Can We believe the precision of DM index measurements?

Statistical Model

Can we believe the precision of DM index measurements?

BY K.J.LEE

August 4, 2017

1 Statistical model

Say the we have data of t_j, f_i, and I_{ij}, where t_j, f_i are the time and frequency at which one measure intensity of I_{ij}. We assume that the spectrum of the source is flat, and pulse profile is Gaussian. That is

$$\langle \delta I_{ij} \delta I_{i' j'} \rangle = \delta_{jj'} \delta_{ii'} \sigma^2$$

where

$$\delta I_{ij} = I_{ij} - \langle I_{ij} \rangle$$

and

$$\langle I_{ij} \rangle = A e^{-\frac{1}{2}\left(\frac{t_i - t(f_j)}{w}\right)^2}$$

the time shifts accroding to DM law is

$$t(f_j) = \alpha f_j^{-\beta}$$

so

$$\langle I_{ij} \rangle = A e^{-\frac{1}{2}\left(\frac{t_i - \alpha f_j^{-\beta}}{w}\right)^2}$$

The total probability model becomes

$$\Lambda \sim \frac{1}{\sqrt{N_f N_t} \sigma} \exp\left(-\frac{1}{2} \sum_{i,j} \left(\frac{I_{ij} - A e^{-\frac{1}{2}\left(\frac{t_i - \alpha f_j^{-\beta}}{w}\right)^2}}{\sigma} \right)^2 \right)$$

Using the CRB, we have

$$\langle \delta\lambda_k \delta\lambda_l \rangle = \sum_{i,j} \frac{\partial f}{\partial \lambda_k} \frac{1}{\sigma^2} \frac{\partial f}{\partial \lambda_l}$$

where

$$f = A e^{-\frac{1}{2}\left(\frac{t_i - \alpha f_j^{-\beta}}{w}\right)^2}$$

Fisher Information Matrix:

$[I(\theta)]_{ij} = \langle \frac{\partial}{\partial \theta_i} \log f \cdot \frac{\partial}{\partial \theta_j} \log f \rangle$

$f(x;\theta)$

we have

$$\frac{\partial f}{\partial A} = e^{-\frac{1}{2}\left(\frac{t_i - \alpha f_j^{-\beta}}{w}\right)^2}$$

$$\frac{\partial f}{\partial \alpha} = A e^{-\frac{1}{2}\left(\frac{t_i - \alpha f_j^{-\beta}}{w}\right)^2} \frac{1}{w^2} (t - \alpha f_j^{-\beta}) f^{-\beta}$$

$$\frac{\partial f}{\partial \beta} = A e^{-\frac{1}{2}\left(\frac{t_i - \alpha f_j^{-\beta}}{w}\right)^2} \frac{\alpha \log(f_j)}{w^2} (t - \alpha f_j^{-\beta}) f^{-\beta}$$

So the Fisher information matrix is

$$B = \frac{1}{\sigma^2} \begin{pmatrix} \sum_{ij} e^{-\left(\frac{t_i - \alpha f_j^{-\beta}}{w}\right)^2} & \sum_{ij} \frac{1}{w^2} (t - \alpha f_j^{-\beta}) f^{-\beta} e^{-\left(\frac{t_i - \alpha f_j^{-\beta}}{w}\right)^2} & \sum_{ij} \frac{\alpha \log(f_j)}{w^2} (t - \alpha f_j^{-\beta}) f^{-\beta} e^{-\left(\frac{t_i - \alpha f_j^{-\beta}}{w}\right)^2} \\ & \sum_{ij} \frac{1}{w^4} (t - \alpha f_j^{-\beta})^2 f^{-2\beta} e^{-\left(\frac{t_i - \alpha f_j^{-\beta}}{w}\right)^2} & \sum_{ij} \frac{\alpha \log(f_j)}{w^4} (t - \alpha f_j^{-\beta})^2 f^{-2\beta} e^{-\left(\frac{t_i - \alpha f_j^{-\beta}}{w}\right)^2} \\ & & \sum_{ij} \frac{\alpha^2 \log^2(f_j)}{w^4} (t - \alpha f_j^{-\beta})^2 f^{-2\beta} e^{-\left(\frac{t_i - \alpha f_j^{-\beta}}{w}\right)^2} \end{pmatrix}$$

对称的.

$$N_f = \frac{f_h - f_l}{2\Delta f}$$

Replace the summation using integral that

$$\sum_{ij} e^{-\left(\frac{t_i-\alpha f_j^{-\beta}}{w}\right)^2} \overset{\frac{1}{\Delta f}\int df}{=} N_f \frac{1}{\Delta T}\int e^{-\left(\frac{t_i-\alpha f_j^{-\beta}}{w}\right)^2} dt = \sqrt{\pi}\frac{w}{\Delta T}N_f = \sqrt{\pi}N_{t,\mathrm{eff}}N_f$$

where $N_{t,\mathrm{eff}} = w/\Delta T$ is the effective number of data points in time domain.
and similarly

$$\sum_{ij}\frac{1}{w^2}(t-\alpha f_j^{-\beta})f^{-\beta}e^{-\left(\frac{t_i-\alpha f_j^{-\beta}}{w}\right)^2} = 0$$

$$\sum_{ij}\frac{\alpha\log(f_j)}{w^2}(t-\alpha f_j^{-\beta})f^{-\beta}e^{-\left(\frac{t_i-\alpha f_j^{-\beta}}{w}\right)^2} = 0$$

$$\sum_{ij}\frac{1}{w^4}(t-\alpha f_j^{-\beta})^2 f^{-2\beta}e^{-\left(\frac{t_i-\alpha f_j^{-\beta}}{w}\right)^2} = \frac{\sqrt{\pi}}{2w^2}\frac{N_{t,\mathrm{eff}}}{\Delta f}X$$

$$\sum_{ij}\frac{\alpha^2\log^2(f_j)}{w^4}(t-\alpha f_j^{-\beta})^2 f^{-2\beta}e^{-\left(\frac{t_i-\alpha f_j^{-\beta}}{w}\right)^2} \simeq \frac{\sqrt{\pi}}{2w^2}N_{t,\mathrm{eff}}\alpha^2\frac{1}{\Delta f}Y$$

$$\sum_{ij}\frac{\alpha\log(f_j)}{w^4}(t-\alpha f_j^{-\beta})^2 f^{-2\beta}e^{-\left(\frac{t_i-\alpha f_j^{-\beta}}{w}\right)^2} = \frac{\sqrt{\pi}}{2w^2}N_{t,\mathrm{eff}}\frac{\alpha}{\Delta f}Z$$

$$B = \frac{1}{\sigma^2}\begin{pmatrix} \sqrt{\pi}N_fN_t & 0 & 0 \\ 0 & BX & B\alpha Z \\ 0 & B\alpha Z & B\alpha^2 Y \end{pmatrix}$$

The error becomes

$$\langle\delta A^2\rangle = \frac{\sigma^2}{\sqrt{\pi}N_fN_t}$$

$$\langle\delta\alpha^2\rangle = \frac{\sigma^2 Y}{BXY - BZ^2}$$

$$\langle\delta\beta^2\rangle = \frac{2w^2\Delta f}{N_{t,\mathrm{eff}}\sqrt{\pi}}\frac{\sigma^2}{\alpha^2}\frac{X}{XY-Z^2}$$

$$\delta\beta = 2^{1/2}\pi^{-1/4}\frac{w}{\alpha}N_{t,\mathrm{eff}}^{-1/2}\frac{\sigma}{A}\underset{N_f^{-1/2}}{\Delta f^{1/2}}\left(\frac{f_h^{2\beta}f_l^{2\beta}(f_h^{2\beta}f_l - f_hf_l^{2\beta})(-1+2\beta)^3}{(f_h^{2\beta}f_l - f_hf_l^{2\beta})^2 - f_h^{2\beta+1}f_l^{2\beta+1}(1-2\beta)^2\log^2\frac{f_h}{f_l}}\right)^{1/2}$$

that is

$$\delta\beta = 2^{1/2}\pi^{-1/4}\frac{w_{\mathrm{ms}}}{4.15_{\,[\mathrm{ms}]}\,\mathrm{DM}}\mathrm{BW}^{1/2}N_{t,\mathrm{eff}}^{-1/2}\frac{\sigma}{A}N_f^{-1/2}\left(\frac{f_h^{2\beta}f_l^{2\beta}(f_h^{2\beta}f_l - f_hf_l^{2\beta})(-1+2\beta)^3}{(f_h^{2\beta}f_l - f_hf_l^{2\beta})^2 - f_h^{2\beta+1}f_l^{2\beta+1}(1-2\beta)^2\log^2\frac{f_h}{f_l}}\right)^{1/2}$$

$$\delta\beta = 2^{1/2}\pi^{-1/4}\frac{w_{\mathrm{ms}}}{4.15_{\,[\mathrm{ms}]}\,\mathrm{DM}}\mathrm{BW}^{1/2}\mathrm{SNR}^{-1/2}\left(\frac{f_h^{2\beta}f_l^{2\beta}(f_h^{2\beta}f_l - f_hf_l^{2\beta})(-1+2\beta)^3}{(f_h^{2\beta}f_l - f_hf_l^{2\beta})^2 - f_h^{2\beta+1}f_l^{2\beta+1}(1-2\beta)^2\log^2\frac{f_h}{f_l}}\right)^{1/2}$$

The derivative part is rather easy to deal with, but the summation part is rather difficult. Under Professor Lee's conduct, we transfer the summation into integrate for t and f. We calculate the integrate with t first and it will simplify the integration dramatically.

Integration Calculate

First let's calculate the Gaussian Integral:

$$\int_0^\infty e^{-(\frac{t_i - \alpha f_j^{-2\beta}}{\omega})^2} dt = \frac{1}{2}\sqrt{\pi}\omega$$

Now let's see the first two integration, we can get its value just by a glance of it, for its even or odd quality, of course, it equals 0.
Now the third one, we can use $x = t - \alpha f_j^{-2\beta}$, so we get

$$\int_0^\infty x^2 e^{-\frac{x^2}{\omega^2}} dx$$

through integration by parts we get

$$-\frac{\omega^2}{2} x^2 e^{-\frac{x^2}{\omega^2}} |_0^\infty + \omega^2 \int_0^\infty e^{-\frac{x^2}{\omega^2}} dx$$

it is obviously the first part is 0 and the second is $\frac{1}{2}\sqrt{\pi}\omega^3$
we ignore the constant and f parts while do this integration, then let's plus these parts we get

$$\frac{f^{-2\beta}}{\omega^4 \Delta T} * \frac{1}{2}\sqrt{\pi}\omega^3$$

Now let's integrate f from f_l to f_h

$$\int_{f_l}^{f_h} \frac{f^{-2\beta}}{\omega^4 \Delta T} * \frac{1}{2}\sqrt{\pi}\omega^3 df$$

it's easy to compute, the Gaussian Integration dramatically simplify the integration. It is similar for the upcoming summations.