The Sinusoidal Source

$$v = V_m \cos(\omega t + \phi)$$

The sinusoidal source of either current or voltage which varies as a sinusoid with respect to time. The equation above is the general expression for the voltage of an alternating source where, V_m is the amplitude between which the voltage is bounded. The term ω gives the radial frequency of the function, in radians per second, and given by the expression $\omega = 2\pi f$, where f is the frequency. Since frequency is given as the inverse of the period, $f = \frac{1}{T}$, this expression can be given as, $\omega = \frac{2\pi}{T}$.

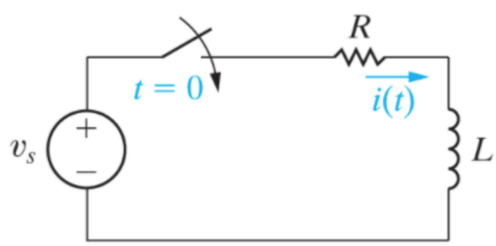
The angle ϕ gives the phase angle of the function for the source and is responsible for the translation of the value of the function at time $t = 0$.

$$V_{rms} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} V_m^2 \cos^2(\omega t + \phi)\ dt}$$

This equation is used to calculate the mean value of the voltage. This is done by integrating the voltage squared and multiplying by the inverse of the difference of the bounds. The value V_{rms} is used in the calculation of the power generated across an alternating source, however, for this course it will be sufficient to know that the value under the radical will equal $V_m^2/2$, giving a value for V_{rms} of,

$$V_{rms} = \frac{V_m}{\sqrt{2}}.$$

The Sinusoidal Response



For this particular circuit, the voltage at time $t = 0$ will be given by the general for of the equation of an alternating source. Writing Kirchoff's Voltage Rule for this configuration we get,

$$Ri + L\frac{di}{dt} = V_m \cos(\omega t\ +\ \phi).$$

Solving this differential equation in order to obtain the expression for current as a function of time, $i(t)$, we get,

$$i(t) = \frac{-V_m}{\sqrt{R^2\ +\ \omega^2 L^2}}\cos(\phi - \theta)e^{-\left(\frac{R}{L}\right)t}\ +\ \frac{V_m}{\sqrt{R^2\ +\ \omega^2 L^2}}\cos(\omega t + \phi - \theta)$$

where,

$$\theta = \tan^{-1}\left(\frac{\omega L}{R}\right)$$

The first term is this expression is referred to as the homogeneous solution and the second, the particular solution. The difference between these solutions is extrapolated upon in linear differential equations.

The Phasor

$$e^{\pm j\theta} = \cos\theta \pm j\sin\theta$$

The Phasor is a complex number which is rooted in Euler's Identity, shown above and with its use, we attain a different method by which the cosine and sine function are defined. The cosine being the real part and the sine being the imaginary part of the exponential function.

$$\cos\theta = \mathscr{R}\{e^{j\theta}\}$$

and,

$$\sin\theta = \mathscr{R}\{e^{j\theta}\}$$

This leads to the function for the alternating voltage source to be written as,

$$\begin{aligned} v &= V_m \cos(\omega t + \phi) \\ &= V_m \mathscr{R}\{e^{j(\omega t + \phi)}\} \\ &= V_m \mathscr{R}\{e^{j\omega t} e^{j\phi}\} \end{aligned}$$

Since V_m is a constant, it can be moved inside the argument of $\mathscr{R}$ without altering the expression and the equation for voltage becomes,

$$v = \mathscr{R}\{V_m e^{j\phi} e^{j\omega t}\}.$$

Phasor Transform

$$\mathbf{V} = V_m e^{j\phi} = \mathscr{P}\{V_m \cos(\omega t + \phi)\}$$

This is the polar form of a phasor however it can also be expressed in rectangular form with some manipulation.

$$\mathbf{V} = V_m \cos\phi \;+\; jV_m \sin\phi$$

For the polar form it can be seen that the phasor takes the form $Ae^{j\phi}$, where A is the amplitude and the angle can be expression using the angle notation, $A\underline{/\phi^\circ}$.

$$A\underline{/\phi^\circ} \equiv Ae^{j\phi}$$

Inverse Phasor Transform

$$\mathscr{P}^{-1}\{V_m e^{j\theta}\} = \mathscr{R}\{V_m e^{j\theta} e^{j\omega t}\} = V_m \cos(\omega t + \theta^\circ)$$

Example 9.5 Adding Cosines Using Phasers

If $y_1 = 20\cos(\omega t - 30°)$ and $y_2 = 40\cos(\omega t + 60°)$, express $y = y_1 + y_2$ as a single sinusoidal function.

$$y = 20\cos(\omega t - 30°) + 40\cos(\omega t + 60°)$$

Using Euler's identity, the right hand side of the expression can be rewritten as,

$$\begin{aligned} y &= \mathscr{R}\left\{20e^{-j30°}e^{j\omega}\right\} + \mathscr{R}\left\{40e^{j60°}e^{j\omega}\right\}. \\ &= \mathscr{R}\left\{20e^{-j30°}e^{j\omega} + 40e^{j60°}e^{j\omega}\right\} \end{aligned}$$

The term $e^{j\omega}$ can be factored out and the expression becomes,

$$y = \mathscr{R}\left\{\left(20e^{-j30°} + 40e^{j60°}\right)e^{j\omega}\right\}$$

and with that, the two phasors can be added using angle notation,

$$\begin{aligned} 20\underline{/-30°} + 40\underline{/60°} &= (17.32 - j10) + (20 + j34.64) \\ &= 37.32 + j24.64 \\ &= \boxed{44.72\underline{/33.43°}.} \end{aligned}$$

With that, the sum of the two functions can be given as,

$$y = \mathscr{R}\left\{44.72e^{j33.43°}e^{j\omega}\right\}$$

or,

$$\boxed{y = 44.72\cos(\omega t + 33.43°).}$$

Passive Circuit Elements in the Frequency Domain

The V-I Relationship for a Resistor

$$\begin{aligned} v &= R\left[I_m\cos(\omega t + \theta_i)\right] \\ &= RI_m\cos(\omega t + \theta_i), \\ &= RI_m\underline{/\theta_i} \end{aligned}$$

From Ohm's Law it can be deduced that current can be defined as the following where I_m is the maximum amplitude of the current, and θ_i is the phase angle of the current.

The phasor transform of this voltage can be written as,

$$\mathbf{V} = RI_me^{j\theta_i} = RI_m\underline{/\theta_i}.$$

Since the term $I_m\underline{/\theta_i}$ is the phasor representation of the current across the resistor, the relationship between the phasor current and the phasor voltage can be written simply as,

$$\mathbf{V} = R\mathbf{I}.$$

The V-I Relationship for an Inductor

$$v = L\frac{di}{dt} = -\omega L I_m \sin(\omega t + \theta_i).$$

This is the general equation for the voltage-current relationship across an inductor. By differentiating the phasor equation for current defined previously, a relationship with the voltage in phasor form can be determined.

Replacing the sine with a cosine function,

$$v = -\omega L I_m \cos(\omega t + \theta_i - 90),$$

the phasor representation can be written as,

$$\begin{aligned}\mathbf{V} &= -\omega L I_m e^{j(\theta_i - 90^\circ)} \\ &= -\omega L I_m e^{j\theta_i} e^{-j90^\circ}\end{aligned}$$

and since $e^{-j90^\circ} = -j$,

$$\begin{aligned}\mathbf{V} &= j\omega L I_m e^{j\theta_i} \\ &= j\omega L I_m \underline{/\theta_i}\end{aligned}$$

Once again, the term $I_m\underline{/\theta_i}$ defines the phasor expression for current therefore, the expression for the voltage across an inductor of an alternating source can be written as,

$$\mathbf{V} = j\omega L\mathbf{I}.$$

The V-I Relationship for a Capacitor

$$v = \frac{1}{C}\int i(t)dt = \frac{1}{C}\int I_m \cos(\omega t + \theta_i)\ dt = \frac{1}{\omega C}\ I_m \sin(\omega t + \theta_i)$$

Here once again, the sine function can be converted into a cosine,

$$v = \frac{I_m}{\omega C}\cos(\omega t + \theta_i - 90^\circ),$$

and again, writing this as an exponential,

$$\begin{aligned}\mathbf{V} &= \frac{1}{\omega C}\ I_m e^{j(\theta_i - 90^\circ)} \\ &= \frac{1}{\omega C}\ I_m e^{j\theta_i} e^{-j90^\circ}\end{aligned}$$

and once again, since $e^{-j90^\circ} = -j$,

$$\mathbf{V} = \frac{-j}{\omega C}\ I_m\underline{/\theta_i}.$$

Now, since $I_m\underline{/\theta_i}$ defines the phasor $\mathbf{I}$ it can be substituted in but also, to clean up the expression, we can bring the j to the denominator. Since the complex number is a radical, we can in a sense, "reverse-rationalize" it and bring it to the denominator. The final expression for the relationship between voltage an current across a capacitor is given by,

$$\mathbf{V} = \frac{1}{j\omega C}\mathbf{I}.$$

Impedance

$$\mathbf{V} = Z\mathbf{I}$$

The impedance is given by the term Z in this equation and can be thought of as the "resistance" across a specific circuit element.

Circuit Element	Impedance	Reactance
Resistor	R	–
Inductor	$j\omega L$	ωL
Capacitor	$-j/\omega C$	$-1/\omega C$

By looking at this table, it can be seen that the value of Z is determined by the expression that is multiplying the phasor $\mathbf{I}$ in the voltage-current relationship equations. The impedance can be calculated using these expressions for the respective elements, and has a value given in Ohms (Ω).

Kirchoff's Laws in the Frequency Domain

Voltage Law

$$\mathbf{V}_1 + \mathbf{V}_1 + \ldots + \mathbf{V}_n = 0$$

Around a closed loop, the voltage phasors across each of the elements in the loop will add to equal zero.

Current Law

$$\mathbf{I}_1 + \mathbf{I}_1 + \ldots + \mathbf{I}_n = 0$$

At a junction, the current phasors traveling in from each of the paths of the junction will add to equal zero.

Steady State Voltage Drop

Given that,

$$\mathbf{V} = 5.59\underline{/71.565^\circ}\ V,$$

in order to determine the steady state voltage across the elements in a loop being supplied by a sinusoidal source, the inverse phasor transform can be applied.
If $\omega = 1000 rad/s$:

$$v_{ss}(t) = \mathscr{P}^{-1}\{\mathbf{V}\} = \mathscr{P}^{-1}\{5.59\underline{/71.565^\circ}\},$$

and since the inverse phaser transform is defined as,

$$\mathscr{P}^{-1}\{V_m e^{j\theta}\} = \mathscr{R}\{V_m e^{j\theta} e^{j\omega t}\} = V_m \cos(\omega t + \theta^\circ),$$

the steady state voltage is given by,

$$v_{ss}(t) = 5.59\cos(1000t + 71.565^\circ)\ V.$$

Series-Parallel Simplifications

Series Combinations

$$Z_{eq} = Z_1 + Z_2 + \ldots + Z_n$$

Parallel Combinations

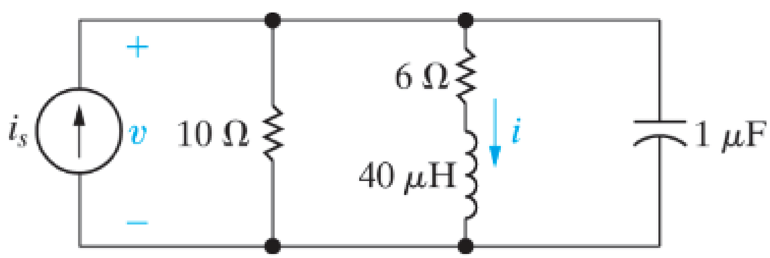
$$Z_{eq} = \left(Z_1^{-1} + Z_2^{-1} + \ldots + Z_n^{-1}\right)^{-1}$$

Voltage Division

$$V_j = \frac{Z_j}{Z_{eq}} V_s$$

Example 9.9 Combining Impedances

For $i_s = 8\cos(200,000t)\ A$ in the circuit:

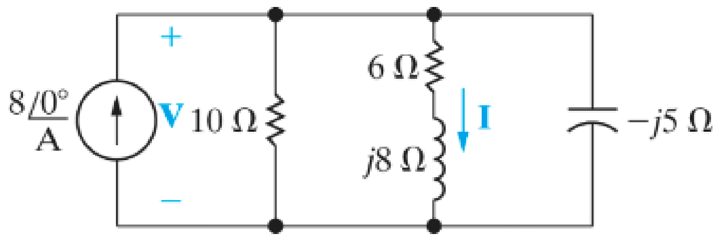


1. Construct the frequency-domain equivalent circuit.
2. Find the equivalent admittance to the right of the current source.
3. Use the equivalent admittance to find the phasor voltage **V**.
4. Find the phasor current **I**, using current division.
5. Find the steady-state expressions for v and i.

For the first objective, the current circuit can be transformed into its frequency equivalent by using the equations for the individual elements to determine their impedance.

For the $40\mu H$ inductor, the impedance will be equal to, $j(200,000)(40 * 10^{-6}) = j8\ \Omega$. The same can be done for the capacitor, $1/j(200,000)(1 * 10^{-6}) = -5j\ \Omega$. The impedance for the resistor is simply R, the resistance of the resistor, 6 Ω.

The current source can be written using angle notation as, $8\underline{/0^\circ}$, and the final circuit will resemble,



The second objective can be met by finding the admittances of the three branches in the circuit and then summing those values to find the equivalent admittance. Since admittance is defined as the inverse of impedance, for the right most branch, the impedance will be,

$$Y_1 = \frac{1}{-j5} = j0.2\ S,$$

for the middle branch,

$$Y_2 = \frac{1}{6+j8} = 0.06 - j0.08\ S,$$

and the final branch, consisting of only a resistor, the admittance will be,

$$Y_3 = \frac{1}{10} = 0.1\ S.$$

Admittance is given in Seimens and in order to find the total admittance for the three branches, the individual values must be summed together,

$$\begin{aligned} Y_{eq} &= Y_1 + Y_2 + Y_3 \\ &= 0.16 + j0.12 \\ &= \boxed{0.2\underline{/36.87^\circ}\ S.} \end{aligned}$$

To find the phasor voltage two characteristics of the circuit are required, Z_{eq}, the equivalent impedance, and **I**, the phasor current. Since the admittance is the inverse of the impedance, to find the equivalent impedance, all that needs to be done is the inverse of the admittance found in the previous part.

$$Z_{eq} = \frac{1}{Y_{eq}} = \boxed{5\underline{/-36.87^\circ}\ \Omega}$$

Since the current was found to be $8\underline{/0^\circ}$,

$$\mathbf{V} = (5\underline{/-36.87^\circ})(8\underline{/0^\circ}) = \boxed{40\underline{/-36.87^\circ}\ V.}$$

The phasor for the branch current can be given as,

$$\mathbf{I} = \frac{5\underline{/-36.87^\circ}}{6+j8}(8\underline{/0^\circ}) = \boxed{4\underline{/-90^\circ}\ A.}$$

Using the respective phasors, the expressions for voltage and current as a function of time can then be written as,

$$v(t) = 40\cos(200,000t - 36.87^\circ)\ V,$$

$$i(t) = 4\cos(200,000t - 90^\circ)\ A.$$