Solutions to exercises

(1) Bayes' theorem example

— Suppose you test positive for a rare disease (1 in 10,000 people have the disease on average). The probability that the test comes out positive if you have the disease is $0.95 = p(+|H)$.

The probability that the test comes out positive if you don't have the disease is $0.01 = p(+|\bar{H})$.

What is the probability that you have the disease?

want to determine

$p(H \mid +)$ = prob. that you have the disease given that you tested +

$$= \frac{p(+ \mid H)\, p(H)}{p(+)}$$

where $p(+ \mid H) = 0.95$

$p(H) = 0.0001$

$p(+) = p(+ \mid H)\, p(H) + p(+ \mid \bar{H})\, p(\bar{H})$

$= 0.95 \times 0.0001 + 0.01 \times 0.999$

$\approx 0.0001 + 0.01$

≈ 0.01

Thus, $p(H \mid +) \approx \dfrac{0.95 \times 0.0001}{0.01}$

$= 0.95 \times 0.01$

≈ 0.01 — so $\frac{1}{100}$ instead of $\frac{1}{10{,}000}$

(2) Frequentist vs. Bayesian analyses for a simple example

$$p(d \mid M_0) = \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^N \exp\left[-\frac{1}{2\sigma^2}\sum_i d_i^2\right]$$

$$p(d \mid a, M_1) = \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^N \exp\left[-\frac{1}{2\sigma^2}\sum_i (d_i - a)^2\right]$$

Maximum likelihood estimator of a:

$$0 = \left.\frac{dp}{da}\right|_{a=\hat{a}} = \left.(\;)\exp[\;]\,\left(-\frac{1}{2\sigma^2}\right)2\sum_i (d_i - a)\right|_{a=\hat{a}}$$

$$\Leftrightarrow \quad 0 = \left.\sum_i (d_i - a)\right|_{a=\hat{a}}$$

$$= \sum_i d_i - \hat{a}\,N$$

$$\rightarrow \quad \boxed{\hat{a} = \frac{1}{N}\sum_i d_i}$$

$$\sum_i (d_i - a)^2 = \sum_i \left(d_i^2 + a^2 - 2a d_i\right)$$

$$= \sum_i d_i^2 + N a^2 - 2a \sum_i d_i$$

$$= N\left(\frac{1}{N}\sum_i d_i^2\right) + N a^2 - 2aN\left(\frac{1}{N}\sum_i d_i\right)$$

$$= N\left[\frac{1}{N}\sum_i d_i^2 + a^2 - 2a\hat{a}\right]$$

Now $\mathrm{Var}[d] = \frac{1}{N}\sum_i (d_i - \bar{d})^2$

$$= \frac{1}{N}\sum_i (d_i - \hat{a})^2$$

$$= \frac{1}{N}\sum_i \left(d_i^2 + \hat{a}^2 - 2\hat{a} d_i\right)$$

$$= \frac{1}{N}\sum_i d_i^2 + \hat{a}^2 - 2\hat{a}^2$$

$$= \frac{1}{N}\sum_i d_i^2 - \hat{a}^2$$

$$\rightarrow \sum_i (d_i - a)^2 = N\left[\mathrm{Var}[d] + (a - \hat{a})^2\right]$$

thus,

$$p(d|a, M_1) = \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^N \exp\left[-\frac{1}{2\sigma^2}\sum_i (d_i - a)^2\right]$$

$$= \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^N \exp\left[-\frac{N}{2\sigma^2}\left(\mathrm{Var}[d] + (a-\hat{a})^2\right)\right]$$

$$= \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^N \exp\left[-\frac{\mathrm{Var}[d]}{2\sigma_{\hat{a}}^2}\right] \underbrace{\exp\left[-\frac{1}{2\sigma_{\hat{a}}^2}(a-\hat{a})^2\right]}_{\text{parameter } a \text{ only enters here}}$$

Evidence:

$$p(d|M_1) = \int_0^\infty da\; p(d|a, M_1)\; p(a|M_1)$$

$$p(a|M_1) = \begin{cases} \frac{1}{a_{max}} & 0 \le a \le a_{max} \\ 0 & a < a_{max} \end{cases}$$

$$= \frac{1}{a_{max}} \int_0^{a_{max}} da\; p(d|a, M_1)$$

$$= \frac{1}{a_{max}} \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^N \exp\left[\frac{-\mathrm{Var}[d]}{2\sigma_{\hat{a}}^2}\right] \int_0^{a_{max}} da\, \exp\left[-\frac{1}{2\sigma_{\hat{a}}^2}(a-\hat{a})^2\right]$$

Now:

$$\int_0^{a_{max}} da \, \exp\left[-\frac{1}{2\sigma_{\hat{a}}^2}(a-\hat{a})^2\right] = \int_{-\hat{a}}^{a_{max}-\hat{a}} dx \, \exp\left[-\frac{x^2}{2\sigma_{\hat{a}}^2}\right]$$

Let $x = a - \hat{a}$, $dx = d$

$a = 0,\ a_{max} \rightarrow x = -\hat{a},\ a_{max} - \hat{a}$

Let $t = \dfrac{x}{\sqrt{2}\sigma_{\hat{a}}} \rightarrow dt = \dfrac{dx}{\sqrt{2}\sigma_{\hat{a}}}$

$$= \int_{-\frac{\hat{a}}{\sqrt{2}\sigma_{\hat{a}}}}^{(a_{max}-\hat{a})/\sqrt{2}\sigma_{\hat{a}}} dt \, \sqrt{2}\sigma_{\hat{a}} \, e^{-t^2}$$

$$= \sqrt{2}\sigma_{\hat{a}} \int_{-\frac{\hat{a}}{\sqrt{2}\sigma_{\hat{a}}}}^{\frac{a_{max}-\hat{a}}{\sqrt{2}\sigma_{\hat{a}}}} dt \, e^{-t^2}$$

$$= \sqrt{2}\sigma_{\hat{a}} \left[\int_{-\frac{\hat{a}}{\sqrt{2}\sigma_{\hat{a}}}}^{0} + \int_{0}^{(a_{max}-\hat{a})/\sqrt{2}\sigma_{\hat{a}}}\right] dt \, e^{-t^2}$$

Write in terms of $\mathrm{erf}(z)$:

$$\mathrm{erf}(z) \equiv \frac{2}{\sqrt{\pi}} \int_0^z dt \, e^{-t^2}$$

$$\int_0^{a_{max}} da \; \exp\left[-\frac{1}{2\sigma_{\hat{a}}^2}(a-\hat{a})^2\right] = \sqrt{2}\,\sigma_{\hat{a}} \left[\int_0^{(a_{max}-\hat{a})/\sqrt{2}\sigma_{\hat{a}}} - \int_0^{-a_{max}/\sqrt{2}\sigma_{\hat{a}}}\right] dt\, e^{-t^2}$$

$$= \sqrt{2}\,\sigma_{\hat{a}} \frac{\sqrt{\pi}}{2}\left[\operatorname{erf}\left(\frac{a_{max}-\hat{a}}{\sqrt{2}\,\sigma_{\hat{a}}}\right) + \operatorname{erf}\left(\frac{\hat{a}}{\sqrt{2}\,\sigma_{\hat{a}}}\right)\right]$$

thus,

$$p(d|M_1) = \frac{1}{a_{max}}\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^N \exp\left(-\frac{Var[d]}{2\sigma_{\hat{a}}^2}\right)\frac{\sigma_{\hat{a}}\sqrt{\pi}}{\sqrt{2}}\left[\operatorname{erf}\left(\frac{a_{max}-\hat{a}}{\sqrt{2}\,\sigma_{\hat{a}}}\right) + \operatorname{erf}\left(\frac{\hat{a}}{\sqrt{2}\,\sigma_{\hat{a}}}\right)\right]$$

$$= \frac{\exp\left(-\frac{Var[d]}{2\sigma_{\hat{a}}^2}\right)\left[\operatorname{erf}\left(\frac{a_{max}-\hat{a}}{\sqrt{2}\,\sigma_{\hat{a}}}\right) + \operatorname{erf}\left(\frac{\hat{a}}{\sqrt{2}\,\sigma_{\hat{a}}}\right)\right]}{\sqrt{2}\,(\sqrt{2\pi})^N \sigma^N \frac{\sqrt{N}}{\sigma\sqrt{\pi}}\, a_{max}}$$

$$= \boxed{\frac{\exp\left(-\frac{Var[d]}{2\sigma_{\hat{a}}^2}\right)\left[\operatorname{erf}\left(\frac{a_{max}-\hat{a}}{\sqrt{2}\,\sigma_{\hat{a}}}\right) + \operatorname{erf}\left(\frac{\hat{a}}{\sqrt{2}\,\sigma_{\hat{a}}}\right)\right]}{2 a_{max}(\sqrt{2\pi}\,\sigma)^{N-1}\sqrt{N}}}$$

Posterior distribution

$$p(a \mid d, M_1) = \frac{p(d \mid a, M_1)\; p(a \mid M_1)}{p(d \mid M_1)}$$

$$= \frac{\left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^N \cancel{\exp\left(-\frac{\mathrm{Var}[d]}{2\sigma_{\hat{a}}^2}\right)} \exp\left(-\frac{1}{2}\frac{(a-\hat{a})^2}{\sigma_{\hat{a}}^2}\right) \cancel{\frac{1}{a_{max}}}}{\dfrac{\cancel{\exp\left(\frac{-\mathrm{Var}[d]}{2\sigma_{\hat{a}}^2}\right)} \left[\mathrm{erf}(\quad) + \mathrm{erf}(\quad)\right]}{2\cancel{a_{max}}\left(\sqrt{2\pi}\,\sigma\right)^{N-1}\sqrt{N}}}$$

$$= \frac{1}{\sqrt{2\pi}}\frac{\sqrt{N}}{\sigma} \exp\left(-\frac{1}{2}\frac{(a-\hat{a})^2}{\sigma_{\hat{a}}^2}\right) 2\left[\mathrm{erf}(\,) + \mathrm{erf}(\,)\right]^{-1}$$

$$\rightarrow \boxed{p(a \mid d, M_1) = \frac{1}{\sqrt{2\pi}\,\sigma_{\hat{a}}} \exp\left(-\frac{1}{2}\frac{(a-\hat{a})^2}{\sigma_{\hat{a}}^2}\right) 2\left[\mathrm{erf}\left(\frac{a_{max}-\hat{a}}{\sqrt{2}\,\sigma_{\hat{a}}}\right) + \mathrm{erf}\left(\frac{\hat{a}}{\sqrt{2}\,\sigma_{\hat{a}}}\right)\right]^{-1}}$$

Truncated gaussian on $[0, a_{max}]$

Bayes' Factor:

$$B_{10}(d) \equiv \frac{p(d|M_1)}{p(d|M_0)}$$

doesn't have any free parameters

$$= \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^N \exp\left[-\frac{\sum_i d_i^2}{2\sigma^2}\right]$$

$$= \frac{\exp\left(-\frac{\mathrm{Var}[d]}{2\sigma_{\hat{a}}^2}\right)\frac{1}{2a_{max}\,(\sqrt{2\pi}\sigma)^{N-1}\sqrt{N}}\left[\mathrm{erf}(\ )+\mathrm{erf}(\ )\right]}{\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^N \exp\left(-\frac{\sum_i d_i^2}{2\sigma^2}\right)}$$

cancels

Recall: $\mathrm{Var}[d] = \frac{1}{N}\sum_i d_i^2 - \hat{a}^2$

$$\rightarrow \exp\left(-\frac{\mathrm{Var}[d]}{2\sigma_{\hat{a}}^2}\right) = \exp\left(-\frac{1}{2N\sigma_{\hat{a}}^2}\sum_i d_i^2\right)\exp\left(\frac{+\hat{a}^2}{2\sigma_{\hat{a}}^2}\right)$$

$$= \exp\left(-\frac{\sum_i d_i^2}{2\sigma^2}\right)\exp\left(\frac{\hat{a}^2}{2\sigma_{\hat{a}}^2}\right)$$

$$B_{10}(d) = \frac{\exp\left(\frac{\hat{a}^2}{2\sigma_{\hat{a}}^2}\right)\left[\mathrm{erf}(\,) + \mathrm{erf}(\,)\right]}{2a_{max}} \frac{(\sqrt{2\pi}\,\sigma)^N}{(\sqrt{2\pi}\,\sigma)^{N-1}\sqrt{N}}$$

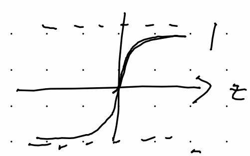
$$= \exp\left(\frac{\hat{a}^2}{2\sigma_{\hat{a}}^2}\right) \frac{\left[\mathrm{erf}(\,) + \mathrm{erf}(\,)\right]}{2} \left(\frac{\sqrt{2\pi}\,\sigma_{\hat{a}}}{a_{max}}\right)$$

$$= \exp\left(\frac{\hat{a}}{2\sigma_{\hat{a}}^2}\right)\left(\frac{\sqrt{2\pi}\,\sigma_{\hat{a}}}{a_{max}}\right) \frac{1}{2}\left[\mathrm{erf}\left(\frac{a_{max}-\hat{a}}{\sqrt{2}\,\sigma_{\hat{a}}}\right) + \mathrm{erf}\left(\frac{\hat{a}}{\sqrt{2}\,\sigma_{\hat{a}}}\right)\right]$$

Note:

$$B_{10}(d) \simeq \exp\left(\frac{\hat{a}^2}{2\sigma_{\hat{a}}^2}\right)\left(\frac{\sqrt{2\pi}\,\sigma_{\hat{a}}}{a_{max}}\right)$$

If $\hat{a}$ tightly peaked away from 0 and a_{max}



$\mathrm{erf}(z)$

Note: $\Lambda(d)$ = frequentist detection statistic

$$= 2 \ln \Lambda_{ML}(d)$$

$$= 2 \ln \left[\frac{p(d \mid a, \mathcal{M}_1)\big|_{a = a_{ML}}}{\underbrace{p(d \mid \mathcal{M}_0)}_{\text{no parameters}}} \right]$$

$$p(d \mid a, \mathcal{M}_1) = \left(\frac{1}{\sqrt{2\pi}\,\sigma} \right)^N \exp\left(- \frac{\mathrm{Var}[d]}{2 \sigma_{\hat{a}}^2} \right) \exp\left(- \frac{(a - \hat{a})^2}{2 \sigma_{\hat{a}}^2} \right)$$

$$\longrightarrow \quad p(d \mid a = \underbrace{\hat{a}}_{\text{max likelihood estimator}}) = \left(\frac{1}{\sqrt{2\pi}\,\sigma} \right)^N \exp\left(\frac{-\mathrm{Var}[d]}{2 \sigma_{\hat{a}}^2} \right)$$

Thus,

$$\Lambda_{ML}(d) = \frac{\cancel{\left(\frac{1}{\sqrt{2\pi}\,\sigma} \right)^N} \exp\left(- \frac{\mathrm{Var}[d]}{2 \sigma_{\hat{a}}^2} \right)}{\cancel{\left(\frac{1}{\sqrt{2\pi}\,\sigma} \right)^N} \exp\left(- \frac{\sum_i d_i^2}{2 \sigma^2} \right)}$$

$$\Lambda_{ML}(d) \equiv \frac{\exp\left(-\frac{Var[d]}{2\sigma_{\hat{a}}^2}\right)}{\exp\left(-\frac{\sum_i d_i^2}{2\sigma^2}\right)}$$

Recall: $Var[d] = \frac{1}{N}\sum_i d_i^2 - \hat{a}^2$

$$\rightarrow \quad \exp\left(-\frac{Var[d]}{2\sigma_{\hat{a}}^2}\right) = \exp\left(-\frac{\sum_i d_i^2}{2N\sigma_{\hat{a}}^2}\right)\exp\left(\frac{+\hat{a}^2}{2\sigma_{\hat{a}}^2}\right)$$

$$= \exp\left(-\frac{\sum_i d_i^2}{2\sigma^2}\right)\exp\left(\frac{\hat{a}^2}{2\sigma_{\hat{a}}^2}\right)$$

so

$$\Lambda_{ML}(d) = \exp\left(\frac{\hat{a}^2}{2\sigma_{\hat{a}}^2}\right)$$

$$\Lambda(d) \equiv 2\ln\Lambda_{ML}(d) = \frac{\hat{a}^2}{\sigma_{\hat{a}}^2} = \boxed{\frac{N\hat{a}^2}{\sigma^2}}$$

$\leftarrow$ squared SNR

For informative data:

$$B_{10}(d) \simeq \exp\left(\frac{+\hat{a}^2}{2\sigma_{\hat{a}}^2}\right)\left(\frac{\sqrt{2\pi}\,\sigma_{\hat{a}}}{a_{max}}\right)$$

$$2\ln B_{10}(d) \simeq \frac{\hat{a}}{\sigma_{\hat{a}}^2} + 2\ln\left(\frac{\sqrt{2\pi}\,\sigma_{\hat{a}}}{a_{max}}\right)$$

$$= 2\ln\Lambda_{ML}(d) + 2\ln\left(\frac{\sqrt{2\pi}\,\sigma_{\hat{a}}}{a_{max}}\right)$$

$$= \Lambda(d) + 2\ln\underbrace{\left(\frac{\sqrt{2\pi}\,\sigma_{\hat{a}}}{a_{max}}\right)}_{\text{Occam Factor} \sim \frac{\Delta V_1}{V_1}}$$

Sampling distribution of frequentist detection statistic

$$\Lambda(d) = \frac{N\hat{a}^2}{\sigma^2} = \left(\frac{\sqrt{N}\,\bar{d}}{\sigma}\right)^2 \equiv \rho^2 \quad \text{where } \rho \equiv \frac{\sqrt{N}\,\bar{d}}{\sigma}$$

Now: ρ is gaussian distributed being an average of d_i (gauss)

(i) In the absence of a signal: $\langle \rho \rangle = 0$

$$\mathrm{Var}(\rho) = \frac{N}{\sigma^2}\underbrace{\mathrm{Var}(\bar{d})}_{\frac{\sigma^2}{N}} = 1$$

(ii) In the presence of a signal: $\langle \rho \rangle = \frac{\sqrt{N}\,a}{\sigma} \equiv \mu$

$\mathrm{Var}(\rho) = 1$ (since signal is deterministic)

Central chi-square with 1 DOF:

$$p(\Lambda \mid M_0) = \frac{1}{\sqrt{2}\,\underbrace{\Gamma(\frac{1}{2})}_{\pi}}\,\Lambda^{-1/2}\,e^{-\Lambda/2} = \boxed{\frac{1}{\sqrt{2\pi\Lambda}}\,e^{-\Lambda/2}}$$

$p(\Lambda \mid a, \mathcal{M}_1)$ = non-central chi-square distribution with one DOF $\lambda = \mu^2 = \dfrac{Na^2}{\sigma^2}$

$$= \frac{1}{2} e^{-(\Lambda + \lambda)/2} \left(\frac{\Lambda}{\lambda}\right)^{-\frac{1}{4}} \cdot I_{-\frac{1}{2}}\left(\sqrt{\lambda \Lambda}\right)$$

$$= \frac{1}{2\sqrt{\Lambda}} \left[\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\sqrt{\Lambda} - \sqrt{\lambda})^2} + \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\sqrt{\Lambda} + \sqrt{\lambda})^2} \right]$$

$$= \boxed{\frac{1}{\sqrt{2\pi\Lambda}} \, \frac{1}{2} \left[e^{-\frac{1}{2}(\sqrt{\Lambda} - \sqrt{\lambda})^2} + e^{-\frac{1}{2}(\sqrt{\Lambda} + \sqrt{\lambda})^2} \right]}$$

for $0 \leq \Lambda < \infty$

where $\lambda = \mu^2 = \dfrac{Na^2}{\sigma^2}$

3. LIGO BBH rate calculation:

$$R_0 = 10 - 200 \ \mathrm{Gpc}^{-3}\ \mathrm{yr}^{-1} \quad \text{(local rate estimate)}$$

$$r = R_0 \underbrace{\frac{4}{3}\pi\, d_0^3(z)}_{\text{comoving volume}}$$

where

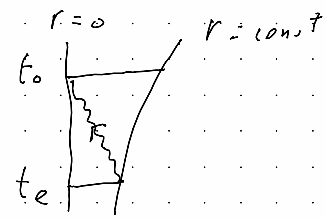
$d_0(z)$ = proper distance today to a source which emitted GWs at redshift z

FRW: $$ds^2 = -c^2 dt^2 + a^2(t)\left[dr^2 + S_k^2(r)\, d\Omega^2\right]$$

$$\rightarrow \quad d_0(z) = a(t_0)\int_0^r dr'$$

$$= a(t_0)\, r$$

$$= r \quad (\text{taking } a(t_0) = 1)$$



Radial photon: $$ds^2 = 0 = -c^2 dt^2 + a^2(t)\, dr^2$$

$$\rightarrow \quad c\,dt = a(t)\, dr$$

$$dr = \frac{c\,dt}{a(t)}$$

Thus,

$$d_0(z) = \int_0^{r} dr'$$

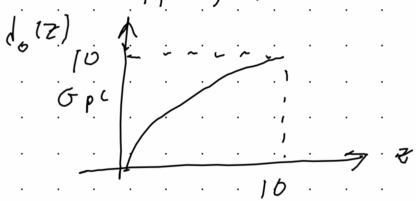
$$= \int_{t_e}^{t_0} \frac{c\,dt'}{a(t')} \qquad 1+z = \frac{1}{a(t)}$$

$$= \int_z^0 c\,(1+z')\left(\frac{dt'}{dz'}\right) dz'$$

Now: $\dfrac{dt}{dz} = \dfrac{-1}{(1+z)\,H_0\,E(z)}$, $E(z) = \sqrt{\Omega_m (1+z)^3 + \Omega_\Lambda}$

$$\rightarrow d_0(z) = \frac{c}{H_0}\int_z^0 \frac{-\cancel{(1+z')}\,dz'}{\cancel{(1+z')}\,E(z)} = \frac{c}{H_0}\int_0^z \frac{dz'}{\sqrt{\Omega_m (1+z)^3 + \Omega_\Lambda}}$$

Do the integral



$\rightarrow d_0(z=10) \approx 10\,\text{Gpc}$

$\rightarrow r = R_0 \frac{4}{3}\pi\,(10\,\text{Gpc})^3$

<u>Now</u>: L160 local rate estimate: $R_0 = 10 - 200 \ Gpc^{-3} yr^{-1}$

<u>$R_0 = 10$:</u>

$$r \simeq 10 \ \cancel{Gpc^{-3}} \ yr^{-1} \ \frac{4}{\cancel{3}} \cancel{\pi} \left(10 \ \cancel{Gpc}\right)^3$$

$$\simeq 4 \times 10^4 \ \frac{1}{yr} \underbrace{\left(\frac{1 \ yr}{\pi \times 10^7 \ \cancel{s}}\right)\left(\frac{3600 \ \cancel{s}}{hr}\right)}_{\simeq 10^{-4} \frac{1}{hr}}$$

$$\simeq \boxed{4 \ \frac{\text{events}}{hr}}$$

<u>$R_0 = 200$:</u> 20 x larger

$$r \simeq \frac{80 \text{ events}}{hr} \simeq \boxed{\frac{1 \text{ event}}{\text{minute}}}$$

④ Relationship between $S_h(f)$ and $\Omega_{gw}(f)$:

$$h_{ab}(t,\vec{x}) = \int_{-\infty}^{\infty} df \int d^2\Omega_{\hat{k}} \sum_A h_A(f,\hat{k})\, e^A_{ab}(\hat{k})\, e^{i2\pi f(t-\hat{k}\cdot\vec{x}/c)}$$

$$\langle h_A(f,\hat{k})\rangle = 0$$

$$\langle h_A(f,\hat{k})\, h^*_{A'}(f',\hat{k}')\rangle = \frac{1}{16\pi} S_h(f)\, \delta(f-f')\, \delta_{AA'}\, \delta^2(\hat{k},\hat{k}')$$

$$\Omega_{gw}(f) = \frac{1}{\rho_c}\frac{d\rho_{gw}}{d\ln f} = \frac{f}{\rho_c}\frac{d\rho_{gw}}{df}$$

$$\rho_{gw} = \frac{c^2}{32\pi G}\langle \dot{h}_{ab}(t,\vec{x})\, \dot{h}^{ab}(t,\vec{x})\rangle$$

$$= \frac{c^2}{32\pi G}\int_{-\infty}^{\infty} df \int_{-\infty}^{\infty} df' \int d^2\Omega_{\hat{k}} \int d^2\Omega_{\hat{k}'} \sum_A \sum_{A'}$$

$$\langle h_A(f,\hat{k})\, h^*_{A'}(f',\hat{k}')\rangle\, e^A_{ab}(\hat{k})\, e^{A'ab}(\hat{k}')$$

$$(i2\pi f)(-i2\pi f')\, e^{i2\pi f(t-\hat{k}\cdot\vec{x}/c)}\, e^{-i2\pi f'(t-\hat{k}'\cdot\vec{x}/c)}$$

$$= \frac{c^2}{32\pi G}\int_{-\infty}^{\infty} df \int d^2\Omega_{\hat{k}} \sum_A e^A_{ab}(\hat{k})\, e^{Aab}(\hat{k})\, 4\pi^2 f^2\, \frac{1}{16\pi} S_h(f)$$

Now:

$$e^{+}_{ab}(\hat{n}) = l_a l_b - m_a m_b$$
$$e^{\times}_{ab}(\hat{n}) = l_a m_b + m_a l_b$$

$\{\hat{n}, \hat{l}, \hat{m}\}$ right-hand set of unit vectors

$$e^{+}_{ab}(\hat{n})\, e^{+ab}(\hat{n}) = (\hat{l}\cdot\hat{l})^2 - 2\cancel{(\hat{l}\cdot\hat{m})^2}^{\,0} + (\hat{m}\cdot\hat{m})^2 = 2$$
$$e^{\times}_{ab}(\hat{n})\, e^{\times ab}(\hat{n}) = 2(\hat{l}\cdot\hat{l})(\hat{m}\cdot\hat{m}) + 2\cancel{(\hat{l}\cdot\hat{m})}^{\,0} = 2$$

thus,

$$\rho_{gw} = \frac{c^2}{32\pi G}\,\cancel{4}\,\cancel{\pi}\,2\cdot\cancel{4}\,\frac{1}{\cancel{16\pi}}\int d^2\Omega_{\hat{n}}\int_{-\infty}^{\infty} df\; f^2 S_h(f)$$
$$= \frac{c^2}{32\,G}\cdot 4\pi\int_{-\infty}^{\infty} df\; f^2 S_h(f)$$
$$= \frac{\pi c^2}{8G}\int_{-\infty}^{\infty} df\; f^2 S_h(f)$$

Now:

$$\rho_c = \frac{3H_0^2 c^2}{8\pi G} \longrightarrow \left(\frac{\pi c^2}{8G}\right)\cdot\frac{\pi}{\pi} = \frac{\pi^2 c^2}{8\pi G} = \frac{\pi^2 \rho_c}{3H_0^2}$$

Thus,

$$\rho_{gw} = \frac{\pi^2 \rho_c}{3H_0^2}\int_{-\infty}^{\infty} df\; f^2 S_h(f)$$
$$= \frac{2\pi^2}{3H_0^2}\,\rho_c\int_{0}^{\infty}\frac{df}{f}\; f^3 S_h(f)$$

Compare to

$$\rho_{gw} = \int_0^\infty df \left(\frac{d\rho_{gw}}{df}\right) = \frac{2\pi^2}{3H_0^2}\rho_c \int_0^\infty \frac{df}{f} f^3 S_h(f)$$

Thus,

$$\frac{d\rho_{gw}}{df} = \frac{2\pi^2}{3H_0^2}\rho_c \frac{f^3 S_h(f)}{f}$$

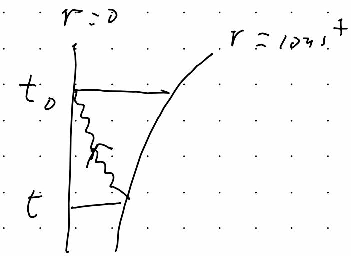
$$\rightarrow \quad \Omega_{gw}(f) = \frac{f}{\rho_c}\frac{d\rho_{gw}}{df} = \frac{2\pi^2}{3H_0^2} f^3 S_h(f)$$

or

$$\boxed{S_h(f) = \frac{3H_0^2}{2\pi^2}\frac{\Omega_{gw}(f)}{f^3}}$$

5. $\frac{dt}{dz}$ calculation and Phinney formula in terms of $R(z)$

a) Friedmann equation



$$\left(\frac{\dot a}{a}\right)^2 = H_0^2\left(\frac{\Omega_m}{a^3} + \Omega_\Lambda\right)$$

$$\rightarrow \quad \frac{\dot a}{a} = H_0\sqrt{\frac{\Omega_m}{a^3} + \Omega_\Lambda}$$

$$1+z = \frac{a(t_0)}{a(t)} = \frac{1}{a(t)}$$ where $a(t_0) \equiv 1$ (t_0 = today)
and t = time of emission

Rewrite Friedmann's equation in terms of z

$$LHS = \frac{\dot a}{a} = \frac{1}{a}\frac{da}{dt} = (1+z)\frac{d}{dt}\left(\frac{1}{1+z}\right)$$

$$= (1+z)\frac{-1}{(1+z)^2}\frac{dz}{dt} = \frac{-1}{1+z}\frac{dz}{dt}$$

$$RHS = H_0\sqrt{\frac{\Omega_m}{a^3} + \Omega_\Lambda} = H_0\sqrt{\Omega_m(1+z)^3 + \Omega_\Lambda} \equiv H_0 E(z)$$

Thus, $$-\frac{1}{1+z}\frac{dz}{dt} = H_0 E(z) \rightarrow \boxed{\frac{dt}{dz} = \frac{-1}{(1+z)H_0 E(z)} \quad \text{where } E(z) \equiv \sqrt{\Omega_m(1+z)^3 + \Omega_\Lambda}}$$

b) Phinney formula in terms of number density $n(z)$:

$$\Omega_{gw}(f) = \frac{1}{\rho_c}\int_0^\infty dz\, n(z)\, \frac{1}{1+z}\left(f_s \frac{dE_{gw}}{df_s}\right)\Bigg|_{f_s = f(1+z)}$$

Now: $n(z)\,dz = R(z)\,|dt|$

$n(z)$ ↳ number density; $R(z)$ ↳ rate density

$$\rightarrow n(z) = R(z)\left|\frac{dt}{dz}\right| = \frac{R(z)}{(1+z)H_0 E(z)}$$

Thus,

$$\Omega_{gw}(f) = \frac{1}{\rho_c}\int_0^\infty dz\, \frac{R(z)}{(1+z)H_0E(z)}\left(\frac{1}{1+z}\right)\left(f_s\frac{dE_{gw}}{df_s}\right)\Bigg|_{f_s = f(1+z)}$$

$$= \frac{1}{\rho_c H_0}\int_0^\infty dz\, \frac{R(z)}{E(z)}\,\frac{1}{(1+z)^{\cancel{2}}}\, f\cancel{(1+z)}\left(\frac{dE_{gw}}{df_s}\right)\Bigg|_{f_s = f(1+z)}$$

$$= \frac{f}{\rho_c H_0}\int_0^\infty dz\, \frac{R(z)}{(1+z)E(z)}\left(\frac{dE_{gw}}{df_s}\right)\Bigg|_{f_s = f(1+z)}$$

(6.) optimal Filtering for cross-correlation statistic

$$\hat{S}_h \simeq \int_{-\infty}^{\infty} df \int_{-\infty}^{\infty} df' \, \delta_T(f-f') \, \tilde{d}_1(f) \, \tilde{d}_2^*(f') \, \tilde{Q}^*(f')$$

Expected value

$$\mu \equiv \langle \hat{S}_h \rangle$$

$$= \int_{-\infty}^{\infty} df \int_{-\infty}^{\infty} df \, \delta_T(f-f') \, \langle \tilde{d}_1(f) \, \tilde{d}_2^*(f') \rangle \, \tilde{Q}^*(f')$$

For uncorrelated noise

$$\langle \tilde{d}_1(f) \, \tilde{d}_2^*(f') \rangle = \langle \tilde{h}_1(f) \, \tilde{h}_2^*(f') \rangle$$

$$= \frac{1}{2} \delta(f-f') \, \Gamma_{12}(f) \, S_h(f)$$

$$\rightarrow \quad \mu = \frac{1}{2} \int_{-\infty}^{\infty} df \, \underbrace{\delta_T(0)}_{T} \, \Gamma_{12}(f) \, S_h(f) \, \tilde{Q}^*(f)$$

$$= \frac{T}{2} \int_{-\infty}^{\infty} df \, \Gamma_{12}(f) \, S_h(f) \, \tilde{Q}^*(f)$$

Variance:

$$\sigma^2 \equiv \langle (\hat{S}_h - \langle \hat{S}_h \rangle)^2 \rangle$$

$$= \langle \hat{S}_h^2 \rangle - \mu^2$$

$$= \int df \int df' \int dp \int dp' \, \delta_T(f-f') \, \delta_T(p-p') \, \tilde{Q}^*(f') \, \tilde{Q}(p')$$

$$\Big(\langle \tilde{d}_1(f) \tilde{d}_2^*(f') \tilde{d}_1^*(p) \tilde{d}_2(p') \rangle$$

$$- \langle \tilde{d}_1(f) \tilde{d}_2^*(f') \rangle \langle \tilde{d}_1^*(p) \tilde{d}_2(p') \rangle \Big)$$

Use $\langle abcd \rangle = \langle ab \rangle \langle cd \rangle + \langle ac \rangle \langle bd \rangle + \langle ad \rangle \langle bc \rangle$ for zero-mean Gaussian random variables

so

$$\langle d_1 d_2^* d_1^* d_2 \rangle - \langle d_1 d_2^* \rangle \langle d_1^* d_2 \rangle$$

$$= \langle \tilde{d}_1(f) \tilde{d}_1^*(p) \rangle \langle \tilde{d}_2^*(f') \tilde{d}_2(p') \rangle + \underbrace{\langle \tilde{d}_1(f) \tilde{d}_2(p') \rangle \langle \tilde{d}_2^*(f') \tilde{d}_1^*(p) \rangle}_{\text{for uncorrelated noise, this is proportional to } S_h^2(f)}$$

$$\approx \frac{1}{2} \delta(f-p) \, P_1(f) \, \frac{1}{2} \delta(f'-p') \, P_2(f')$$

total autocorrelated power

$P_1(f) = P_{n_1}(f) + P_{gw}(f) \approx P_{n_1}(f)$ for weak signals

Thus,

$$\sigma^2 \approx \frac{1}{4}\int_{-\infty}^{\infty} df \int_{-\infty}^{\infty} df'\, \delta_T(f-f')\, \underbrace{\delta_T(f-f')}_{\approx\, \delta(f-f')}\, \tilde{Q}^*(f)\, \tilde{Q}(f')\, P_1(f) P_2(f)$$

$$= \frac{1}{4}\int_{-\infty}^{\infty} df\, \underbrace{\delta_T(0)}_{T}\, |\tilde{Q}(f)|^2\, P_1(f) P_2(f)$$

$$= \frac{T}{4}\int_{-\infty}^{\infty} df\, |\tilde{Q}(f)|^2\, P_1(f) P_2(f)$$

$$\rightarrow \mathrm{SNR} = \frac{\mu}{\sigma} = \frac{\frac{T}{\cancel{2}}\int_{-\infty}^{\infty} df\, \Gamma_{12}(f)\, S_h(f)\, \tilde{Q}^*(f)}{\sqrt{\frac{T}{\cancel{4}}\int_{-\infty}^{\infty} df\, |\tilde{Q}(f)|^2\, P_1(f) P_2(f)}}$$

$$= \sqrt{T}\, \frac{\int_{-\infty}^{\infty} df\, \frac{\Gamma_{12}(f)\, S_h(f)}{P_1(f) P_2(f)}\, \tilde{Q}^*(f)\, P_1(f) P_2(f)}{\sqrt{\int_{-\infty}^{\infty} df\, \tilde{Q}(f)\, \tilde{Q}^*(f)\, P_1(f) P_2(f)}}$$

$$= \sqrt{T}\, \left(\frac{\Gamma_{12} S_h}{P_1 P_2}, Q\right) \Big/ \sqrt{(Q, Q)}$$

where $(A,B) \equiv \int_{-\infty}^{\infty} df\, A(f)\, B^*(f)\, P_1(f)\, P_2(f)$

$\rightarrow$ $SNR = \sqrt{T}\, \dfrac{(A, \tilde{Q})}{\sqrt{(\tilde{Q}, \tilde{Q})}}$ where $A(f) \equiv \dfrac{\Gamma_{12}(f)\, S_h(f)}{P_1(f)\, P_2(f)}$

<u>Recall:</u> $\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta$ $\rightarrow$ $\dfrac{\vec{A} \cdot \vec{B}}{\sqrt{\vec{B} \cdot \vec{B}}} = |\vec{A}| \cos\theta$

$\vec{B} \cdot \vec{B} = |\vec{B}|^2$

is maximized by have $\theta = 0$

$\rightarrow$ $\vec{A}$ should point in same direction as B.

Thus, SNR is maximized when $\tilde{Q}(f) \propto \dfrac{\Gamma_{12}(f)\, S_h(f)}{P_1(f)\, P_2(f)}$

$\propto \dfrac{\Gamma_{12}(f)\, H(f)}{P_1(f)\, P_2(f)}$

just differ by overall amplitude

7. (4) $p(d \mid a, \sigma) \propto \exp\left[-\frac{1}{2} \sum_{i=1}^{N} \frac{(d_i - a)^2}{\sigma_i^2} \right]$

Maximize likelihood $\Longleftrightarrow$ Maximize ln (likelihood)

Define $L(a) = \ln[p(d \mid a, \sigma)] = -\frac{1}{2} \sum_{i=1}^{N} \frac{(d_i - a)^2}{\sigma_i^2}]$

$$0 = \left.\frac{dL}{da}\right|_{a=\hat{a}} = - \sum_i \left.\frac{2(d_i - a)(-1)}{\sigma_i^2}\right|_{a=\hat{a}}$$

$$= 2 \sum_i \frac{(d_i - \hat{a})}{\sigma_i^2}$$

$$= 2\left(\sum_i \frac{d_i}{\sigma_i^2} - \hat{a} \sum_i \frac{1}{\sigma_i^2} \right)$$

$$\longrightarrow \boxed{\hat{a} = \sum_i \frac{d_i}{\sigma_i^2} \Big/ \sum_i \frac{1}{\sigma_i^2}}$$

(b) $p(d|A,C) \propto \exp\left[-\frac{1}{2}(d-MA)^\dagger C^{-1}(d-MA)\right]$

Dagger $\equiv$ complex-conjugate transpose

allows for complex data d, parameter A

<u>Define</u>: $L(A,A^\dagger) = \ln p(d|A,C)$

$$= -\frac{1}{2}(d-MA)^\dagger C^{-1}(d-MA)$$

Treat $A, A^\dagger$ as independent variables for the variation

$$\delta L = \frac{1}{2}\delta A^\dagger M^\dagger C^{-1}(d-MA) + \frac{1}{2}(d-MA)^\dagger C^{-1}M\,\delta A$$

$\delta L = 0$ for $\delta A^\dagger$ $\rightarrow$ $0 = M^\dagger C^{-1}(d-M\hat{A})$

$$= M^\dagger C^{-1}d - M^\dagger C^{-1}M\hat{A}$$

$$\rightarrow \quad \hat{A} = (M^\dagger C^{-1}M)^{-1}M^\dagger C^{-1}d$$

$$= F^{-1}X$$

where $F \equiv M^\dagger C^{-1}M$ (Fisher matrix)

$X \equiv M^\dagger C^{-1}d$ ("dirty map")

$\delta \mathcal{L} = 0$ for δA

$$\frac{1}{2}(d - M\hat{A})^{+} C^{-1} M = 0$$

$$d^{+} C^{-1} M - \hat{A}^{+} M^{+} C^{-1} M = 0$$

Take $+$ of above equation using $(C^{-1})^{+} = C^{-1}$

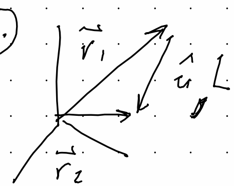
$$\rightarrow \quad M^{+} C^{-1} d - M^{+} C^{-1} M \hat{A} = 0$$

$$\rightarrow \quad \hat{A} = (M^{+} C^{-1} M)^{-1} M^{+} C^{-1} d$$

$$= F^{-1} X$$

(as before)

8.



$$r(t) = \Delta T(t) = \frac{1}{2c} u^a u^b \int_0^L ds\, h_{ab}(t(s), \vec{x}(s))$$

$$\text{where} \quad t(s) = \left(t - \frac{L}{c}\right) + \frac{s}{c}$$

$$\vec{x}(s) = \vec{r}_1 + s\hat{u}$$

$$s \in [0, L]$$

plane-wave expansion

$$h_{ab}(t, \vec{x}) = \int_{-\infty}^{\infty} df \int d^2\Omega_{\hat{k}} \sum_A h_A(f, \hat{k})\, e^A_{ab}(\hat{k})\, e^{i2\pi f(t - \hat{k}\cdot\vec{x}/c)}$$

Replace $t, \vec{x}$ by $t(s), \vec{x}(s)$

Exponential becomes

$$e^{i2\pi f\left(\left(t - \frac{L}{c}\right) + \frac{s}{c} - \frac{\hat{k}}{c}\cdot(\vec{r}_1 + s\hat{u})\right)}$$

$$= e^{i2\pi f\left(\left(t - \frac{L}{c}\right) - \frac{\hat{k}}{c}\cdot\vec{r}_1\right)}\, e^{i2\pi f \frac{s}{c}(1 - \hat{k}\cdot\hat{u})}$$

$$= e^{i2\pi f\left(t_1 - \frac{\hat{k}\cdot\vec{r}_1}{c}\right)}\, e^{i2\pi f \frac{s}{c}(1 - \hat{k}\cdot\hat{u})}$$

Do integral

$$\int_0^L ds\, e^{i2\pi f \frac{s}{c}(1-\hat{k}\cdot\hat{u})} = \frac{c}{i2\pi f(1-\hat{k}\cdot\hat{u})} e^{i2\pi f\frac{s}{c}(1-\hat{k}\cdot\hat{u})}\Big|_0^L$$

$$= \frac{c}{(i2\pi f)}\frac{1}{(1-\hat{k}\cdot\hat{u})}\left(e^{i2\pi f\frac{L}{c}(1-\hat{k}\cdot\hat{u})} - 1\right)$$

Thus,

$$\Delta T(t) = \frac{1}{2c}\int df\int d^2\Omega_{\hat{k}} \sum_A h_A(f,\hat{k}) e^A_{ab}(\hat{k}) \frac{c}{(i2\pi f)}\frac{1}{(1-\hat{k}\cdot\hat{u})}$$

$$\Big[\underbrace{e^{i2\pi f(t_1 - \hat{k}\cdot\vec{r}_1/c)}\, e^{i2\pi f\frac{L}{c}(1-\hat{k}\cdot\hat{u})}}_{e^{i2\pi f\left(\underbrace{(t_1+\frac{L}{c})}_{t_2} - \frac{\hat{k}}{c}\cdot\underbrace{(\vec{r}_1+\hat{u}L)}_{\vec{r}_2}\right)}} - e^{i2\pi f(t_1 - \hat{k}\cdot\vec{r}_1/c)}\Big]$$

$$= \frac{1}{2}\int df\int d^2\Omega_{\hat{k}} \sum_A h_A(f,\hat{k}) e^A_{ab}(\hat{k}) \left(\frac{1}{i2\pi f}\right)\left(\frac{1}{1-\hat{k}\cdot\hat{u}}\right)$$

$$\Big[e^{i2\pi f(t_2 - \hat{k}\cdot\vec{r}_2/c)} - e^{i2\pi (t_1 - \hat{k}\cdot\vec{r}_1/c)}\Big]$$

NOTE:

$e^{i2\pi f(t_2 - \hat{k}\cdot\vec{r}_2/c)}$ corresponds to reception of pulse at $\vec{r}_2$ at time t_2

$e^{i2\pi f(t_1 - \hat{k}\cdot\vec{r}_1/c)}$ corresponds to emission of pulse at $\vec{r}_1$ at time t_1

Rewrite:

$$\Delta T(t) = \int df \int d\Omega_k \sum_A h_A(f,\hat{k}) \left(\frac{1}{i2\pi f}\right)\left(\frac{1}{1-\hat{k}\cdot\hat{u}}\right) e^A_{ab}(\hat{k})\, u^a u^b$$

$$e^{i2\pi f t}\, e^{-i2\pi f \hat{k}\cdot\vec{r}_2/c}\left[1 - e^{-i2\pi f t}\, e^{+i2\pi f \hat{k}\cdot\vec{r}_2/c}\, \underbrace{e^{i2\pi f \overset{t - \frac{L}{c}}{\uparrow}{t_1}}\, e^{-i2\pi f \hat{k}\cdot\vec{r}_1/c}}\right]$$

$$= e^{-i2\pi f \frac{L}{c}}\, e^{i2\pi f \hat{k}\cdot(\vec{r}_2 - \vec{r}_1)/c}$$

$$= e^{-i2\pi f \frac{L}{c}\left(1 - \hat{k}\cdot\underbrace{\frac{(\vec{r}_2 - \vec{r}_1)}{L}}_{\hat{u}}\right)},$$

$$= e^{-i2\pi f \frac{L}{c}(1 - \hat{k}\cdot\hat{u})}$$

Thus,

$$\Delta T(t) = \int_{-\infty}^{\infty} df \, e^{i2\pi f t} \int d^2\Omega_k \sum_A h_A(f,\hat{k}) \left(\frac{1}{i2\pi f}\right) \frac{u^a u^b e^A_{ab}(\hat{k})}{1-\hat{k}\cdot\hat{u}} \, e^{-i2\pi f \hat{k}\cdot\vec{r}_2/c} \left[1 - e^{-i2\pi f \frac{L}{c}(1-\hat{k}\cdot\hat{u})}\right]$$

So

$$\boxed{R_A(f,\hat{k}) = \frac{1}{i2\pi f} \frac{u^a u^b e^A_{ab}(\hat{k})}{2(1-\hat{k}\cdot\hat{u})} \left[1 - e^{-i2\pi f \frac{L}{c}(1-\hat{k}\cdot\hat{u})}\right] e^{-i2\pi f \hat{k}\cdot\vec{r}_2/c}}$$

<u>NOTE</u>: For pulsar timing $\hat{p} = -\hat{u}$, take $\vec{r}_2 = \vec{0}$ (SSB)

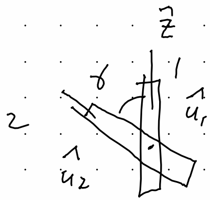
$\frac{1}{i2\pi f} \rightarrow 1$ for redshift $\left(\frac{\Delta\nu}{\nu}\right)$ measurements

$$\boxed{R_A(f,\hat{k}) = \underbrace{\frac{1}{2}\left(\frac{p^a p^b e^A_{ab}(\hat{k})}{1+\hat{k}\cdot\hat{p}}\right)}_{\equiv F^A(\hat{k})} \Big[\underbrace{1}_{\text{Earth term}} - \underbrace{e^{-i2\pi f \frac{L}{c}(1+\hat{k}\cdot\hat{p})}}_{\text{pulsar term}}\Big]}$$

(9.) ORF for colocated electric dipole antennae

$$r_I(t) = \hat{u}_I \cdot \vec{E}(t, \vec{x}=0)$$

$$\vec{E}(t,\vec{x}) = \int df \int d^2\Omega_k \sum_{\alpha=1,2} \tilde{E}_\alpha(f,\hat{k})\, \hat{\epsilon}_\alpha(\hat{k})\, e^{i2\pi f(t-\hat{k}\cdot\vec{x}/c)}$$



Take: $\hat{u}_1 = \hat{z}$

$\hat{u}_2 = \sin\gamma\, \hat{x} + \cos\gamma\, \hat{z}$

$\hat{\epsilon}_1(\hat{k}) = -\hat{\phi}$

$\hat{\epsilon}_2(\hat{k}) = -\hat{\theta}$

Now: $r_I(t) = \hat{u}_I \cdot \vec{E}(t, \vec{x}=0)$

$$= \int_{-\infty}^{\infty} df \int d^2\Omega_k \sum_{\alpha=1,2} \tilde{E}_\alpha(f,\hat{k})\, \hat{u}_I \cdot \hat{\epsilon}_\alpha(\hat{k})\, e^{i2\pi f t}$$

$$= \int_{-\infty}^{\infty} df\, e^{i2\pi f t} \int d^2\Omega_k \sum_\alpha \tilde{E}_\alpha(f,\hat{k})\, \hat{u}_I \cdot \hat{\epsilon}_\alpha(\hat{k})$$

so $R_I^\alpha(f,\hat{k}) = \hat{u}_I \cdot \hat{\epsilon}_\alpha(\hat{k})$

$$\hat{u}_1 = \hat{z} \qquad \hat{u}_2 = \sin\gamma\,\hat{x} + \cos\gamma\,\hat{z}$$

$$\hat{\epsilon}_1(\hat{k}) = -\hat{\phi} = \sin\phi\,\hat{x} - \cos\phi\,\hat{y}$$

$$\hat{\epsilon}_2(\hat{k}) = -\hat{\theta} = -\cos\theta\cos\phi\,\hat{x} - \cos\theta\sin\phi\,\hat{y} + \sin\theta\,\hat{z}$$

$$\hat{k} = -\sin\theta\cos\phi\,\hat{x} - \sin\theta\sin\phi\,\hat{y} - \cos\theta\,\hat{z}$$

Thus,

$$\hat{u}_1 \cdot \hat{\epsilon}_1(\hat{k}) = -\hat{z}\cdot\hat{\phi} = 0$$

$$\hat{u}_2 \cdot \hat{\epsilon}_1(\hat{k}) = -(\sin\gamma\,\hat{x} + \cos\gamma\,\hat{z})\cdot\hat{\phi}$$

$$= \sin\phi\sin\gamma$$

$$\hat{u}_1 \cdot \hat{\epsilon}_2(\hat{k}) = -\hat{z}\cdot\hat{\theta} = \sin\theta$$

$$\hat{u}_2 \cdot \hat{\epsilon}_2(\hat{k}) = -(\sin\gamma\,\hat{x} + \cos\gamma\,\hat{z})\cdot\hat{\theta}$$

$$= -\sin\gamma\cos\theta\cos\phi + \sin\theta\cos\gamma$$

$$\Gamma_{12}(f) = \frac{1}{8\pi}\int d^2\Omega_{\hat{k}} \sum_{\alpha} R_1^{\alpha}(f,\hat{k}) R_2^{\alpha}(f,\hat{k})$$

$$= \frac{1}{8\pi}\int d^2\Omega_{\hat{k}} \sum_{\alpha} (\hat{u}_1 \cdot \hat{\epsilon}_{\alpha}(\hat{k}))(\hat{u}_2 \cdot \hat{\epsilon}_{\alpha}(\hat{k}))$$

$$= \frac{1}{8\pi}\int d^2\Omega_{\hat{k}} \Big[\big(\cancelto{0}{\hat{u}_1 \cdot \hat{\epsilon}_1(\hat{k})}\big)(\hat{u}_2 \cdot \hat{\epsilon}_1(\hat{k})) + (\hat{u}_1 \cdot \hat{\epsilon}_2(\hat{k}))(\hat{u}_2 \cdot \hat{\epsilon}_2(\hat{k})) \Big]$$

$$= \frac{1}{8\pi}\int d^2\Omega_{\hat{k}} \big[\sin\theta \, (-\sin\gamma\cos\theta\cos\phi + \cos\gamma\sin\theta) \big]$$

$$= -\frac{1}{8\pi}\int d^2\Omega_{\hat{k}} \sin\theta\cos\theta \cancelto{0}{\cos\phi} \sin\gamma + \frac{1}{8\pi}\int d^2\Omega_{\hat{k}} \sin^2\theta\cos\gamma$$

$$= \frac{1}{8\pi}\int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta)\,(1-\cos^2\theta)\cos\gamma$$

$$= \frac{\cancel{2}\pi}{\cancel{8}\pi}\int_{-1}^{1} dx\,(1-x^2) = \frac{1}{4}\left(x - \frac{x^3}{3}\right)\Big|_{-1}^{1}\cos\gamma = \frac{1}{4}\left(\frac{4}{3}\right)\cos\gamma$$

$$= \boxed{\frac{1}{3}\cos\gamma}$$

(10) ML estimators

$$p(d \mid S_{n_1}, S_{n_2}, S_h) = \frac{1}{\sqrt{\det(2\pi C)}} \exp\left[-\frac{1}{2} d^T C^{-1} d\right]$$

$$d = \begin{bmatrix} d_1 \\ \hline d_2 \end{bmatrix} = \begin{bmatrix} d_{11} \\ d_{12} \\ \vdots \\ d_{1N} \\ d_{21} \\ d_{22} \\ \vdots \\ d_{2N} \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} d \\ d \\ \vdots \\ d \end{matrix}}\right\} d_1 \\ \left.\vphantom{\begin{matrix} d \\ d \\ \vdots \\ d \end{matrix}}\right\} d_2 \end{matrix}, \qquad C = \left[\begin{array}{c|c} S_1 \mathbb{1}_{N\times N} & S_h \mathbb{1}_{N\times N} \\ \hline S_h \mathbb{1}_{N\times N} & S_2 \mathbb{1}_{N\times N} \end{array}\right]$$

$$S_1 = S_h + S_{n_1}$$
$$S_2 = S_h + S_{n_2}$$

$$C^{-1} = \frac{1}{(S_1 S_2 - S_h^2)} \left[\begin{array}{c|c} S_2 \mathbb{1}_{N\times N} & -S_h \mathbb{1}_{N\times N} \\ \hline -S_h \mathbb{1}_{N\times N} & S_1 \mathbb{1}_{N\times N} \end{array}\right]$$

$$= \frac{1}{\left(1 - \frac{S_h^2}{S_1 S_2}\right)} \left[\begin{array}{c|c} \frac{1}{S_1} \mathbb{1}_{N\times N} & \frac{-S_h}{S_1 S_2} \mathbb{1}_{N\times N} \\ \hline -\frac{S_h}{S_1 S_2} \mathbb{1}_{N\times N} & \frac{1}{S_2} \mathbb{1}_{N\times N} \end{array}\right]$$

Thus,

$$p(d|S_{n_1}, S_{n_2}, S_h) = \frac{1}{(2\pi)^N (S_1 S_2 - S_h^2)^{N/2}} \exp\left[-\frac{1}{2}\frac{1}{\left(1-\frac{S_h^2}{S_1 S_2}\right)}\left(\frac{1}{S_1}\sum_i d_{1i}^2 + \frac{1}{S_2}\sum_i d_{2i}^2 - \frac{2S_h}{S_1 S_2}\sum_i d_{1i}d_{2i}\right)\right]$$

$$= \frac{1}{(2\pi)^N (S_1 S_2 - S_h^2)^{N/2}} \exp\left[-\frac{N}{2}\frac{1}{\left(1-\frac{S_h^2}{S_1 S_2}\right)}\left(\frac{\hat{C}_{11}}{S_1} + \frac{\hat{C}_{22}}{S_2} - \frac{2S_h\hat{C}_{12}}{S_1 S_2}\right)\right]$$

where $\hat{C}_{11} = \frac{1}{N}\sum_i d_{1i}^2$, etc.

Max likelihood $\Longleftrightarrow$ Max ln(likelihood)

$$\mathcal{L}(S_{n_1}, S_{n_2}, S_h) = \ln\left[p(d|S_{n_1}, S_{n_2}, S_h)\right]$$

$$= -N\ln 2\pi - \frac{N}{2}\ln(S_1 S_2 - S_h^2) - \frac{N}{2}\frac{1}{\left(1-\frac{S_h^2}{S_1 S_2}\right)}\left(\frac{\hat{C}_{11}}{S_1} + \frac{\hat{C}_{22}}{S_2} - \frac{2S_h\hat{C}_{12}}{S_1 S_2}\right)$$

$$= \text{const} - \frac{N}{2}\left[\ln(S_1 S_2 - S_h^2) + \frac{1}{\left(1-\frac{S_h^2}{S_1 S_2}\right)}\left(\frac{\hat{C}_{11}}{S_1} + \frac{\hat{C}_{22}}{S_2} - \frac{2S_h\hat{C}_{12}}{S_1 S_2}\right)\right]$$

Want to show that $\hat{C}_{11}$, $\hat{C}_{22}$, $\hat{C}_{12}$ are the ML estimators of S_1, S_2, S_h; Take partial derivative wrt S_1, S_2, S_h

$$0 = \frac{\partial \mathcal{L}}{\partial S_1}$$

$$= -\frac{N}{2}\left[\frac{1}{(S_1S_2 - S_h^2)} S_2 - \frac{1}{\left(1 - \frac{S_h^2}{S_1S_2}\right)^2} \frac{S_h^2}{S_1^2 S_2}\left(\frac{\hat{C}_{11}}{S_1} + \frac{\hat{C}_{22}}{S_2} - \frac{2S_h\hat{C}_{12}}{S_1S_2}\right)\right.$$

$$\left. + \frac{1}{\left(1 - \frac{S_h^2}{S_1S_2}\right)}\left(-\frac{\hat{C}_{11}}{S_1^2} + \frac{2S_h\hat{C}_{12}}{S_1^2S_2}\right)\right]$$

$$= -\frac{N}{2}\left[\frac{1}{(S_1S_2 - S_h^2)} S_2 - \frac{1}{(S_1S_2 - S_h^2)^2} S_2 S_h^2\left(\frac{\hat{C}_{11}}{S_1} + \frac{\hat{C}_{22}}{S_2} - \frac{2S_h\hat{C}_{12}}{S_1S_2}\right)\right.$$

$$\left. + \frac{1}{(S_1S_2 - S_h^2)} S_2\left(-\frac{\hat{C}_{11}}{S_1} + \frac{2S_h\hat{C}_{12}}{S_1S_2}\right)\right]$$

multiply through by $-\frac{2}{N}(S_1S_2 - S_h^2)^2 \frac{1}{S_2}$:

$$0 = (s_1 s_2 - s_h^2) - s_h^2\left(\frac{\hat{c}_{11}}{s_1} + \frac{\hat{c}_{22}}{s_2} - \frac{2 s_h \hat{c}_{12}}{s_1 s_2}\right) + (s_1 s_2 - s_h^2)\left(\frac{-\hat{c}_{11}}{s_1} + \frac{2 s_h \hat{c}_{12}}{s_1 s_2}\right)$$

$$= (s_1 s_2 - s_h^2)\left[1 - \frac{\hat{c}_{11}}{s_1} + \frac{2 s_h \hat{c}_{12}}{s_1 s_2}\right] - s_h^2\left(\frac{\hat{c}_{11}}{s_1} + \frac{\hat{c}_{22}}{s_2} - \frac{2 s_h \hat{c}_{12}}{s_1 s_2}\right)$$

substitute $\hat{c}_{11}$, $\hat{c}_{22}$, $\hat{c}_{12}$ for s_1, s_2, s_h on RHS:

$$\text{RHS} = (\hat{c}_{11}\hat{c}_{22} - \hat{c}_{12}^2)\left(\cancel{1} - \cancel{1} + \frac{2\hat{c}_{12}^2}{\hat{c}_{11}\hat{c}_{22}}\right) - \hat{c}_{12}^2\left(1 + 1 - 2\frac{\hat{c}_{12}^2}{\hat{c}_{11}\hat{c}_{22}}\right)$$

$$= 2\hat{c}_{12}^2\left(1 - \frac{\hat{c}_{12}^2}{\hat{c}_{11}\hat{c}_{22}}\right) - 2\hat{c}_{12}^2\left(1 - \frac{\hat{c}_{12}^2}{\hat{c}_{11}\hat{c}_{22}}\right)$$

$$= 0 \checkmark$$

Same analysis with $s_2 \longleftrightarrow s_1$ gives $\left.\dfrac{\partial \mathcal{L}}{\partial s_2}\right|_{\substack{s_1 = \hat{c}_{11} \\ s_2 = \hat{c}_{22} \\ s_h = \hat{c}_{12}}} = 0$

Finally, consider

$$0 = \frac{\partial \mathcal{L}}{\partial s_h}$$

$$= -\frac{N}{2}\left[\left(\frac{1}{s_1 s_2 - s_h^2}\right)(-2s_h) - \frac{1}{\left(1-\frac{s_h^2}{s_1 s_2}\right)^2}\left(\frac{-2s_h}{s_1 s_2}\right)\left(\frac{\hat{c}_{11}}{s_1} + \frac{\hat{c}_{22}}{s_2} - \frac{2 s_h \hat{c}_{12}}{s_1 s_2}\right)\right.$$

$$\left. + \frac{1}{\left(1-\frac{s_h^2}{s_1 s_2}\right)}\left(-\frac{2\hat{c}_{12}}{s_1 s_2}\right)\right]$$

Now multiply through by $-\frac{2}{N}(s_1 s_2 - s_h^2)^2\left(-\frac{1}{2}\right)$:

$$0 = (s_1 s_2 - s_h^2)s_h - s_h s_1 s_2\left(\frac{\hat{c}_{11}}{s_1} + \frac{\hat{c}_{22}}{s_2} - \frac{2 s_h \hat{c}_{12}}{s_1 s_2}\right) + (s_1 s_2 - s_h^2)\hat{c}_{12}$$

$$= (s_1 s_2 - s_h^2)(s_h + \hat{c}_{12}) - s_h s_1 s_2\left(\frac{\hat{c}_{11}}{s_1} + \frac{\hat{c}_{22}}{s_2} - \frac{2 s_h \hat{c}_{12}}{s_1 s_2}\right)$$

Substitute $\hat{c}_{11}$, $\hat{c}_{22}$, $\hat{c}_{12}$ for σ_1, σ_2, σ_h:

$$RHS = (\hat{c}_{11}\hat{c}_{22} - \hat{c}_{12}^2)(\hat{c}_{12} + \hat{c}_{12}) - \hat{c}_{12}\hat{c}_{11}\hat{c}_{22}\left(1 + 1 - \frac{2\hat{c}_{12}^2}{\hat{c}_{11}\hat{c}_{22}}\right)$$

$$= 2\hat{c}_{12}(\hat{c}_{11}\hat{c}_{22} - \hat{c}_{12}^2) - 2\hat{c}_{12}\hat{c}_{11}\hat{c}_{22}\left(1 - \frac{\hat{c}_{12}^2}{\hat{c}_{11}\hat{c}_{22}}\right)$$

$$= 0 \checkmark$$

Thus, $\hat{c}_{11}$, $\hat{c}_{22}$, $\hat{c}_{12}$ are ML estimators of σ_1, σ_2, σ_h

(11.) ML detection statistic: (same white GWB, white noise model in 2 colocated and co-aligned detectors as before)

$$p(d \mid S_{n_1}, S_{n_2}, \mathcal{M}_0) = \frac{1}{\sqrt{\det(2\pi C_n)}} \exp\left[-\frac{1}{2} d^T C_n^{-1} d\right]$$

$$p(d \mid S_{n_1}, S_{n_2}, S_h, \mathcal{M}_1) = \frac{1}{\sqrt{\det(2\pi C)}} \exp\left[-\frac{1}{2} d^T C^{-1} d\right]$$

where $d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$, $\quad C_n = \begin{bmatrix} S_{n_1} \mathbb{1}_{N\times N} & 0_{N\times N} \\ 0_{N\times N} & S_{n_2} \mathbb{1}_{N\times N} \end{bmatrix}$

$$C = C_n + S_h \begin{bmatrix} \mathbb{1}_{N\times N} & \mathbb{1} \\ \mathbb{1} & \mathbb{1} \end{bmatrix} = \begin{bmatrix} (S_{n_1} + S_h)\mathbb{1}_{N\times N} & S_h \mathbb{1}_{N\times N} \\ S_h \mathbb{1}_{N\times N} & (S_{n_2}+S_h)\mathbb{1}_{N\times N} \end{bmatrix}$$

$$S_1 \equiv S_{n_1} + S_h \qquad S_2 \equiv S_{n_2} + S_h \rightarrow \quad = \begin{bmatrix} S_1 \mathbb{1}_{N\times N} & S_h \mathbb{1}_{N\times N} \\ S_h \mathbb{1}_{N\times N} & S_2 \mathbb{1}_{N\times N} \end{bmatrix}$$

NOTE:

$$C_h^{-1} = \begin{bmatrix} \frac{1}{S_{h_1}} \mathbb{1}_{N\times N} & 0_{N\times N} \\ 0_{N\times N} & \frac{1}{S_{h_2}} \mathbb{1}_{N\times N} \end{bmatrix}$$

$$C^{-1} = \frac{1}{(S_1 S_2 - S_h^2)} \begin{bmatrix} S_2 \mathbb{1}_{N\times N} & -S_h \mathbb{1}_{N\times N} \\ -S_h \mathbb{1}_{N\times N} & S_1 \mathbb{1}_{N\times N} \end{bmatrix}$$

$$= \frac{1}{\left(1 - \frac{S_h^2}{S_1 S_2}\right)} \begin{bmatrix} \frac{1}{S_1} \mathbb{1}_{N\times N} & -\frac{S_h}{S_1 S_2} \mathbb{1}_{N\times N} \\ -\frac{S_h}{S_1 S_2} \mathbb{1}_{N\times N} & \frac{1}{S_2} \mathbb{1}_{N\times N} \end{bmatrix}$$

Arguments of exponentials:

$$-\frac{1}{2} d^T C_n^{-1} d = -\frac{1}{2}\left(\frac{1}{S_{n1}}\sum_i d_{1i}^2 + \frac{1}{S_{n2}}\sum_i d_{2i}^2\right)$$

$$= -\frac{N}{2}\left(\frac{\hat{C}_{11}}{S_{n1}} + \frac{\hat{C}_{22}}{S_{n2}}\right)$$

$$-\frac{1}{2} d^T C^{-1} d = -\frac{N}{2}\frac{1}{\left(1-\frac{S_h^2}{S_1 S_2}\right)}\left[\frac{\hat{C}_{11}}{S_1} + \frac{\hat{C}_{22}}{S_2} - \frac{2 S_h}{S_1 S_2}\hat{C}_{12}\right]$$

(see previous problem)

where $\hat{C}_{11} = \frac{1}{N}\sum_i d_{1i}^2$, $\hat{C}_{22} = \frac{1}{N}\sum_i d_{2i}^2$, $\hat{C}_{12} = \frac{1}{N}\sum d_{1i} d_{2i}$

As shown in the previous problem these data combinations are ML estimators of S_{n1}, S_{n2} for M_0, and S_1, S_2, S_h for M_1

Detection statistic:

$$\Lambda_{ML}(d) = \frac{\max_{s_{n_1}, s_{n_2}, s_h} p(d \mid s_{n_1}, s_{n_2}, s_h, \mathcal{M}_1)}{\max_{s_{n_1}, s_{n_2}} p(d \mid s_{n_1}, s_{n_2}, \mathcal{M}_0)}$$

$$\text{numerator} = \frac{1}{(2\pi)^N \left(\hat{C}_{11}\hat{C}_{22} - \hat{C}_{12}^2\right)^{N/2}} \exp\left[-\frac{1}{2} \frac{N}{\left(1 - \frac{\hat{C}_{12}^2}{\hat{C}_{11}\hat{C}_{22}}\right)} \underbrace{\left(\frac{\hat{C}_{11}}{\hat{C}_{11}} + \frac{\hat{C}_{22}}{\hat{C}_{22}} - \frac{2\hat{C}_{12}^2}{\hat{C}_{11}\hat{C}_{12}}\right)}_{2\left(1 - \frac{\hat{C}_{12}^2}{\hat{C}_{11}\hat{C}_{22}}\right)}\right]$$

$$= \frac{1}{(2\pi)^N \left(\hat{C}_{11}\hat{C}_{22} - \hat{C}_{12}^2\right)^{N/2}} \exp[-N]$$

$$\text{denominator} = \frac{1}{(2\pi)^N \left(\hat{C}_{11}\hat{C}_{22}\right)^{N/2}} \exp\left[-\frac{N}{2}\left(\frac{\hat{C}_{11}}{\hat{C}_{11}} + \frac{\hat{C}_{22}}{\hat{C}_{22}}\right)\right]$$

$$= \frac{1}{(2\pi)^N \left(\hat{C}_{11}\hat{C}_{22}\right)^{N/2}} \exp[-N]$$

Thus,

$$\Lambda_{ML}(d) = \frac{\dfrac{1}{\cancel{(2\pi)^N}\left(\hat{C}_{11}\hat{C}_{22}-\hat{C}_{12}^2\right)^{N/2}} \cancel{\exp[-N]}}{\dfrac{1}{\cancel{(2\pi)^N}\left(\hat{C}_{11}\hat{C}_{22}\right)^{N/2}} \cancel{\exp[-N]}}$$

$$= \left(\frac{\hat{C}_{11}\hat{C}_{22}}{\hat{C}_{11}\hat{C}_{22}-\hat{C}_{12}^2}\right)^{N/2}$$

$$= \frac{1}{\left(1-\dfrac{\hat{C}_{12}^2}{\hat{C}_{11}\hat{C}_{22}}\right)^{N/2}}$$

$$2\ln\Lambda_{ML}(d) = 2\ln\left[\left(1-\frac{\hat{C}_{12}^2}{\hat{C}_{11}\hat{C}_{22}}\right)^{-N/2}\right]$$

$$= -N\ln\left(1-\hat{C}_{12}^2/\hat{C}_{11}\hat{C}_{22}\right)$$

$$\simeq N\frac{\hat{C}_{12}^2}{\hat{C}_{11}\hat{C}_{22}}$$

assuming weak signal approx using $\ln(1+x)\simeq x$ for $|x|\ll 1$

(12) Perform marginalization integral:

$$p(d \mid S_{n_1}, S_{n_2}, S_h) = \int_{-\infty}^{\infty} dh \; p_n(d-h \mid S_{n_1}, S_{n_2})\, p(h \mid S_h)$$

where $p_n(d-h \mid S_{n_1}, S_{n_2}) = \dfrac{1}{2\pi\sqrt{S_{n_1} S_{n_2}}} \exp\left[-\dfrac{1}{2}\left(\dfrac{(d_1-h)^2}{S_{n_1}} + \dfrac{(d_2-h)^2}{S_{n_2}}\right)\right]$

$$p(h \mid S_h) = \frac{1}{\sqrt{2\pi S_h}} \exp\left[-\frac{1}{2}\frac{h^2}{S_h}\right]$$

Integrand on RHS:

$$= \frac{1}{(2\pi)^{3/2}\sqrt{S_{n_1} S_{n_2} S_h}} \exp\left[-\frac{1}{2}\left(\frac{(d_1-h)^2}{S_{n_1}} + \frac{(d_2-h)^2}{S_{n_2}} + \frac{h^2}{S_h}\right)\right]$$

<u>Now</u>: $[\;\;] = -\dfrac{1}{2}\left(\dfrac{d_1^2+h^2-2d_1h}{S_{n_1}} + \dfrac{d_2^2+h^2-2d_2h}{S_{n_2}} + \dfrac{h^2}{S_h}\right)$

$$= -\frac{1}{2}\left[h^2\left(\frac{1}{S_{n_1}} + \frac{1}{S_{n_2}} + \frac{1}{S_h}\right) - 2h\left(\frac{d_1}{S_{n_1}} + \frac{d_2}{S_{n_2}}\right) + \left(\frac{d_1^2}{S_{n_1}} + \frac{d_2^2}{S_{n_2}}\right)\right]$$

$$= -\frac{1}{2}\left[A h^2 - 2hB + D \right]$$

$$= -\frac{A}{2}\left[h^2 - 2h\frac{B}{A} + \frac{D}{A} \right]$$

where $A = \frac{1}{S_{n_1}} + \frac{1}{S_{n_2}} + \frac{1}{S_h}$

$$= \frac{S_{n_2} S_h + S_{n_1} S_h + S_{n_1} S_{n_2}}{S_{n_1} S_{n_2} S_h}$$

$$= \frac{S_{n_1} S_{n_2} + S_h \left(S_{n_1} + S_{n_2} \right)}{S_{n_1} S_{n_2} S_h}$$

$$= \frac{\det C}{\left(\det C_n\right) \cdot S_h}$$ where $C = \begin{bmatrix} S_{n_1} + S_h & S_h \\ S_h & S_{n_2} + S_h \end{bmatrix}$

$$C_n = \begin{bmatrix} S_{n_1} & 0 \\ 0 & S_{n_2} \end{bmatrix}$$

$$B = \frac{d_1}{s_{h_1}} + \frac{d_2}{s_{h_2}}$$

$$D = \frac{d_1^2}{s_{h_1}} + \frac{d_2^2}{s_{h_2}}$$

Complete the square:

$$[\;] = -\frac{A}{2}\left[\left(h - \frac{B}{A}\right)^2 - \frac{B^2}{A^2} + \frac{D}{A}\right]$$

$$= -\frac{A}{2}\left[\left(h - \frac{B}{A}\right)^2 - \left(\frac{B^2 - AD}{A^2}\right)\right]$$

Now!

$$\int_{-\infty}^{\infty} dh \, \exp\left[-\frac{A}{2}\left(h - \frac{B}{A}\right)^2\right] = \sqrt{2\pi}\,\frac{1}{\sqrt{A}}$$

using $\frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} dx \, e^{-\frac{(x-\mu)^2}{2\sigma^2}} = 1$ (for a Gaussian distribution)

Thus,

$$p(d \mid S_{n_1}, S_{n_2}, S_h) = \frac{1}{(2\pi)^{\cancel{3/2}} \sqrt{S_{n_1} S_{n_2} S_h}} \cancel{\sqrt{2\pi}} \frac{1}{\sqrt{A}} \exp\left[+ \frac{A}{2} \left(\frac{B^2 - AD}{A^2} \right) \right]$$

$$= \frac{1}{2\pi \sqrt{\det C}} \exp\left[-\frac{1}{2} \left(\frac{AD - B^2}{A} \right) \right]$$

Argument of exponential:

$$-\frac{1}{2}\left(\frac{AD - B^2}{A}\right) = -\frac{1}{2}\left(\frac{S_{n_1} S_{n_2} S_h}{\det C}\right)\left(\frac{S_{n_1} S_{n_2} + S_h (S_{n_1} + S_{n_2})}{S_{n_1} S_{n_2} S_h} \left(\frac{d_1^2}{S_{n_1}} + \frac{d_2^2}{S_{n_2}} \right) - \left(\frac{d_1}{S_{n_1}} + \frac{d_2}{S_{n_2}} \right)^2 \right)$$

$$= -\frac{1}{2}\left(\frac{1}{\det C}\right)\left[\left(S_{n_1} S_{n_2} + S_h (S_{n_1} + S_{n_2})\right)\left(\frac{d_1^2}{S_{n_1}} + \frac{d_2^2}{S_{n_2}} \right) - S_{n_1} S_{n_2} S_h \left(\frac{d_1^2}{S_{n_1}^2} + \frac{d_2^2}{S_{n_2}^2} + \frac{2 d_1 d_2}{S_{n_1} S_{n_2}} \right) \right]$$

$$= -\frac{1}{2}\left(\frac{1}{\det C}\right)\left(d_1^2\left(S_{h_2} + S_h + \frac{\cancel{S_h S_{h_2}}}{\cancel{S_{h_1}}} - \frac{\cancel{S_{h_2} S_h}}{\cancel{S_{h_1}}}\right)\right.$$

$$\left. + d_2^2\left(S_{h_1} + S_h + \frac{\cancel{S_h S_{h_1}}}{\cancel{S_{h_2}}} - \frac{\cancel{S_{h_1} S_h}}{\cancel{S_{h_2}}}\right) - 2 S_h d_1 d_2\right)$$

$$= -\frac{1}{2}\left(\frac{1}{\det C}\right)\left(d_1^2 (S_{h_2} + S_h) + d_2^2 (S_{h_1} + S_h) - 2 S_h d_1 d_2\right)$$

$$= -\frac{1}{2}\left(d_1^2\left(\frac{S_{h_2} + S_h}{\det C}\right) + d_2^2\left(\frac{S_{h_1} + S_h}{\det C}\right) + 2 d_1 d_2\left(\frac{-S_h}{\det C}\right)\right)$$

$$= -\frac{1}{2}\sum_{I,J=1}^{2} d_I \left(C^{-1}\right)_{IJ} d_J$$

where $(C^{-1})_{IJ}$ are the matrix component of the inverse to

$$C = \begin{bmatrix} S_{h_1} + S_h & S_h \\ S_h & S_{h_2} + S_h \end{bmatrix}, \quad C^{-1} = \frac{1}{\det C}\begin{bmatrix} S_{h_2} + S_h & -S_h \\ -S_h & S_{h_1} + S_h \end{bmatrix}$$