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Project #1

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Introduction

So far in this course we have examined three separate types of circuits, the RC, RL, and RLC configurations. For the tasks of this project we will focus on RC circuits, which contain resistors and one capacitor. The resistor is a simple passive device which does not possess the ability to store energy, it can only dissipate it. The capacitor on the other hand is an energy storing device, as it stores charge, and its equations reflect that by being time dependent. Ohm's Law is able to define the behavior of a resistor quite simply with a time invariant linear relationship, however, the behavior of the capacitor will vary with respect to time. Conceptually, to understands this we can examine a parallel plate capacitor with current flowing into one end.

As the current flows into one of the plates of the capacitor, electrons will flow into and deposit onto the plate. Since the plates of the capacitor are separated by some distance, there will be nowhere for the electrons to go and they must deposit on the face of the plate. Current can be defined as the flow of change per unit of time and for this reason, as time goes on there will be more and more charge that accumulates on the plate. The principle that like charges repel will dictate it becoming increasingly difficult to deposit more charge, causing the flow rate of charge to keep decreasing over time. This will be seen in the math as a decaying relationship between the current that flows into and the voltage that is across the capacitor.

With this basic concept in mind, as we look at the RC circuit, we can see how the equation to describe the configuration will be differential and of the first order. There are two cases which are common to observe with these circuits, the natural and the step response. The natural response occurs when there is no source connected and therefore the capacitor holds all the voltage and it is dissipated by the resistor. On the other hand, we have the step response, which is akin to the charging of the capacitor. This charging and discharging can be linked together with the equations, and are useful to understand in the analysis of large scale circuits which possess both responses.

Task 1

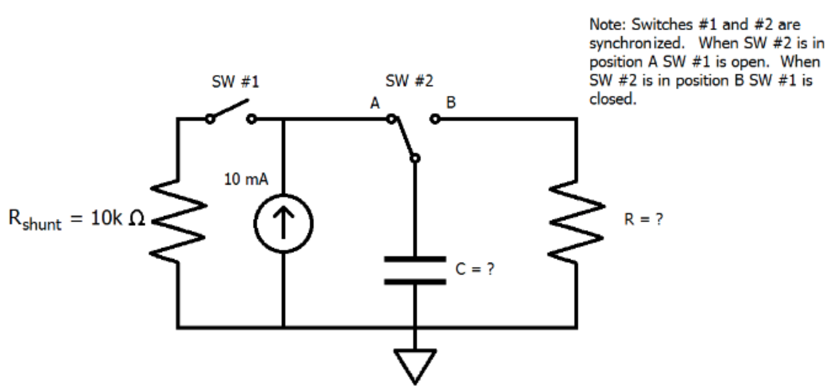

Figure 1: Circuit Schematic

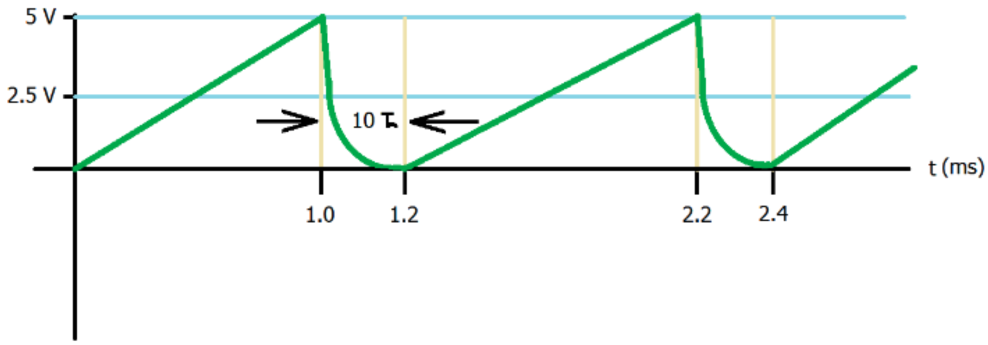

Figure 2: Waveform

Task: Calculate the values of C and R needed to produce the waveform shown above.

In order to find the value of C we can employ the equation relating the current and voltage in a capacitor,

$$i(t) = C\,\frac{dv}{dt}$$

For the 1 millisecond that switch 2 is in position A, the circuit is simplified to a capacitor connected to a current source. Since the current source has a constant value of 10 mA, the expression for $i(t)$ is constant and does not change over time t. Along with this, from the graph of the waveform, the slope of the linear rise can be found to be $\frac{5-0}{1\times10^{-3}-0}$ from $t = 0$ ms to $t = 1$ ms. This is the value of $\frac{dv}{dt}$ in the equation since the derivative of voltage would be its rate of change with respect to time, the slope of v vs. t. From here, we can plug into the equation where C is the only unknown,

$$10 \times 10^{-3} = C\left(\frac{5}{1 \times 10^{-3}}\right)$$

Solving for C we get,

$$C = \frac{(1 \times 10^{-3})(10 \times 10^{-3})}{5} = 2 \times 10^{-6} = \boxed{2\ \mu F}$$

Now for the value of the resistor. We can see from the waveform that the value of time for which switch 2 is in position B, 0.2 ms, is equivalent to 10τ, where τ is the time constant. Since this is an RC circuit, the value of τ is given by R multiplied by C, and since we have the value of C, we can find R.

$$\begin{aligned}
\tau &= \frac{0.2 \times 10^{-3}}{10} = 2 \times 10^{-5} = RC \\
RC &= R(2 \times 10^{-6}) = 2 \times 10^{-5} \\
R &= \frac{2 \times 10^{-5}}{2 \times 10^{-6}} = \boxed{10\Omega}
\end{aligned}$$

Task 2

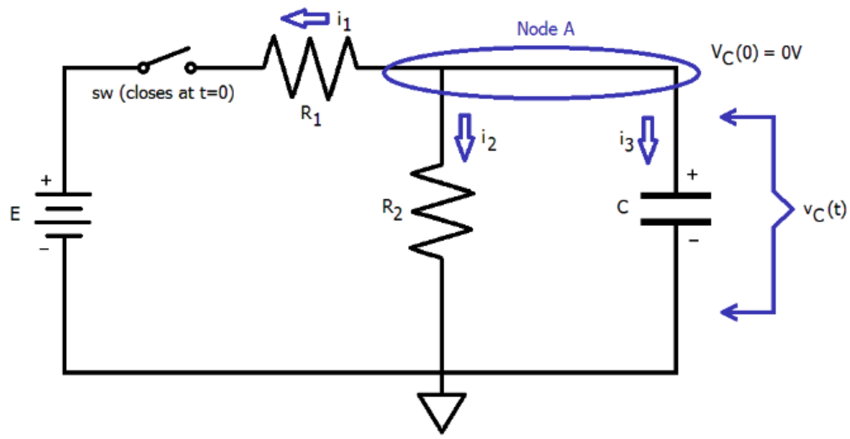

Figure 3: Circuit Schematic

Task: Derive the general formula for $v_c(t)$ for $t \geq 0$ in terms of the given variables.

This circuit will yield the step response of an RC circuit since at time $t = 0$, the switch closes and the voltage source is connected to the circuit. Since the switch has been open for a long time, the initial conditions for voltage and current are zero,

$$v(0) = 0,\ i(0) = 0.$$

At node A, we can sum together the currents and set them equal to zero since according to KCL and nodal analysis, the sum of all currents going into or out of a node must equal zero.

$$i_1 + i_2 + i_3 = 0.$$

To define each of these currents we can see that node A will be at a voltage, v_c. In addition, i_3 is simply the current through the capacitor which can be related by the equation $C\frac{dv_c}{dt}$. Using this information, $i_1 = \frac{v_c - E}{R_1}$, $i_2 = \frac{v_c}{R_2}$, and $i_3 = C\frac{dv_c}{dt}$. We can now plug these currents into the equation and begin solving for v_c.

$$\frac{v_c - E}{R_1} + \frac{v_c}{R_2} + C\,\frac{dv_c}{dt} = 0.$$

To get the derivative term by itself, we can divide the equation by C and rearrange to get this into the standard form for a linear ordinary differential equation.

$$\frac{dv_c}{dt} + \frac{v_c - E}{R_1 C} + \frac{v_c}{R_2 C} = 0$$

$$\frac{dv_c}{dt} + \left(\frac{1}{R_1 C} + \frac{1}{R_2 C}\right) v_c = \frac{E}{R_1 C}$$

$$v_c' + \left(\frac{1}{R_1 C} + \frac{1}{R_2 C}\right) v_c = \frac{E}{R_1 C}$$

This equation can be solved as any other linear ODE with the use of the integration factor, $e^{\int p(t)dt}$. Here, we will define it as μ.

$$p(t) = \left(\frac{1}{R_1C} + \frac{1}{R_2C}\right) = \delta$$

$$\int p(t)\ dt = \left(\frac{1}{R_1C} + \frac{1}{R_2C}\right) t = \delta t$$

$$\mu = e^{\left(\frac{1}{R_1C} + \frac{1}{R_2C}\right)t} = \mu^{\delta t}$$

Since $p(t)$ here has a constant value, we will call it delta to clear up the derivation. The equation can then be written as,

$$(\mu v_c)' = \mu \frac{E}{R_1C}$$

$$v_c e^{\delta t} = \frac{E}{R_1C} \int e^{\delta t}\ dt$$

Solving the integral and isolating the term v_c,

$$v_c(t) = \frac{E}{R_1C}\ e^{-\delta t}\left(\frac{1}{\delta}e^{\delta t} + k\right)$$

$$v_c(t) = \frac{E}{R_1C}\left(\frac{1}{\delta} + ke^{-\delta t}\right)$$

Here, k is the arbitrary constant of integration and can be solved for plugging in the initial condition, $v_c(0) = 0$,

$$0 = \frac{E}{R_1C}\left(\frac{1}{\delta} + ke^{0}\right)$$

and since $e^0 = 1$,

$$k = -\frac{1}{\delta}$$

With that, the capacitor voltage can be written by substituting in the expression for delta and rearranging, and the derivation is finished.

$$v_c(t) = \frac{E}{R_1C}\left(\frac{1}{\delta} - \frac{1}{\delta}e^{-\delta t}\right)$$

$$v_c(t) = \frac{E}{R_1C}\frac{1}{\delta}\left(1 - e^{-\delta t}\right)$$

$$v_c(t) = \frac{E}{R_1C}\left(\frac{1}{R_1C} + \frac{1}{R_2C}\right)^{-1}\left(1 - e^{-\left(\frac{1}{R_1C} + \frac{1}{R_2C}\right)t}\right)$$

$$\boxed{v_c(t) = \frac{ER_2}{R_1 + R_2}\left(1 - e^{-\left(\frac{R_1+R_2}{R_1R_2C}\right)t}\right)}$$