(1.) LIGO BBH rate calculation:

$R_0 = 10 - 200 \ Gpc^{-3} \ yr^{-1}$ (local rate estimate)

$$r = R_0 \underbrace{\frac{4}{3}\pi d_0^3(z)}_{\text{comoving volume}}$$

where

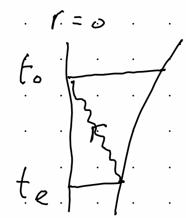
$d_0(z)$ = proper distance today to a source which emitted GWs at redshift z

FRW: $ds^2 = -c^2 dt^2 + a^2(t)\left[dr^2 + S_k^2(r)\, d\Omega^2\right]$

$$\rightarrow \quad d_0(z) = a(t_0)\int_0^r dr'$$

$$= a(t_0)\, r$$

$$= r \quad (\text{taking } a(t_0)=1)$$



Radial photon: $ds^2 = 0 = -c^2 dt^2 + a^2(t)\, dr^2$

$$\rightarrow \quad c\,dt = a(t)\, dr$$

$$dr = \frac{c\,dt}{a(t)}$$

Thus,

$$d_0(z) = \int_0^r dr'$$

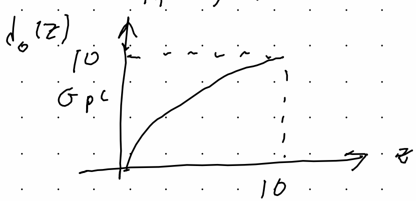
$$= \int_{t_e}^{t_0} \frac{c\,dt'}{a(t')} \qquad\qquad 1+z = \frac{1}{a(t)}$$

$$= \int_z^0 c\,(1+z')\left(\frac{dt'}{dz'}\right) dz'$$

<u>Now</u>: $$\frac{dt}{dz} = \frac{-1}{(1+z)\,H_0\,E(z)}\,, \qquad E(z) = \sqrt{\Omega_m(1+z)^3 + \Omega_\Lambda}$$

$$\rightarrow d_0(z) = \frac{c}{H_0}\int_z^0 \frac{-\cancel{(1+z')}\,dz'}{\cancel{(1+z')}\,E(z)} = \frac{c}{H_0}\int_0^z \frac{dz'}{\sqrt{\Omega_m(1+z)^3 + \Omega_\Lambda}}$$

Do the integral



$$\rightarrow d_0(z=10) \approx 10\ \mathrm{Gpc}$$

$$\rightarrow r = R_0\,\frac{4}{3}\pi\,(10\ \mathrm{Gpc})^3$$

Now: L1.60 local rate estimate: $R_0 = 10 - 200 \ Gpc^{-3} yr^{-1}$

$R_0 = 10$:

$$r \simeq 10 \cancel{Gpc^{-3}} yr^{-1} \frac{4}{3}\pi (10 \cancel{Gpc})^3$$

$$\simeq 4 \times 10^4 \frac{1}{yr} \underbrace{\left(\frac{1 yr}{\pi \times 10^7 \cancel{s}}\right)\left(\frac{3600 \cancel{s}}{hr}\right)}_{\simeq 10^{-4} \frac{1}{hr}}$$

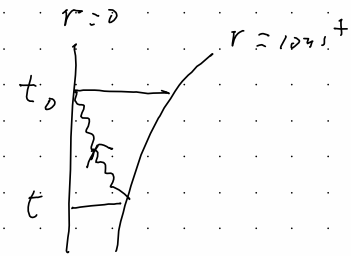
$$\simeq \boxed{\frac{4 \text{ events}}{hr}}$$

$R_0 = 200$: 20 x larger

$$r \simeq \frac{80 \text{ events}}{hr} \simeq \boxed{\frac{1 \text{ event}}{\text{minute}}}$$

(2.) $\frac{dt}{dz}$ calculation and Phinney formula in terms of $R(z)$

a) Friedmann equation



$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left(\frac{\Omega_m}{a^3} + \Omega_\Lambda\right)$$

$$\rightarrow \quad \frac{\dot{a}}{a} = H_0\sqrt{\frac{\Omega_m}{a^3} + \Omega_\Lambda}$$

$$1+z = \frac{a(t_0)}{a(t)} = \frac{1}{a(t)}$$ where $a(t_0) \equiv 1$ (t_0 = today)

and t = time of emission

Rewrite Friedmann's equation in terms of z

$$\text{LHS} = \frac{\dot{a}}{a} = \frac{1}{a}\frac{da}{dt} = (1+z)\frac{d}{dt}\left(\frac{1}{1+z}\right)$$

$$= (1+z)\frac{-1}{(1+z)^2}\frac{dz}{dt} = \frac{-1}{1+z}\frac{dz}{dt}$$

$$\text{RHS} = H_0\sqrt{\frac{\Omega_m}{a^3} + \Omega_\Lambda} = H_0\sqrt{\Omega_m(1+z)^3 + \Omega_\Lambda} \equiv H_0 E(z)$$

Thus,

$$-\frac{1}{1+z}\frac{dz}{dt} = H_0 E(z) \rightarrow \boxed{\frac{dt}{dz} = \frac{-1}{(1+z)H_0 E(z)} \quad \text{where } E(z) \equiv \sqrt{\Omega_m(1+z)^3 + \Omega_\Lambda}}$$

b) Phinney formula in terms of number density $n(z)$:

$$\Omega_{gw}(f) = \frac{1}{\rho_c}\int_0^\infty dz\, n(z)\, \frac{1}{1+z}\left(f_s \frac{dE_{gw}}{df_s}\right)\Bigg|_{f_s = f(1+z)}$$

<u>Now</u>: $\underbrace{n(z)}_{\text{number density}}\, dz = \underbrace{R(z)}_{\text{rate density}}\, |dt|$

$$\rightarrow\ n(z) = R(z)\left|\frac{dt}{dz}\right| = \frac{R(z)}{(1+z)H_0 E(z)}$$

Thus,

$$\Omega_{gw}(f) = \frac{1}{\rho_c}\int_0^\infty dz\, \frac{R(z)}{(1+z)H_0E(z)}\left(\frac{1}{1+z}\right)\left(f_s \frac{dE_{gw}}{df_s}\right)\Bigg|_{f_s = f(1+z)}$$

$$= \frac{1}{\rho_c H_0}\int_0^\infty dz\, \frac{R(z)}{E(z)}\frac{1}{(1+z)^{\cancel{2}}}\, f\cancel{(1+z)}\left(\frac{dE_{gw}}{df_s}\right)\Bigg|_{f_s = f(1+z)}$$

$$= \frac{f}{\rho_c H_0}\int_0^\infty dz\, \frac{R(z)}{(1+z)E(z)}\left(\frac{dE_{gw}}{df_s}\right)\Bigg|_{f_s = f(1+z)}$$

(3.) Relationship between $S_h(f)$ and $\Omega_{gw}(f)$:

$$h_{ab}(t,\vec{x}) = \int_{-\infty}^{\infty} df \int d^2\Omega_{\hat{k}} \sum_A h_A(f,\hat{k})\, e^A_{ab}(\hat{k})\, e^{i2\pi f(t-\hat{k}\cdot\vec{x}/c)}$$

$$\langle h_A(f,\hat{k})\rangle = 0$$

$$\langle h_A(f,\hat{k})\, h^*_{A'}(f',\hat{k}')\rangle = \frac{1}{16\pi} S_h(f)\, \delta(f-f')\, \delta_{AA'}\, \delta^2(\hat{k},\hat{k}')$$

$$\Omega_{gw}(f) = \frac{1}{\rho_c}\frac{d\rho_{gw}}{d\ln f} = \frac{f}{\rho_c}\frac{d\rho_{gw}}{df}$$

$$\rho_{gw} = \frac{c^2}{32\pi G}\langle \dot{h}_{ab}(t,\vec{x})\, \dot{h}^{ab}(t,\vec{x})\rangle$$

$$= \frac{c^2}{32\pi G}\int_{-\infty}^{\infty} df \int_{-\infty}^{\infty} df' \int d^2\Omega_{\hat{k}} \int d^2\Omega_{\hat{k}'} \sum_A \sum_{A'}$$

$$\langle h_A(f,\hat{k})\, h^*_{A'}(f',\hat{k}')\rangle\, e^A_{ab}(\hat{k})\, e^{A'ab}(\hat{k}')$$

$$(i2\pi f)(-i2\pi f')\, e^{i2\pi f(t-\hat{k}\cdot\vec{x}/c)}\, e^{-i2\pi f'(t-\hat{k}'\cdot\vec{x}/c)}$$

$$= \frac{c^2}{32\pi G}\int_{-\infty}^{\infty} df \int d^2\Omega_{\hat{k}} \sum_A e^A_{ab}(\hat{k})\, e^{Aab}(\hat{k})\, 4\pi^2 f^2\, \frac{1}{16\pi} S_h(f)$$

Now:
$$e^{+}_{ab}(\hat{n}) = l_a l_b - m_a m_b \qquad \{\hat{n}, \hat{l}, \hat{m}\} \text{ right-hand set of unit vectors}$$
$$e^{\times}_{ab}(\hat{n}) = l_a m_b + m_a l_b$$

$$e^{+}_{ab}(\hat{n})\, e^{+ab}(\hat{n}) = (\hat{l}\cdot\hat{l})^2 - 2\cancel{(\hat{l}\cdot\hat{m})^2}^{\,0} + (\hat{m}\cdot\hat{m})^2 = 2$$
$$e^{\times}_{ab}(\hat{n})\, e^{\times ab}(\hat{n}) = 2(\hat{l}\cdot\hat{l})(\hat{m}\cdot\hat{m}) + 2\cancel{(\hat{l}\cdot\hat{m})}^{\,0} = 2$$

thus,
$$\rho_{gw} = \frac{c^2}{32\pi G}\, \cancel{4}\cancel{\pi}\cdot\cancel{4}\, \frac{1}{\cancel{16\pi}} \int d^2\Omega_{\hat{n}} \int_{-\infty}^{\infty} df\, f^2 S_h(f)$$
$$= \frac{c^2}{32G}\cdot 4\pi \int_{-\infty}^{\infty} df\, f^2 S_h(f)$$
$$= \frac{\pi c^2}{8G} \int_{-\infty}^{\infty} df\, f^2 S_h(f)$$

Now!
$$\rho_c = \frac{3H_0^2 c^2}{8\pi G} \longrightarrow \left(\frac{\pi c^2}{8G}\right)\cdot\frac{\pi}{\pi} = \frac{\pi^2 c^2}{8\pi G} = \frac{\pi^2 \rho_c}{3H_0^2}$$

Thus,
$$\rho_{gw} = \frac{\pi^2 \rho_c}{3H_0^2} \int_{-\infty}^{\infty} df\, f^2 S_h(f)$$
$$= \frac{2\pi^2}{3H_0^2}\,\rho_c \int_0^{\infty} \frac{df}{f}\, f^3 S_h(f)$$

Compare to

$$\rho_{gw} = \int_0^\infty df \left(\frac{d\rho_{gw}}{df}\right) = \frac{2\pi^2}{3H_0^2}\rho_c \int_0^\infty \frac{df}{f} f^3 S_h(f)$$

Thus, $\displaystyle \frac{d\rho_{gw}}{df} = \frac{2\pi^2}{3H_0^2}\rho_c \frac{f^3 S_h(f)}{f}$

$$\rightarrow \Omega_{gw}(f) = \frac{f}{\rho_c}\frac{d\rho_{gw}}{df} = \frac{2\pi^2}{3H_0^2} f^3 S_h(f)$$

or $\displaystyle \boxed{S_h(f) = \frac{3H_0^2}{2\pi^2}\frac{\Omega_{gw}(f)}{f^3}}$