Problem 1.1

a) Determine whether M and N are positive definite or not

$$M = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \qquad N = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

A matrix is positive definite if all the n upper left determinants are positive.

M

$$\det[1] = 1$$

$$\det\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = 1 - 2\cdot 2 = -3 \qquad -3 < 0, \text{ hence } M \text{ is not}$$

positive definite

N

$$\det[2] = 2$$

$$\det\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = 4 - ((-1)\cdot(-1)) = 3$$

$$\det\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{matrix} 2 & -1 \\ -1 & 2 \\ 0 & -1 \end{matrix} = 8 - 2 - 2 = 4$$

$2, 3, 4 > 0$, hence M is positive definite

b) $f(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 6xy - 6x \\ 3x^2 + 3y^2 - 6y \end{bmatrix}$$

$$\nabla f = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 6xy - 6x \\ 3x^2 + 3y^2 - 6y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$$

$$\begin{cases} xy = x \\ x^2 = y(2-y) \end{cases}$$

Solutions:

$$\left.\begin{array}{l} A_1(0,0) \\ A_2(1,1) \\ A_3(-1,1) \\ A_4(0,2) \end{array}\right\}$$

$$H_f = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 6y-6 & 6x \\ 6x & 6y-6 \end{bmatrix}$$

Computing H_f for A_1 : $(-6x_1 - 6x_2)(x_1 \ x_2)$

$-6x_1^2 - 6x_2^2$

$\rightarrow H_f(0,0) = \begin{bmatrix} -6 & 0 \\ 0 & -6 \end{bmatrix}$ is negative definite

because $H_f(0,0)$ is a diagonal matrix with negative values on the diagonal entries.

$\rightarrow H_f(0,2) = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$ is positive definite

for the same reason as above, but this time it's the case with positive values.

$$\rightarrow H_f(1,1) = \begin{bmatrix} 0 & 6 \\ 6 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 0 & 6 \\ 6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6x_2 & 6x_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \boxed{12 x_2 x_1}$$

$$\downarrow$$

Indefinite $\forall \vec{x} \in \mathbb{R}^2$

A_2 neither max nor min point

$$\rightarrow H_f(-1,1) = \begin{bmatrix} 0 & -6 \\ -6 & 0 \end{bmatrix}$$

$$x^t H_f(-1,1)\, x = \boxed{-12 x_2 x_1}$$

$$\downarrow$$

Indefinite $\forall \vec{x} \in \mathbb{R}^2$

A_3 neither max nor min point

Problem 2

$$X^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 4 \end{bmatrix} \quad \vec{y} = \begin{bmatrix} 2 \\ 3 \\ 3 \\ 4 \end{bmatrix}$$

$$\omega = (X^T X)^{-1} X^T y \iff X^T X \omega = X^T y$$

- $$X^T X = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 2 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 6 & 9 \\ 6 & 10 & 15 \\ 9 & 15 & 25 \end{bmatrix}$$

- $$X^T y = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 12 \\ 19 \\ 30 \end{bmatrix}$$

- Solving the linear system. $(X^T X)\vec{\omega} = (X^T \vec{y})$

$$\left[\begin{array}{ccc|c} 4 & 6 & 9 & 12 \\ 6 & 10 & 15 & 19 \\ 9 & 15 & 25 & 30 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 4 & 6 & 9 & 12 \\ 0 & 1 & 1.5 & 1 \\ 0 & 1.5 & 4.75 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 4 & 6 & 9 & 12 \\ 0 & 1 & 3/2 & 1 \\ 0 & 0 & 5/2 & 3/2 \end{array}\right]$$

By back substitution we have:

$$\omega_2 = \frac{3}{5} \quad \omega_1 = \frac{1}{10} \quad \omega_0 = \frac{3}{2} . \quad \vec{\omega} = \begin{bmatrix} 3/2 \\ 1/10 \\ 3/5 \end{bmatrix} \text{ and } \vec{x}_T = \begin{bmatrix} 1 \\ 1.5 \\ 1.5 \end{bmatrix}$$

$$f_\omega(\vec{X}_T) = \omega^t x = \frac{3}{2} + \frac{3}{20} + \frac{9}{10} = \boxed{\frac{51}{20}}$$