32G}\cdot 4\pi \int_{-\infty}^{\infty} df\, f^2 S_h(f)$$

$$= \frac{\pi c^2}{8G} \int_{-\infty}^{\infty} df\, f^2 S_h(f)$$

Now: $\rho_c = \dfrac{3H_0^2 c^2}{8\pi G} \longrightarrow \left(\dfrac{\pi c^2}{8G}\right)\cdot\dfrac{\pi}{\pi} = \dfrac{\pi^2 c^2}{8\pi G} = \dfrac{\pi^2 \rho_c}{3H_0^2}$

Thus, $$\rho_{gw} = \frac{\pi^2 \rho_c}{3H_0^2} \int_{-\infty}^{\infty} df\, f^2 S_h(f)$$

$$= \frac{2\pi^2}{3H_0^2}\,\rho_c \int_0^{\infty} \frac{df}{f}\, f^3 S_h(f)$$

Compare to

$$\rho_{gw} = \int_0^\infty df \left(\frac{d\rho_{gw}}{df}\right) = \frac{2\pi^2}{3H_0^2}\,\rho_c \int_0^\infty \frac{df}{f}\, f^3 S_h(f)$$

Thus, $\displaystyle \frac{d\rho_{gw}}{df} = \frac{2\pi^2}{3H_0^2}\,\rho_c\,\frac{f^3 S_h(f)}{f}$

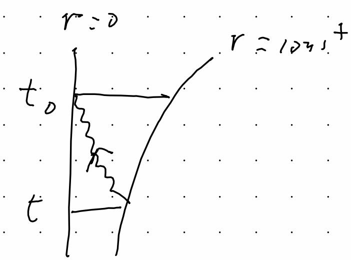
$$\rightarrow \quad \Omega_{gw}(f) = \frac{f}{\rho_c}\frac{d\rho_{gw}}{df} = \frac{2\pi^2}{3H_0^2}\, f^3 S_h(f)$$

or $$\boxed{S_h(f) = \frac{3H_0^2}{2\pi^2}\,\frac{\Omega_{gw}(f)}{f^3}}$$

5. $\frac{dt}{dz}$ calculation and Phinney formula in terms of $R(z)$

a) Friedmann equation

$$\left(\frac{\dot a}{a}\right)^2 = H_0^2\left(\frac{\Omega_m}{a^3} + \Omega_\Lambda\right)$$




$$\rightarrow \quad \frac{\dot a}{a} = H_0\sqrt{\frac{\Omega_m}{a^3} + \Omega_\Lambda}$$

$1 + z = \frac{a(t_0)}{a(t)} = \frac{1}{a(t)}$ where $a(t_0) \equiv 1$ ($t_0$ = today)

and $t$ = time of emission

Rewrite Friedmann's equation in terms of $z$

$$LHS = \frac{\dot a}{a} = \frac{1}{a}\frac{da}{dt} = (1+z)\frac{d}{dt}\left(\frac{1}{1+z}\right)$$

$$= (1+z)\frac{-1}{(1+z)^2}\frac{dz}{dt} = \frac{-1}{1+z}\frac{dz}{dt}$$

$$RHS = H_0\sqrt{\frac{\Omega_m}{a^3} + \Omega_\Lambda} = H_0\sqrt{\Omega_m(1+z)^3 + \Omega_\Lambda} \equiv H_0 E(z)$$

Thus, $-\frac{1}{1+z}\frac{dz}{dt} = H_0 E(z) \rightarrow \boxed{\frac{dt}{dz} = \frac{-1}{(1+z)H_0 E(z)} \text{ where } E(z) \equiv \sqrt{\Omega_m(1+z)^3 + \Omega_\Lambda}}$

b) Phinney formula in terms of number density $n(z)$:

$$\Omega_{gw}(f) = \frac{1}{\rho_c} \int_0^\infty dz\, n(z) \frac{1}{1+z} \left( f_s \frac{dE_{gw}}{df_s} \right) \Bigg|_{f_s = f(1+z)}$$

Now: $\underbrace{n(z)}_{\text{number density}}\, dz = \underbrace{R(z)}_{\text{rate density}} |dt|$

$$\rightarrow \quad n(z) = R(z) \left| \frac{dt}{dz} \right| = \frac{R(z)}{(1+z) H_0 E(z)}$$

Thus,

$$\Omega_{gw}(f) = \frac{1}{\rho_c} \int_0^\infty dz \frac{R(z)}{(1+z) H_0 E(z)} \left( \frac{1}{1+z} \right) \left( f_s \frac{dE_{gw}}{df_s} \right) \Bigg|_{f_s = f(1+z)}$$

$$= \frac{1}{\rho_c H_0} \int_0^\infty dz \frac{R(z)}{E(z)} \frac{1}{(1+z)^{\cancel{2}}} f \cancel{(1+z)} \left( \frac{dE_{gw}}{df_s} \right) \Bigg|_{f_s = f(1+z)}$$

$$= \frac{f}{\rho_c H_0} \int_0^\infty dz \frac{R(z)}{(1+z) E(z)} \left( \frac{dE_{gw}}{df_s} \right) \Bigg|_{f_s = f(1+z)}$$