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J-integral concept for multi-phase materials

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The well-known J-integral concept is systematically extended to multi-phase materials in various ways, and, the extended two-dimensional J-integral vector is applied in numerical calculations. The results of these computations show that J_k retains its characteristic features, namely its path-independency and its physical meaning of an energy-release rate, for strongly heterogeneous materials, if we take additional terms into account. At last, the suitability of the extended J-integral concept as a method to treat the problem of the crack propagation in multi-phase materials is assessed.

1. Introduction

For safety and economy reasons the possibility of the existence of cracks is taken into account by modern concepts in the most branches of industry (engineering industry, power station industry, aircraft industry, building and construction industry, etc.). So, one can observe a rising interest in the field of mechanics of materials and a widely spread attention to fracture mechanics as a fundamental pillar.

Initially, fracture mechanics was restricted to situations where the structure remains predominantly linear elastic up to the fracture event (LEFM). An extension to the non-linear range and to situations where plastic material behaviour plays a great role (PYFM) has followed. The latter is still an open research field.

One aim of fracture mechanics is to provide methods by which one can transfer results from laboratory type specimen to full scale structures. Connected with this is a great interest in the problem of predicting the failure of a structure due to great cracks without making an experiment and thereby destroying the structure.

Attempts to solve this problem differ in the way of crack-tip modelling. Three levels of complexity in crack-tip modelling are given in ref. [1]. On the first one, the crack-tip region is assumed to obey the same constitutive law as is assumed for the rest of the body, and therefore an additional condition to the usual equations of continuum mechanics (motion, kinematics, constitutive behaviour) is needed. It is called crack growth law or fracture criterion. Since the complexity increases rapidly with each additional level, it is of great interest to treat the problem of cracks by a fracture criterion. One-parameter crack-growth conditions are widely spread and should suffice, if the fracture process zone is small enough [1]. It is important to find criteria based on theoretical considerations, which are valid in very general cases, because of the great variety of phenomena of cracks and fracture.

Fracture criteria based on generalized J-integral expressions fulfil the mentioned conditions. Since the discovery of the path-independent J-integral by Eshelby [2–4] and Rice [5,6], path-independent integrals have been widely used in fracture mechanics analyses. The J-integral concept bypasses the mathematical difficulties to find a detailed solution of the boundary value problem by an approximate analysis. If the stresses and strains are calculated in a numerical way, e.g. using the Finite-Element-Method, one can evaluate the J-integral along a contour far away from the crack tip, where the errors are not so great, due to its path-independency. Then, the J-integral provides a value, which can be interpreted as an average value of the concentrated stresses at the crack-tip [5], as the energy-release rate considering an infinitesimal small quasi-static crack advance in the plane of the crack [6] or as the force on the crack-tip in direction of the crack [2–4] for non-linearly elastic materials.

Due to J's physical meaning of an energy-release rate, fracture initiation has been reasonably well characterized in terms of a critical material-dependent value J_c of the J-integral. Two further stages of the failure sequence follow the initiation: stable crack growth and instability [7]. Another attempt to describe these further stages by the J-integral is to compare the curve "J versus crack-tip advance Δl" and the curve "J_{mat} versus crack-tip advance Δl" – where J_{mat} is the material resistance against crack advance – analogously to the R-curve concept for linear elasticity [8]. Stable crack growth occurs, if

$$J > J_c \text{ and } \frac{dJ}{dl} < \frac{dJ_{mat}}{dl}.$$

For the point of instability,

$$\frac{dJ}{dl} = \frac{dJ_{mat}}{dl}$$

is valid (fig. 1).

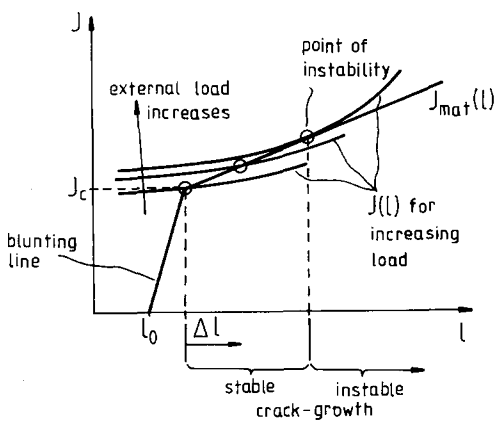

Fig. 1. Failure sequence.

In this paper we investigate whether the J-integral concept in principle can be applied to multi-phase materials, because it is more realistic to assume that phase boundaries exist in a material than presupposing homogeneity. Examples are inclusions, different phases and crystals in a material, welded materials, composites and so on. So, this study deals with the attempt to apply the J-integral concept to a further class of materials.

2. Extension of the J-integral to multi-phase materials

In this chapter, the two-dimensional J-integral vector shall be extended to multi-phase materials. For this, we start from three different points:

1) Rice's fundamental work;
2) the derivation of path-independent integrals by an overall energy-time-rate balance considering a control volume with time-dependent boundaries;
3) Eshelby's fundamental work;

and we will get the same results in every way.

2.1. *Rice's derivation of J as an energy-release rate for an infinitesimal small, quasi-static crack advance by an energy-balance*

Rice has considered two bodies of elastic material containing a notch or a void [6] to obtain the energy-release rate. Each body is subjected to the same system of loads and the two bodies are identical in every feature but one: the notches differ in size.

We can extend Rice's derivation to inhomogeneity realizing that the strain energy density not only depends on the strains but also on the spatial coordinates, so that for two-dimensional cases

$$W = W(\epsilon_{mn}, x_1, x_2) \tag{1}$$

is valid.

First we consider inhomogeneous material behaviour which can be described by continuous functions in the spatial coordinates excluding phase boundaries. Using eq. (1) and following Rice's derivation, we obtain the energy-release rate:

$$J = \int_\Gamma \left(W \, \mathrm{d}x_2 - T_i \frac{\partial u_1}{\partial x_1} \mathrm{d}\Gamma \right) \underbrace{- \int_A \left(\frac{\partial W}{\partial x_1} \right)_{\text{expl.}} \mathrm{d}A}_{\text{additional term}} \tag{2}$$

The first term of the right side of eq. (2) is the well-known J-integral according to Rice in which Γ is a curve surrounding the crack tip in a counterclockwise sense, W is the strain energy density, T_i are the components of the traction vector defined according to the outward drawn normal along Γ and u_i are the components of the displacement vector. The domain integral in eq. (2) is an additional term by which the influence of the inhomogeneities is described. The integrand is a so-called explicit derivative, defined by:

$$\left(\frac{\partial W}{\partial x_1} \right)_{\text{expl.}} := \left. \frac{\partial W(\epsilon_{mn}, x_1, x_2)}{\partial x_1} \right|_{\substack{\epsilon_{mn} = \text{const.} \\ x_2 = \text{const.}}} \tag{3}$$

and A is the area enclosed by Γ.

To include phase boundaries we replace each of them by a thin layer of thickness ϵ (fig. 2(a)). Within this layer and on its boundaries the material properties are assumed to vary continuously. Then, we can evaluate the domain integral in eq. (2) for the limiting case $\epsilon \to 0$ by which a phase boundary is described. For details see ref. [9]. The J-integral then becomes:

$$J = \int_\Gamma \left(W \, \mathrm{d}x_2 - T_i \frac{\partial u_i}{\partial x_1} \mathrm{d}\Gamma \right) - \int_{\Gamma_\mathrm{p}} \left(W \, \mathrm{d}x_2 - T_i \frac{\partial u_i}{\partial x_1} \mathrm{d}\Gamma \right) \tag{4}$$

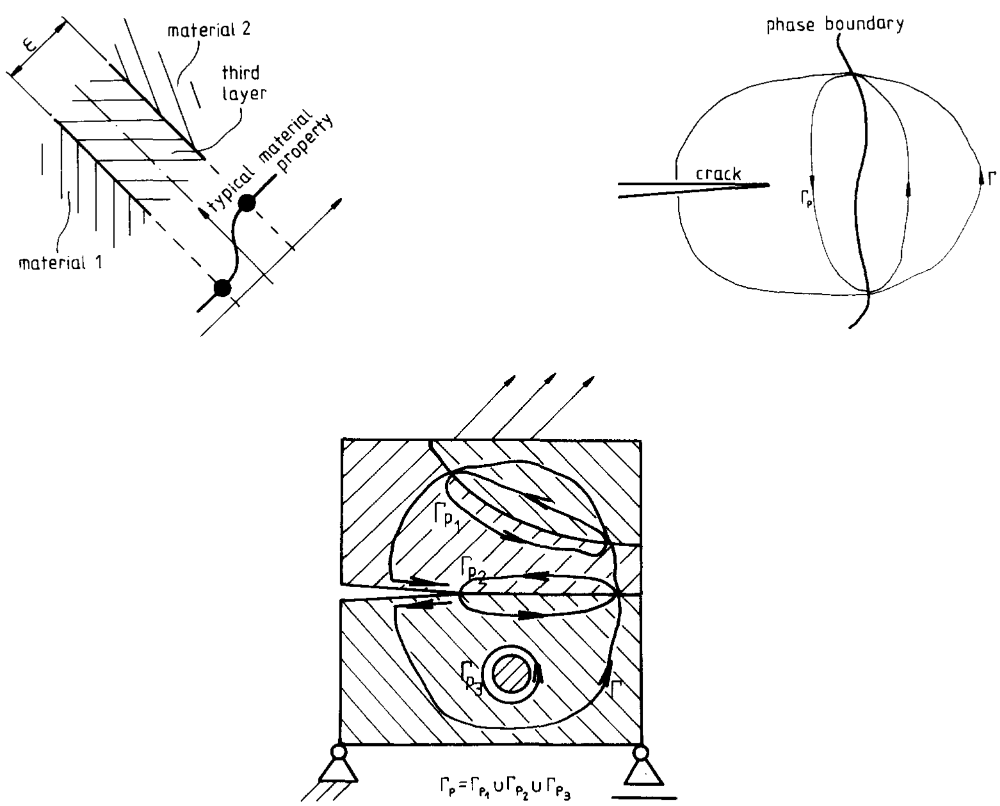

Fig. 2. (a) Interface of dissimilar materials replaced by a thin layer. (b) Path Γ_p. (c) Contour Γ and Γ_p.

where Γ_p is a path embracing that part of phase boundary which is enclosed by Γ (fig. 2(b)).

2.2. *Derivation of both components of the 2D J-integral vector for multi-phase materials regarding a moving crack tip*

There exist several concepts dealing with dynamically propagating cracks (e.g. [10–13]). The corresponding dynamic energy-release rates and path-independent integrals can be derived by an over-all energy-time rate balance regarding a control volume with time dependent boundaries similar to balances in thermo- and fluiddynamics. By this way the second component J_2 of the known 2D J-integral vector can be introduced very elegantly when neglecting dynamic effects in the resulting equations. In the following, J according to Rice is called J_1 to indicate J being the first component of the vector.

The extension of the concept to inhomogeneity can be done in the same way as before:

1) We realize that the strain energy density additionally depends on the spatial coordinates;
2) We replace the phase boundary by a thin layer and evaluate the terms associated with inhomogeneity for vanishing thickness of the layer.

Following this line, the two components of the 2D J-integral vector for multi-phase materials are obtained:

$$J_1 = \int_\Gamma \left(W\, \mathrm{d}x_2 - T_i \frac{\partial u_i}{\partial x_1} \mathrm{d}\Gamma \right) - \int_{\Gamma_p} \left(W\, \mathrm{d}x_2 - T_i \frac{\partial u_i}{\partial x_1} \mathrm{d}\Gamma \right), \tag{5}$$

$$J_2 = \int_\Gamma \left(-W\, \mathrm{d}x_1 - T_i \frac{\partial u_i}{\partial x_2} \mathrm{d}\Gamma \right) - \int_{\Gamma_p} \left(-W\, \mathrm{d}x_1 - T_i \frac{\partial u_i}{\partial x_2} \mathrm{d}\Gamma \right). \tag{6}$$

For the evaluation of J_2, Γ has to start and end at opposite points of the flat crack surfaces *at the crack tip*. Otherwise, J_2 is not path independent.

2.3. *Eshelby's derivation of the force on defects from the viewpoint of solid state physics*

It is trivial to extend the J-integral vector to phase boundaries regarding Eshelby's derivation [2], and there is no mathematical operation necessary to do this.

Eshelby considers two bodies – an original and its exact replica – containing defects. By an imaginary redistribution of material he has derived an expression which has to be evaluated along a contour Γ for the computation of the energy-release rate associated with a propagation of *all defects inside* Γ. Eshelby's expression is totally identical with the J-integral vector.

To extend the J_k-integral to multi-phase materials, we define

defect 1 := crack tip, and

defect 2 := phase boundary.

Then, a path Γ including both defects delivers the energy-release rate associated with a translation of the crack tip plus the phase boundary. So, we have to subtract from the original J_k evaluated along Γ the energy-release rate associated with a propagation of the phase boundary, which can be calculated by evaluating original J_k along Γ_p (fig. 2(b)), to obtain the energy-release rate associated only with a propagation of the crack. This is exactly the extension of the J-integral vector to multi-phase materials we have described before and is independently presented in ref. [17] without derivation of the theoretical results.

In fig. 2(c) we see the example of a structure and the paths Γ and Γ_p.

3. Numerical examples

Our aim is to illustrate that the J-integral vector retains its two important features, its path-independency and its physical interpretation of an energy-release rate, in numerical calculations concerning multi-phase materials when taking the additional terms into account.

For the evaluation of stresses and strains we use a finite element code for two-dimensional problems described in ref. [14]. After the calculation of the stresses and strains a subroutine is called which computes the J-integral vector using a numerical integration process.

We consider two different problems: a plate containing a crack and a circular inclusion in front of the crack-tip (fig. 3(a)) and a plate with a branched interface crack (fig. 3(h)). All our computations are restricted to plane strain situations and are in the range of linear elasticity. Therefore, we can compare the J-integral (extended or non-extended) with the energy-release rate $\mathscr{G}$ evaluated analytically, e.g..

Plate containing a crack and a circular inclusion

The J_k evaluation is carried out for various paths. Some paths contain no phase boundary, others cross the inclusion and there are also contour paths which contain the inclusion completely (fig. 3(a)).

For pure tension the extended J_1- and Rice's

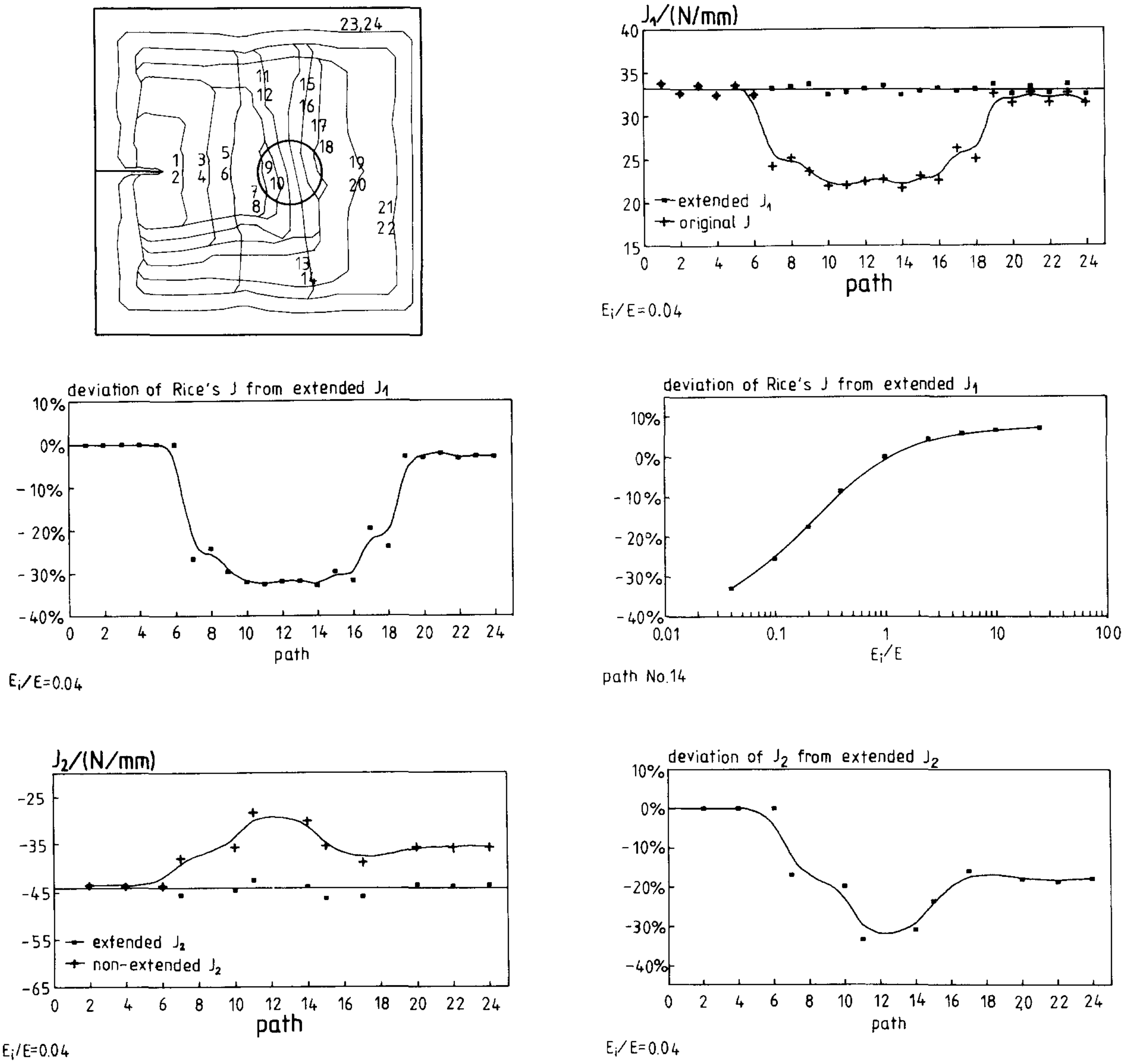

Fig. 3. (a) Plate containing a crack and an inclusion: nonextended contour paths Γ. (b). (c). (d). (e). (f).

original J-integral were evaluated along all paths and typical results are shown in fig. 3(b), which demonstrates the path-independency of the extended J_1. As expected, Rice's J and the extended J_1 agree for path 1–6, because these paths do not include any phase boundary. However, regarding path 7–18 which cross the inclusion, Rice's J deviates extremely. The maximal absolute value of the percentage deviation of Rice's J from extended J_1 is higher than 30% (fig. 3(c)). This deviation decreases for an evaluation along paths containing the inclusion completely and is about -2% or -3%. Such small values can be neglected taking into account the errors caused by the numerical process.

Figure 3(d) shows the influence of the material mismatch. The difference between extended and non-extended J_1 increases, of course, for increasing differences in the Young's moduli of the bulk material (E) and the inclusion (E_i). Furthermore, we see that the potential danger of the crack would be under-estimated by Rice's J considering a soft inclusion in a hard matrix and would be over-estimated for a hard inclusion in a softer matrix.

Typical elastic finite element solutions in case of mixed-mode loading are shown in figs. 3(e)–(g). Figure 3(e) demonstrates that the path-independency of the extended J_2 is quite good, whereas the non-extended J_2 deviates from the extended J_2 considering paths which contain the complete phase boundary or only parts of it. The maximal absolute value of the percentage deviation is higher than 30% (fig. 3(f)).

Solutions obtained from various Young's moduli ratios E_i/E are shown in fig. 3(g). It is not unexpected that as the difference in the material properties increases, so does the difference be-

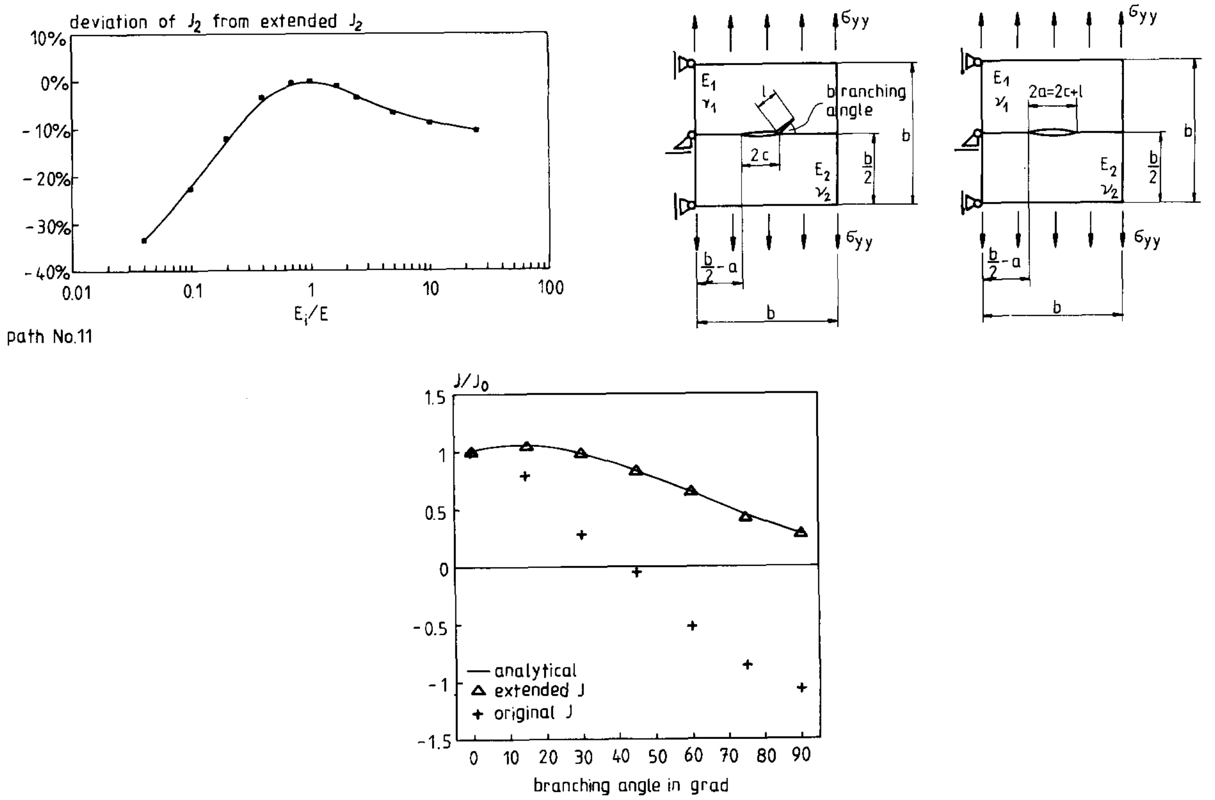

Fig. 3 (continued). (g). (h) Branching interface crack. On the left the evaluation of J, on the right the evaluation of reference value J_0. (i).

tween extended and non-extended J_2. But here, the original J_2 is lower than the extended J_2 for all ratios E_i/E, so that we cannot predict in general, whether the original J_k-integral underestimates or over-estimates the danger of a crack in a multiphase material.

A branched interface crack

An "analytical" solution for the problem of a finite length crack branching of the interface between two bonded dissimilar isotropic materials is presented in ref. [15]. The solution is obtained by a Green's function technique based on distributing dislocation singularities along the branch. Results are presented in terms of the ratio of the energy-release rate of a branched interface crack to the energy-release rate of a straight interface crack with the same total length. Since we admit only linear elastic deformations, our results from a finite element analysis and an evaluation of the extended J-integral (first component of the vector) can be compared directly with the energy-release rate presented in ref. [15] expecting the influence of the chosen finite dimensions of the plate (fig. 3(h) to be negligible [9].

The structures and the loading for which we have evaluated the first component of the J-integral vector (extended and non-extended) – called simply J and not J_1 in the following diagrams – are depicted in fig. 3(h). In fig. 3(i) the finite element solution is shown in comparison with the "analytical" solution presented in ref. [15] ($b = 500$ mm, $2a = 50$ mm, $1/c = 0.1$, $E_1 = 2.1 \times 10^5$ MPa, $E_2/E_1 = 3$, $\nu_1 = \nu_2 = 0.3$, $\sigma_{yy} = 420$ MPa). We have obtained similar diagrams for various length ratios $1/c$ and Young's moduli ratios E_2/E_1. The evaluation of the extended J gives again quite good results which agree with the work of D.J. Mukai et al. [15]. On the other hand, to evaluate original J disregarding the additional terms for multi-phase materials is not recommendable, because, then, a lot of J values are negative. Furthermore, we can gather from fig. 3(i) that the extended J agrees with the original J in case of a non-branched interface crack (branching angle = 0). This is corroborated by Park and Earmme [16] who found Rice's J to be also valid for a straight interface crack, for which the additional term of the extended J_1 vanishes.

4. Concluding remarks

The extension of the well-known two-dimensional J-integral vector to multi-phase materials was presented, and, results from numerical calculations have illustrated the path-independency and the physical meaning of the extended J-integral vector in case of inhomogeneity.

The extended J_k-integral not only retains the path-independency and the physical meaning of the original J_k; also the same restrictions are valid. The most important concerns the material behaviour. Strictly speaking, extended J_k and original J_k are applicable only to elastic situations, because J_k changes its physical meaning for incremental flow theory of plasticity, e.g. However, for small scale yielding the extended J_k-integral is applicable just as the original J_k.

The complete fracture criterion for multi-phase materials consists of a comparison of the extended J-integral to the material resistance of the phase containing the crack-tip analogously to the homogeneous case.

At last, we want to discuss the question whether the introduction of the additional terms is a practicable way to take phase boundaries into account in numerical computations. The extension allows to perform calculations in regions where phase boundaries have to be crossed. However, regarding phase boundaries far away from the crack-tip, we can choose a path Γ which contains no phase boundary so that there is no need to evaluate the additional terms. Another disadvantage is that one has to take the inhomogeneities explicitly into account by the additional path Γ_p. This is very uncomfortable when imagining materials with many defects or phase boundaries spread over the whole domain.

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