Suggested exercises for stochastic GW background lectures

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Abstract

Some suggested exercises accompanying the lectures on searches for stochastic gravitational-wave backgrounds.

Contents

1. *Rate estimate of stellar-mass binary black hole mergers:*

   Estimate the total rate (number of events per time) of stellar-mass binary black hole mergers throughout the universe by multiplying LIGO's local rate estimate $R_0 \sim 10$ - $200\ \mathrm{Gpc}^{-3}\,\mathrm{yr}^{-1}$ by the comoving volume out to some large redshift, e.g., $z = 10$. (For this calculation you can ignore any dependence of the rate density with redshift.) You should find a merger rate of $\sim$1 per minute to a few per hour.

   Hint: You will need to do numerically evaluate the following integral for proper distance today as a function of source redshift:

$$d_0(z) = \frac{c}{H_0}\int_0^z \frac{\mathrm{d}z'}{E(z')}\,, \qquad E(z) \equiv \sqrt{\Omega_\mathrm{m}(1+z)^3 + \Omega_\Lambda}\,, \tag{1}$$

   with

$$\Omega_\mathrm{m} = 0.31\,, \qquad \Omega_\Lambda = 0.69\,, \qquad H_0 = 68\ \mathrm{km\,s^{-1}\,Mpc^{-1}}\,. \tag{2}$$

   Doing that integral, you should find what's shown in Figure 1, which you can then evaluate at $z = 10$ to convert R_0 (number of events per comoving volume per time) to total rate (number of events per time) for sources out to redshift $z = 10$.

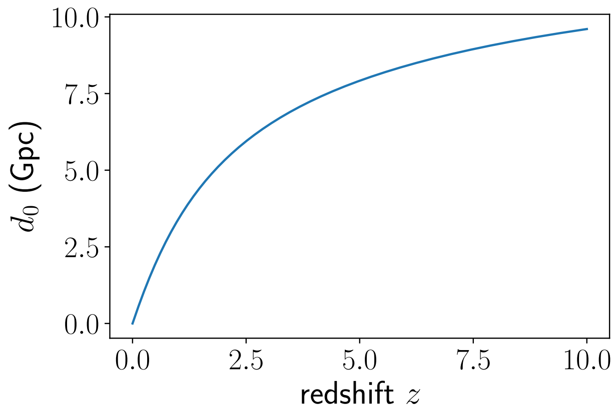

Figure 1:

2. *Relating $S_h(f)$ and $\Omega_\mathrm{gw}(f)$:*

   Derive the relationship

$$S_h(f) = \frac{3H_0^2}{2\pi^2}\frac{\Omega_\mathrm{gw}(f)}{f^3} \tag{3}$$

   between the strain power spectral density $S_h(f)$ and the dimensionless fractional energy density spectrum $\Omega_\mathrm{gw}(f)$. (*Hint*: You will need to use the various definitions of these quantities and also

$$\rho_\mathrm{gw} = \frac{c^2}{32\pi G}\langle \dot{h}_{ab}(t,\vec{x})\dot{h}^{ab}(t,\vec{x})\rangle\,, \tag{4}$$

   which expresses the energy-density in gravitational-waves to the metric perturbations $h_{ab}(t,\vec{x})$.)

3. *Cosmology and the "Phinney formula" for astrophysical backgrounds:*

   (a) Using the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left(\frac{\Omega_\mathrm{m}}{a^3} + \Omega_\Lambda\right) \tag{5}$$

for a spatially-flat FRW spacetime with matter and cosmological constant, and the relationship

$$1 + z = \frac{1}{a(t)}\,, \qquad a(t_0) \equiv 1 \quad (t_0 \equiv \text{today})\,, \tag{6}$$

between redshift z and scale factor $a(t)$, derive

$$\frac{\mathrm{d}t}{\mathrm{d}z} = -\frac{1}{(1+z)H_0 E(z)}\,, \qquad E(z) = \sqrt{\Omega_{\mathrm{m}}(1+z)^3 + \Omega_\Lambda}\,. \tag{7}$$

(b) Using this result for $\mathrm{d}t/\mathrm{d}z$, show that

$$\Omega_{\mathrm{gw}}(f) = \frac{f}{\rho_{\mathrm{c}} H_0} \int_0^\infty \mathrm{d}z\, R(z)\, \frac{1}{(1+z)E(z)} \left(\frac{\mathrm{d}E_{\mathrm{gw}}}{\mathrm{d}f_{\mathrm{s}}}\right)\Bigg|_{f_{\mathrm{s}}=f(1+z)} \tag{8}$$

in terms of the rate density $R(z)$ as measured in the source frame (number of events per comoving volume per time interval in the source frame). (*Hint*: The expression for $\mathrm{d}t/\mathrm{d}z$ from part (a) will allow you to go from the "Phinney formula" for $\Omega_{\mathrm{gw}}(f)$ written in terms of the number density $n(z)$,

$$\Omega_{\mathrm{gw}}(f) = \frac{1}{\rho_c} \int_0^\infty \mathrm{d}z\, n(z)\, \frac{1}{1+z} \left(f_{\mathrm{s}} \frac{\mathrm{d}E_{\mathrm{gw}}}{\mathrm{d}f_{\mathrm{s}}}\right)\Bigg|_{f_{\mathrm{s}}=f(1+z)}\,, \tag{9}$$

to one in terms of the rate density $R(z)$, where $n(z)\,\mathrm{d}z = R(z)\,|\mathrm{d}t|_{t=t(z)}$. Note: Both of the above expressions for $\Omega_{\mathrm{gw}}(f)$ assume that there is only one type of source, described by some set of average source parameters. If there is more than one type of source, one must sum the contributions of each source to $\Omega_{\mathrm{gw}}(f)$.)