J. Mech. Phys. Solids, 1972, Vol. 20, pp. 353 to 366. Pergamon Press. Printed in Great Britain.

THE FRACTURE MECHANICS OF SLIT-LIKE CRACKS IN ANISOTROPIC ELASTIC MEDIA

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SUMMARY

USING THE method of continuously distributed dislocations, the problem of a slit-like crack in an arbitrarily-anisotropic linear elastic medium stressed uniformly at infinity is formulated and solved. The crack faces may be either freely-slipping or loaded by arbitrary equal and opposite tractions. If there is no net dislocation content in the crack, then the tractions and stress concentrations on the plane of the crack are independent of the elastic constants and the anisotropy; the same is true of the elastic stress intensity factors. The crack extension force depends on anisotropy only through the inverse matrix elements K_{mg}^{-1}, where $[K]$ is the pre-logarithmic energy factor matrix for a single dislocation parallel to the crack front. Numerical results for crack extension forces are presented for three media of cubic symmetry.

1. INTRODUCTION

OF PRIMARY importance in elastic fracture mechanics is the theoretical determination of the total mechanical energy (elastic strain energy plus the decrease in potential energy of the external loading mechanism) of a cracked solid and the negative of its derivative with respect to crack length, i.e. the so-called *crack extension force G*. The experimental determination of G or the fracture toughness is made indirectly by measuring the load at which an instability (pop-in) first appears in the load-displacement curve for a cracked specimen. If the theoretical expression for crack extension force as a function of applied stress, elastic constants, and crack and specimen geometry is available, the measured load at instability can then be converted into an experimental value of G.

Using laser holographic interferometry, DUDDERAR and O'REGAN (1971) have measured the strain field near crack tips in polymethylmethacrylate (PMMA) specimens and have compared their results with those predicted by available two-dimensional isotropic crack solutions. Deviations between the two sets of results can be partially attributed to the finite thickness of the PMMA specimens. A knowledge of the elastic stress intensity factors is sufficient to predict theoretically the locus of the interference fringes in the vicinity of the crack tip for comparison with the fringes observed experimentally. More recently, interferometric studies are being conducted on anisotropic polymeric specimens†, and thus it seems prudent and

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desirable to investigate the fracture mechanics of cracks in anisotropic media so as to produce results for stress intensity and crack extension force which are computationally convenient for use by experimentalists.

It is generally known that the energy per unit length of an infinite straight dislocation line in an anisotropic linear elastic medium can be written in the form

$$E = K_{mg} b_m b_g \ln (R/r_0)$$

$$(m, g = 1, 2, 3; \text{ with summation over repeated subscripts implied}), \tag{1}$$

where R is of the order of the external specimen linear dimensions and r_0 is the dislocation core or inner cut-off radius ($r_0 \approx 2$ Å). **b** is the Burgers vector of the dislocation, and the K_{mg}, which are components of a positive-definite symmetric second-rank tensor, depend only on the direction of the dislocation line and the elastic constants C_{ijmn} of the medium (see, for example, MALÉN and LOTHE (1970) and BARNETT and SWANGER (1971)). BARNETT and SWANGER (1971) have developed formulae for the K_{mg} which may be evaluated rapidly and accurately by simple numerical integration for media of arbitrary elastic anisotropy. For an isotropic medium, K_{mg} is diagonal in an (x_1, x_2, x_3) cartesian coordinate system whose x_3-axis is coincident with the dislocation line with

$$K_{11} = K_{22} = \mu/4\pi(1-\nu), K_{33} = \mu/4\pi, \tag{2}$$

where μ is the shear modulus and ν is Poisson's ratio.

It is also well-established (BILBY and ESHELBY, 1968) that a slit-like crack in a linear elastic solid subjected to external loads can be represented by a continuous distribution of straight dislocations of infinitesimal Burgers vectors appropriately partitioned along the cracked region, the dislocation lines being parallel to the crack front. When recast in such a form, the calculation of the crack stress field involves first solving a singular integral equation for the dislocation distribution. Solutions obtained for cracks in isotropic solids using this technique are legion and have been reviewed in detail by BILBY and ESHELBY (1968).

BILBY and ESHELBY (1968) have shown how the crack extension force may be easily calculated if the dislocation distribution simulating the crack (or alternatively the displacement discontinuity across the crack faces) is known, a result which is valid for arbitrary elastic anisotropy. In this paper we use the continuous dislocation distribution approach to derive a simple expression for the crack extension force associated with a freely-slipping slit-like crack loaded by constant tractions at infinity in an anisotropic medium. We shall show that the determination of the crack extension force involves only a knowledge of the inverse matrix components ${K_{mg}}^{-1}$ for a single dislocation parallel to the crack front, so that G may be easily computed without really solving in detail an elastostatic boundary-value problem. Furthermore, the evaluation of G may be performed in a laboratory frame of reference so that there is no need to transform the elastic constant tensor to "crack-coordinates", a fact which greatly facilitates numerical computations.

The technique is then extended to include slit-like cracks whose faces are not traction-free. The following interesting universal result is derived: when the relative displacement of the crack faces vanishes at the ends of the crack, the tractions and stress concentrations on the plane of the crack are independent of the elastic anisotropy of the medium, i.e. they are identical with those predicted by isotropic elastic theory; the same is true of the elastic stress intensity factors, a result first derived by STROH (1958) using a method different from that to be presented here. This rather unique result was not extracted by CLEMENTS (1971) and WILLIS (1971) as limiting cases of their solutions for interfacial cracks in bimetallic anisotropic media.

2. THE FREELY-SLIPPING CRACK

Consider a slit-like crack ($|x_1| < c$, $x_2 = 0$, $-\infty < x_3 < \infty$) in an infinite anisotropic linear elastic medium (Fig. 1). Let X_1, X_2, X_3 denote that rectangular

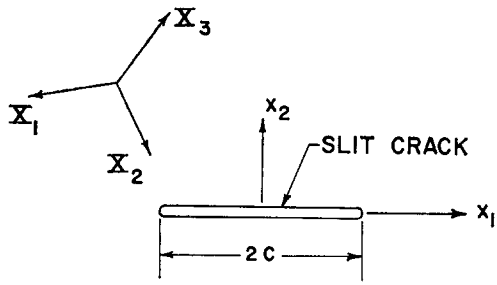

FIG. 1. Crack coordinates for the slit-like crack. The X_i are laboratory coordinates.

cartesian reference frame (the laboratory frame) in which the elastic constant tensor C_{ijmn} is displayed in its simplest form, e.g. for a medium of cubic symmetry the X_i axes would coincide with the cube directions. For the present we shall perform all calculations in terms of "crack coordinates", i.e. the x_i. As $x_1^2+x_2^2 \to \infty$, the state of stress is uniform and is specified by

$$\lim_{x_1{}^2+x_2{}^2\to\infty} p_{ij} = p_{ij}^A, \tag{3}$$

where p_{ij} denotes a component of the (symmetric) stress tensor. We assume small strains so that the elastic strain e_{ij} is given by

$$2e_{ij} = u_{i,j}+u_{j,i}, \tag{4}$$

where u_i represents a component of displacement and the subscript j means partial differentiation with respect to x_j. Hooke's law is

$$p_{ij} = C_{ijmn}u_{m,n}, \tag{5}$$

and in the absence of body forces elastic equilibrium demands that

$$p_{ij,j} = 0. \tag{6}$$

For a freely-slipping crack the boundary condition

$$p_{ij}n_j = 0, \tag{7}$$

must be satisfied on the faces of the crack (**n** is the unit normal at a point on the crack surface; our convention is such that on the upper crack surface, $x_2 = 0+$, $n_1 = n_3 = 0$ and $n_2 = -1$, while on the lower crack surface, $x_2 = 0-$, $n_1 = n_3 = 0$ and $n_2 = +1$), so that $p_{12} = p_{22} = p_{32} = 0$ for $|x_1| < c$, $x_2 = 0$, $|x_3| < \infty$.

STROH (1958) has treated this problem by the method of Fourier transforms and dual integral equations. We choose instead to use the method of continuous distributions of dislocations, which seems to us to be more straightforward and to facilitate computational convenience. Following BILBY and ESHELBY (1968) we write the solution to the elastostatic problem in the form

$$u_i = u_i^A + u_i^D,\ e_{ij} = e_{ij}^A + e_{ij}^D,\ p_{ij} = p_{ij}^A + p_{ij}^D. \tag{8}$$

The D-fields will turn out to be the elastic fields due to linear superpositions of three types of straight dislocations distributed along the cracked region (Fig. 2). The A-fields

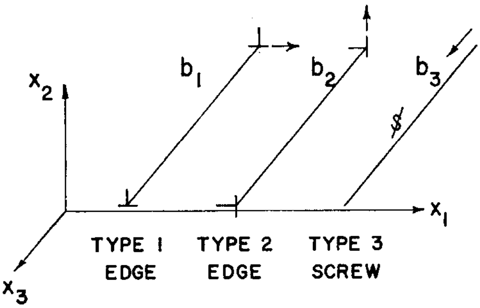

FIG. 2. The three types of dislocations used to simulate the slit-like crack. All dislocation lines are parallel to the crack front; a dislocation of type s has a Burgers vector in the x_s-direction.

refer to the uniformly stressed uncracked solid, so that, excluding arbitrary rigid-body terms,

$$u_i^A = e_{ij}^A x_j,\ e_{ij}^A = S_{ijmn} p_{mn}^A, \tag{9}$$

where the S_{ijmn} are the elastic compliances of the medium. The A-fields are everywhere continuous. We note that

$$p_{ij}^D \to 0 \quad \text{as} \quad x_1^2 + x_2^2 \to \infty, \tag{10}$$

since the stress field of a single straight dislocation has this property. Since both the A-fields and the D-fields are admissible solutions to the equilibrium and compatibility equations, we can complete the solution by choosing p_{ij}^D so that (7) is satisfied, i.e. so that

$$p_{12}^D = -p_{12}^A = -T_1^A,\ p_{22}^D = -p_{22}^A = -T_2^A,\ p_{32}^D = -p_{32}^A = -T_3^A, \quad \text{for } |x_1| < c,\ x_2 = 0. \tag{11}$$

We now construct the D-field by an integral superposition of three types of straight dislocations (Fig. 2), the dislocations being parallel to the x_3-axis and lying in the region $|x_1| < c$, $x_2 = 0$. If we denote by $b_s f^{(s)}(t)dt$ (no sum on s) the amount of Burgers vector in the x_s-direction distributed between $x_1 = t$ and $x_1 = t + dt$, then

$$p_{ij}^D(x_1, x_2) = \sum_{s=1}^{3} \int_{-c}^{c} f^{(s)}(t) p_{ij}^{(s)}(x_1, x_2; t, 0)dt, \tag{12}$$

where $p_{ij}^{(s)}(x_1, x_2; t, 0)$ is the stress field at (x_1, x_2) due to a single straight dislocation of type "s" piercing the planes $x_3 = \text{const.}$ at the point $(t, 0)$. Thus, equations (11) may be written as

$$\sum_{s=1}^{3} \int_{-c}^{c} f^{(s)}(t) p_{i2}^{(s)}(x_1, 0; t, 0)dt = -T_i^A\ (i = 1, 2, 3),\ |x_1| < c. \tag{13}$$

In an infinite homogeneous medium,

$$p_{ij}^{(s)}(x_1, 0; t, 0) = p_{ij}^{(s)}(x_1 - t), \tag{14}$$

and from the theory of elastic Green's functions (WILLIS, 1970, and BARNETT and SWANGER, 1971) it is known that $p_{ij}^{(s)}(x_1-t)$ is proportional to $(x_1-t)^{-1}$.

Now consider a single dislocation parallel to the x_3-axis and piercing the planes $x_3 = \text{const.}$ at $(t, 0)$. Let its Burgers vector have components b_1, b_2, b_3. The energy per unit length of the dislocation can be written as

$$E = K_{mg} b_m b_g \ln (R/r_0) = K_{mg} b_m b_g \int_{t+r_0}^{t+R} (x_1-t)^{-1} dx_1. \tag{15}$$

But the formula

$$E = \tfrac{1}{2} \sum_{s=1}^{3} \int_{t+r_0}^{t+R} p_{m2}^{(s)}(x_1-t) b_m dx_1, \tag{16}$$

is also valid. If the dislocation is of type 1 (i.e. $s = 1$, $b_2 = b_3 = 0$), a comparison of (15) and (16) yields

$$p_{12}^{(1)}(x_1-t) = 2K_{11} b_1/(x_1-t). \tag{17}$$

Similar reasoning shows that

$$p_{22}^{(2)}(x_1-t) = 2K_{22} b_2/(x_1-t),\ p_{32}^{(2)}(x_1-t) = 2K_{33} b_3/(x_1-t). \tag{18}$$

Equations (17) and (18) also follow from applying BROWN's (1967) theorem to an infinite straight dislocation line.

If we then consider the case $s = 1, 2$ and $b_3 = 0$ and note that

$$p_{22}^{(1)}(x_1-t) b_2 = p_{12}^{(2)}(x_1-t) b_1, \tag{19}$$

since one is able to calculate the interaction elastic strain energy (ESHELBY, 1956) between a dislocation of type 1 and a dislocation of type 2 by either of two methods, a comparison of (15) and (16) shows that

$$p_{22}^{(1)}(x_1-t) = 2K_{21} b_1/(x_1-t),\ p_{12}^{(2)}(x_1-t) = 2K_{12} b_2/(x_1-t). \tag{20}$$

Equation (19) essentially follows from Betti's reciprocal theorem. In general,

$$p_{i2}^{(s)}(x_1-t) = 2K_{is} b_s/(x_1-t) \qquad \text{(no sum on } s\text{)}. \tag{21}$$

If we use (21) and define

$$F_s(t) = b_s f^{(s)}(t) \qquad \text{(no sum on } s\text{)}, \tag{22}$$

then (13) may be expressed concisely as

$$2K_{ij} \int_{-c}^{c} (x_1-t)^{-1} F_j(t)\, dt = -T_i^A \qquad (|x_1| < c) \tag{23}$$

or

$$\int_{-c}^{c} (x_1-t)^{-1} F_j(t)\, dt = -\tfrac{1}{2} K_{ji}^{-1} T_i^A \qquad (|x_1| < c), \tag{24}$$

where we now sum over all indices and the integrals in question (as will be the case with all singular integrals appearing in this work) are defined by their *Cauchy principal values* (BILBY and ESHELBY, 1968). The inverse matrix components K_{ji}^{-1} are given by

$$K_{ji}^{-1} = K_{ij}^{-1} = \varepsilon_{jmn} \varepsilon_{irs} K_{mr} K_{ns} / 2\varepsilon_{\alpha\beta\lambda} K_{1\alpha} K_{2\beta} K_{3\lambda}, \tag{25}$$

so that

$$K_{ji}^{-1} K_{is} = K_{ji} K_{is}^{-1} = \delta_{js}, \tag{26}$$

where δ_{js} and ε_{jmn} are the Kronecker delta and the alternating tensor, respectively.

We expect stress singularities at the ends of the crack, so that we require $F_j(\pm c)$ to be unbounded with a weak singularity. Furthermore, if we demand that there be no relative displacement of the crack faces at $x_1 = \pm c$, then

$$\int_{-c}^{c} F_j(t)dt = 0; \tag{27}$$

we remark here that implicit in our analysis has been the convention that the displacement discontinuity across the crack faces at $(x_1, 0)$ is given by

$$\Delta u_i(x_1) = u_i^D(x_1, x_2) \Big|_{x_2=0+}^{x_2=0-} = \int_{-c}^{x_1} F_j(t)dt. \tag{28}$$

The solution of (24) is (MUSKHELISHVILI, 1953)

$$F_j(t) = K_{ji}^{-1} T_i^A \frac{t}{2\pi(c^2-t^2)^{1/2}}. \tag{29}$$

A rather remarkable result can be extracted when one considers the tractions p_{i2} acting on the plane of the crack $(|x_1| > c,\ x_2 = 0)$. Now,

$$p_{i2} \Big|_{x_2=0} = p_{i2}^A + 2 \sum_{s=1}^{3} \int_{-c}^{c} \frac{F_s(t)}{b_s} \frac{K_{is} b_s}{(x_1 - t)} dt \qquad (|x_1| > c). \tag{30}$$

Since

$$T_i^A = p_{i2}^A, \tag{31}$$

using (26) and (29) yields

$$\begin{aligned} p_{i2} \Big|_{x_2=0} &= p_{i2}^A \left(1 + \frac{1}{\pi} \int_{-c}^{c} \frac{t\,dt}{(c^2-t^2)^{1/2}(x_1-t)}\right) \\ &= p_{i2}^A \frac{|x_1|}{(x_1^2-c^2)^{1/2}} \qquad (|x_1| > c). \end{aligned} \tag{32}$$

As $x_1 \to \pm c$, (32) reduces to the usual isotropic expression for stress concentration $p_{i2} \approx p_{i2}^A(c/2r)^{1/2}$, where $r = |x_1| - c$. Hence, the tractions and stress concentrations on the plane of the crack are independent of the elastic constants and the anisotropy of the medium, i.e. they are identical with those for a crack in an isotropic medium loaded by stresses p_{ij}^A at infinity. This result was first extracted by STROH (1958); STROH (1958, eq. (107)) is clearly equivalent to the limit of (32) as $x_1 \to \pm c$. In Section 3 we shall show that this result also holds when the faces of the crack are symmetrically, but otherwise arbitrarily, stressed by self-equilibrating tractions provided that (27) is satisfied.

The energy of deformation and the crack extension force may now be easily calculated. According to BILBY and ESHELBY (1968) the difference in total mechanical energy per unit length in x_3 between the stressed cracked solid and the uncracked solid stressed homogeneously by $p_{ij} = p_{ij}^A$ is

$$\Delta E = \frac{1}{2} \int_{-c}^{c} p_{i2}^A \Delta u_i^D dx_1. \tag{33}$$

(The formula given by BILBY and ESHELBY (1968, p. 145, eq. (115)) is the negative of our expression because they define their Δu_i^c as the negative of our Δu_i^D). Combining (28), (29), and (33) we obtain

$$\Delta E = -\tfrac{1}{8} c^2 p_{i2}^A p_{s2}^A \quad \mathrm{K}_{is}^{-1}. \tag{34}$$

Equation (34) restates the expression for ΔE previously derived by STROH (1958, eq. (112)). Using STROH (1958, eqs. (50), (56)) and comparing with our equation (1) clearly shows that $4\pi K_{ij}{}^{-1}$ is equivalent to his matrix B_{ij}. The crack extension force G is then given by

$$G = -\frac{\partial(\Delta E)}{\partial c} = \tfrac{1}{4}cp_{i2}^{A}p_{s2}^{A}K_{is}^{-1}. \tag{35}$$

The appropriate Griffith criterion for brittle fracture in the anisotropic medium is obtained by requiring that G be greater than or equal to 4γ, where γ is the surface energy associated with the plane $x_2 = 0$. Thus, the applied stress state required to propagate the crack catastrophically is determined from

$$p_{i2}^{A}p_{s2}^{A}K_{is}{}^{-1} \geq 16\gamma/c. \tag{36}$$

Using equation (2) allows one to recover the isotropic Griffith criterion

$$(1-\nu)\{(p_{12}^{A})^2+(p_{22}^{A})^2\}+(p_{32}^{A})^2 \geq 4\mu\gamma/\pi c. \tag{37}$$

One immediately apparent effect of anisotropy upon the crack extension force is the introduction of coupling between the stresses p_{12}^{A}, p_{22}^{A}, p_{32}^{A} due to the non-diagonal nature of $K_{is}{}^{-1}$.

In Appendix I we give a summary of the technique of BARNETT and SWANGER (1971) for calculating K_{is}, and hence $K_{is}{}^{-1}$ using (25), in the laboratory frame of reference, i.e. the X_i. If we refer all field quantities to the X_i-frame, then (35) becomes

$$G = \tfrac{1}{4}cp_{ij}^{A}n_j p_{sm}^{A}n_m K_{is}{}^{-1}, \tag{38}$$

where the n_m are components of the unit normal to the crack surface in the X_i-frame. A simple example of the influence of anisotropy upon the crack extension force is shown in Fig. 3. We have considered cleavage cracks lying in an $(X_1 - X_2)$ or (001)

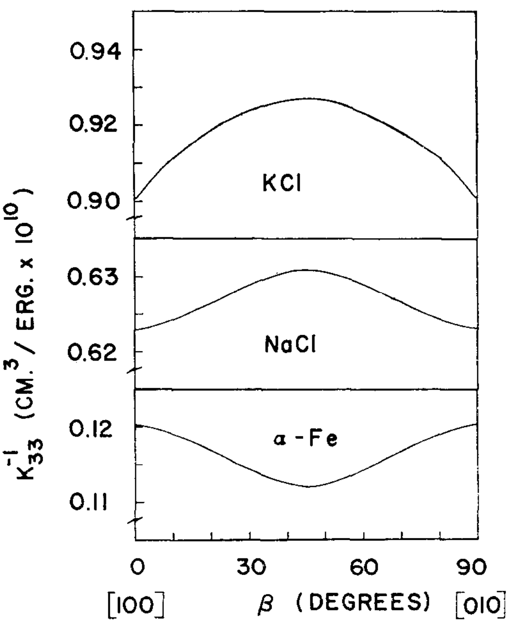

FIG. 3. The variation of $K_{33}{}^{-1}$ with β, the angle between the front of a slit crack in an (001) plane and the [100] cube direction, for KCl, NaCl, and α-Fe.

plane in three cubic media, KCl, NaCl, and α-Fe stressed only by the far-field tension p^A normal to the cleavage plane, so that in the laboratory frame

$$G = \tfrac{1}{4}(p^A)^2 c K_{33}^{-1}. \tag{39}$$

Figure 3 shows the variation of K_{33}^{-1} with β (the angle between the X_1-axis, i.e. the [100] direction, and the crack front). The effect of anisotropy is appreciable only in α-Fe, in which K_{33}^{-1} decreases by about 8 per cent as the crack front varies from a cube edge to a cube face diagonal. Using Voigt average elastic constants, the isotropic prediction for K_{33}^{-1} in α-Fe yields $K_{33}^{-1} = 0{\cdot}103 \times 10^{-10}$ cm^3/erg, which differs by between 10 and 20 per cent from our anisotropic calculations.

3. Crack Faces Loaded by Tractions

The extension of the results for a freely-slipping crack to the case of a crack whose faces are loaded by tractions $-R_i(x_1)$ is easily done. The same analysis suffices if we replace the boundary-condition (7) by

$$p_{ij}n_j = \begin{cases} -R_i(x_1) & \text{for} \quad x_2 = 0+, \\ R_i(x_1) & \text{for} \quad x_2 = 0-, \end{cases} \; |x_1| < c, \tag{40}$$

so that we assume that the tractions on the upper and lower crack faces are equal in magnitude and opposite in direction, i.e. the loading is symmetrical about $x_2 = 0$ and thus is self-equilibrating. For example, if one wished to model a crack loaded by internal gas pressure P we would choose $R_1 = R_3 = 0$ and $R_2 = -P$. Hence, the distribution functions $F_j(t)$ must now satisfy

$$\int_{-c}^{c} (x_1 - t)^{-1} F_j(t)\, dt = -\tfrac{1}{2} K_{ij}^{-1}\{T_i^A - R_i(x_1)\}, \qquad |x_1| < c. \tag{41}$$

We again require that (27) be satisfied when $F_j(\pm c)$ is unbounded. The integral equation may be easily solved once we specify $R_i(x_1)$. For our purposes it is sufficient to note that we can write

$$2F_j(t) = K_{ji}^{-1} Q_i(t), \tag{42}$$

where $Q_i(t)$ is *independent* of the anisotropy of the medium and satisfies

$$\int_{-c}^{c} (x_1 - t)^{-1} Q_i(t)\, dt = -\{T_i^A - R_i(x_1)\}, \qquad |x_1| < c. \tag{43}$$

If $F_j(\pm c)$ is to be unbounded, then (MUSKHELISHVILI, 1953)

$$Q_i(t) = \frac{T_i^A t}{\pi(c^2 - t^2)^{1/2}} + \frac{1}{\pi^2(c^2 - t^2)^{1/2}} \int_{-c}^{c} \frac{R_i(x_1)(c^2 - x_1^2)^{1/2}}{x_1 - t}\, dx_1. \tag{44}$$

If one wishes to consider cracks with no stress singularities at $x_1 = \pm c$, such as equilibrium cracks (BARENBLATT, 1961) and simple models of cracks relaxed by plastic deformation (DUGDALE, 1960, and BILBY, COTTRELL, and SWINDEN, 1963), then $F_j(\pm c)$ must vanish with

$$Q_i(t) = \pi^{-2}(c^2 - t^2)^{1/2} \int_{-c}^{c} (c^2 - x_1^2)^{-1/2}(x_1 - t)^{-1} R_i(x_1)\, dx_1, \tag{45}$$

provided that the subsidiary condition

$$\int_{-c}^{c} (c^2 - s^2)^{-1/2} R_i(s)\, ds = \pi T_i^A, \tag{46}$$

is satisfied.

The tractions p_{i2} acting on the plane of the crack ($|x_1| > c$, $x_2 = 0$) are then given by (32) which, using (26) and (42), reduces to

$$p_{i2}\Big|_{x2=0} = p_{i2}^A + \int_{-c}^{c} (x_1-t)^{-1}Q_i(t)dt, \qquad |x_1| > c. \tag{47}$$

Since $Q_i(t)$ depends only on the loading $T_i^A - R_i(x_1)$, the tractions and also the stress concentrations (if any) on the plane of the crack are *independent* of the elastic anisotropy [see STROH (1958, eq. (106))]. In Section 4 we shall use (47) to calculate stress intensity factors and the crack extension force for slit-cracks whose stress fields are singular at $x_1 = \pm c$.

We complete this section by remarking that when (27) is not satisfied, i.e. when there exists a net dislocation content in the crack so that

$$\int_{-c}^{c} F_j(t)dt = N_j, \tag{48}$$

one must add to $F_j(t)$ as given by (29) or (42) and (44) the term

$$F_j^*(t) = N_j/\pi(c^2-t^2)^{1/2}. \tag{49}$$

In this situation the tractions on the plane of the crack are no longer independent of the anisotropy, since the distribution (49) induces extra tractions

$$p_{i2}^*\Big|_{x_2=0} = 2K_{is}N_s(x_1^2-c^2)^{-1/2}\,\mathrm{sgn}\,(x_1), \qquad |x_1| > c. \tag{50}$$

4. STRESS INTENSITY FACTORS AND THE CRACK EXTENSION FORCE

It is common in elastic fracture mechanics to consider the stress intensity factor associated with a given crack external loading configuration. The three stress intensity factors k_i are most simply defined by noting that if the stresses are singular at the tip $x_1 = c$ of the slit-like crack then

$$p_{i2}\Big|_{x_1\to c, x_2=0} = k_i/(2\pi r)^{1/2} + \text{non-singular terms}, \tag{51}$$

where $r = x_1 - c$, so that

$$k_i(c) = \lim_{x_1\to c} (2\pi r)^{1/2} p_{i2}\Big|_{x2=0} \quad x_1 > c. \tag{52}$$

When (27) is satisfied, the use of (44) and (47) yields

$$k_i(c) = p_{i2}^A(\pi c)^{1/2} + \lim_{x_1\to c}\left[(2\pi r)^{1/2}\int_{-c}^{c}\frac{dt}{x_1-t}\,\frac{1}{\pi^2(c^2-t^2)^{1/2}}\int_{-c}^{c}\frac{R_i(s)(c^2-s^2)^{1/2}ds}{s-t}\right]. \tag{53}$$

Interchanging the order of integration and noting that

$$\left.\begin{aligned}
&\frac{1}{(x_1-t)(s-t)} = \frac{1}{s-x_1}\left(\frac{1}{x_1-t}-\frac{1}{s-t}\right),\\
&\int_{-c}^{c}\frac{dt}{(c^2-t^2)^{1/2}(s-t)} = 0, \qquad |s| < c \text{ (Cauchy principal value)},\\
&\int_{-c}^{c}\frac{dt}{(c^2-t^2)^{1/2}(x_1-t)} = \frac{\pi}{(x_1^2-c^2)^{1/2}}\,\mathrm{sgn}\,(x_1), \qquad |x_1| > c,
\end{aligned}\right\} \tag{54}$$

the limit of (53) as $x_1 \to c$ gives

$$k_i(c) = p_{i2}^A(\pi c)^{1/2} - (\pi c)^{-1/2}\int_{-c}^{c} R_i(s)\left(\frac{c+s}{c-s}\right)^{1/2} ds. \tag{55}$$

In a similar fashion one deduces that at the tip $x_1 = -c$,

$$k_i(-c) = p_{i2}^A(\pi c)^{1/2} - (\pi c)^{-1/2} \int_{-c}^{c} R_i(s) \left(\frac{c-s}{c+s}\right)^{1/2} ds. \tag{56}$$

Thus, $k_i(\pm c)$ is *independent* of the anisotropy of the medium. We now proceed to show that the crack extension force is easily calculated only from a knowledge of k_i and K_{ij}^{-1}.

Using a formulation developed by BILBY and ESHELBY (1968, eq. (112); see also eq. (102)) when the crack tip $x_1 = c$ extends to $x_1 = c + \delta c$, the change in total mechanical energy is

$$\delta E = \frac{1}{2} \int_{c}^{c+\delta c} dx_1 \, . \, p_{i2}(x_1, 0) \int_{-c}^{x_1 - \delta c} F_i(t) dt. \tag{57}$$

Using (44), (51), and (55) and letting $\delta c \to 0$ (see Appendix II), the crack extension force is found to be

$$G = -\lim_{\delta c \to 0} \frac{\delta E}{\delta c} = \frac{1}{8\pi} k_i k_m K_{im}^{-1}, \tag{58}$$

which is the desired result. Had we considered the crack tip at $x_1 = -c$ extending from $-c$ to $-(c+\delta c)$ we would have obtained (58) with k_i given by (56). Since in deriving (58) we considered extension of only one end of the crack, in this instance the proper Griffith criterion for brittle fracture is $G = 2\gamma$.

Equations (55) and (56) may also be used to derive formulae for the applied stress at which an equilibrium crack (BARENBLATT, 1961) becomes mobile. In this instance we interpret the $R_i(x_1)$ as restraining stresses acting on the crack surfaces due to atomic cohesive forces; usually one imagines that $R_i(x_1)$ differs appreciably from zero only in the regions $c - d < |x_1| < c$, where $d \ll c$ and d is independent of c. Such a crack will propagate when $k_i(\pm c) \geq 0$, i.e. when

$$p_{i2}^A \geq \frac{1}{\pi c} \int_{-c}^{c} R_i(s) \left(\frac{c \pm s}{c \mp s}\right)^{1/2} ds, \tag{59}$$

where the upper and lower signs correspond to the tips $x_1 = c$ and $x_1 = -c$ respectively. Equation (59) is equivalent to BILBY and ESHELBY (1968, eqs. (132), (136)). The fracture criterion (59) depends on anisotropy only through the dependence of $R_i(s)$, i.e. the cohesive forces, on anisotropy.

The beauty of the expression (58) for crack extension force is that the stress intensity factors k_i need be calculated only once using either (55) or (56) for a given crack configuration, since anisotropic effects appear only through the K_{ij}^{-1}. The determination of G may also be executed in laboratory coordinates for ease. This would be most easily done by calculating K_{ij}^{-1} in the laboratory frame and computing $k_i(\text{lab}) = A_{im} k_m(\text{crack frame})$, where A_{im} is the cosine of the angle between the X_i- and the x_m-directions. It also seems that the simplicity of (58) reflects a desirable unification of dislocation field theory and continuum fracture mechanics.

5. CONCLUSIONS

We have not attempted here to extend the details of the many existing isotropic crack solutions (i.e. specific loadings T_i^A and $R_i(x_1)$) to the anisotropic case, since it is now clear how this may be done with relative ease. Rather we have tried to point out how important quantities such as energy of deformation, crack extension force, and crack plane tractions, stress concentrations, and stress intensity factors may be extracted with a minimum of effort. For the sake of completeness we add that even the problem of N slit-like cracks of different lengths, each loaded differently, lying in the plane $x_2 = 0$ is easily treated within the present framework. One only need solve the standard integral equation (MUSKHELISHVILI, 1953)

$$\int_D (x_1 - t)^{-1} Q_i(t)\, dt = -\{T_i^A - R_i(x_1)\}, \qquad x_1 \mathscr{E} D, \tag{60}$$

where D is the interval consisting of the N finite segments corresponding to the slit cracks.

Intuitively we would expect that the consideration of the effect of elastic anisotropy upon crack extension force will usually amount to a correction of less than 30 per cent to the value predicted by isotropic theory. One might expect anisotropic effects to be of considerably greater importance in dynamic crack propagation. It has been shown that straight dislocations moving at uniform velocities become unstable at a critical velocity which depends sensitively on the material anisotropy (MALÉN, 1970); instabilities have, in fact, been predicted at zero velocity (HEAD, 1967). In view of the equivalent representation of a crack by dislocation distributions, it is tempting to speculate that such instabilities could be related to the occurrence of bifurcation or deflection during dynamic crack propagation, although other effects such as those due to strain rate may also play an important role in such phenomena. To exactly what extent these dislocation instabilities have any physical significance in crack problems should provide an interesting topic for future investigation.

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REFERENCES

BARENBLATT, G. I. 1961 *Problems of Continuum Mechanics*, p. 21. SIAM, Philadelphia.

BARNETT, D. M. and SWANGER, L. A. 1971 *Phys. Status Solidi (b)* **48**, 419.

BILBY, B. A., COTTRELL, A. H. and SWINDEN, K. H. 1963 *Proc. Roy. Soc. (London)* **A279**, 304.

BILBY, B. A. and ESHELBY, J. D. 1968 *Fracture: An Advanced Treatise* (edited by LIEBOWITZ, H.), Vol. 1: Microscopic and Macroscopic Fundamentals, p. 29. Academic Press, New York and London.

BROWN. L. M. 1967 *Phil. Mag.* **15**, 363.

CLEMENTS, D. L. 1971 *Int. J. Eng. Sci.* **9**, 256.

DUDDERAR, T. D. and O'REGAN, R. 1971 *Expl Mech.* **11**, 1.

DUGDALE, D. S. 1960 *J. Mech. Phys. Solids* **8**, 100.

ESHELBY, J. D. 1956 *Solid State Physics* (edited by SEITZ, F. and TURNBULL, D.) Vol. 3, p. 79.

HEAD, A. K. 1967 *Phys. Status Solidi* **19**, 185.
HEARMON, R. F. S. 1961 *Applied Anisotropic Elasticity*, p. 68. Oxford University Press.
MALÉN, K. 1970 *Fundamental Aspects of Dislocation Theory*, (edited by SIMMONS, J. A., DEWIT, R., and BULLOUGH, R.) p. 23. Bureau of Standards Special Publication 317, Vol. I. National U.S. Govt. Printing Office, Washington, D.C.
MALÉN, K. and LOTHE, J. 1970 *Phys. Status Solidi* **39**, 287.
MUSKHELISHVILI, N. I. 1953 *Singular Integral Equations*. Noordhoff, Groningen.
STROH, A. N. 1958 *Phil. Mag*. **3**, 625.
WILLIS, J. R. 1970 *Phil. Mag*. **21**, 931.
1971 *J. Mech. Phys. Solids* **19**, 353.

APPENDIX I

EVALUATION OF K_{mg}

BARNETT and SWANGER (1971) have shown that the K_{mg} may be written as

$$K_{mg} = \frac{1}{8\pi^2} \varepsilon_{pjw}\, t_j \,(C_{ngip} C_{wmrs} + C_{nmip} C_{wgrs}) \int_0^\pi z_s (dz_n/d\psi) M_{ir}^{-1} d\psi, \tag{AI.1}$$

where $\mathbf{t}$ is a unit vector in the direction of the dislocation line, $\mathbf{z}$ is any unit vector perpendicular to $\mathbf{t}$, and ψ is an angular variable in the plane $\mathbf{z}\cdot\mathbf{t} = 0$. The symmetric Christoffel stiffness matrix M_{ir} and its inverse M_{ir}^{-1} are defined by

$$M_{ir} = C_{ijrs} z_j z_s, \qquad M_{ir}^{-1} = \varepsilon_{imn}\varepsilon_{rsj} M_{ms} M_{nj}/2\varepsilon_{\alpha\beta\lambda} M_{\alpha 1} M_{\beta 2} M_{\lambda 3}. \tag{AI.2}$$

The K_{mg} are easily evaluated rapidly and extremely accurately by numerical Romberg integration combined with a trapezoidal rule. If one wishes to evaluate K_{mg} for a given direction $\mathbf{t}$, this is most easily done by using spherical polar coordinates relative to the laboratory X_i-frame so that

$$t_1 = \sin\varphi \cos\theta, \qquad t_2 = \sin\varphi \sin\theta, \qquad t_3 = \cos\varphi. \tag{AI.3}$$

Then, any unit vector $\mathbf{z}$ normal to $\mathbf{t}$ may be written as

$$z_m = a_m \cos\psi + d_m \sin\psi, \tag{AI.4}$$

where

$$\left.\begin{aligned} a_1 &= \sin\theta, & a_2 &= -\cos\theta, & a_3 &= 0, \\ d_1 &= \cos\varphi\cos\theta, & d_2 &= \cos\varphi \sin\theta, & d_3 &= -\sin\varphi, \end{aligned}\right\} \tag{AI.5}$$

with

$$dz_m/d\psi = -a_m \sin\psi + d_m \cos\psi. \tag{AI.6}$$

For a given $\mathbf{t}$ (fixed θ, φ), (AI.1) may be numerically evaluated in a straightforward fashion. K_{mg}^{-1} is then computed using (25).

If one wishes to examine the variation of K_{mg} or K_{mg}^{-1} in a fixed plane, it is more convenient to work in terms of $\mathbf{n}$, the unit normal to the plane. Denoting the direction cosines of $\mathbf{n}$ relative to the laboratory frame by

$$n_1 = \sin\varphi\cos\theta, \qquad n_2 = \sin\varphi\sin\theta, \qquad n_3 = \cos\varphi, \tag{AI.7}$$

and using (AI.5) to define two orthogonal vectors **a** and **d** which are both normal to **n**, we can write

$$t_m = a_m \cos \beta + d_m \sin \beta, \tag{AI.8}$$

where β is the angle between **a** and **t**. Simple geometry shows that any unit vector **z** normal to **t** may be written as

$$z_m = n_m \cos \psi - (dt_m/d\beta) \sin \psi, \tag{AI.9}$$

with

$$(dt_m/d\beta) = -a_m \sin \beta + d_m \cos \beta, \qquad (dz_m/d\psi) = -n_m \sin \psi - (dt_m/d\beta) \cos \psi. \tag{AI.10}$$

The variation of K_{mg} with β may now be evaluated using (AI.1). The advantage of this method for calculating K_{mg} and $K_{mg}{}^{-1}$ is the ability to utilize the laboratory reference frame in which the C_{ijmn} array assumes its simplest form.

For media of cubic symmetry we note that in laboratory coordinates (AI.2) reduces to

$$M_{11}{}^{-1} = [e(e+f) - efz_1^2 + (f^2-1)(z_2 z_3)^2]/[(c_{12}+c_{44})\Delta], \tag{AI.11}$$

$$M_{12}{}^{-1} = -[z_1 z_2\{e+(f-1)z_3^2\}]/[(c_{12}+c_{44})\Delta], \tag{AI.12}$$

where

$$e = c_{44}/(c_{12}+c_{44}), \qquad f = (c_{11}-c_{44})/(c_{12}+c_{44}), \tag{AI.13}$$

$$\Delta = e^2(e+f) + e(f^2-1)(z_1^2 z_2^2 + z_1^2 z_3^2 + z_2^2 z_3^2) + (f-1)^2(f+2)(z_1 z_2 z_3)^2, \tag{AI.14}$$

with the remaining $M_{ir}{}^{-1}$ obtainable by cyclic permutation. For an isotropic medium, $f = 1$.

Appendix II

Evaluation of the Crack Extension Force

We wish to evaluate

$$G = -\lim_{\delta c \to 0} \frac{\delta E}{\delta c},$$

where

$$\delta E = \frac{1}{4} \int_c^{c+\delta c} p_{i2}(x_1, 0) dx_1 \int_{-c}^{x_1-\delta c} K_{im}{}^{-1} Q_m(t) dt \tag{AII.1}$$

$$= \frac{1}{2} (8\pi)^{-1/2} k_i K_{im}{}^{-1} \int_c^{c+\delta c} (x_1 - c)^{-1/2} dx_1 \int_{-c}^{x_1-\delta c} Q_m(t) dt, \tag{AII.2}$$

with k_i given by (55). If we let $s = x_1 - c$ and integrate by parts, noting that

$$\int_{-c}^{c} Q_m(t) dt = 0$$

when (27) is satisfied, then

$$\delta E = -(8\pi)^{-1/2} k_i K_{im}{}^{-1} \int_0^{\delta c} s^{1/2} Q_m(c - \delta c + s) ds. \tag{AII.3}$$

The further substitution $s=(1-\lambda)\delta c$ reduces (AII.3) to

$$-\frac{\delta E}{\delta c}=(8\pi)^{-1/2}k_iK_{im}{}^{-1}(\delta c)^{1/2}\int\limits_0^1(1-\lambda)^{1/2}Q_m(c-\lambda\delta c)d\lambda. \tag{AII.4}$$

For the crack with singular stresses at $x_1=\pm c$,

$$Q_m(t)=\frac{1}{\pi}p_{i2}^A\frac{t}{(c^2-t^2)^{1/2}}+\frac{1}{\pi^2(c^2-t^2)^{1/2}}\int\limits_{-c}^{c}\frac{R_i(s)(c^2-s^2)^{1/2}}{s-t}ds. \tag{AII.5}$$

As $\delta c\to 0$ the first term on the right-hand side of (AII.5) yields a contribution to $-\delta E/\delta c$ given by

$$(8\pi)^{-1/2}k_iK_{im}{}^{-1}p_{i2}^A\frac{1}{\pi}(\delta c)^{1/2}\left(\frac{c}{2\delta c}\right)^{1/2}\int\limits_0^1\left(\frac{1-\lambda}{\lambda}\right)^{1/2}d\lambda=\frac{1}{8\pi}k_iK_{im}{}^{-1}p_{i2}^A(\pi c)^{1/2}. \tag{AII.6}$$

Interchanging the order of integration, the second term on the right-hand side of (AII.5) yields a contribution to $-\delta E/\delta c$ given by

$$\lim_{\delta c\to 0}\left[(8\pi)^{-1/2}k_iK_{im}{}^{-1}\frac{1}{\pi^2}(\delta c)^{1/2}\left(\frac{1}{2c\delta c}\right)^{1/2}\int\limits_{-c}^{c}R_i(s)(c^2-s^2)^{1/2}ds\times\right.$$
$$\left.\times\int\limits_0^1\left(\frac{1-\lambda}{\lambda}\right)^{1/2}\frac{d\lambda}{c-s+\lambda\delta c}\right]$$
$$=-\frac{1}{8\pi}k_iK_{im}{}^{-1}\frac{1}{(\pi c)^{1/2}}\int\limits_{-c}^{c}R_i(s)\left(\frac{c+s}{c-s}\right)^{1/2}ds. \tag{AII.7}$$

In obtaining (AII.7) we have noted that

$$\lim_{\delta c\to 0}\int\limits_0^1\left(\frac{1-\lambda}{\lambda}\right)^{1/2}\frac{d\lambda}{s-c+\lambda\delta c}=-\frac{\pi}{2(c-s)}. \tag{AII.8}$$

Comparing (AII.6) and (AII.7) with (55) yields the crack extension force in the form

$$G=\frac{1}{8\pi}k_iK_{im}{}^{-1}k_m. \tag{AII.9}$$

Equation (AII.9) is also valid for extension of the crack tip at $x_1=-c$ if we use k_i as given by (56).