The Inductor

$$v(t) = L\frac{di}{dt}$$

The voltage across an inductor is given by the equation above and from this the expression for the current across an inductor can also be derived by separating variables and integrating both sides.

$$i(t) = \frac{1}{L}\int v(t)\ dt = \frac{1}{L}\int_{-\infty}^{t} v(\tau)\ d\tau$$

One thing to note is the behavior of on inductor in the presence of a dc power supply. If the current across the inductor is constant, the voltage across it will be zero. This implicates that an inductor in the presence of a constant dc current source will behave the same as a short circuit in its place. This fact also makes it clear that the current across an inductor cannot change instantaneously since the derivative at that point would not exist. The **current** in an inductor **must be continuous.**

For the equation calculating current, it is not ideal to integrate for an infinite amount of time since the inductor will reach a steady state at some point as well as the fact that the interval of time that is of interest in not all of time. To rewrite this, the integral can be written as going from the initial time to some time t and an initial condition can be specified which is added on to the end of the expression.

$$i(t) = \frac{1}{L}\int_{t_0}^{t} v(\tau)\ d\tau + i(t_0)$$

Most of the time this requires the functions to be defined as a piecewise functions, and it is imperative to make explicit that for an inductor, current cannot be discontinuous however, voltage can.

Power and Energy

The power stored in an inductor is calculated using $p = iv$ and since $v = L\frac{di}{dt}$, the power is given as,

$$p = Li\frac{di}{dt}.$$

The energy in an inductor is given by,

$$w = \frac{1}{2}Li^2.$$

The Capacitor

$$i(t) = C\frac{dv}{dt}$$

For a capacitor, somewhat the opposite is given by the equations, the current is the capacitance multiplied with the derivative of the voltage across the capacitor. Examining this the same as with the inductor, here it can be seen that since the voltage is being differentiated, the function for the voltage, **cannot be discontinuous.**

For the current, the same is done, separation of variables and then integration, and the following expression is received,

$$v(t) = \frac{1}{C}\int_{t_0}^{t} i(\tau)\ d\tau + v(t_0).$$

The same can be said for this conversion and the same rules apply. In this case, the current can be given as a discontinuous function however, the voltage cannot.

Power and Energy

Power is defined much the same way, since $p = vi$,

$$p = Cv\frac{dv}{dt}$$

Energy is also much the same exert this time,

$$w = \frac{1}{2}Cv^2$$

Appendix G Integrals

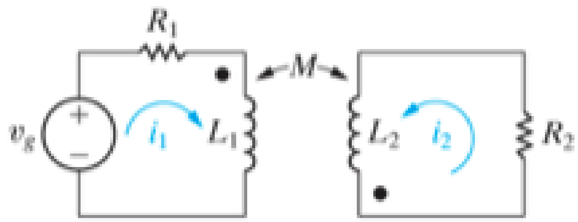
$$\int xe^{ax}\ dx = \frac{e^{ax}}{a^2}(ax-1)$$

$$\int x^2e^{ax}\ dx = \frac{e^{ax}}{a^3}(a^2x^2-2ax+2)$$

$$\int x\sin(ax)\ dx = \frac{1}{a^2}\sin(ax) - \frac{x}{a}\cos(ax)$$

$$\int x\cos(ax)\ dx = \frac{1}{a^2}\cos(ax) - \frac{x}{a}\sin(ax)$$

Mutual Inductance



The inductors in this circuit are magnetically coupled due to them being in close proximity with one another. This causes the original mesh equations to be altered to accommodate the voltage induced by the opposing inductor. The original mesh equations written for the independent circuits would be, $-v_g + i_1R_1 + L_1\frac{di_1}{dt} = 0$ for the one on the left, and for the right, $i_2R_2 + L_2\frac{di_2}{dt} = 0$.

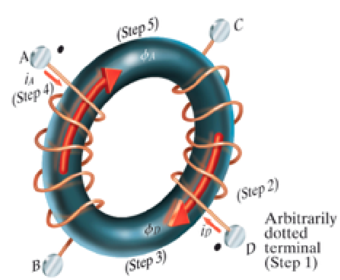
Now since these inductors are close to each other and each induce voltage on the other, another term must be added to each of their equations. For the circuit on the left, since the current i_2 travels into the negative polar side of the inductor, indicated by the dot, the term for the voltage induced will be negative. This gives the new equation,

$$-v_g + i_1R_1 + L_1\frac{di_1}{dt} - M\frac{di_2}{dt} = 0.$$

The same must be done for the second equation and in this case, since i_1 travels into the positive end of L_1, the positive emf is generated at the positive end of the other inductor. Since the current i_2 is entering the negative end of that inductor, the term in the second equation will be negative.

$$i_2R_2 + L_2\frac{di_2}{dt} - M\frac{di_1}{dt} = 0$$

Dot Placement



The placement of the dots for the positive and negative ends of the inductor is dependent on the direction of the flux that is traveling through the coil of the circuit. To determine which ends of the inductor are paired, use the right hand rule following the coils of the inductor with the curling of your fingers. The direction your thumb points is the direction the flux is traveling through the circuit. Do the same for the rest of the ends and which ever ones produce flux that travels in the same direction, are the ones that are coupled and that is where the dots should be placed.

Derivation of Mutual Inductance

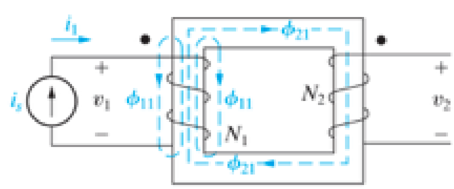
$$\varepsilon = \frac{d\lambda}{dt}$$

The concept of mutual inductance stems as a result of Faraday's Law where, an emf is induced according to the rate of change of the magnetic flux within a coil. In simpler terms, if there is a change in the amount of flux generated within an electrically conductive material, there will be a voltage generated as a result.

The term λ represents flux linkage and is given by the equation, $\lambda = \phi N$. The magnitude of the flux is given by the equation $\phi = \mathcal{P}Ni$, where N is the number of turns the coil makes and $\mathcal{P}$ is the permeance of the region the flux is intersecting. Plugging these expressions into Faraday's Law, the equation for the current within an inductor can be derived.

$$\begin{aligned} v &= \frac{d\lambda}{dt} = \frac{d(N\phi)}{dt}, \\ &= N\frac{d\phi}{dt} = N\frac{d}{dt}(\mathcal{P}Ni), \\ &= N^2\mathcal{P}\frac{di}{dt} = L\frac{di}{dt}. \end{aligned}$$

It can be determined from here that the term for inductance, L, is given by $N^2\mathcal{P}$. Further, we can derive the term for mutual inductance, M, which is actually responsible for the coupling of the two coils.



The total flux in the first coil can be written as the sum of the flux generated by the coil and the flux traveling through the medium,

$$\phi_1 = \phi_{11} + \phi_{21}.$$

Now, each of the flux terms can be written as their full expressions,

$$\phi_1 = \mathcal{P}_1 N_1 i_1,$$

$$\phi_{11} = \mathcal{P}_{11} N_1 i_1,$$

$$\phi_{21} = \mathcal{P}_{21} N_1 i_1.$$

Here, $\mathcal{P}_1$ is the permeance of space occupied by the flux ϕ_1, $\mathcal{P}_{11}$ is the permeance of the space through which ϕ_{11} travels, and $\mathcal{P}_{21}$ is the permeance of ϕ_{21}. Using this, the equation relating the permeances of each of the fluxes can be written as,

$$\mathcal{P}_1 = \mathcal{P}_{11} + \mathcal{P}_{21}.$$

With this, Faraday's Law can be employed and with the same methodology as with the derivation of self-inductance, the term for mutual inductance can be determined.

$$\begin{aligned} v_1 &= \frac{d\lambda_1}{dt} = \frac{d(N_1\phi_1)}{dt} = N_1\frac{d}{dt}(\phi_{11} + \phi_{21}) \\ &= N_1^2(\mathcal{P}_{11} + \mathcal{P}_{21})\frac{di_1}{dt} = N_1^2\mathcal{P}_1\frac{di_1}{dt} = L_1\frac{di_1}{dt}. \end{aligned}$$

For v_2 the same can be written, as this one was the derivation of the term for the self-inductance of the first inductor, the second equation will give the term for the induced voltage.

$$\begin{aligned} v_2 &= \frac{d\lambda_2}{dt} = \frac{d(N_2\phi_{21})}{dt} = N_2\frac{d}{dt}(\mathcal{P}_{21}N_1 i_1) \\ &= N_2N_1\mathcal{P}_{21}\frac{di_1}{dt} = M_{21}\frac{di_1}{dt}. \end{aligned}$$

Now it can be seen that the term M is given by the product of the number of coils in each of the inductors, as well as the permeance of the medium through which the flux is traveling for both inductors.

$$M_{21} = N_2N_1\mathcal{P}_{21}.$$

The subscript on M signifies that this is the inductance as it relates to the voltage induced in coil 2 by the current in coil 1. For non-magnetic materials, the permeances of both are equal, which means that the subscript does not matter and the term can be written as just simply, M.

In terms of Self-Inductance

$$L_1 = N_1^2\mathcal{P}_1,$$

$$L_2 = N_2^2\mathcal{P}_2,$$

$$L_1L_2 = N_1^2N_2^2\mathcal{P}_1\mathcal{P}_2.$$

These are the self-inductances for each of the two coils as well as their product. Using previous knowledge, this product can be rewritten in terms of the two individual permeances.

$$L_1L_2 = N_1^2N_2^2(\mathcal{P}_{11} + \mathcal{P}_{21})(\mathcal{P}_{22} + \mathcal{P}_{12}).$$

Since for linear systems $\mathcal{P}_{21} = \mathcal{P}_{12}$, this expression can be simplified and written as,

$$L_1 L_2 = (N_1 N_2 \mathcal{P}_{12})^2 \left(1 + \frac{\mathcal{P}_{11}}{\mathcal{P}_{12}}\right) \left(1 + \frac{\mathcal{P}_{22}}{\mathcal{P}_{12}}\right)$$

Here it can be seen that the definition for M is present. Along with that, the self-inductance terms can be found by rearranging the product terms and with the use of a coefficient, k, the equation can be written as,

$$\frac{1}{k^2} = \left(1 + \frac{\mathcal{P}_{11}}{\mathcal{P}_{12}}\right) \left(1 + \frac{\mathcal{P}_{22}}{\mathcal{P}_{12}}\right),$$

rearranging,

$$M^2 = k^2 L_1 L_2,$$
$$M = k\sqrt{L_1 L_2}.$$

The coefficient k is known as the *coefficient of coupling* and it is defined within the range,

$$0 \leq k \leq 1.$$

Energy Stored in Coupled Coils

$$w(t) = \frac{1}{2} L_1 i_1^2 + \frac{1}{2} L_2 i_2^2 \pm M i_1 i_2.$$