Probability and Statistical Modeling

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Chapter 1

Concept of Probability

1.1 Introduction

Probability is the likelihood or chance of something happening.
In an experiment in which all outcomes are equally likely, the probability of an event E is,

$$P(E) = \frac{\text{number of favourable outcomes}}{\text{total number of outcomes}} = \frac{n(E)}{n(S)}$$

- ➢ $0 \leq P(E) \leq 1$
- ➢ $P(E) = 0$ is an impossible event
- ➢ $P(E) = 1$ is a certain event
- ➢ Sum of all probabilities of an experiment must total 1
- ➢ P(E does not occur) = 1 - P(E) (Complementary Probability)

1.2 Terminologies

- ➢ Statistical event

  It is defined as any subset of the given outcome set that is of interest

- ➢ Statistical experiment

It is described as any situation, specially set up or occuring naturally, which can be performed, enacted or otherwise considered in order to gain useful information

➢ Sample Space

Sample space is the collection of all possible outcomes
Examples: 6 faces of a die, 52 cards of a bridge deck

➢ Complement Rule

The complement of an event E is the collection of all possible elementary events not contained in event E. The complement of event E is represented by $\overline{E}$.

$$P(\overline{E}) = 1 - P(E)$$
$$or, P(E) + P(\overline{E}) = 1$$

1.3 Events

➢ Combined Events

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

➢ Mutually Exclusive Events

Two events of the same experiment are said to be mutually exclusive if their respective events do not overlap.

♦ If E_1 occurs, then E_2 cannot occur

♦ If E_1 and E_2 have no common elements

➢ Not mutually exclusive

If two or more events occur at one time.

➢ Independent events

Two events are said to be independent if the occurance (or not) of one of the events will in no way affect the occurence (or not) of other.

Alternatively, two events that are defined on two physically different experiments are said to be independent.

$$
\begin{aligned}
E_1 &= \text{heads on one flip of fair coin} \\
E_2 &= \text{heads on second flip of same coin}
\end{aligned}
$$

Result of second flip does not depend on the result of the first flip.

- Dependent events

$$
\begin{aligned}
E_1 &= \text{rain forecasted on the news} \\
E_2 &= \text{take umbrella to work}
\end{aligned}
$$

Probability of the second event is affected by the occurence of the first event.

- Conditional event

  One of the outcomes of which is influenced by the outcomes of another event.

  If A and B are two events not necessarily from the same experiment, then the conditional probability that A occurs, given that B has already occured, is written,

$$
\begin{aligned}
P(\text{A, given B}) &= P(A \mid B) = \frac{P(\text{A and B})}{P(B)} \\
P(\text{B, given A}) &= P(B \mid A) = \frac{P(\text{A and B})}{P(A)}
\end{aligned}
$$

1.4 Rules of Probability

- Addition Rule

  If A and B are not mutually exclusive events, then

$$
\begin{aligned}
P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\
P(A \cup B \cup C) &= P(A) + P(B) + P(C) - P(A \cap B) \\
&\quad - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)
\end{aligned}
$$

  If A and B are mutually exclusive events, then

$$
\begin{aligned}
P(A \cup B) &= P(A) + P(B) \\
P(A \cup B \cup C) &= P(A) + P(B) + P(C)
\end{aligned}
$$

- ➢ Multiplication Rule

  If A and B are two events, then the probability of P(A and B), i.e, probability that A and B occur can be calculated as below:

  Independent events

$$P(A \cap B) = P(A) \times P(B)$$

  Dependent events

$$P(A \cap B) = P(A) \times P(B \setminus A)$$

  Probability of event B given that A has occured

$$P(B \mid A) = \frac{P(\text{A and B})}{P(A)}$$

1.5 Tree Diagram

It augments the fundamental principle of counting by exhibiting all possible outcomes of a sequence of events where each event can occur in a finite number of ways.

1.6 Contingency Tables

- ➢ A table used to classify sample observations according to two or more identifiable characteristics
- ➢ It is a cross tabulation that simultaneously summarizes two variables of interest and their relationship
- ➢ Example:

  A survey of 150 adults classified each as to gender and the number of movies attended last month. Each respondent is classified according to two criteria - the number of movies attended and gender.

Gender			
Movies Attended	Men	Women	Total
0	20	40	60
1	40	30	70
2 or more	10	10	20
Total	70	80	150

1.7 Posterior Probability

➢ Bayes Theorem

➢ It is a formula which can be thought of as 'reversing' conditional probability. That is, it finds a conditional probability (A|B) given, among other things, its inverse (B|A).

➢ If A and B are two events of an experiment, then

$$P(A|B) = \frac{P(A) \times P(B|A)}{P(B)}$$

1.8 Examples

Example 1.1

A group of 20 university students contain eight who are in the first year of study. A student is picked at random. Find the probability that the student is not in the first year of study.

Given,
$n(S) = 20$
$n(E) = 20 - 8 = 12$
Now,
$P(E) = \frac{n(E)}{n(S)} = \frac{12}{20} = \frac{3}{5}$

Example 1.2

If a card is drawn from a deck of playing cards, what is the probability of getting a red or an ace?

Sample Space, $n(S) = 52$
No. of red cards, $n(\text{Red}) = 26$
No. of ace, $n(\text{Ace}) = 4$
No. of red aces, $n(\text{Red and Ace}) = 2$
Now,

$$\begin{aligned} P(\text{Red or Ace}) &= P(\text{Red}) + P(\text{Ace}) - P(\text{Red and Ace}) \\ &= \frac{26}{52} + \frac{4}{52} - \frac{2}{52} \\ &= \frac{28}{52} \\ &= \frac{7}{13} \end{aligned}$$

Example 1.3

In a class of 20 children, 4 of the boys and 3 of the eleven girls are in the athletics team. A child from the class is chosen at random. Find the probability that the child chosen is
(a) in the athletics team
(b) a girl
(c) a girl member of the athletics team
(d) a girl member or in the athletics team

Given,
Total number of children, $n(S) = 20$
Number of boys, $n(B) = 4$
Number of girls, $n(G) = 3$

(a)
Probability that the children is in the athletics team,
$n(A) = n(B) + n(G) = 4 + 3 = 7$
$P(A) = \dfrac{n(E)}{n(S)} = \dfrac{7}{20}$

(b)
Probability that the children is a girl,

$P(G) = \frac{n(E)}{n(S)} = \frac{11}{20}$

(c)

Probability that the children is a girl member of the athletics team,

$P(G \cap A) = \frac{3}{20}$

(d)

Probability that the children is a girl member or in the athletics team,

$P(G \cup A) = P(A) + P(G) - P(G \cap A)$

$P(G \cup A) = \frac{7}{20} + \frac{11}{20} - \frac{3}{20}$

$P(G \cup A) = \frac{15}{20} = \frac{3}{4}$

Example 1.4

In a race in which there are no dead heats, the probability that John wins is 0.3, the probability that Paul wins is 0.2 and the probabiity that Mark wins is 0.4. Find the probability that

(a) John or Mark wins

(b) John or Paul or Mark wins

(c) someone else wins

Given,

Probability that John wins, P(J) = 0.3

Probability that Paul wins, P(P) = 0.2

Probability that Mark wins, P(M) = 0.4

(a)

Probability that John or Mark wins,

$P(J \cup M) = P(J) + P(M)$

$P(J \cup M) = 0.3 + 0.4$

$(J \cup M) = 0.7$

(b)

Probability that John or Paul or Mark wins,
$P(J \cup P \cup M) = P(J) + P(P) + P(M)$
$P(J \cup P \cup M) = 0.3 + 0.2 + 0.4$
$P(J \cup P \cup M) = 0.9$

(c)
Probability that someone else wins,
$P(E) = 1 - P(J \cup P \cup M)$
$P(E) = 1 - 0.9$
$P(E) = 0.1$

Example 1.5

A card is drawn from a bag containing 5 red cards numbered 1 to 5 and 3 green cards numbered 1 to 3. Find the probability that the card is
(a) a green card or a red card
(b) a green card or an even number

Given,
Total number of cards, $n(S) = 8$
No. of red cards, $n(R) = 5$
No. of green cards, $n(G) = 3$
Even numbered cards, $n(E) = 3$

(a)
Probability that the card is green or red,
$P(G \cup R) = P(G) + P(R)$
$P(G \cup R) = \dfrac{n(G)}{n(S)} + \dfrac{n(R)}{n(S)}$
$P(G \cup R) = \dfrac{3}{8} + \dfrac{5}{8}$
$P(G \cup R) = 1$

(b)
Probability that the card is green or an even number,
$P(G \cup E) = P(G) + P(E)$

$$P(G \cup E) = \frac{n(E)}{n(S)} + \frac{n(E)}{n(S)}$$
$$P(G \cup E) = \frac{3}{8} + \frac{3}{8}$$
$$P(G \cup E) = \frac{3}{4}$$

Example 1.6

When a die was thrown, the score was an odd number. What is the probability that it was a prime number?

Number of sample space, n(S) = 6
Number of odd numbers, n(O) = 3
Number of prime numbers, n(P) = 4

Now,

Probability that the number was prime,

$$P(P|O) = \frac{P(P \cap O)}{P(O)}$$

$$P(P|O) = \frac{P(P) \times P(O)}{P(O)}$$

$$P(P|O) = \frac{\frac{4}{6} \times \frac{3}{6}}{\frac{3}{6}}$$

$$P(P|O) = \frac{2}{3}$$

Example 1.7

In a certain college,

65% of the students are full time students
55% of the students are female
35% of the students are male full time students

Find the probability that a students chosen at random

a) from all the students in the college is a part time student
b) from all the students in the college is female and a part time student
c) from all the female students in the college is a part time student

Gender			
Type	Male	Female	Total
Full-time	35	30	65
Part-time	10	25	35
Total	45	55	100

n(S) = 100
n(M) = 45
n(F) = 55
n(Fp) = 25
a) Probability that the student is a part time student,

$$\begin{aligned} P(\text{Part-time}) &= \frac{n(\text{Part-time})}{n(S)} \\ &= \frac{35}{100} \\ &= \frac{7}{20} \end{aligned}$$

b) Probability that the student is female and a part time student,

$$\begin{aligned} P(Fp) &= \frac{n(Fp)}{n(S)} \\ &= \frac{25}{100} \\ &= \frac{1}{4} \end{aligned}$$

c) Probability that the student is part time from female students,

$$\begin{aligned} P(\text{Part-time}) &= \frac{n(Fp)}{n(F)} \\ &= \frac{25}{55} \\ &= \frac{5}{11} \end{aligned}$$