Suggested exercises for stochastic GW background lectures

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Abstract

Some suggested exercises accompanying the lectures on searches for stochastic gravitational-wave backgrounds.

Contents

1. *Practical application of Bayes' theorem*

   Suppose on your last visit to the doctor's office you took a test for some rare disease. This type of disease occurs in only 1 out of 10,000 people, as determined by a random sample of the population. The test that you took is rather effective in that it can correctly identify the presence of the disease 95% of the time, but it gives false positives 1% of the time.

   Suppose the test came up positive. What is the probability that you have the disease?

2. *Comparing frequentist and Bayesian analyses for a constant signal in white noise*

   Consider a constant amplitude signal a in white Gaussian noise with variance σ^2 described by the likelihood functions

$$p(d|\mathcal{M}_0) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^N \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{N} d_i^2\right] \tag{1}$$

   Perform both frequentist and Bayesian analyses for the above signal and noise models, doing both analytic and numerical calculations of relevant quantities.

3. *Rate estimate of stellar-mass binary black hole mergers:*

   Estimate the total rate (number of events per time) of stellar-mass binary black hole mergers throughout the universe by multiplying LIGO's local rate estimate $R_0 \sim 10$ - $200\ \mathrm{Gpc}^{-3}\,\mathrm{yr}^{-1}$ by the comoving volume out to some large redshift, e.g., $z = 10$. (For this calculation you can ignore any dependence of the rate density with redshift.) You should find a merger rate of $\sim$1 per minute to a few per hour.

   Hint: You will need to do numerically evaluate the following integral for proper distance today as a function of source redshift:

$$d_0(z) = \frac{c}{H_0}\int_0^z \frac{\mathrm{d}z'}{E(z')}\,, \qquad E(z) \equiv \sqrt{\Omega_\mathrm{m}(1+z)^3 + \Omega_\Lambda}\,, \tag{2}$$

   with

$$\Omega_\mathrm{m} = 0.31\,, \qquad \Omega_\Lambda = 0.69\,, \qquad H_0 = 68\ \mathrm{km\,s^{-1}\,Mpc^{-1}}\,. \tag{3}$$

   Doing that integral, you should find what's shown in Figure 1, which you can then evaluate at $z = 10$ to convert R_0 (number of events per comoving volume per time) to total rate (number of events per time) for sources out to redshift $z = 10$.

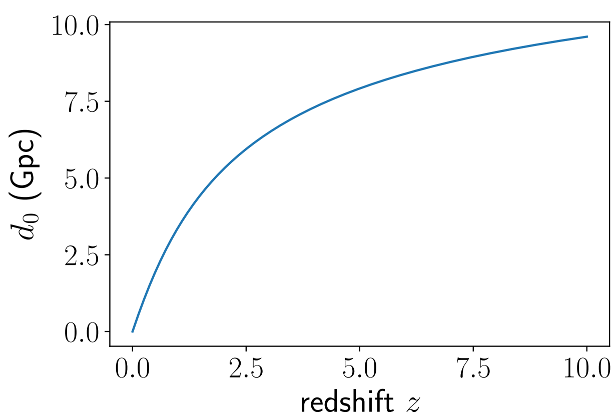

Figure 1:

4. *Relating $S_h(f)$ and $\Omega_\mathrm{gw}(f)$:*

Derive the relationship

$$S_h(f) = \frac{3H_0^2}{2\pi^2}\frac{\Omega_{\rm gw}(f)}{f^3} \tag{4}$$

between the strain power spectral density $S_h(f)$ and the dimensionless fractional energy density spectrum $\Omega_{\rm gw}(f)$. (*Hint*: You will need to use the various definitions of these quantities and also

$$\rho_{\rm gw} = \frac{c^2}{32\pi G}\langle \dot{h}_{ab}(t,\vec{x})\dot{h}^{ab}(t,\vec{x})\rangle\,, \tag{5}$$

which expresses the energy-density in gravitational-waves to the metric perturbations $h_{ab}(t,\vec{x})$.)

5. *Cosmology and the "Phinney formula" for astrophysical backgrounds:*

(a) Using the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left(\frac{\Omega_{\rm m}}{a^3} + \Omega_\Lambda\right) \tag{6}$$

for a spatially-flat FRW spacetime with matter and cosmological constant, and the relationship

$$1+z = \frac{1}{a(t)}\,, \qquad a(t_0)\equiv 1 \quad (t_0 \equiv \text{today})\,, \tag{7}$$

between redshift z and scale factor $a(t)$, derive

$$\frac{{\rm d}t}{{\rm d}z} = -\frac{1}{(1+z)H_0E(z)}\,, \qquad E(z) = \sqrt{\Omega_{\rm m}(1+z)^3+\Omega_\Lambda}\,. \tag{8}$$

(b) Using this result for ${\rm d}t/{\rm d}z$, show that

$$\Omega_{\rm gw}(f) = \frac{f}{\rho_{\rm c}H_0}\int_0^\infty {\rm d}z\, R(z)\,\frac{1}{(1+z)E(z)}\left(\frac{{\rm d}E_{\rm gw}}{{\rm d}f_{\rm s}}\right)\bigg|_{f_{\rm s}=f(1+z)} \tag{9}$$

in terms of the rate density $R(z)$ as measured in the source frame (number of events per comoving volume per time interval in the source frame). (*Hint*: The expression for ${\rm d}t/{\rm d}z$ from part (a) will allow you to go from the "Phinney formula" for $\Omega_{\rm gw}(f)$ written in terms of the number density $n(z)$,

$$\Omega_{\rm gw}(f) = \frac{1}{\rho_c}\int_0^\infty {\rm d}z\, n(z)\,\frac{1}{1+z}\left(f_{\rm s}\,\frac{{\rm d}E_{\rm gw}}{{\rm d}f_{\rm s}}\right)\bigg|_{f_{\rm s}=f(1+z)}\,, \tag{10}$$

to one in terms of the rate density $R(z)$, where $n(z)\,{\rm d}z = R(z)\,|{\rm d}t|_{t=t(z)}$. Note: Both of the above expressions for $\Omega_{\rm gw}(f)$ assume that there is only one type of source, described by some set of average source parameters. If there is more than one type of source, one must sum the contributions of each source to $\Omega_{\rm gw}(f)$.)