Instantaneous Power

In general, power is equal to vi, where v is the voltage and i is the current. In this case, since the voltage and the current are given by sinusoidal functions,

$$v = V_m \cos(\omega t + \theta_v),$$

$$i = I_m \cos(\omega t + \theta_i)$$

power can be represented as,

$$p = V_m I_m \cos(\omega t + \theta_v) \cos(\omega t + \theta_i).$$

Now instead of dealing with both the phase angles for the current and the voltage, the expressions can be written as,

$$v = V_m \cos(\omega t + \theta_v - \theta_i),$$

$$i = I_m \cos(\omega t),$$

$$p = V_m I_m \cos(\omega t + \theta_v - \theta_i) \cos(\omega t).$$

Using the trigonometric identity,

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}(\cos(\alpha - \beta) + \cos(\alpha + \beta)),$$

where $\alpha = \omega t + \theta_v - \theta_i$ and $\beta = \omega t$,

$$p = \frac{V_m I_m}{2} \cos(\theta_v - \theta_i) + \frac{V_m I_m}{2} \cos(2\omega t + \theta_v - \theta_i).$$

Using the identity,

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta),$$

the expression can be expanded to be written as,

$$\boxed{p = \frac{V_m I_m}{2} \cos(\theta_v - \theta_i) + \frac{V_m I_m}{2} \cos(\theta_v - \theta_i)\cos(2\omega t) - \frac{V_m I_m}{2} \sin(\theta_v - \theta_i)\sin(2\omega t).}$$

Average and Reactive Power

The expression for instantaneous power seems daunting however, it can be broken into parts where each part is significant in its own regard,

$$p = P + P\cos(2\omega t) - Q\sin(2\omega t).$$

Average (Real) Power:

$$P = \frac{V_m I_m}{2} \cos(\theta_v - \theta_i)$$

Reactive Power:

$$Q = \frac{V_m I_m}{2} \sin(\theta_v - \theta_i)$$

Here, P gives the average power while Q gives the reactive power. The average power is referred to as the "real" power since it is power that is transformed from electric to non-electric energy within the circuit.

Power for Purely Resistive Networks

$$p = P + P\cos(2\omega t)$$

For a purely resistive circuit the phases of the current and the voltage are equal, $\theta_v = \theta_i$, and with that the expression simplifies to the one above. Power cannot be extracted from a purely resistive network since resistors by nature dissipate energy in the form of heat.

Power for Purely Inductive Networks

$$p = -Q\sin(2\omega t)$$

For a circuit containing only inductors the current lags the voltage by 90° meaning, $\theta_v - \theta_i = 90°$. With that, the expression simplifies to the one above as the cosine terms will be equal to zero.

In a purely inductive circuit, the average power is zero as no energy is transformed from electric to non-electric, rather, the energy is continually transferred between the source that is driving the circuit and the coils of the inductors.

The power in a purely inductive circuit is said to be reactive since the inductor itself is a reactive element in that, its impedance is completely reactive.

The value of Average Power is given in *watts*, W, while Reactive power is given in *volt-ampere reactive*, VAR. This is due to the fact that both measurements of power are dimensionally identical.

Power for Purely Capacitive Networks

$$p = -Q\sin(2\omega t)$$

With capacitors, the current leads the voltage by 90° meaning $\theta_v - \theta_i = -90°$. This leads once again to the expression for the Real part of the instantaneous power to equal zero, leaving purely the reactive power.

The Power and Reactive Factors

$$\text{pf} = \cos(\theta_v - \theta_i),$$

$$\text{rf} = \sin(\theta_v - \theta_i)$$

The angle $\theta_v - \theta_i$ is used in both calculations of the average and reactive powers and for this reason, it is referred to as the power factor angle. The cosine of this angle in called the *power factor* and the sine of the angle is known as the *reactive factor*.

Despite having the value of the power factor, the power factor angle cannot be determined since $\cos(\theta_v - \theta_i) = \cos(-(\theta_v - \theta_i))$. To completely describe the angle therefore, the phrases

lagging and **leading** are used when describing the current and voltage in the circuit. A lagging power factor means that the current lags the voltage, the load is inductive. A leading power factor means that the current is leading the voltage, the load is capacitive.

The rms Values in Power Calculations

$$P = \frac{V_{rms}^2}{R},$$

$$P = I_{rms}^2 R$$

The rms value is also referred to as the effective value of the sinusoidal voltage or current. Given an equivalent resistive load, R, and an equivalent time period for the voltage and current, the rms value of a sinusoidal source delivers the same energy to R as does a *dc* source of the same value.

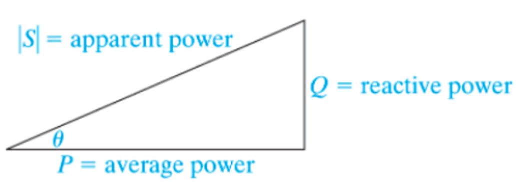
Complex Power

$$S = P + jQ$$

Complex Power has the same units as average and reactive power. To distinguish it from the other two, the units given to it are *volt-amps*, VA.

Quantity	Units
Average Power	watts
Reactive Power	volt-amps reactive
Complex Power	volt-amps

The Power Triangle



From the definitions of the different forms of power,

$$\frac{Q}{P} = \frac{V_m I_m/2 \ \sin(\theta_v - \theta_i)}{V_m I_m/2 \ \cos(\theta_v - \theta_i)}$$

$$\frac{Q}{P} = \tan(\theta_v - \theta_i)$$

Apparent Power

$$|S| = \sqrt{P^2 + Q^2}$$

From the triangle it can be seen that the angle θ is equal to the power factor angle, $\theta_v - \theta_i$. This relationship allows for the definition of apparent power, the magnitude of complex power.

The apparent power, or volt-amp, requirement of a device designed to convert electrical energy to a non-electrical is more useful than the average power requirement. The apparent power represents the volt-amp capacity required to supply the average power used by the device. From the power triangle it can be seen that unless the power factor angle is 0°, the volt-amp capacity required is larger than the average power that will be used by the device.

Power Calculations

$$\begin{aligned} S &= \frac{V_m I_m}{2}\cos(\theta_v - \theta_i) + j\frac{V_m I_m}{2}\sin(\theta_v - \theta_i) \\ &= \frac{V_m I_m}{2}\left[\cos(\theta_v - \theta_i) + j\sin(\theta_v - \theta_i)\right] \\ &= \frac{V_m I_m}{2}\, e^{j(\theta_v - \theta_i)} = \frac{V_m I_m}{2}\, \underline{/\theta_v - \theta_i} \end{aligned}$$

Using the rms values for both the current and the voltage,

$$\begin{aligned} S &= V_{rms} I_{rms} \underline{/\theta_v - \theta_i} \\ &= V_{rms} I_{rms} e^{j(\theta_v - \theta_i)} \\ &= V_{rms} e^{j\theta_v} I_{rms} e^{-j\theta_i} \end{aligned}$$

Complex Power, Alternate Form

$$S = \mathbf{V}_{rms}\mathbf{I}^*_{rms}$$

Above is the formula to calculate the apparent power of a network given the rms values of the voltage and the current. It is defined as the phasor of the voltage multiplied by the complex conjugate of the current. If the values are not rms, the same derivation technique yields the formula,

$$S = \frac{1}{2}\mathbf{V}\mathbf{I}^*.$$

Using the same principles, other formulas can be defined for the apparent, reactive, and average power.
Since,

$$\mathbf{V}_{rms} = Z\mathbf{I}_{rms},$$

$$\begin{aligned} S &= Z\mathbf{I}_{rms}\mathbf{I}_{rms}^{*} \\ &= |\mathbf{I}_{rms}|^2 Z \\ &= |\mathbf{I}_{rms}|^2 (R + jX) \\ &= |\mathbf{I}_{rms}|^2 R + j|\mathbf{I}_{rms}|^2 X \end{aligned}$$

From this it can be seen that the values of P and Q can be given as,

$$P = |\mathbf{I}_{rms}|^2 R = \frac{1}{2}\mathbf{I}_m^2 R$$

$$Q = |\mathbf{I}_{rms}|^2 X = \frac{1}{2}\mathbf{I}_m^2 R$$

Using the other general formula for power,

$$S = \mathbf{V}_{rms}\left(\frac{\mathbf{V}_{rms}}{Z}\right)^{*} = \frac{|\mathbf{V}_{rms}|^2}{Z^{*}}$$

From this, as before, for a purely resistive element,

$$P = \frac{|\mathbf{V}_{rms}|^2}{R},$$

and for a purely reactive element,

$$Q = \frac{|\mathbf{V}_{rms}|^2}{X}.$$

Maximum Power Transfer

For a resistive circuit, it was found that in order for there to be a maximum amount of power transferred to the load, the Thevenin's resistance must be equal to the load resistance. For a circuit consisting of both resistive and reactive elements, the condition that must be met is for maximum power is,

$$Z_L = Z_{TH}^{*}$$

Maximum Average Power Absorbed

$$P_{max} = \frac{|V_{TH}|^2 R_L}{4R_L^2} = \frac{1}{4}\frac{|V_{TH}|^2}{R_L}$$