6. optimal Filtering for cross-correlation statistic

$$\hat{S}_h \simeq \int_{-\infty}^{\infty} df \int_{-\infty}^{\infty} df' \, \delta_T(f-f') \, \tilde{d}_1(f) \, \tilde{d}_2^*(f') \, \tilde{Q}^*(f')$$

<u>Expected value</u>

$$\mu \equiv \langle \hat{S}_h \rangle$$

$$= \int_{-\infty}^{\infty} df \int_{-\infty}^{\infty} df \, \delta_T(f-f') \, \langle \tilde{d}_1(f) \, \tilde{d}_2^*(f') \rangle \, \tilde{Q}^*(f')$$

For uncorrelated noise

$$\langle \tilde{d}_1(f) \, \tilde{d}_2^*(f') \rangle = \langle \tilde{h}_1(f) \, \tilde{h}_2^*(f') \rangle$$

$$= \frac{1}{2} \delta(f-f') \, \Gamma_{12}(f) \, S_h(f)$$

$$\rightarrow \quad \mu = \frac{1}{2} \int_{-\infty}^{\infty} df \, \underbrace{\delta_T(0)}_{T} \, \Gamma_{12}(f) \, S_h(f) \, \tilde{Q}^*(f)$$

$$= \frac{T}{2} \int_{-\infty}^{\infty} df \, \Gamma_{12}(f) \, S_h(f) \, \tilde{Q}^*(f)$$

Variance:

$$\sigma^2 \equiv \left\langle \left(\hat{S}_h - \langle \hat{S}_h \rangle\right)^2 \right\rangle$$

$$= \langle \hat{S}_h^2 \rangle - \mu^2$$

$$= \int df \int df' \int dp \int dp'\, \delta_T(f-f')\, \delta_T(p-p')\, \tilde{Q}^*(f')\, \tilde{Q}(p')$$

$$\Big(\langle \tilde{d}_1(f)\, \tilde{d}_2^*(f')\, \tilde{d}_1^*(p)\, \tilde{d}_2(p') \rangle$$

$$- \langle \tilde{d}_1(f)\, \tilde{d}_2^*(f') \rangle \langle \tilde{d}_1^*(p)\, \tilde{d}_2(p') \rangle \Big)$$

Use $\langle abcd \rangle = \langle ab \rangle\langle cd \rangle + \langle ac \rangle \langle bd \rangle + \langle ad \rangle \langle bc \rangle$
for zero-mean Gaussian random variables

so

$$\langle d_1 d_2^* d_1^* d_2 \rangle - \langle d_1 d_2^* \rangle \langle d_1^* d_2 \rangle$$

$$= \langle \tilde{d}_1(f)\, \tilde{d}_1^*(p) \rangle \langle \tilde{d}_2^*(f')\, \tilde{d}_2(p') \rangle + \underbrace{\langle \tilde{d}_1(f)\, \tilde{d}_2(p') \rangle \langle \tilde{d}_2^*(f')\, \tilde{d}_1^*(p) \rangle}_{\substack{\text{for uncorrelated noise, this is} \\ \text{proportional to } S_h^2(f)}}$$

$$\approx \frac{1}{2}\delta(f-p)\, P_1(f)\, \frac{1}{2}\delta(f'-p')\, P_2(f')$$

totals autocorrelated power

$P_1(f) = P_{n_1}(f) + P_{gw}(f) \approx P_{n_1}(f)$ for weak signals

Thus,

$$\sigma^2 \approx \frac{1}{4}\int_{-\infty}^{\infty} df \int_{-\infty}^{\infty} df' \, \delta_T(f-f') \underbrace{\delta_T(f-f')}_{\approx\, \delta(f-f')} \tilde{Q}^*(f)\, \tilde{Q}(f')\, P_1(f) P_2(f')$$

$$= \frac{1}{4}\int_{-\infty}^{\infty} df \, \underbrace{\delta_T(0)}_{T} |\tilde{Q}(f)|^2 \, P_1(f) P_2(f)$$

$$= \frac{T}{4}\int_{-\infty}^{\infty} df \, |\tilde{Q}(f)|^2 P_1(f) P_2(f)$$

$$\rightarrow \mathrm{SNR} = \frac{\mu}{\sigma} = \frac{\frac{T}{\cancel{2}}\int_{-\infty}^{\infty} df\, \Gamma_{12}(f) S_h(f) \tilde{Q}^*(f)}{\sqrt{\frac{T}{\cancel{4}}\int_{-\infty}^{\infty} df\, |\tilde{Q}(f)|^2 P_1(f) P_2(f)}}$$

$$= \sqrt{T}\, \frac{\int_{-\infty}^{\infty} df\, \frac{\Gamma_{12}(f) S_h(f)}{P_1(f) P_2(f)} \tilde{Q}^*(f)\, P_1(f) P_2(f)}{\sqrt{\int_{-\infty}^{\infty} df\, \tilde{Q}(f)\, \tilde{Q}^*(f)\, P_1(f) P_2(f)}}$$

$$= \sqrt{T}\, \left(\frac{\Gamma_{12} S_h}{P_1 P_2}, Q\right) \Big/ \sqrt{(Q,Q)}$$

where $(A, B) \equiv \int_{-\infty}^{\infty} df\, A(f) B^*(f) P_1(f) P_2(f)$

$\rightarrow$ $SNR = \sqrt{T} \dfrac{(A, \tilde{Q})}{\sqrt{(\tilde{Q}, \tilde{Q})}}$ where $A(f) \equiv \dfrac{\Gamma_{12}(f) S_h(f)}{P_1(f) P_2(f)}$

<u>Recall:</u> $\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta$
$\vec{B} \cdot \vec{B} = |\vec{B}|^2$

$\rightarrow$ $\dfrac{\vec{A} \cdot \vec{B}}{\sqrt{\vec{B} \cdot \vec{B}}} = |\vec{A}| \cos\theta$

is maximized by have $\theta = 0$

$\rightarrow$ $\vec{A}$ should point in same direction as B.

Thus, SNR is maximized when $\tilde{Q}(f) \propto \dfrac{\Gamma_{12}(f) S_h(f)}{P_1(f) P_2(f)}$

$\propto \dfrac{\Gamma_{12}(f)\, H(f)}{P_1(f)\, P_2(f)}$

just differ by overall amplitude

7. (a) $p(d \mid a, \sigma) \propto \exp\left[-\frac{1}{2} \sum_{i=1}^{N} \frac{(d_i - a)^2}{\sigma_i^2} \right]$

Maximize likelihood $\Longleftrightarrow$ Maximize $\ln$ (likelihood)

Define $L(a) = \ln\left[p(d \mid a, \sigma) \right] = -\frac{1}{2} \sum_{i=1}^{N} \frac{(d_i - a)^2}{\sigma_i^2}]$

$$0 = \left.\frac{dL}{da}\right|_{a=\hat{a}} = - \sum_i \left.\frac{2(d_i - a)(-1)}{\sigma_i^2}\right|_{a=\hat{a}}$$

$$= 2 \sum_i \frac{(d_i - \hat{a})}{\sigma_i^2}$$

$$= 2\left(\sum_i \frac{d_i}{\sigma_i^2} - \hat{a} \sum_i \frac{1}{\sigma_i^2} \right)$$

$$\longrightarrow \boxed{\hat{a} = \sum_i \frac{d_i}{\sigma_i^2} \Big/ \sum_i \frac{1}{\sigma_i^2}}$$

(b) $p(d|A,C) \propto \exp\left[-\frac{1}{2}(d-MA)^\dagger C^{-1}(d-MA)\right]$

Dagger $\equiv$ complex-conjugate transpose

allows for complex data d, parameter A

<u>Define</u>: $L(A,A^\dagger) = \ln p(d|A,C)$

$$= -\frac{1}{2}(d-MA)^\dagger C^{-1}(d-MA)$$

Treat $A, A^\dagger$ as independent variables for the variation

$$\delta L = \frac{1}{2}\delta A^\dagger M^\dagger C^{-1}(d-MA) + \frac{1}{2}(d-MA)^\dagger C^{-1} M\,\delta A$$

$\delta L = 0$ for $\delta A^\dagger$ $\rightarrow$ $0 = M^\dagger C^{-1}(d-M\hat{A})$

$$= M^\dagger C^{-1} d - M^\dagger C^{-1} M\hat{A}$$

$$\rightarrow \quad \hat{A} = (M^\dagger C^{-1} M)^{-1} M^\dagger C^{-1} d$$

$$= F^{-1} X$$

where $F \equiv M^\dagger C^{-1} M$ (Fisher matrix)

$X \equiv M^\dagger C^{-1} d$ ("dirty map")

$\delta L = 0$ for δA

$$\frac{1}{2}(d - M\hat{A})^{+} C^{-1} M = 0$$

$$d^{+} C^{-1} M - \hat{A}^{+} M^{+} C^{-1} M = 0$$

Take $+$ of above equation using $(C^{-1})^{+} = C^{-1}$

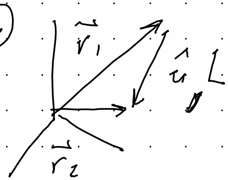
$$\rightarrow \quad M^{+} C^{-1} d - M^{+} C^{-1} M \hat{A} = 0$$

$$\rightarrow \quad \hat{A} = (M^{+} C^{-1} M)^{-1} M^{+} C^{-1} d$$

$$= F^{-1} X$$

(as before)

8.



$$r(t) = \Delta T(t) = \frac{1}{2c} u^a u^b \int_0^L ds\, h_{ab}(t(s), \vec{x}(s))$$

where
$$t(s) = \left(t - \frac{L}{c}\right) + \frac{s}{c}$$
$$\vec{x}(s) = \vec{r}_1 + s\hat{u}$$
$$s \in [0, L]$$

plane-wave expansion

$$h_{ab}(t, \vec{x}) = \int_{-\infty}^{\infty} df \int d^2\Omega_{\hat{k}} \sum_A h_A(f, \hat{k})\, e^A_{ab}(\hat{k})\, e^{i2\pi f(t - \hat{k}\cdot\vec{x}/c)}$$

Replace $t, \vec{x}$ by $t(s), \vec{x}(s)$

Exponential becomes

$$e^{i2\pi f\left(\left(t - \frac{L}{c}\right) + \frac{s}{c} - \frac{\hat{k}\cdot}{c}(\vec{r}_1 + s\hat{u})\right)}$$

$$= e^{i2\pi f\left(\left(t - \frac{L}{c}\right) - \frac{\hat{k}}{c}\cdot\vec{r}_1\right)}\, e^{i2\pi f\frac{s}{c}\left(1 - \hat{k}\cdot\hat{u}\right)}$$

$$= e^{i2\pi f\left(t_1 - \frac{\hat{k}\cdot\vec{r}_1}{c}\right)}\, e^{i2\pi f\frac{s}{c}(1 - \hat{k}\cdot\hat{u})}$$

Do integral

$$\int_0^L ds\, e^{i2\pi f \frac{s}{c}(1-\hat{k}\cdot\hat{u})} = \frac{c}{i2\pi f(1-\hat{k}\cdot\hat{u})} e^{i2\pi f \frac{s}{c}(1-\hat{k}\cdot\hat{u})}\Big|_0^L$$

$$= \frac{c}{(i2\pi f)}\frac{1}{(1-\hat{k}\cdot\hat{u})}\left(e^{i2\pi f\frac{L}{c}(1-\hat{k}\cdot\hat{u})} - 1\right)$$

Thus,

$$\Delta T(t) = \frac{1}{2c}\int df \int d^2\Omega_{\hat{k}} \sum_A h_A(f,\hat{k}) e^A_{ab}(\hat{k}) \frac{c}{(i2\pi f)}\frac{1}{(1-\hat{k}\cdot\hat{u})}$$

$$\Big[\underbrace{e^{i2\pi f(t_1 - \frac{\hat{k}\cdot\vec{r}_1}{c})} e^{i2\pi f\frac{L}{c}(1-\hat{k}\cdot\hat{u})}}_{e^{i2\pi f\left(\underbrace{(t_1+\frac{L}{c})}_{t_2} - \frac{\hat{k}}{c}\cdot\underbrace{(\vec{r}_1+\hat{u}L)}_{\vec{r}_2}\right)}} - e^{i2\pi f(t_1 - \frac{\hat{k}\cdot\vec{r}_1}{c})}\Big]$$

$$= \frac{1}{2}\int df \int d^2\Omega_{\hat{k}} \sum_A h_A(f,\hat{k}) e^A_{ab}(\hat{k}) \left(\frac{1}{i2\pi f}\right)\left(\frac{1}{1-\hat{k}\cdot\hat{u}}\right)$$

$$\Big[e^{i2\pi f(t_2 - \hat{k}\cdot\vec{r}_2/c)} - e^{i2\pi(t_1 - \hat{k}\cdot\vec{r}_1/c)}\Big]$$

NOTE:

$e^{i2\pi f\left(t_2 - \hat{k}\cdot\vec{r}_2/c\right)}$ corresponds to reception of pulse at $\vec{r}_2$ at time t_2

$e^{i2\pi f\left(t_1 - \hat{k}\cdot\vec{r}_1/c\right)}$ corresponds to emission of pulse at $\vec{r}_1$ at time t_1

Rewrite!

$$\Delta T(t) = \int df \int d\Omega_{k} \sum_A h_A(f, \hat{k}) \left(\frac{1}{i2\pi f}\right)\left(\frac{1}{1-\hat{k}\cdot\hat{u}}\right) e^A_{ab}(\hat{k})\, u^a u^b$$

$$\underline{\underline{e^{i2\pi f t}}}\ \underline{\underline{e^{-i2\pi f \hat{k}\cdot\vec{r}_2/c}}} \left[1 - e^{-i2\pi f t}\, e^{+i2\pi f \hat{k}\cdot\vec{r}_2/c}\, \underbrace{e^{i2\pi f \overset{t - \frac{L}{c}}{t_1}}\, e^{-i2\pi f \hat{k}\cdot\vec{r}_1/c}}\right]$$

$$= e^{-i2\pi f \frac{L}{c}}\, e^{i2\pi f \hat{k}\cdot(\vec{r}_2 - \vec{r}_1)/c}$$

$$= e^{-i2\pi f \frac{L}{c}\left(1 - \hat{k}\cdot\underbrace{\frac{(\vec{r}_2 - \vec{r}_1)}{L}}_{\hat{u}}\right)},$$

$$= e^{-i2\pi f \frac{L}{c}\left(1 - \hat{k}\cdot\hat{u}\right)}$$

Thus,

$$\Delta T(t) = \int_{-\infty}^{\infty} df\, e^{i2\pi f t} \int d^2\Omega_{\hat{k}} \sum_A h_A(f, \hat{k}) \left(\frac{1}{i2\pi f}\right) \frac{u^a u^b e^A_{ab}(\hat{k})}{1 - \hat{k}\cdot\hat{u}}$$

$$e^{-i2\pi f \frac{\hat{k}\cdot\vec{r}_1}{c}} \left[1 - e^{-i2\pi f \frac{L}{c}(1 - \hat{k}\cdot\hat{u})} \right]$$

So

$$\boxed{R_A(f,\hat{k}) = \frac{1}{i2\pi f} \frac{u^a u^b e^A_{ab}(\hat{k})}{2(1 - \hat{k}\cdot\hat{u})} \left[1 - e^{-i2\pi f \frac{L}{c}(1 - \hat{k}\cdot\hat{u})} \right] e^{-i2\pi f \frac{\hat{k}\cdot\vec{r}_2}{c}}}$$

Note: For pulsar timing $\hat{p} = -\hat{u}$, take $\vec{r}_2 = \vec{0}$ (SSB)

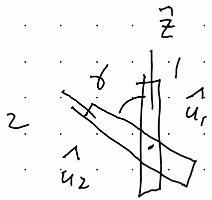
$\frac{1}{i2\pi f} \to 1$ for redshift $\left(\frac{\Delta\nu}{\nu}\right)$ measurements

$$\boxed{R_A(f,\hat{k}) = \underbrace{\frac{1}{2}\left(\frac{p^a p^b e^A_{ab}(\hat{k})}{1 + \hat{k}\cdot\hat{p}}\right)}_{\equiv F^A(\hat{k})} \Big[\underbrace{1}_{\text{Earth term}} - \underbrace{e^{-i2\pi f \frac{L}{c}(1 + \hat{k}\cdot\hat{p})}}_{\text{pulsar term}} \Big]}$$

(9.) ORF for colocated electric dipole antenna

$$r_I(t) = \hat{u}_I \cdot \vec{E}(t, \vec{x}=0)$$

$$\vec{E}(t,\vec{x}) = \int df \int d^2\Omega_{\hat{k}} \sum_{\alpha=1,2} \tilde{E}_\alpha(f,\hat{k})\, \hat{\epsilon}_\alpha(\hat{k})\, e^{i2\pi f(t - \hat{k}\cdot\vec{x}/c)}$$



<u>Take</u>: $\hat{u}_1 = \hat{z}$

$\hat{u}_2 = \sin\gamma\, \hat{x} + \cos\gamma\, \hat{z}$

$\hat{\epsilon}_1(\hat{k}) = -\hat{\phi}$

$\hat{\epsilon}_2(\hat{k}) = -\hat{\theta}$

<u>Now</u>: $r_I(t) = \hat{u}_I \cdot \vec{E}(t, \vec{x}=0)$

$$= \int_{-\infty}^{\infty} df \int d^2\Omega_{\hat{k}} \sum_{\alpha=1,2} \tilde{E}_\alpha(f,\hat{k})\, \hat{u}_I \cdot \hat{\epsilon}_\alpha(\hat{k})\, e^{i2\pi f t}$$

$$= \int_{-\infty}^{\infty} df\, e^{i2\pi f t} \int d^2\Omega_{\hat{k}} \sum_\alpha \tilde{E}_\alpha(f,\hat{k})\, \hat{u}_I \cdot \hat{\epsilon}_\alpha(\hat{k})$$

so $R_I^\alpha(f,\hat{k}) = \hat{u}_I \cdot \hat{\epsilon}_\alpha(\hat{k})$

$$\hat{u}_1 = \hat{z} \qquad \hat{u}_2 = \sin\gamma\, \hat{x} + \cos\gamma\, \hat{z}$$

$$\hat{\epsilon}_1(\hat{k}) = -\hat{\phi} = \sin\phi\, \hat{x} - \cos\phi\, \hat{y}$$

$$\hat{\epsilon}_2(\hat{k}) = -\hat{\theta} = -\cos\theta\cos\phi\, \hat{x} - \cos\theta\sin\phi\, \hat{y} + \sin\theta\, \hat{z}$$

$$\hat{k} = -\sin\theta\cos\phi\, \hat{x} - \sin\theta\sin\phi\, \hat{y} - \cos\theta\, \hat{z}$$

Thus,

$$\hat{u}_1 \cdot \hat{\epsilon}_1(\hat{k}) = -\hat{z}\cdot\hat{\phi} = 0$$

$$\hat{u}_2 \cdot \hat{\epsilon}_1(\hat{k}) = -(\sin\gamma\, \hat{x} + \cos\gamma\, \hat{z})\cdot\hat{\phi}$$

$$= \sin\phi\sin\gamma$$

$$\hat{u}_1 \cdot \hat{\epsilon}_2(\hat{k}) = -\hat{z}\cdot\hat{\theta} = \sin\theta$$

$$\hat{u}_2 \cdot \hat{\epsilon}_2(\hat{k}) = -(\sin\gamma\, \hat{x} + \cos\gamma\, \hat{z})\cdot\hat{\theta}$$

$$= -\sin\gamma\cos\theta\cos\phi + \sin\theta\cos\gamma$$

$$\Gamma_{12}(f) = \frac{1}{8\pi} \int d^2\Omega_{\hat{k}} \sum_{\alpha} R_1^{\alpha}(f,\hat{k}) R_2^{\alpha}(f,\hat{k})$$

$$= \frac{1}{8\pi} \int d^2\Omega_{\hat{k}} \sum_{\alpha} \left(\hat{u}_1 \cdot \hat{\epsilon}_{\alpha}(\hat{k})\right)\left(\hat{u}_2 \cdot \hat{\epsilon}_{\alpha}(\hat{k})\right)$$

$$= \frac{1}{8\pi} \int d^2\Omega_{\hat{k}} \Big[\cancelto{0}{\left(\hat{u}_1 \cdot \hat{\epsilon}_1(\hat{k})\right)} \left(\hat{u}_2 \cdot \hat{\epsilon}_1(\hat{k})\right) + \left(\hat{u}_1 \cdot \hat{\epsilon}_2(\hat{k})\right)\left(\hat{u}_2 \cdot \hat{\epsilon}_2(\hat{k})\right) \Big]$$

$$= \frac{1}{8\pi} \int d^2\Omega_{\hat{k}} \left[\sin\theta \left(-\sin\gamma \cos\theta \cos\phi + \cos\gamma \sin\theta\right)\right]$$

$$= -\frac{1}{8\pi} \int d^2\Omega_{\hat{k}} \sin\theta \cos\theta \cancelto{0}{\cos\phi} \sin\gamma + \frac{1}{8\pi} \int d^2\Omega_{\hat{k}} \sin^2\theta \cos\gamma$$

$$= \frac{1}{8\pi} \int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta) \left(1-\cos^2\theta\right) \cos\gamma$$

$$= \frac{2\pi}{8\pi} \int_{-1}^{1} dx \left(1-x^2\right) = \frac{1}{4} \left(x - \frac{x^3}{3}\right)\Big|_{-1}^{1} \cos\gamma = \frac{1}{4}\left(\frac{4}{3}\right) \cos\gamma$$

$$= \boxed{\frac{1}{3}\cos\gamma}$$