10. ML estimators

$$p(d \mid S_{n_1}, S_{n_2}, S_h) = \frac{1}{\sqrt{\det(2\pi C)}} \exp\left[-\frac{1}{2} d^T C^{-1} d\right]$$

$$d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} d_{11} \\ d_{12} \\ \vdots \\ d_{1N} \\ d_{21} \\ d_{22} \\ \vdots \\ d_{2N} \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} a \\ a \\ a \\ a \end{matrix}}\right\} d_1 \\ \left.\vphantom{\begin{matrix} a \\ a \\ a \\ a \end{matrix}}\right\} d_2 \end{matrix}, \quad C = \begin{bmatrix} S_1 \mathbb{1}_{N\times N} & S_h \mathbb{1}_{N\times N} \\ S_h \mathbb{1}_{N\times N} & S_2 \mathbb{1}_{N\times N} \end{bmatrix}$$

$$S_1 = S_h + S_{n_1}$$
$$S_2 = S_h + S_{n_2}$$

$$C^{-1} = \frac{1}{(S_1 S_2 - S_h^2)} \begin{bmatrix} S_2 \mathbb{1}_{N\times N} & -S_h \mathbb{1}_{N\times N} \\ -S_h \mathbb{1}_{N\times N} & S_1 \mathbb{1}_{N\times N} \end{bmatrix}$$

$$= \frac{1}{\left(1 - \frac{S_h^2}{S_1 S_2}\right)} \begin{bmatrix} \frac{1}{S_1} \mathbb{1}_{N\times N} & \frac{-S_h}{S_1 S_2} \mathbb{1}_{N\times N} \\ -\frac{S_h}{S_1 S_2} \mathbb{1}_{N\times N} & \frac{1}{S_2} \mathbb{1}_{N\times N} \end{bmatrix}$$

Thus,

$$p(d|S_{n_1}, S_{n_2}, S_h) = \frac{1}{(2\pi)^N (S_1 S_2 - S_h^2)^{N/2}} \exp\left[-\frac{1}{2}\frac{1}{\left(1-\frac{S_h^2}{S_1 S_2}\right)}\right.$$

$$\left.\left(\frac{1}{S_1}\sum_i d_{1i}^2 + \frac{1}{S_2}\sum_i d_{2i}^2 - \frac{2 S_h}{S_1 S_2}\sum_i d_{1i} d_{2i}\right)\right]$$

$$= \frac{1}{(2\pi)^N (S_1 S_2 - S_h^2)^{N/2}} \exp\left[-\frac{N}{2}\frac{1}{\left(1-\frac{S_h^2}{S_1 S_2}\right)}\left(\frac{\hat{C}_{11}}{S_1} + \frac{\hat{C}_{22}}{S_2} - \frac{2 S_h \hat{C}_{12}}{S_1 S_2}\right)\right]$$

where $\hat{C}_{11} = \frac{1}{N}\sum_i d_{1i}^2$, etc.

Max likelihood $\Longleftrightarrow$ Max $\ln$(likelihood)

$$\mathcal{L}(S_{n_1}, S_{n_2}, S_h) = \ln\left[p(d|S_{n_1}, S_{n_2}, S_h)\right]$$

$$= -N\ln 2\pi - \frac{N}{2}\ln(S_1 S_2 - S_h^2) - \frac{N}{2}\frac{1}{\left(1-\frac{S_h^2}{S_1 S_2}\right)}\left(\frac{\hat{C}_{11}}{S_1} + \frac{\hat{C}_{22}}{S_2} - \frac{2 S_h \hat{C}_{12}}{S_1 S_2}\right)$$

$$= \text{const} - \frac{N}{2}\left[\ln(S_1 S_2 - S_h^2) + \frac{1}{\left(1-\frac{S_h^2}{S_1 S_2}\right)}\left(\frac{\hat{C}_{11}}{S_1} + \frac{\hat{C}_{22}}{S_2} - \frac{2 S_h \hat{C}_{12}}{S_1 S_2}\right)\right]$$

Want to show that $\hat{C}_{11}, \hat{C}_{22}, \hat{C}_{12}$ are the ML estimators of S_1, S_2, S_h; Take partial derivative wrt S_1, S_2, S_h

$$0 = \frac{\partial \mathcal{L}}{\partial S_1}$$

$$= -\frac{N}{2}\left[\frac{1}{(S_1 S_2 - S_h^2)} S_2 - \frac{1}{\left(1 - \frac{S_h^2}{S_1 S_2}\right)^2} \frac{S_h^2}{S_1^2 S_2}\left(\frac{\hat{C}_{11}}{S_1} + \frac{\hat{C}_{22}}{S_2} - 2\frac{S_h \hat{C}_{12}}{S_1 S_2}\right) + \frac{1}{\left(1 - \frac{S_h^2}{S_1 S_2}\right)}\left(\frac{-\hat{C}_{11}}{S_1^2} + \frac{2 S_h \hat{C}_{12}}{S_1^2 S_2}\right)\right]$$

$$= -\frac{N}{2}\left[\frac{1}{(S_1 S_2 - S_h^2)} S_2 - \frac{1}{(S_1 S_2 - S_h^2)^2} S_2 S_h^2 \left(\frac{\hat{C}_{11}}{S_1} + \frac{\hat{C}_{22}}{S_2} - 2\frac{S_h \hat{C}_{12}}{S_1 S_2}\right) + \frac{1}{(S_1 S_2 - S_h^2)} S_2 \left(-\frac{\hat{C}_{11}}{S_1} + 2\frac{S_h \hat{C}_{12}}{S_1 S_2}\right)\right]$$

multiply through by $-\frac{2}{N}(S_1 S_2 - S_h^2)^2 \frac{1}{S_2}$:

$$0 = (s_1 s_2 - s_h^2) - s_h^2\left(\frac{\hat{c}_{11}}{s_1} + \frac{\hat{c}_{22}}{s_2} - \frac{2 s_h \hat{c}_{12}}{s_1 s_2}\right) + (s_1 s_2 - s_h^2)\left(\frac{-\hat{c}_{11}}{s_1} + \frac{2 s_h \hat{c}_{12}}{s_1 s_2}\right)$$

$$= (s_1 s_2 - s_h^2)\left[1 - \frac{\hat{c}_{11}}{s_1} + \frac{2 s_h \hat{c}_{12}}{s_1 s_2}\right] - s_h^2\left(\frac{\hat{c}_{11}}{s_1} + \frac{\hat{c}_{22}}{s_2} - \frac{2 s_h \hat{c}_{12}}{s_1 s_2}\right)$$

substitute $\hat{c}_{11}$, $\hat{c}_{22}$, $\hat{c}_{12}$ for s_1, s_2, s_h on RHS:

$$\text{RHS} = (\hat{c}_{11}\hat{c}_{22} - \hat{c}_{12}^2)\left(\cancel{1} - \cancel{1} + \frac{2\hat{c}_{12}^2}{\hat{c}_{11}\hat{c}_{22}}\right) - \hat{c}_{12}^2\left(1 + 1 - 2\frac{\hat{c}_{12}^2}{\hat{c}_{11}\hat{c}_{22}}\right)$$

$$= 2\hat{c}_{12}^2\left(1 - \frac{\hat{c}_{12}^2}{\hat{c}_{11}\hat{c}_{22}}\right) - 2\hat{c}_{12}^2\left(1 - \frac{\hat{c}_{12}^2}{\hat{c}_{11}\hat{c}_{22}}\right)$$

$$= 0 \checkmark$$

Same analysis with $s_2 \leftrightarrow s_1$ gives $\left.\frac{\partial \mathcal{L}}{\partial s_2}\right|_{\substack{s_1 = \hat{c}_{11} \\ s_2 = \hat{c}_{22} \\ s_h = \hat{c}_{12}}} = 0$

Finally, consider

$$0 = \frac{\partial \mathcal{L}}{\partial s_h}$$

$$= -\frac{N}{2}\left[\left(\frac{1}{s_1 s_2 - s_h^2}\right)(-2s_h) - \frac{1}{\left(1 - \frac{s_h^2}{s_1 s_2}\right)^2}\left(\frac{-2s_h}{s_1 s_2}\right)\left(\frac{\hat{c}_{11}}{s_1} + \frac{\hat{c}_{22}}{s_2} - \frac{2 s_h \hat{c}_{12}}{s_1 s_2}\right)\right.$$

$$\left. + \frac{1}{\left(1 - \frac{s_h^2}{s_1 s_2}\right)}\left(-\frac{2\hat{c}_{12}}{s_1 s_2}\right)\right]$$

Now multiply through by $-\frac{2}{N}(s_1 s_2 - s_h^2)^2\left(-\frac{1}{2}\right)$.

$$0 = (s_1 s_2 - s_h^2)s_h - s_h s_1 s_2\left(\frac{\hat{c}_{11}}{s_1} + \frac{\hat{c}_{22}}{s_2} - \frac{2 s_h \hat{c}_{12}}{s_1 s_2}\right) + (s_1 s_2 - s_h^2)\hat{c}_{12}$$

$$= (s_1 s_2 - s_h^2)(s_h + \hat{c}_{12}) - s_h s_1 s_2\left(\frac{\hat{c}_{11}}{s_1} + \frac{\hat{c}_{22}}{s_2} - \frac{2 s_h \hat{c}_{12}}{s_1 s_2}\right)$$

Substitute $\hat{c}_{11}$, $\hat{c}_{22}$, $\hat{c}_{12}$ for $\sigma_1, \sigma_2, \sigma_4$:

$$RHS = (\hat{c}_{11}\hat{c}_{22} - \hat{c}_{12}^2)(\hat{c}_{12} + \hat{c}_{12}) - \hat{c}_{12}\hat{c}_{11}\hat{c}_{22}\left(1 + 1 - \frac{2\hat{c}_{12}^2}{\hat{c}_{11}\hat{c}_{22}}\right)$$

$$= 2\hat{c}_{12}(\hat{c}_{11}\hat{c}_{22} - \hat{c}_{12}^2) - 2\hat{c}_{12}\hat{c}_{11}\hat{c}_{22}\left(1 - \frac{\hat{c}_{12}^2}{\hat{c}_{11}\hat{c}_{22}}\right)$$

$$= 0 \checkmark$$

Thus, $\hat{c}_{11}$, $\hat{c}_{22}$, $\hat{c}_{12}$ are ML estimators of $\sigma_1, \sigma_2, \sigma_4$

11. ML detection statistic: (same white GWB, white noise model in 2 colocated and co-aligned detectors as before)

$$p(d \mid S_{n_1}, S_{n_2}, \mathcal{M}_0) = \frac{1}{\sqrt{\det(2\pi C_n)}} \exp\left[-\frac{1}{2} d^T C_n^{-1} d\right]$$

$$p(d \mid S_{n_1}, S_{n_2}, S_h, \mathcal{M}_1) = \frac{1}{\sqrt{\det(2\pi C)}} \exp\left[-\frac{1}{2} d^T C^{-1} d\right]$$

where $d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$, $\quad C_n = \begin{bmatrix} S_{n_1} \mathbb{1}_{N\times N} & 0_{N\times N} \\ 0_{N\times N} & S_{n_2} \mathbb{1}_{N\times N} \end{bmatrix}$

$$C = C_n + S_h \begin{bmatrix} \mathbb{1}_{N\times N} & \mathbb{1} \\ \mathbb{1} & \mathbb{1} \end{bmatrix} = \begin{bmatrix} (S_{n_1}+S_h)\mathbb{1}_{N\times N} & S_h \mathbb{1}_{N\times N} \\ S_h \mathbb{1}_{N\times N} & (S_{n_2}+S_h)\mathbb{1}_{N\times N} \end{bmatrix}$$

$S_1 \equiv S_{n_1} + S_h$
$S_2 \equiv S_{n_2} + S_h$ $\rightarrow$ $= \begin{bmatrix} S_1 \mathbb{1}_{N\times N} & S_h \mathbb{1}_{N\times N} \\ S_h \mathbb{1}_{N\times N} & S_2 \mathbb{1}_{N\times N} \end{bmatrix}$

NOTE:

$$C_h^{-1} = \left[\begin{array}{c|c} \frac{1}{S_{h_1}} \mathbb{1}_{N\times N} & O_{N\times N} \\ \hline O_{N\times N} & \frac{1}{S_{h_2}} \mathbb{1}_{N\times N} \end{array}\right]$$

$$C^{-1} = \frac{1}{(S_1 S_2 - S_h^2)} \left[\begin{array}{c|c} S_2 \mathbb{1}_{N\times N} & -S_h \mathbb{1}_{N\times N} \\ \hline -S_h \mathbb{1}_{N\times N} & S_1 \mathbb{1}_{N\times N} \end{array}\right]$$

$$= \frac{1}{\left(1 - \frac{S_h^2}{S_1 S_2}\right)} \left[\begin{array}{c|c} \frac{1}{S_1} \mathbb{1}_{N\times N} & -\frac{S_h}{S_1 S_2} \mathbb{1}_{N\times N} \\ \hline -\frac{S_h}{S_1 S_2} \mathbb{1}_{N\times N} & \frac{1}{S_2} \mathbb{1}_{N\times N} \end{array}\right]$$

Arguments of exponentials:

$$-\frac{1}{2} d^T C_h^{-1} d = -\frac{1}{2}\left(\frac{1}{s_{h1}} \sum_i d_{1i}^2 + \frac{1}{s_{h2}} \sum_i d_{2i}^2 \right)$$

$$= -\frac{N}{2}\left(\frac{\hat{c}_{11}}{s_{h1}} + \frac{\hat{c}_{22}}{s_{h2}} \right)$$

$$-\frac{1}{2} d^T C^{-1} d = -\frac{N}{2} \frac{1}{\left(1 - \frac{s_h^2}{s_1 s_2}\right)} \left[\frac{\hat{c}_{11}}{s_1} + \frac{\hat{c}_{22}}{s_2} - \frac{2 s_h}{s_1 s_2} \hat{c}_{12} \right]$$

(see previous problem)

where $\hat{c}_{11} = \frac{1}{N} \sum_i d_{1i}^2$, $\hat{c}_{22} = \frac{1}{N} \sum_i d_{2i}^2$, $\hat{c}_{12} = \frac{1}{N} \sum d_{1i} d_{2i}$

As shown in the previous problem these data combinations are ML estimators of s_{h1}, s_{h2} for M_0, and s_1, s_2, s_h for M_1

Detection statistic:

$$\Lambda_{ML}(d) = \frac{\max_{S_{n1}, S_{n2}, S_h} p(d \mid S_{n1}, S_{n2}, S_h, \mathcal{M}_1)}{\max_{S_{n1}, S_{n2}} p(d \mid S_{n1}, S_{n2}, \mathcal{M}_0)}$$

$$\text{numerator} = \frac{1}{(2\pi)^N (\hat{C}_{11}\hat{C}_{22} - \hat{C}_{12}^2)^{N/2}} \exp\left[-\frac{1}{2} \frac{N}{\left(1 - \frac{\hat{C}_{12}^2}{\hat{C}_{11}\hat{C}_{22}}\right)} \underbrace{\left(\frac{\hat{C}_{11}}{\hat{C}_{11}} + \frac{\hat{C}_{22}}{\hat{C}_{22}} - \frac{2\hat{C}_{12}^2}{\hat{C}_{11}\hat{C}_{12}}\right)}_{2\left(1 - \frac{\hat{C}_{12}^2}{\hat{C}_{11}\hat{C}_{22}}\right)}\right]$$

$$= \frac{1}{(2\pi)^N (\hat{C}_{11}\hat{C}_{22} - \hat{C}_{12}^2)^{N/2}} \exp[-N]$$

$$\text{denominator} = \frac{1}{(2\pi)^N (\hat{C}_{11}\hat{C}_{22})^{N/2}} \exp\left[-\frac{N}{2}\left(\frac{\hat{C}_{11}}{\hat{C}_{11}} + \frac{\hat{C}_{22}}{\hat{C}_{22}}\right)\right]$$

$$= \frac{1}{(2\pi)^N (\hat{C}_{11}\hat{C}_{22})^{N/2}} \exp[-N]$$

Thus,

$$\Lambda_{ML}(d) = \frac{\dfrac{1}{\cancel{(2\pi)^N} (\hat{C}_{11}\hat{C}_{22} - \hat{C}_{12}^2)^{N/2}} \cancel{\exp[-N]}}{\dfrac{1}{\cancel{(2\pi)^N} (\hat{C}_{11}\hat{C}_{22})^{N/2}} \cancel{\exp[-N]}}$$

$$= \left(\frac{\hat{C}_{11}\hat{C}_{22}}{\hat{C}_{11}\hat{C}_{22} - \hat{C}_{12}^2} \right)^{N/2}$$

$$= \frac{1}{\left(1 - \dfrac{\hat{C}_{12}^2}{\hat{C}_{11}\hat{C}_{22}} \right)^{N/2}}$$

$$2 \ln \Lambda_{ML}(d) = 2 \ln \left[\left(1 - \frac{\hat{C}_{12}^2}{\hat{C}_{11}\hat{C}_{22}} \right)^{-N/2} \right]$$

$$= -N \ln \left(1 - \hat{C}_{12}^2 / \hat{C}_{11}\hat{C}_{22} \right)$$

$$\approx N \frac{\hat{C}_{12}^2}{\hat{C}_{11}\hat{C}_{22}}$$ assuming weak signal approx using $\ln(1+x) \approx x$ for $|x| \ll 1$

(12) Perform marginalization integral:

$$p(d \mid S_{n_1}, S_{n_2}, S_h) = \int_{-\infty}^{\infty} dh \; p_n(d-h \mid S_{n_1}, S_{n_2}) \, p(h \mid S_h)$$

where $p_n(d-h \mid S_{n_1}, S_{n_2}) = \dfrac{1}{2\pi\sqrt{S_{n_1} S_{n_2}}} \exp\left[-\dfrac{1}{2}\left(\dfrac{(d_1-h)^2}{S_{n_1}} + \dfrac{(d_2-h)^2}{S_{n_2}}\right)\right]$

$$p(h \mid S_h) = \frac{1}{\sqrt{2\pi S_h}} \exp\left[-\frac{1}{2}\frac{h^2}{S_h}\right]$$

Integrand on RHS:

$$= \frac{1}{(2\pi)^{3/2}\sqrt{S_{n_1} S_{n_2} S_h}} \exp\left[-\frac{1}{2}\left(\frac{(d_1-h)^2}{S_{n_1}} + \frac{(d_2-h)^2}{S_{n_2}} + \frac{h^2}{S_h}\right)\right]$$

Now: $[\;] = -\dfrac{1}{2}\left(\dfrac{d_1^2+h^2-2d_1h}{S_{n_1}} + \dfrac{d_2^2+h^2-2d_2h}{S_{n_2}} + \dfrac{h^2}{S_h}\right)$

$$= -\frac{1}{2}\left[h^2\left(\frac{1}{S_{n_1}} + \frac{1}{S_{n_2}} + \frac{1}{S_h}\right) - 2h\left(\frac{d_1}{S_{n_1}} + \frac{d_2}{S_{n_2}}\right) + \left(\frac{d_1^2}{S_{n_1}} + \frac{d_2^2}{S_{n_2}}\right)\right]$$

$$= -\frac{1}{2}\left[A h^2 - 2hB + D \right]$$

$$= -\frac{A}{2}\left[h^2 - 2h\frac{B}{A} + \frac{D}{A} \right]$$

where $A = \frac{1}{S_{n_1}} + \frac{1}{S_{n_2}} + \frac{1}{S_h}$

$$= \frac{S_{n_2} S_h + S_{n_1} S_h + S_{n_1} S_{n_2}}{S_{n_1} S_{n_2} S_h}$$

$$= \frac{S_{h_1} S_{n_2} + S_h (S_{n_1} + S_{n_2})}{S_{n_1} S_{n_2} S_h}$$

$$= \frac{\det C}{(\det C_n) \cdot S_h}$$

where $C = \begin{bmatrix} S_{n_1} + S_h & S_h \\ S_h & S_{n_2} + S_h \end{bmatrix}$

$C_n = \begin{bmatrix} S_{n_1} & 0 \\ 0 & S_{n_2} \end{bmatrix}$

$$B = \frac{d_1}{s_{h_1}} + \frac{d_2}{s_{h_2}}$$

$$D = \frac{d_1^2}{s_{h_1}} + \frac{d_2^2}{s_{h_2}}$$

Complete the square:

$$[\ ] = -\frac{A}{2}\left[\left(h - \frac{B}{A}\right)^2 - \frac{B^2}{A^2} + \frac{D}{A}\right]$$

$$= -\frac{A}{2}\left[\left(h - \frac{B}{A}\right)^2 - \left(\frac{B^2 - AD}{A^2}\right)\right]$$

Now:

$$\int_{-\infty}^{\infty} dh \exp\left[-\frac{A}{2}\left(h - \frac{B}{A}\right)^2\right] = \sqrt{2\pi}\,\frac{1}{\sqrt{A}}$$

using $\frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty} dx\, e^{-\frac{(x-\mu)^2}{2\sigma^2}} = 1$ (For a Gaussian distribution)

Thus,

$$p(d \mid S_{n_1}, S_{n_2}, S_h) = \frac{1}{(2\pi)^{\cancel{3/2}} \sqrt{S_{n_1} S_{n_2} S_h}} \cancel{\sqrt{2\pi}} \frac{1}{\sqrt{A}} \exp\left[+ \frac{A}{2} \left(\frac{B^2 - AD}{A^2} \right) \right]$$

$$= \frac{1}{2\pi \sqrt{\det C}} \exp\left[-\frac{1}{2} \left(\frac{AD - B^2}{A} \right) \right]$$

Argument of exponential:

$$-\frac{1}{2}\left(\frac{AD - B^2}{A} \right) = -\frac{1}{2} \left(\frac{S_{n_1} S_{n_2} S_h}{\det C} \right) \left(\frac{S_{n_1} S_{n_2} + S_h (S_{n_1} + S_{n_2})}{S_{n_1} S_{n_2} S_h} \left(\frac{d_1^2}{S_{n_1}} + \frac{d_2^2}{S_{n_2}} \right) - \left(\frac{d_1}{S_{n_1}} + \frac{d_2}{S_{n_2}} \right)^2 \right)$$

$$= -\frac{1}{2} \left(\frac{1}{\det C} \right) \left[\left(S_{n_1} S_{n_2} + S_h (S_{n_1} + S_{n_2}) \right) \left(\frac{d_1^2}{S_{n_1}} + \frac{d_2^2}{S_{n_2}} \right) - S_{n_1} S_{n_2} S_h \left(\frac{d_1^2}{S_{n_1}^2} + \frac{d_2^2}{S_{n_2}^2} + \frac{2 d_1 d_2}{S_{n_1} S_{n_2}} \right) \right]$$

$$= -\frac{1}{2}\left(\frac{1}{\det C}\right)\Bigg(d_1^2\left(S_{n_2} + S_h + \cancel{\frac{S_h S_{n_2}}{S_{n_1}}} - \cancel{\frac{S_{n_2} S_h}{S_{n_1}}}\right)$$

$$+ d_2^2\left(S_{n_1} + S_h + \cancel{\frac{S_h S_{n_1}}{S_{n_2}}} - \cancel{\frac{S_{n_1} S_h}{S_{n_2}}}\right) - 2 S_h d_1 d_2\Bigg)$$

$$= -\frac{1}{2}\left(\frac{1}{\det C}\right)\left(d_1^2\left(S_{n_2} + S_h\right) + d_2^2\left(S_{n_1} + S_h\right) - 2 S_h d_1 d_2\right)$$

$$= -\frac{1}{2}\left(d_1^2\left(\frac{S_{n_2} + S_h}{\det C}\right) + d_2^2\left(\frac{S_{n_1} + S_h}{\det C}\right) + 2 d_1 d_2\left(\frac{-S_h}{\det C}\right)\right)$$

$$= -\frac{1}{2}\sum_{I,J=1}^{2} d_I \left(C^{-1}\right)_{IJ} d_J$$

where $(C^{-1})_{IJ}$ are the matrix component of the inverse to

$$C = \begin{bmatrix} S_{n_1} + S_h & S_h \\ S_h & S_{n_2} + S_h \end{bmatrix}, \quad C^{-1} = \frac{1}{\det C}\begin{bmatrix} S_{n_2} + S_h & -S_h \\ -S_h & S_{n_1} + S_h \end{bmatrix}$$