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Micromechanics of Composites

ME EN 7540 **Handout 1: Basics** **Fall 2006**

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1 Introduction

The main aim of micromechanics is to find the properties of a continuum material point based on the microstructure. We assume that the microstructure can be described as a continuum even though it is heterogeneous and may contain voids.

The properties at a material point are computed using an averaging procedure. Many such averaging procedures exist, some of which assume the existence of a *representative volume element (RVE)*. A RVE is usually chosen such that it represents adequately the local microstructure of the macroscopic scale continuum (in a statistical sense). [1]

We follow the following principle stated by Hill (1972):
"Experimental determinations of mechanical behaviour rest ultimately on measured loads or mean displacements over pairs of opposite faces of a representative cube. **Macro-variables intended for constitutive laws should thus be capable of definition in terms of surface data alone**, *either directly or indirectly. It is not necessary, by any means, that macro-variables so defined should be unweighted volume averages of their macroscopic counterparts. However, variables that do have this special property are naturally the easiest to handle analytically in the transition between levels. Accordingly, we approach the construction of macro-variables by first identifying some relevant averages that depend uniquely on surface data."*

In this handout, the main focus is on defining average quantities that may be used to describe the macroscopic constitutive behavior of composites. The average quantities are then shown to depend solely on boundary data. Both infinitesimal and finite deformations are considered.

2 Governing Equations

The equations that govern the motion of a solid include the balance laws for mass, momentum, and energy. Kinematic equations and constitutive relations are needed to complete the system of equations. Physical restrictions on the form of the constitutive relations are imposed by an entropy inequality that expresses the second law of thermodynamics in mathematical form.

The balance laws express the idea that the rate of change of a quantity (mass, momentum, energy) in a volume must arise from three causes:

1. the physical quantity itself flows through the surface that bounds the volume,
2. there is a source of the physical quantity on the surface of the volume, or/and,
3. there is a source of the physical quantity inside the volume.

Let Ω be an RVE and let $\partial\Omega$ be the surface of the RVE. Let the position vector of a point in space be given by $\mathbf{x}$. We can write

$$\mathbf{x} = x_1\,\mathbf{e}_1 + x_2\,\mathbf{e}_2 + x_3\,\mathbf{e}_3 \qquad \Longrightarrow \qquad \mathbf{x} = x_i\,\mathbf{e}_i \tag{1}$$

where $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ are the unit vectors that define a Cartesian coordinate system.

Let $f(\mathbf{x}, t)$ be the physical quantity that is flowing through the RVE. Let $g(\mathbf{x}, t)$ be sources on the surface of the RVE and let $h(\mathbf{x}, t)$ be sources inside the RVE. Let $\mathbf{n}(\mathbf{x}, t)$ be the outward unit normal to the surface $\partial\Omega$. Let $\mathbf{v}(\mathbf{x}, t)$ be the velocity of the physical particles that carry the physical quantity that is flowing. Also, let the speed at which the bounding surface $\partial\Omega$ is moving be u_n (in the direction $\mathbf{n}$).

Then, balance laws can be expressed in the general form

$$\frac{d}{dt}\left[\int_\Omega f(\mathbf{x}, t)\;\mathrm{dV}\right] = \int_{\partial\Omega} f(\mathbf{x}, t)[u_n(\mathbf{x}, t) - \mathbf{v}(\mathbf{x}, t)\cdot\mathbf{n}(\mathbf{x}, t)]\;\mathrm{dA} + \int_{\partial\Omega} g(\mathbf{x}, t)\;\mathrm{dA} + \int_\Omega h(\mathbf{x}, t)\;\mathrm{dV}\,. \tag{2}$$

Note that the functions $f(\mathbf{x}, t)$, $g(\mathbf{x}, t)$, and $h(\mathbf{x}, t)$ can be scalar valued, vector valued, or tensor valued - depending on the physical quantity that the balance equation deals with.

[1] See the introductory chapters from the following for a detailed description of a RVE: S. Nemat-Nasser and M. Hori, 1993, *Micromechanics: Overall Properties of Heterogeneous Materials*, North-Holland ; G. W. Milton, 2002, *The Theory of Composites*, Cambridge University Press ; S. Torquato, 2002, *Random Heterogeneous Materials*, Springer.

It can be shown that the balance laws of mass, momentum, and energy can be written as (see Appendix):

$$\begin{aligned}
\dot{\rho} + \rho\, \boldsymbol{\nabla} \cdot \mathbf{v} &= 0 && \text{Balance of Mass} \\
\rho\, \dot{\mathbf{v}} - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} - \rho\, \mathbf{b} &= 0 && \text{Balance of Linear Momentum} \\
\boldsymbol{\sigma} &= \boldsymbol{\sigma}^T && \text{Balance of Angular Momentum} \\
\rho\, \dot{e} - \boldsymbol{\sigma} : (\boldsymbol{\nabla}\mathbf{v}) + \boldsymbol{\nabla} \cdot \mathbf{q} - \rho\, s &= 0 && \text{Balance of Energy.}
\end{aligned} \tag{3}$$

In the above equations $\rho(\mathbf{x}, t)$ is the mass density (current), $\dot{\rho}$ is the material time derivative of ρ, $\mathbf{v}(\mathbf{x}, t)$ is the particle velocity, $\dot{\mathbf{v}}$ is the material time derivative of $\mathbf{v}$, $\boldsymbol{\sigma}(\mathbf{x}, t)$ is the Cauchy stress tensor, $\mathbf{b}(\mathbf{x}, t)$ is the body force density, $e(\mathbf{x}, t)$ is the internal energy per unit mass, $\dot{e}$ is the material time derivative of e, $\mathbf{q}(\mathbf{x}, t)$ is the heat flux vector, and $s(\mathbf{x}, t)$ is an energy source per unit mass.

The gradient and divergence operators are defined such that

$$\boldsymbol{\nabla}\mathbf{v} = \sum_{i,j=1}^{3} \frac{\partial v_i}{\partial x_j} \mathbf{e}_i \otimes \mathbf{e}_j = v_{i,j} \mathbf{e}_i \otimes \mathbf{e}_j \; ; \;\; \boldsymbol{\nabla} \cdot \mathbf{v} = \sum_{i=1}^{3} \frac{\partial v_i}{\partial x_i} = v_{i,i} \; ; \;\; \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = \sum_{i,j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j}\, \mathbf{e}_i = \sigma_{ij,j}\, \mathbf{e}_i \; . \tag{4}$$

The contraction operation is given as

$$\boldsymbol{A} : \boldsymbol{B} = \sum_{i,j=1}^{3} A_{ij}\, B_{ij} = A_{ij}\, B_{ij} \; . \tag{5}$$

3 Infinitesimal Deformations

Let us first consider only infinitesimal strains and linear elasticity. If we assume that the RVE is small enough, we can neglect inertial and body forces. In addition, if the RVE is in equilibrium, conservation of mass automatically holds. Then the equations that govern the motion of the RVE can be written as:

$$\boxed{\begin{aligned}
\boldsymbol{\varepsilon} &= \frac{1}{2}(\boldsymbol{\nabla}\mathbf{u} + \boldsymbol{\nabla}\mathbf{u}^T) && \text{Strain-Displacement Relations} \\
\boldsymbol{\sigma} &= \mathsf{C} : \boldsymbol{\varepsilon} && \text{Stress-Strain Relations} \\
\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} &= 0 && \text{Balance of Linear Momentum} \\
\boldsymbol{\sigma} &= \boldsymbol{\sigma}^T && \text{Balance of Angular Momentum} \\
\rho\, \dot{e} &= \boldsymbol{\sigma} : (\boldsymbol{\nabla}\mathbf{v}) - \boldsymbol{\nabla} \cdot \mathbf{q} + \rho\, s && \text{Balance of Energy.}
\end{aligned}} \tag{6}$$

In the above equations, $\boldsymbol{\varepsilon}(\mathbf{x})$ is the strain tensor (small strain), $\mathbf{u}(\mathbf{x})$ is the displacement vector, and $\mathsf{C}(\mathbf{x})$ is the fourth-order tensor of elastic moduli at the point $\mathbf{x} \in \Omega$.

To get a unique solution of the governing equations, we need boundary conditions on $\partial\Omega$. These boundary conditions may be in the form of applied *tractions*:

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \bar{\mathbf{t}} \; ; \qquad \sigma_{ij}\, n_j = \bar{t}_i \tag{7}$$

where $\mathbf{n}(\mathbf{x})$ is the outward normal vector to the surface $\partial\Omega$ and $\bar{\mathbf{t}}(\mathbf{x})$ is the applied traction.

Alternatively, the boundary conditions may be in the form of applied *displacements*:

$$\mathbf{u} = \bar{\mathbf{u}} \; ; \qquad u_i = \bar{u}_i \tag{8}$$

where $\bar{\mathbf{u}}(\mathbf{x})$ is the applied displacement.

We usually assume that the portions of the boundary on which tractions and displacements are applied are nonoverlapping, i.e., $\partial\Omega = \partial\Omega_t \cup \partial\Omega_u$ and $\partial\Omega_t \cap \partial\Omega_u = \varnothing$.

If we need to solve the energy equation, we also have to specify heat flux or specified temperature boundary conditions on the RVE.

3.1 Average Strain in a RVE

The average strain tensor is defined as

$$\boxed{\langle \boldsymbol{\varepsilon} \rangle := \frac{1}{2}\left(\langle \boldsymbol{\nabla}\mathbf{u} \rangle + \langle \boldsymbol{\nabla}\mathbf{u} \rangle^T\right)} \tag{9}$$

where the average displacement gradient is

$$\boxed{\langle \boldsymbol{\nabla}\mathbf{u} \rangle = \frac{1}{V}\int_{\Omega} \boldsymbol{\nabla}\mathbf{u}\,\mathrm{dV}\,.} \tag{10}$$

We would like to find the relation between the average strain in a RVE and the applied displacements at the boundary of the RVE. To do that, recall the relation (see Appendix)

$$\int_{\Omega} \boldsymbol{\nabla}\mathbf{v}\,\mathrm{dV} = \int_{\partial\Omega} \mathbf{v} \otimes \mathbf{n}\,\mathrm{dA}$$

where $\mathbf{v}$ is a vector field on Ω and $\mathbf{n}$ is the normal to $\partial\Omega$. Using this relation, we get

$$\langle \boldsymbol{\nabla}\mathbf{u} \rangle = \frac{1}{V}\int_{\partial\Omega} \mathbf{u} \otimes \mathbf{n}\,\mathrm{dA}\,. \tag{11}$$

Hence,

$$\langle \boldsymbol{\nabla}\mathbf{u} \rangle^T = \frac{1}{V}\int_{\partial\Omega} \mathbf{n} \otimes \mathbf{u}\,\mathrm{dA}\,.$$

Plugging these into the definition of average strain, we get

$$\boxed{\langle \boldsymbol{\varepsilon} \rangle := \frac{1}{2V}\int_{\partial\Omega} (\mathbf{u} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{u})\,\mathrm{dA}\,.} \tag{12}$$

This implies that *the average strain is completely defined in terms of the applied displacements at the boundary!* Also, the average strain tensor is *symmetric* by virtue of its definition.

We can define the average rotation tensor (which represents an infinitesimal rotation) in an analogous manner. The rotation tensor is given by

$$\boldsymbol{\omega} = \frac{1}{2}(\boldsymbol{\nabla}\mathbf{u} - \boldsymbol{\nabla}\mathbf{u}^T)\,. \tag{13}$$

Therefore, the *average rotation* can be defined as

$$\langle \boldsymbol{\omega} \rangle := \frac{1}{2}\left(\langle \boldsymbol{\nabla}\mathbf{u} \rangle - \langle \boldsymbol{\nabla}\mathbf{u} \rangle^T\right)\,. \tag{14}$$

In terms of the applied boundary displacements,

$$\langle \boldsymbol{\omega} \rangle = \frac{1}{2V}\int_{\partial\Omega} (\mathbf{u} \otimes \mathbf{n} - \mathbf{n} \otimes \mathbf{u})\,\mathrm{dA}\,. \tag{15}$$

3.1.1 The effect of rigid body motions on the average strain

Let us consider a rigid body displacement given by (see Appendix)

$$\mathbf{u}(\mathbf{x}) = \mathbf{c} + \boldsymbol{\omega} \cdot \mathbf{x}$$

where $\mathbf{c}$ is a constant translation and $\boldsymbol{\omega}$ is a second-order skew symmetric tensor representing an infinitesimal rotation. Then,

$$\langle \boldsymbol{\nabla}\mathbf{u} \rangle = \frac{1}{V}\int_{\partial\Omega} \mathbf{u}\otimes\mathbf{n}\,\mathrm{dA} = \frac{1}{V}\int_{\partial\Omega} (\mathbf{c}+\boldsymbol{\omega}\cdot\mathbf{x})\otimes\mathbf{n}\,\mathrm{dA} = \frac{1}{V}\int_{\partial\Omega} \mathbf{c}\otimes\mathbf{n}\,\mathrm{dA} + \frac{1}{V}\int_{\partial\Omega} (\boldsymbol{\omega}\cdot\mathbf{x})\otimes\mathbf{n}\,\mathrm{dA}\,.$$

Recall that

$$(\boldsymbol{A}\cdot\mathbf{b})\otimes\mathbf{c} = \boldsymbol{A}\cdot(\mathbf{b}\otimes\mathbf{c})$$

where $\boldsymbol{A}$ is a second-order tensor and $\mathbf{b}$ and $\mathbf{c}$ are vectors. Therefore,

$$\langle \boldsymbol{\nabla}\mathbf{u} \rangle = \mathbf{c}\otimes\left(\frac{1}{V}\int_{\partial\Omega} \mathbf{n}\,\mathrm{dA}\right) + \boldsymbol{\omega}\cdot\left(\frac{1}{V}\int_{\partial\Omega} \mathbf{x}\otimes\mathbf{n}\,\mathrm{dA}\right)\,.$$

From the divergence theorem,

$$\int_{\Omega} \boldsymbol{\nabla}\cdot\boldsymbol{A}\,\mathrm{dV} = \int_{\partial\Omega} \boldsymbol{A}\cdot\mathbf{n}\,\mathrm{dA}$$

where $\boldsymbol{A}$ is a second-order tensor field and $\mathbf{n}$ is the unit outward normal vector to $\partial\Omega$. Hence,

$$\int_{\partial\Omega} \mathbf{n}\,\mathrm{dA} = \int_{\partial\Omega} \mathbf{1}\cdot\mathbf{n}\,\mathrm{dA} = \int_{\Omega} \boldsymbol{\nabla}\cdot\mathbf{1}\,\mathrm{dV} = \mathbf{0}\,.$$

We also have (see appendix),

$$\int_{\Omega} \boldsymbol{\nabla}\mathbf{v}\,\mathrm{dV} = \int_{\partial\Omega} \mathbf{v}\otimes\mathbf{n}\,\mathrm{dA}$$

where $\mathbf{v}$ is a vector and $\mathbf{n}$ is the unit outward normal to $\partial\Omega$. Therefore,

$$\int_{\partial\Omega} \mathbf{x}\otimes\mathbf{n}\,\mathrm{dA} = \int_{\Omega} \boldsymbol{\nabla}\mathbf{x}\,\mathrm{dV} = \int_{\Omega} \mathbf{1}\,\mathrm{dV} = V\,\mathbf{1}\,.$$

We then have

$$\langle \boldsymbol{\nabla}\mathbf{u} \rangle = \mathbf{c}\otimes\mathbf{0} + \boldsymbol{\omega}\cdot\mathbf{1} = \boldsymbol{\omega}\,.$$

Since $\boldsymbol{\omega}$ is a skew-symmetric second-order tensor we have

$$\boldsymbol{\omega} = -\boldsymbol{\omega}^T\,.$$

Therefore,

$$\langle \boldsymbol{\varepsilon} \rangle = \frac{1}{2}\left(\langle \boldsymbol{\nabla}\mathbf{u} \rangle + \langle \boldsymbol{\nabla}\mathbf{u} \rangle^T\right) = \frac{1}{2}(\boldsymbol{\omega}+\boldsymbol{\omega}^T) = \mathbf{0}\,.$$

Hence, the *average strain is not affected by rigid body motions.* However, for simplicity, we assume that the displacement field in a RVE does not contain any rigid body motions.

3.2 Average Displacement in a RVE

The average displacement in a RVE may be defined as

$$\boxed{\langle \mathbf{u} \rangle := \frac{1}{V}\int_{\Omega} \mathbf{u}(\mathbf{x})\,\mathrm{dV}\,.} \tag{16}$$

We would like to find the relation between the average displacement in a RVE and the applied displacements at the boundary of the RVE. To do that, recall the identity

$$\boldsymbol{\nabla}\cdot(\mathbf{v}\otimes\mathbf{w}) = \mathbf{v}\cdot(\boldsymbol{\nabla}\cdot\mathbf{w}) + (\boldsymbol{\nabla}\mathbf{v})\cdot\mathbf{w}$$

where $\mathbf{v}$ and $\mathbf{w}$ are two vector fields.

If we choose $\mathbf{v}$ such that $\boldsymbol{\nabla}\mathbf{v} = \mathbf{1}$ in the above identity, then we can get an equation for $\mathbf{w}$, i.e.,

$$\boldsymbol{\nabla} \cdot (\mathbf{v} \otimes \mathbf{w}) = \mathbf{v} \cdot (\boldsymbol{\nabla} \cdot \mathbf{w}) + \mathbf{1} \cdot \mathbf{w} = \mathbf{v} \cdot (\boldsymbol{\nabla} \cdot \mathbf{w}) + \mathbf{w} \; .$$

Now, $\boldsymbol{\nabla}\mathbf{v} = \mathbf{1}$ if $\mathbf{v} = \mathbf{x}$. Therefore,

$$\boldsymbol{\nabla} \cdot (\mathbf{x} \otimes \mathbf{w}) = \mathbf{x} \cdot (\boldsymbol{\nabla} \cdot \mathbf{w}) + \mathbf{w} \qquad \Longrightarrow \qquad \mathbf{w} = \boldsymbol{\nabla} \cdot (\mathbf{x} \otimes \mathbf{w}) - \mathbf{x} \cdot (\boldsymbol{\nabla} \cdot \mathbf{w}) \; .$$

Using the above in the expression for the average displacement, we have

$$\langle \mathbf{u} \rangle = \frac{1}{V} \int_{\Omega} [\boldsymbol{\nabla} \cdot (\mathbf{x} \otimes \mathbf{u}) - \mathbf{x} \cdot (\boldsymbol{\nabla} \cdot \mathbf{u})] \, \mathrm{dV} \; .$$

Applying the divergence theorem to the first term on the right, we get

$$\boxed{\langle \mathbf{u} \rangle = \frac{1}{V} \int_{\partial\Omega} (\mathbf{x} \otimes \mathbf{u}) \cdot \mathbf{n} \, \mathrm{dV} - \frac{1}{V} \int_{\Omega} \mathbf{x} \cdot (\boldsymbol{\nabla} \cdot \mathbf{u}) \, \mathrm{dV} \; .}$$

There are two terms in the above expression: the first is a boundary term while the second requires information from the interior of the body. Hence, in general, the *average displacement of a RVE cannot be determined solely on the basis of boundary displacements.*

3.2.1 Incompressible materials

In the material is incompressible, the balance of mass gives us

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0 \; .$$

In that case,

$$\langle \mathbf{u} \rangle = \frac{1}{V} \int_{\partial\Omega} (\mathbf{x} \otimes \mathbf{u}) \cdot \mathbf{n} \, \mathrm{dV} \; .$$

It's only in this special case that the average displacement in the RVE can be expressed in terms of boundary displacements.

3.3 Average Stress in a RVE

Let the average stress in the RVE be defined as

$$\boxed{\langle \boldsymbol{\sigma} \rangle := \frac{1}{V} \int_{\Omega} \boldsymbol{\sigma}(\mathbf{x}) \, \mathrm{dV}} \tag{17}$$

where V is the volume of Ω.

We would like to find out the relation between the average stress in a RVE and the applied tractions on the boundary of the RVE. To do that, recall the relation (see Appendix)

$$\int_{\partial\Omega} \mathbf{v} \otimes (\boldsymbol{S}^T \cdot \mathbf{n}) \, \mathrm{dA} = \int_{\Omega} [\boldsymbol{\nabla}\mathbf{v} \cdot \boldsymbol{S} + \mathbf{v} \otimes (\boldsymbol{\nabla} \cdot \boldsymbol{S}^T)] \, \mathrm{dV} \; . \tag{18}$$

If we choose $\mathbf{v}$ such that $\boldsymbol{\nabla}\mathbf{v} = \mathbf{1}$, we have

$$\int_{\partial\Omega} \mathbf{v} \otimes (\boldsymbol{S}^T \cdot \mathbf{n}) \, \mathrm{dA} = \int_{\Omega} [\mathbf{1} \cdot \boldsymbol{S} + \mathbf{v} \otimes (\boldsymbol{\nabla} \cdot \boldsymbol{S}^T)] \, \mathrm{dV} = \int_{\Omega} \boldsymbol{S} \, \mathrm{dV} + \int_{\Omega} \mathbf{v} \otimes (\boldsymbol{\nabla} \cdot \boldsymbol{S}^T) \, \mathrm{dV} \; .$$

Therefore,

$$\int_\Omega \boldsymbol{S}\ \mathrm{dV} = -\int_\Omega \mathbf{v} \otimes (\boldsymbol{\nabla} \cdot \boldsymbol{S}^T)\ \mathrm{dV} + \int_{\partial\Omega} \mathbf{v} \otimes (\boldsymbol{S}^T \cdot \mathbf{n})\ \mathrm{dA}\ .$$

If we choose $\boldsymbol{S}$ to be the stress tensor $\boldsymbol{\sigma}$, and involve the symmetry of the stress tensor, we get

$$\int_\Omega \boldsymbol{\sigma}\ \mathrm{dV} = -\int_\Omega \mathbf{v} \otimes (\boldsymbol{\nabla} \cdot \boldsymbol{\sigma})\ \mathrm{dV} + \int_{\partial\Omega} \mathbf{v} \otimes (\boldsymbol{\sigma} \cdot \mathbf{n})\ \mathrm{dA}\ . \tag{19}$$

Now, the divergence of the stress is zero (from the conservation of linear momentum). Therefore,

$$\int_\Omega \boldsymbol{\sigma}\ \mathrm{dV} = \int_{\partial\Omega} \mathbf{v} \otimes (\boldsymbol{\sigma} \cdot \mathbf{n})\ \mathrm{dA}\ .$$

Using the traction boundary condition, we have

$$\int_\Omega \boldsymbol{\sigma}\ \mathrm{dV} = \int_{\partial\Omega} \mathbf{v} \otimes \bar{\mathbf{t}}\ \mathrm{dA}\ . \tag{20}$$

Now $\boldsymbol{\nabla}\mathbf{v} = \boldsymbol{1}$ if $\mathbf{v} = \mathbf{x}$. Therefore, we have

$$\int_\Omega \boldsymbol{\sigma}\ \mathrm{dV} = \int_{\partial\Omega} \mathbf{x} \otimes \bar{\mathbf{t}}\ \mathrm{dA}\ . \tag{21}$$

Hence the average stress is given by

$$\boxed{\langle\boldsymbol{\sigma}\rangle := \frac{1}{V}\int_{\partial\Omega} \mathbf{x} \otimes \bar{\mathbf{t}}\ \mathrm{dA}\ .} \tag{22}$$

This implies that *the average stress is completely determined by the applied tractions!*

3.3.1 Symmetry of the average stress and the effect of rigid body translation

Let us now assume that the applied tractions are *self equilibrating*. Then the resultant forces and moments due to the applied tractions vanish and we have

$$\int_{\partial\Omega} \bar{\mathbf{t}}\ \mathrm{dA} = \boldsymbol{0} \qquad \text{and} \qquad \int_{\partial\Omega} \mathbf{x} \times \bar{\mathbf{t}}\ \mathrm{dA} = \boldsymbol{0}\ . \tag{23}$$

From the moment balance equation above we can show that (see Appendix)

$$\int_{\partial\Omega} \mathbf{x} \otimes \bar{\mathbf{t}}\ \mathrm{dA} = \int_{\partial\Omega} \bar{\mathbf{t}} \otimes \mathbf{x}\ \mathrm{dA}\ . \tag{24}$$

Therefore the average stress tensor $\langle\boldsymbol{\sigma}\rangle$ is *symmetric* if the applied tractions are *self equilibrated*.

Now, if we translate the body by a constant amount $\mathbf{u}_0$ (rigid body translation), we get

$$\langle\bar{\boldsymbol{\sigma}}\rangle = \frac{1}{V}\int_{\partial\Omega} (\mathbf{x} + \mathbf{u}_0) \otimes \bar{\mathbf{t}}\ \mathrm{dA} = \frac{1}{V}\int_{\partial\Omega} [\mathbf{x} \otimes \bar{\mathbf{t}} + \mathbf{u}_0 \otimes \bar{\mathbf{t}}]\ \mathrm{dA}\ .$$

or

$$\langle\bar{\boldsymbol{\sigma}}\rangle = \frac{1}{V}\left[\int_{\partial\Omega} \mathbf{x} \otimes \bar{\mathbf{t}}\ \mathrm{dA} + \mathbf{u}_0 \otimes \cancelto{0}{\int_{\partial\Omega} \bar{\mathbf{t}}\ \mathrm{dA}}\right] = \langle\boldsymbol{\sigma}\rangle$$

Therefore, the average stress is *not affected by a rigid body translation* only if the applied tractions are *self equilibrated*.

We can conclude that **the average stress $\langle\boldsymbol{\sigma}\rangle$ is an acceptable measure of stress in a RVE if the applied tractions are self equilibrated.**

3.4 Average Stress Power in a RVE

The equation for the balance of energy is

$$\rho\,\dot{e} - \boldsymbol{\sigma} : (\boldsymbol{\nabla}\mathbf{v}) + \boldsymbol{\nabla}\cdot\mathbf{q} - \rho\,s = 0\,. \tag{25}$$

If the absence of heat flux or heat sources in the RVE, the equation reduces to

$$\rho\,\dot{e} = \boldsymbol{\sigma} : (\boldsymbol{\nabla}\mathbf{v})\,. \tag{26}$$

The quantity on the right is the stress power density and is a measure of the internal energy density of the material.

The average stress power in a RVE is defined as

$$\boxed{\langle\boldsymbol{\sigma} : \boldsymbol{\nabla}\mathbf{v}\rangle := \frac{1}{V}\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\nabla}\mathbf{v}\,\mathrm{dV}\,.} \tag{27}$$

Note that the quantities $\boldsymbol{\sigma}$ and $\boldsymbol{\nabla}\mathbf{v}$ need not be related in the general case.

The average velocity gradient $\langle\boldsymbol{\nabla}\mathbf{v}\rangle$ is defined as

$$\boxed{\langle\boldsymbol{\nabla}\mathbf{v}\rangle := \frac{1}{V}\int_\Omega \boldsymbol{\nabla}\mathbf{v}\,\mathrm{dV}\,.} \tag{28}$$

To get an expression for the average stress power in terms of the boundary conditions, we use the identity

$$\boldsymbol{\nabla}\cdot(\boldsymbol{S}^T\cdot\mathbf{v}) = \boldsymbol{S} : \boldsymbol{\nabla}\mathbf{v} + (\boldsymbol{\nabla}\cdot\boldsymbol{S})\cdot\mathbf{v}$$

to get

$$\langle\boldsymbol{\sigma} : \boldsymbol{\nabla}\mathbf{v}\rangle = \frac{1}{V}\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\nabla}\mathbf{v}\,\mathrm{dV} = \frac{1}{V}\int_\Omega \left[\boldsymbol{\nabla}\cdot(\boldsymbol{\sigma}^T\cdot\mathbf{v}) - (\boldsymbol{\nabla}\cdot\boldsymbol{\sigma})\cdot\mathbf{v}\right]\,\mathrm{dV}\,.$$

Using the balance of linear momentum ($\boldsymbol{\nabla}\cdot\boldsymbol{\sigma} = 0$), we get

$$\langle\boldsymbol{\sigma} : \boldsymbol{\nabla}\mathbf{v}\rangle = \frac{1}{V}\int_\Omega \boldsymbol{\nabla}\cdot(\boldsymbol{\sigma}^T\cdot\mathbf{v})\,\mathrm{dV}\,.$$

Using the divergence theorem, we have

$$\langle\boldsymbol{\sigma} : \boldsymbol{\nabla}\mathbf{v}\rangle = \frac{1}{V}\int_{\partial\Omega} (\boldsymbol{\sigma}^T\cdot\mathbf{v})\cdot\mathbf{n}\,\mathrm{dV} = \frac{1}{V}\int_{\partial\Omega} (\boldsymbol{\sigma}^T\cdot\mathbf{v})\cdot\mathbf{n}\,\mathrm{dV} = \frac{1}{V}\int_{\partial\Omega} (\boldsymbol{\sigma}\cdot\mathbf{n})\cdot\mathbf{v}\,\mathrm{dV}\,.$$

Now, the surface traction is given by $\bar{\mathbf{t}} = \boldsymbol{\sigma}\cdot\mathbf{n}$. Therefore,

$$\boxed{\langle\boldsymbol{\sigma} : \boldsymbol{\nabla}\mathbf{v}\rangle = \frac{1}{V}\int_{\partial\Omega} \bar{\mathbf{t}}\cdot\mathbf{v}\,\mathrm{dV}\,.} \tag{29}$$

In micromechanics, it is of interest to see how the average stress power of a RVE is related to the product of the average stress $\langle\boldsymbol{\sigma}\rangle$ and the average velocity gradient $\langle\boldsymbol{\nabla}\mathbf{v}\rangle$. While homogenizing a RVE, we would ideally like to have

$$\langle\boldsymbol{\sigma} : \boldsymbol{\nabla}\mathbf{v}\rangle = \langle\boldsymbol{\sigma}\rangle : \langle\boldsymbol{\nabla}\mathbf{v}\rangle\,.$$

However, this is not true in general. We can show that **if the gradient of the velocity is a symmetric tensor (i.e., there is no spin)**, then (see Appendix for proof)

$$\boxed{\langle\boldsymbol{\sigma} : \boldsymbol{\nabla}\mathbf{v}\rangle - \langle\boldsymbol{\sigma}\rangle : \langle\boldsymbol{\nabla}\mathbf{v}\rangle = \frac{1}{V}\int_{\partial\Omega} [\mathbf{v} - \langle\boldsymbol{\nabla}\mathbf{v}\rangle\cdot\mathbf{x}]\cdot[(\boldsymbol{\sigma} - \langle\boldsymbol{\sigma}\rangle)\cdot\mathbf{n}]\,\mathrm{dA}\,.} \tag{30}$$

We can arrive at $\langle\boldsymbol{\sigma} : \boldsymbol{\nabla}\mathbf{v}\rangle = \langle\boldsymbol{\sigma}\rangle : \langle\boldsymbol{\nabla}\mathbf{v}\rangle$ if either of the following conditions is met on the boundary $\partial\Omega$:

1. $\mathbf{v} = \langle \boldsymbol{\nabla} \mathbf{v} \rangle \cdot \mathbf{x}$.
2. $\boldsymbol{\sigma} \cdot \mathbf{n} = \langle \boldsymbol{\sigma} \rangle \cdot \mathbf{n}$.

1. **Linear boundary velocities:**
   If the prescribed velocities on $\partial\Omega$ are a linear function of $\mathbf{x}$, then we can write

$$\mathbf{v}(\mathbf{x}) = \boldsymbol{H} \cdot \mathbf{x} \qquad\qquad \forall\, \mathbf{x} \in \partial\Omega \tag{31}$$

   where $\boldsymbol{H}$ is a constant second-order tensor.

   From the divergence theorem

$$\int_\Omega \boldsymbol{\nabla}\mathbf{a}\, \mathrm{dV} = \int_{\partial\Omega} \mathbf{a} \otimes \mathbf{n}\, \mathrm{dA} \, .$$

   Therefore,

$$\langle \boldsymbol{\nabla}\mathbf{v} \rangle = \frac{1}{V}\int_\Omega \boldsymbol{\nabla}\mathbf{v}\, \mathrm{dV} = \frac{1}{V}\int_{\partial\Omega} \mathbf{v} \otimes \mathbf{n}\, \mathrm{dA} \, .$$

   Hence, on the boundary

$$\mathbf{v} - \langle \boldsymbol{\nabla}\mathbf{v} \rangle \cdot \mathbf{x} = \boldsymbol{H} \cdot \mathbf{x} - \left[\frac{1}{V}\int_{\partial\Omega} (\boldsymbol{F} \cdot \mathbf{x}) \otimes \mathbf{n}\, \mathrm{dA}\right] \cdot \mathbf{x}$$

   Using the identity (see Appendix)

$$(\boldsymbol{A} \cdot \mathbf{a}) \otimes \mathbf{b} = \boldsymbol{A} \cdot (\mathbf{a} \otimes \mathbf{b})$$

   and since $\boldsymbol{F}$ is constant, we get

$$\mathbf{v} - \langle \boldsymbol{\nabla}\mathbf{v} \rangle^T \cdot \mathbf{x} = \boldsymbol{H} \cdot \mathbf{x} - \left[\boldsymbol{F} \cdot \left(\frac{1}{V}\int_{\partial\Omega} \mathbf{x} \otimes \mathbf{n}\, \mathrm{dA}\right)\right] \cdot \mathbf{x} \, .$$

   From the divergence theorem,

$$\int_{\partial\Omega} \mathbf{x} \otimes \mathbf{n}\, \mathrm{dA} = \int_\Omega \boldsymbol{\nabla}\mathbf{x}\, \mathrm{dV} = \int_\Omega \mathbf{1}\, \mathrm{dV} = V\, \mathbf{1} \, .$$

   Therefore,

$$\mathbf{v} - \langle \boldsymbol{\nabla}\mathbf{v} \rangle \cdot \mathbf{x} = \boldsymbol{H} \cdot \mathbf{x} - (\boldsymbol{H} \cdot \mathbf{1}) \cdot \mathbf{x} = \boldsymbol{H} \cdot \mathbf{x} - \boldsymbol{H} \cdot \mathbf{x} = \mathbf{0} \qquad \Longrightarrow \qquad \boxed{\langle \boldsymbol{\sigma} : \boldsymbol{\nabla}\mathbf{v} \rangle = \langle \boldsymbol{\sigma} \rangle : \langle \boldsymbol{\nabla}\mathbf{v} \rangle} \, .$$

2. **Uniform boundary tractions:**
   If the prescribed tractions on the boundary $\partial\Omega$ are uniform, they can be expressed in terms of a constant symmetric second-order tensor $\boldsymbol{H}$ through the relation

$$\bar{t}(\mathbf{x}) = \boldsymbol{H} \cdot \mathbf{n}(\mathbf{x}) \qquad\qquad \forall\, \mathbf{x} \in \partial\Omega \, .$$

   The tractions are related to the stresses at the boundary of the RVE by $\bar{t} = \boldsymbol{\sigma} \cdot \mathbf{n}$.

   The average stress in the RVE is given by

$$\langle \boldsymbol{\sigma} \rangle = \frac{1}{V}\int_{\partial\Omega} \mathbf{x} \otimes \bar{t}\, \mathrm{dA} = \frac{1}{V}\int_{\partial\Omega} \mathbf{x} \otimes (\boldsymbol{H} \cdot \mathbf{n})\, \mathrm{dA} \, .$$

   Using the identity $\mathbf{a} \otimes (\boldsymbol{A} \cdot \mathbf{b}) = (\mathbf{a} \otimes \mathbf{b}) \cdot \boldsymbol{A}^T$ (see Appendix), we have

$$\langle \boldsymbol{\sigma} \rangle = \frac{1}{V}\int_{\partial\Omega} (\mathbf{x} \otimes \mathbf{n}) \cdot \boldsymbol{H}^T\, \mathrm{dA} \, .$$

Since $\boldsymbol{H}$ is constant and symmetric, we have

$$\langle\boldsymbol{\sigma}\rangle = \left(\frac{1}{V}\int_{\partial\Omega} \mathbf{x}\otimes\mathbf{n}\,\mathrm{dA}\right)\cdot\boldsymbol{H}\,.$$

Applying the divergence theorem,

$$\langle\boldsymbol{\sigma}\rangle = \left(\frac{1}{V}\int_{\Omega} \boldsymbol{\nabla}\mathbf{x}\,\mathrm{dV}\right)\cdot\boldsymbol{H} = \mathbf{1}\cdot\boldsymbol{H} = \boldsymbol{H}\,.$$

Therefore,

$$\boldsymbol{\sigma}\cdot\mathbf{n} - \langle\boldsymbol{\sigma}\rangle\cdot\mathbf{n} = \bar{t} - \boldsymbol{H}\cdot\mathbf{n} = \mathbf{0} \qquad \Longrightarrow \qquad \boxed{\langle\boldsymbol{\sigma}:\boldsymbol{\nabla}\mathbf{v}\rangle = \langle\boldsymbol{\sigma}\rangle:\langle\boldsymbol{\nabla}\mathbf{v}\rangle}\,.$$

- **Remark:**
Recall that for small deformations, the displacement gradient $\boldsymbol{\nabla}\mathbf{u}$ can be expressed as

$$\boldsymbol{\nabla}\mathbf{u} = \boldsymbol{\varepsilon} + \boldsymbol{\omega}\,.$$

For small deformations, the time derivative of $\boldsymbol{\nabla}\mathbf{u}$ gives us the velocity gradient $\boldsymbol{\nabla}\mathbf{v}$, i.e.,

$$\boldsymbol{\nabla}\mathbf{v} = \dot{\boldsymbol{\varepsilon}} + \dot{\boldsymbol{\omega}}\,.$$

If $\boldsymbol{\omega} = 0$, we get

$$\boldsymbol{\nabla}\mathbf{v} = \dot{\boldsymbol{\varepsilon}}\,.$$

Hence, for small strains and in the absence of rigid body rotations, the stress power density is given by $\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}$. Then the average stress power is defined as

$$\boxed{\langle\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}\rangle := \frac{1}{V}\int_{\Omega}\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}\,\mathrm{dV}\,.} \tag{32}$$

and the average strain rate is defined as

$$\boxed{\langle\dot{\boldsymbol{\varepsilon}}\rangle := \frac{1}{V}\int_{\Omega}\dot{\boldsymbol{\varepsilon}}\,\mathrm{dV}\,.} \tag{33}$$

In terms of the surface tractions and the applied boundary velocities, we have

$$\boxed{\langle\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}\rangle = \frac{1}{V}\int_{\partial\Omega}\bar{\mathbf{t}}\cdot\dot{\mathbf{u}}\,\mathrm{dV}\,.} \tag{34}$$

For small strains and no rotation, the stress-power difference relation becomes

$$\boxed{\langle\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}\rangle - \langle\boldsymbol{\sigma}\rangle:\langle\dot{\boldsymbol{\varepsilon}}\rangle = \frac{1}{V}\int_{\partial\Omega}[\dot{\mathbf{u}} - \langle\boldsymbol{\nabla}\dot{\mathbf{u}}\rangle\cdot\mathbf{x}]\cdot[(\boldsymbol{\sigma} - \langle\boldsymbol{\sigma}\rangle)\cdot\mathbf{n}]\,\mathrm{dA}\,.} \tag{35}$$

We can arrive at $\langle\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}\rangle = \langle\boldsymbol{\sigma}\rangle:\langle\dot{\boldsymbol{\varepsilon}}\rangle$ if either of the following conditions is met on the boundary $\partial\Omega$:

1. $\dot{\mathbf{u}} = \langle\boldsymbol{\nabla}\dot{\mathbf{u}}\rangle\cdot\mathbf{x}$ $\Longrightarrow$ Linear boundary velocity field.
2. $\boldsymbol{\sigma}\cdot\mathbf{n} = \langle\boldsymbol{\sigma}\rangle\cdot\mathbf{n}$ $\Longrightarrow$ Uniform boundary tractions.

We can also show in an identical manner that

$$\boxed{\langle\boldsymbol{\sigma}:\boldsymbol{\varepsilon}\rangle = \frac{1}{V}\int_{\partial\Omega}\bar{\mathbf{t}}\cdot\mathbf{u}\,\mathrm{dV}\,.} \tag{36}$$

and that, when $\boldsymbol{\nabla}\mathbf{u}$ is symmetric,

$$\boxed{\langle\boldsymbol{\sigma}:\boldsymbol{\varepsilon}\rangle - \langle\boldsymbol{\sigma}\rangle:\langle\boldsymbol{\varepsilon}\rangle = \frac{1}{V}\int_{\partial\Omega}[\mathbf{u} - \langle\boldsymbol{\nabla}\mathbf{u}\rangle\cdot\mathbf{x}]\cdot[(\boldsymbol{\sigma} - \langle\boldsymbol{\sigma}\rangle)\cdot\mathbf{n}]\,\mathrm{dA}\,.} \tag{37}$$

In this case, we can arrive at the relation $\langle\boldsymbol{\sigma}:\boldsymbol{\varepsilon}\rangle = \langle\boldsymbol{\sigma}\rangle:\langle\boldsymbol{\varepsilon}\rangle$ if either of the following conditions is met at the boundary:

1. $\mathbf{u} = \langle\boldsymbol{\nabla}\mathbf{u}\rangle\cdot\mathbf{x}$ $\Longrightarrow$ Linear boundary displacement field.
2. $\boldsymbol{\sigma}\cdot\mathbf{n} = \langle\boldsymbol{\sigma}\rangle\cdot\mathbf{n}$ $\Longrightarrow$ Uniform boundary tractions.

4 Finite Deformations

If a RVE undergoes finite deformations (i.e., large strains and large rotations), then we have to make a distinction between the initial and deformed configuration. Let us assume that the deformation can be described by a map

$$\mathbf{x} = \varphi(\mathbf{X}) = \mathbf{x}(\mathbf{X}) \tag{38}$$

where $\mathbf{X}$ is the position of a point in the RVE in the initial configuration and $\mathbf{x}$ is the location of the same point in the deformed configuration.

The deformation gradient is given by

$$\boldsymbol{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \boldsymbol{\nabla}_0\, \mathbf{x}\,. \tag{39}$$

If we assume that the RVE is small enough, we can neglect inertial and body forces.

Then the equations that govern the motion of the RVE can be written with respect to the reference configuration as

$$\boxed{\begin{array}{ll} \boldsymbol{C} = \boldsymbol{F}^T \cdot \boldsymbol{F} & \text{Strain-deformation Relations} \\ \boldsymbol{P} = \hat{\boldsymbol{P}}(\boldsymbol{C}) & \text{Stress-Strain Relations} \\ \rho\ \det(\boldsymbol{F}) = \rho_0 & \text{Balance of Mass} \\ \boldsymbol{\nabla}_0 \cdot \boldsymbol{P}^T = 0 & \text{Balance of Linear Momentum} \\ \boldsymbol{F} \cdot \boldsymbol{P} = \boldsymbol{P}^T \cdot \boldsymbol{F}^T & \text{Balance of Angular Momentum} \\ \rho_0\ \dot{e} = \boldsymbol{P}^T : \dot{\boldsymbol{F}} - \boldsymbol{\nabla}_0 \cdot \mathbf{q} + \rho_0\ s & \text{Balance of Energy.} \end{array}} \tag{40}$$

In the above $\boldsymbol{C}$ is the right Cauchy-Green deformation tensor, $\boldsymbol{P}$ is the first Piola-Kirchhoff stress tensor, and rho_0 is the mass density in the reference configuration. The first Piola-Kirchhoff stress tensor is related to the Cauchy stress tensor by

$$\boldsymbol{P} = \det(\boldsymbol{F})\ \boldsymbol{F}^{-1} \cdot \boldsymbol{\sigma}\,. \tag{41}$$

The gradient and divergence operators are defined such that

$$\boldsymbol{\nabla}_0\ \mathbf{v} = \sum_{i,j=1}^{3} \frac{\partial v_i}{\partial X_j} \boldsymbol{E}_i \otimes \boldsymbol{E}_j = v_{i,j} \boldsymbol{E}_i \otimes \boldsymbol{E}_j\ ;\quad \boldsymbol{\nabla}_0 \cdot \mathbf{v} = \sum_{i=1}^{3} \frac{\partial v_i}{\partial X_i} = v_{i,i}\ ;\quad \boldsymbol{\nabla}_0 \cdot \boldsymbol{S} = \sum_{i,j=1}^{3} \frac{\partial S_{ij}}{\partial X_j}\ \boldsymbol{E}_i = S_{ij,j}\ \boldsymbol{E}_i \tag{42}$$

where $\mathbf{v}$ is a vector field, BS is a second-order tensor field, and $\boldsymbol{E}_i$ are the components of an orthonormal basis in the reference configuration.

With respect to the deformed configuration, the governing equations are

$$\boxed{\begin{array}{ll} \boldsymbol{b} = \boldsymbol{F} \cdot \boldsymbol{F}^T & \text{Strain-deformation Relations} \\ \boldsymbol{\sigma} = \hat{\boldsymbol{\sigma}}(\boldsymbol{B}) & \text{Stress-Strain Relations} \\ \rho\ \det(\boldsymbol{F}) = \rho_0 & \text{Balance of Mass} \\ \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = 0 & \text{Balance of Linear Momentum} \\ \boldsymbol{\sigma} = \boldsymbol{\sigma}^T & \text{Balance of Angular Momentum} \\ \rho\,\dot{e} = \boldsymbol{\sigma} : (\boldsymbol{\nabla}\mathbf{v}) - \boldsymbol{\nabla} \cdot \mathbf{q} + \rho\, s & \text{Balance of Energy.} \end{array}} \tag{43}$$

Here, $\boldsymbol{b}$ is the left Cauchy-Green deformation tensor, $\boldsymbol{\sigma}$ is the Cauchy stress, and ρ is the mass density in the deformed configuration. The gradient and divergence operators are defined such that

$$\boldsymbol{\nabla}\mathbf{v} = \sum_{i,j=1}^{3} \frac{\partial v_i}{\partial x_j} \mathbf{e}_i \otimes \mathbf{e}_j = v_{i,j} \mathbf{e}_i \otimes \mathbf{e}_j\ ;\quad \boldsymbol{\nabla} \cdot \mathbf{v} = \sum_{i=1}^{3} \frac{\partial v_i}{\partial x_i} = v_{i,i}\ ;\quad \boldsymbol{\nabla} \cdot \boldsymbol{S} = \sum_{i,j=1}^{3} \frac{\partial S_{ij}}{\partial x_j}\ \mathbf{e}_i = S_{ij,j}\ \mathbf{e}_i\,. \tag{44}$$

It is convenient to express all (unweighted) volume average quantities for finite deformation in terms of integrals over the reference volume (Ω_0) and the reference surface ($\partial\Omega_0$).

Note that the strain measures used for finite deformation contain products of the deformation gradient. For example,

$$\boldsymbol{C} = \boldsymbol{F}^T \cdot \boldsymbol{F} \,.$$

A volume average of $\boldsymbol{C}$ may be defined in two ways:

$$\langle \boldsymbol{C} \rangle := \frac{1}{V_0}\int_{\Omega_0} \boldsymbol{C}\,\mathrm{dV} = \frac{1}{V_0}\int_{\Omega_0} \boldsymbol{F}^T \cdot \boldsymbol{F}\,\mathrm{dV} \qquad \text{or} \qquad \overline{\boldsymbol{C}} := \left(\frac{1}{V_0}\int_{\Omega_0} \boldsymbol{F}^T\,\mathrm{dV}\right)\cdot\left(\frac{1}{V_0}\int_{\Omega_0} \boldsymbol{F}^T\,\mathrm{dV}\right) = \langle \boldsymbol{F}\rangle^T \cdot \langle \boldsymbol{F}\rangle \,.$$

The choice of the definition of a macroscopic average quantity is based on physical considerations. Ideally, *such quantities are chosen such that their unweighted volume averages are completely defined by the surface data.* Unweighted average quantities that satisfy these requirements are the deformation gradient $\boldsymbol{F}$, its rate $\dot{\boldsymbol{F}}$, the first Piola-Kirchhoff stress $\boldsymbol{P}$, and its rate $\dot{\boldsymbol{P}}$.

4.1 Average Deformation Gradient in a RVE

The average deformation gradient is defined as

$$\boxed{\langle \boldsymbol{F} \rangle := \frac{1}{V_0}\int_{\Omega_0} \boldsymbol{F}\,\mathrm{dV}} \tag{45}$$

where V_0 is the volume in the reference configuration.

We can express the average deformation gradient in terms of surface quantities by using the divergence theorem. Thus,

$$\langle \boldsymbol{F} \rangle = \frac{1}{V_0}\int_{\Omega_0} \boldsymbol{F}\,\mathrm{dV} = \frac{1}{V_0}\int_{\Omega_0} \boldsymbol{\nabla}_0\,\mathbf{x}\,\mathrm{dV} = \frac{1}{V_0}\int_{\partial\Omega_0} \mathbf{x}\otimes\mathbf{N}\,\mathrm{dA} = \frac{1}{V_0}\int_{\partial\Omega_0} (\mathbf{X}+\mathbf{u})\otimes\mathbf{N}\,\mathrm{dA}$$

where $\mathbf{N}$ is the unit outward normal to the reference surface $\partial\Omega_0$ and $\mathbf{u}(\mathbf{X}) = \mathbf{x} - \mathbf{X}$ is the displacement.

The surface integral can be converted into an integral over the deformed surface using Nanson's formula for areas:

$$\mathrm{d}\mathbf{a} = \det(\boldsymbol{F})\,\boldsymbol{F}^{-T}\,\mathrm{d}\mathbf{A} \qquad \equiv \qquad \mathbf{n}\,\mathrm{da} = \det(\boldsymbol{F})\,\boldsymbol{F}^{-T}\cdot\mathbf{N}\,\mathrm{dA} \quad \Longrightarrow \quad \frac{1}{\det \boldsymbol{F}}\,\boldsymbol{F}^T\cdot\mathbf{n}\,\mathrm{da} = \mathbf{N}\,\mathrm{dA}$$

where da is an element of area on the deformed surface, $\mathbf{n}$ is the outward normal to the deformed surface, and dA is an element of area on the reference surface.

The conservation of mass gives us

$$J := \det(\boldsymbol{F}) = \frac{\rho_0}{\rho} = \frac{V}{V_0} \,. \tag{46}$$

Therefore,

$$\mathbf{x}\otimes\mathbf{N}\,\mathrm{dA} = \mathbf{x}\otimes(\mathbf{N}\,\mathrm{dA}) = \mathbf{x}\otimes\left(\frac{V_0}{V}\,\boldsymbol{F}^T\cdot\mathbf{n}\,\mathrm{da}\right) = \left(\frac{V_0}{V}\right)\,\mathbf{x}\otimes(\boldsymbol{F}^T\cdot\mathbf{n})\,\mathrm{da}$$

Plugging into the surface integral, we have

$$\langle \boldsymbol{F} \rangle = \frac{1}{V_0}\int_{\partial\Omega_0} \mathbf{x}\otimes\mathbf{N}\,\mathrm{dA} = \frac{1}{V_0}\int_{\partial\Omega}\left[\left(\frac{V_0}{V}\right)\,\mathbf{x}\otimes(\boldsymbol{F}^T\cdot\mathbf{n})\right]\,\mathrm{da} = \frac{1}{V}\int_{\partial\Omega} \mathbf{x}\otimes(\boldsymbol{F}^T\cdot\mathbf{n})\,\mathrm{da} \,.$$

Using the identity $\mathbf{a} \otimes (\boldsymbol{A} \cdot \mathbf{b}) = (\mathbf{a} \otimes \mathbf{b}) \cdot \boldsymbol{A}^T$ (see Appendix), we get

$$\langle \boldsymbol{F} \rangle = \frac{1}{V} \int_{\partial\Omega} (\mathbf{x} \otimes \mathbf{n}) \cdot \boldsymbol{F} \, \mathrm{da} \, .$$

Therefore, the average deformation gradient in surface integral form can be written as

$$\boxed{\langle \boldsymbol{F} \rangle = \frac{1}{V_0} \int_{\partial\Omega_0} \mathbf{x} \otimes \mathbf{N} \, \mathrm{dA} = \frac{1}{V} \int_{\partial\Omega} (\mathbf{x} \otimes \mathbf{n}) \cdot \boldsymbol{F} \, \mathrm{da} \, .} \tag{47}$$

Note that there are three more conditions to be satisfied for the average deformation gradient to behave like a macro variable, i.e.,

$$\det\langle \boldsymbol{F} \rangle > 0 \; ; \;\; \langle \boldsymbol{F} \rangle^{-1} = \langle \boldsymbol{F}^{-1} \rangle \; ; \;\; V = V_0 \det\langle \boldsymbol{F} \rangle \, . \tag{48}$$

These considerations and their detailed exploration can be found in Costanzo et al.(2005).

4.2 Average Velocity Gradient in a RVE

The time rate of the deformation gradient is given by

$$\dot{\boldsymbol{F}} = \frac{\partial}{\partial t}[\boldsymbol{F}(\mathbf{X}, t)] = \frac{\partial}{\partial t}\left(\frac{\partial}{\partial \mathbf{X}}[\mathbf{x}(\mathbf{X}, t)]\right) = \frac{\partial}{\partial \mathbf{X}}\left(\frac{\partial}{\partial t}[\mathbf{x}(\mathbf{X}, t)]\right) = \frac{\partial \dot{\mathbf{x}}}{\partial \mathbf{X}} = \boldsymbol{\nabla}_0 \, \dot{\mathbf{x}} \, .$$

The average time rate of the deformation gradient is defined as

$$\boxed{\langle \dot{\boldsymbol{F}} \rangle := \frac{1}{V_0} \int_{\Omega_0} \dot{\boldsymbol{F}} \, \mathrm{dV} \, .} \tag{49}$$

Following the same procedure as in the previous section, we can show that

$$\boxed{\langle \dot{\boldsymbol{F}} \rangle = \frac{1}{V_0} \int_{\partial\Omega_0} \dot{\mathbf{x}} \otimes \mathbf{N} \, \mathrm{dA} = \frac{1}{V} \int_{\partial\Omega} (\dot{\mathbf{x}} \otimes \mathbf{n}) \cdot \boldsymbol{F} \, \mathrm{da} \, .} \tag{50}$$

The velocity gradient ($\boldsymbol{l}$) is given by

$$\boldsymbol{l} = \boldsymbol{\nabla} \mathbf{v} = \dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1}$$

where $\mathbf{v}(\mathbf{x})$ is the velocity.

The average velocity gradient in a RVE is defined as

$$\boxed{\bar{\boldsymbol{l}} := \langle \dot{\boldsymbol{F}} \rangle \cdot \langle \boldsymbol{F} \rangle^{-1} \, .} \tag{51}$$

Note that $\bar{\boldsymbol{l}} = \langle \boldsymbol{l} \rangle = \langle \dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1} \rangle$ only if $\boldsymbol{F} = \mathbf{1}$.

4.3 Average Stress in a RVE

The average nominal (first Piola-Kirchhoff) stress is defined as

$$\boxed{\langle \boldsymbol{P} \rangle = \frac{1}{V_0} \int_{\Omega_0} \boldsymbol{P} \, \mathrm{dV} \, .} \tag{52}$$

Recall the relation (see Appendix)

$$\int_{\partial\Omega} \mathbf{v} \otimes (\boldsymbol{S}^T \cdot \mathbf{n}) \, \mathrm{dA} = \int_{\Omega} [\boldsymbol{\nabla}\mathbf{v} \cdot \boldsymbol{S} + \mathbf{v} \otimes (\boldsymbol{\nabla} \cdot \boldsymbol{S}^T)] \, \mathrm{dV} \,.$$

In the above equation, let the volume integral be over Ω_0 and let the surface integral be over $\partial\Omega_0$. Let the unit outward normal to $\partial\Omega_0$ be $\mathbf{N}$. Let the gradient and divergence operations be with respect to the reference configuration. Also, let $\mathbf{v} \rightarrow \mathbf{X}$ and let $\boldsymbol{S} \rightarrow \boldsymbol{P}$. Then we have

$$\int_{\partial\Omega_0} \mathbf{X}\otimes(\boldsymbol{P}^T\cdot\mathbf{N}) \, \mathrm{dA} = \int_{\Omega_0} [\boldsymbol{\nabla}_0 \, \mathbf{X}\cdot\boldsymbol{P}+\mathbf{X}\otimes(\boldsymbol{\nabla}_0\cdot\boldsymbol{P}^T)] \, \mathrm{dV} = \int_{\Omega_0} [\mathbf{1}\cdot\boldsymbol{P}+\mathbf{X}\otimes(\boldsymbol{\nabla}_0\cdot\boldsymbol{P}^T)] \, \mathrm{dV} = \int_{\Omega_0} [\boldsymbol{P}+\mathbf{X}\otimes(\boldsymbol{\nabla}_0\cdot\boldsymbol{P}^T)] \, \mathrm{dV} \,.$$

If we assume that there are **no inertial forces or body forces**, then $\boldsymbol{\nabla}_0 \cdot \boldsymbol{P}^T = 0$ (from the conservation of linear momentum), and we have

$$\int_{\partial\Omega_0} \mathbf{X} \otimes (\boldsymbol{P}^T \cdot \mathbf{N}) \, \mathrm{dA} = \int_{\Omega_0} \boldsymbol{P} \, \mathrm{dV} = V_0 \, \langle \boldsymbol{P} \rangle \,.$$

Let $\bar{\mathbf{T}}$ be a **self equilibrating** traction that is applied to the RVE, i.e., it does not lead to any inertial forces. Then, Cauchy's law states that $\bar{\mathbf{T}} = \boldsymbol{P}^T \cdot \mathbf{N}$ on $\partial\Omega_0$. Hence we get

$$\boxed{\langle \boldsymbol{P} \rangle = \frac{1}{V_0} \int_{\partial\Omega_0} \mathbf{X} \otimes \bar{\mathbf{T}} \, \mathrm{dA} \,.} \tag{53}$$

Given the above, the average Cauchy stress in the RVE is defined as

$$\boxed{\overline{\boldsymbol{\sigma}} := \frac{1}{\det\langle \boldsymbol{F} \rangle} \, \langle \boldsymbol{F} \rangle \cdot \langle \boldsymbol{P} \rangle \,.} \tag{54}$$

Note that, in general, $\overline{\boldsymbol{\sigma}} \neq \langle \boldsymbol{\sigma} \rangle$.

The Kirchhoff stress is defined as $\boldsymbol{\tau} := \det \boldsymbol{F} \, \boldsymbol{\sigma}$. The average Kirchhoff stress in the RVE is defined as

$$\boxed{\overline{\boldsymbol{\tau}} := \det\langle \boldsymbol{F} \rangle \, \overline{\boldsymbol{\sigma}} = \langle \boldsymbol{F} \rangle \cdot \langle \boldsymbol{P} \rangle \,.} \tag{55}$$

In general, $\overline{\boldsymbol{\tau}} \neq \langle \boldsymbol{\tau} \rangle$.

4.4 Average Stress Power in a RVE

Recall the equation for the balance of energy (with respect to the reference configuration)

$$\rho_0 \, \dot{e} = \boldsymbol{P}^T : \dot{\boldsymbol{F}} - \boldsymbol{\nabla}_0 \cdot \mathbf{q} + \rho_0 \, s \,.$$

The quantity $\boldsymbol{P}^T : \dot{\boldsymbol{F}}$ is the stress power.

The average stress power is defined as

$$\boxed{\langle \boldsymbol{P}^T : \dot{\boldsymbol{F}} \rangle := \frac{1}{V_0} \int_{\partial\Omega_0} \boldsymbol{P}^T : \dot{\boldsymbol{F}} \, \mathrm{dV} \,.} \tag{56}$$

Here $\boldsymbol{P}^T$ is an arbitrary self-equilibrating nominal stress field that satisfies the balance of momentum (without any body forces or inertial forces) and $\dot{\boldsymbol{F}}$ is the time rate of change of $\boldsymbol{F}$. The reference configuration can be arbitrary. Also, the nominal stress and the rate $\dot{\boldsymbol{F}}$ need not be related.

Note that in that case

$$\boxed{\mathrm{tr}(\langle \boldsymbol{P}^T : \dot{\boldsymbol{F}} \rangle) = \frac{1}{V_0}\int_{\partial\Omega_0} \mathrm{tr}(\boldsymbol{P}\cdot\dot{\boldsymbol{F}})\ \mathrm{dV} = \frac{1}{V_0}\int_{\partial\Omega_0} \mathrm{tr}(\dot{\boldsymbol{F}}\cdot\boldsymbol{P})\ \mathrm{dV} = \mathrm{tr}(\langle \dot{\boldsymbol{F}}\cdot\boldsymbol{P}\rangle)\ .} \tag{57}$$

We can express the stress power in terms of boundary tractions and boundary velocities using the relation (see Appendix)

$$\int_{\partial\Omega} \mathbf{v}\otimes(\boldsymbol{S}^T\cdot\mathbf{n})\ \mathrm{dA} = \int_{\Omega} [\boldsymbol{\nabla}\mathbf{v}\cdot\boldsymbol{S} + \mathbf{v}\otimes(\boldsymbol{\nabla}\cdot\boldsymbol{S}^T)]\ \mathrm{dV}\ .$$

In this case, we have $\partial\Omega \to \partial\Omega_0$, $\Omega\to\Omega_0$, $\boldsymbol{\nabla}\to\boldsymbol{\nabla}_0$, $\mathbf{v}\to\dot{\mathbf{x}}$, $\boldsymbol{S}\to\boldsymbol{P}$, and $\mathbf{n}\to\mathbf{N}$. Then

$$\int_{\partial\Omega} \dot{\mathbf{x}}\otimes(\boldsymbol{P}^T\cdot\mathbf{N})\ \mathrm{dA} = \int_{\Omega} [\boldsymbol{\nabla}_0\,\dot{\mathbf{x}}\cdot\boldsymbol{P} + \dot{\mathbf{x}}\otimes(\boldsymbol{\nabla}_0\cdot\boldsymbol{P}^T)]\ \mathrm{dV}\ .$$

Using the balance of linear momentum (in the absence of body and inertial forces), we get

$$\int_{\partial\Omega} \dot{\mathbf{x}}\otimes(\boldsymbol{P}^T\cdot\mathbf{N})\ \mathrm{dA} = \int_{\Omega} \boldsymbol{\nabla}_0\,\dot{\mathbf{x}}\cdot\boldsymbol{P}\ \mathrm{dV}\ .$$

Recalling that

$$\dot{\boldsymbol{F}} = \frac{\partial}{\partial t}\left(\frac{\partial\mathbf{x}}{\partial\mathbf{X}}\right) = \frac{\partial}{\partial\mathbf{X}}\left(\frac{\partial\mathbf{x}}{\partial t}\right) = \boldsymbol{\nabla}_0\,\dot{\mathbf{x}}$$

we then have

$$\int_{\Omega} \dot{\boldsymbol{F}}\cdot\boldsymbol{P}\ \mathrm{dV} = \int_{\partial\Omega} \dot{\mathbf{x}}\otimes(\boldsymbol{P}^T\cdot\mathbf{N})\ \mathrm{dA}\ .$$

If $\bar{\mathbf{T}}$ is a self equilibrating traction applied on the boundary that leads to the stress field $\boldsymbol{P}$, i.e., $\bar{\mathbf{T}} = \boldsymbol{P}^T\cdot\mathbf{N}$, then we have

$$\boxed{\int_{\Omega} \dot{\boldsymbol{F}}\cdot\boldsymbol{P}\ \mathrm{dV} = \int_{\partial\Omega} \dot{\mathbf{x}}\otimes\bar{\mathbf{T}}\ \mathrm{dA}\ .} \tag{58}$$

Note that the fields $\dot{\boldsymbol{F}}$ and $\boldsymbol{P}$ need not be related and hence the velocities $\dot{\mathbf{x}}$ and the tractions $\bar{\mathbf{T}}$ are not related.

If the boundary velocity field $\dot{\mathbf{x}}$ leads to the rate $\dot{\boldsymbol{F}}$, using the identity (see Appendix)

$$\langle\dot{\boldsymbol{F}}\cdot\boldsymbol{P}\rangle - \langle\dot{\boldsymbol{F}}\rangle\cdot\langle\boldsymbol{P}\rangle = \langle(\dot{\boldsymbol{F}} - \langle\dot{\boldsymbol{F}}\rangle)\cdot(\boldsymbol{P} - \langle\boldsymbol{P}\rangle)\rangle$$

we can show that (see Appendix)

$$\begin{aligned}
\langle\dot{\boldsymbol{F}}\cdot\boldsymbol{P}\rangle - \langle\dot{\boldsymbol{F}}\rangle\cdot\langle\boldsymbol{P}\rangle &= \frac{1}{V_0}\int_{\partial\Omega_0} [\dot{\mathbf{x}} - \langle\dot{\boldsymbol{F}}\rangle\cdot\mathbf{X}]\otimes\left\{[\boldsymbol{P} - \langle\boldsymbol{P}\rangle]^T\cdot\mathbf{N}\right\}\ \mathrm{dA} \\
&= \frac{1}{V_0}\int_{\partial\Omega_0} [\dot{\mathbf{x}} - \langle\dot{\boldsymbol{F}}\rangle\cdot\mathbf{X}]\otimes(\boldsymbol{P}^T\cdot\mathbf{N})\ \mathrm{dA} \\
&= \frac{1}{V_0}\int_{\partial\Omega_0} \dot{\mathbf{x}}\otimes\left\{[\boldsymbol{P} - \langle\boldsymbol{P}\rangle]^T\cdot\mathbf{N}\right\}\ \mathrm{dA}\ .
\end{aligned} \tag{59}$$

- **Remark:**
  Using similar arguments, if we assume that $\boldsymbol{F}$ is a deformation that is compatible with an applied boundary displacement $\mathbf{u} = \mathbf{x} - \mathbf{X}$, we can show that

$$\begin{aligned}
\langle\boldsymbol{F}\cdot\boldsymbol{P}\rangle - \langle\boldsymbol{F}\rangle\cdot\langle\boldsymbol{P}\rangle &= \frac{1}{V_0}\int_{\partial\Omega_0} [\mathbf{x} - \langle\boldsymbol{F}\rangle\cdot\mathbf{X}]\otimes\left\{[\boldsymbol{P} - \langle\boldsymbol{P}\rangle]^T\cdot\mathbf{N}\right\}\ \mathrm{dA} \\
&= \frac{1}{V_0}\int_{\partial\Omega_0} [\mathbf{x} - \langle\boldsymbol{F}\rangle\cdot\mathbf{X}]\otimes(\boldsymbol{P}^T\cdot\mathbf{N})\ \mathrm{dA} \\
&= \frac{1}{V_0}\int_{\partial\Omega_0} \mathbf{x}\otimes\left\{[\boldsymbol{P} - \langle\boldsymbol{P}\rangle]^T\cdot\mathbf{N}\right\}\ \mathrm{dA}\ .
\end{aligned} \tag{60}$$

We can arrive at $\langle \dot{\boldsymbol{F}} \cdot \boldsymbol{P} \rangle = \langle \dot{\boldsymbol{F}} \rangle \cdot \langle \boldsymbol{P} \rangle$ or $\langle \boldsymbol{F} \cdot \boldsymbol{P} \rangle = \langle \boldsymbol{F} \rangle \cdot \langle \boldsymbol{P} \rangle$ if either of the following conditions is satisfied at the boundary:

1. $\dot{\mathbf{x}} = \langle \dot{\boldsymbol{F}} \rangle \cdot \mathbf{X}$ or $\mathbf{x} = \langle \boldsymbol{F} \rangle \cdot \mathbf{X}$.
2. $\boldsymbol{P} \cdot \mathbf{N} = \langle \boldsymbol{P} \rangle^T \cdot \mathbf{N}$.

1. **Linear boundary velocities/displacements:**
If a linear velocity field is prescribed on the boundary $\partial\Omega_0$, we can express this field as

$$\dot{\mathbf{x}}(\mathbf{X}, t) = \dot{\boldsymbol{H}}(t) \cdot \mathbf{X} \qquad \forall \mathbf{X} \in \partial\Omega_0 \,. \tag{61}$$

Now,

$$\begin{aligned} \langle \dot{\boldsymbol{F}} \rangle &= \frac{1}{V_0} \int_{\partial\Omega_0} \dot{\mathbf{x}} \otimes \mathbf{N} \,\mathrm{dA} \\ &= \frac{1}{V_0} \int_{\partial\Omega_0} (\dot{\boldsymbol{H}} \cdot \mathbf{X}) \otimes \mathbf{N} \,\mathrm{dA} \\ &= \dot{\boldsymbol{H}} \cdot \left(\frac{1}{V_0} \int_{\partial\Omega_0} \mathbf{X} \otimes \mathbf{N} \,\mathrm{dA} \right) \,. \end{aligned}$$

Recall that

$$\int_{\Omega_0} \boldsymbol{\nabla}_0 \, \mathbf{X} \,\mathrm{dV} = \int_{\partial\Omega_0} \mathbf{X} \otimes \mathbf{N} \,\mathrm{dA} \,.$$

Therefore,

$$\begin{aligned} \langle \dot{\boldsymbol{F}} \rangle &= \dot{\boldsymbol{H}} \cdot \left(\frac{1}{V_0} \int_{\Omega_0} \boldsymbol{\nabla}_0 \, \mathbf{X} \,\mathrm{dA} \right) \\ &= \dot{\boldsymbol{H}} \cdot \left(\frac{1}{V_0} \int_{\Omega_0} \mathbf{1} \,\mathrm{dA} \right) = \dot{\boldsymbol{H}} \,. \end{aligned}$$

Hence,

$$\langle \dot{\boldsymbol{F}} \rangle = \dot{\boldsymbol{H}} \qquad \Longrightarrow \qquad \dot{\mathbf{x}} - \langle \dot{\boldsymbol{F}} \rangle \cdot \mathbf{X} = \mathbf{0} \,. \tag{62}$$

Then,

$$\begin{aligned} \langle \dot{\boldsymbol{F}} \cdot \boldsymbol{P} \rangle - \langle \dot{\boldsymbol{F}} \rangle \cdot \langle \boldsymbol{P} \rangle &= \frac{1}{V_0} \int_{\partial\Omega_0} [\dot{\mathbf{x}} - \langle \dot{\boldsymbol{F}} \rangle \cdot \mathbf{X}] \otimes (\boldsymbol{P}^T \cdot \mathbf{N}) \,\mathrm{dA} \\ &= \frac{1}{V_0} \int_{\partial\Omega_0} [\dot{\mathbf{x}} - \dot{\boldsymbol{H}} \cdot \mathbf{X}] \otimes (\boldsymbol{P}^T \cdot \mathbf{N}) \,\mathrm{dA} = \mathbf{0} \end{aligned}$$

Hence,

$$\boxed{\langle \dot{\boldsymbol{F}} \cdot \boldsymbol{P} \rangle = \langle \dot{\boldsymbol{F}} \rangle \cdot \langle \boldsymbol{P} \rangle \,.} \tag{63}$$

Similarly, if a linear displacement field is prescribed on the boundary such that

$$\mathbf{u}(\mathbf{X}, t) = \boldsymbol{H}(t) \cdot \mathbf{X} - \mathbf{X} \qquad \Longrightarrow \qquad \mathbf{x}(\mathbf{X}) = \boldsymbol{H}(t) \cdot \mathbf{X} \qquad \forall \mathbf{X} \in \partial\Omega_0 \tag{64}$$

we can show that

$$\langle \boldsymbol{F} \rangle = \boldsymbol{H} \qquad \Longrightarrow \qquad \mathbf{x} - \langle \boldsymbol{F} \rangle \cdot \mathbf{X} = \mathbf{0} \,. \tag{65}$$

This leads to the equality

$$\boxed{\langle \boldsymbol{F} \cdot \boldsymbol{P} \rangle = \langle \boldsymbol{F} \rangle \cdot \langle \boldsymbol{P} \rangle \,.} \tag{66}$$

Recall that, the average Kirchhoff stress is given by $\overline{\boldsymbol{\tau}} = \langle \boldsymbol{F} \rangle \cdot \langle \boldsymbol{P} \rangle$. Therefore, if a uniform boundary displacement is prescribed, we have

$$\overline{\boldsymbol{\tau}} = \langle \boldsymbol{F} \rangle \cdot \langle \boldsymbol{P} \rangle = \langle \boldsymbol{F} \cdot \boldsymbol{P} \rangle = \langle \boldsymbol{\tau} \rangle$$

or,

$$\boxed{\bar{\boldsymbol{\tau}} = \langle \boldsymbol{\tau} \rangle \, .} \tag{67}$$

2. **Uniform boundary tractions:**
A uniform boundary traction field in the reference configuration can be represented as

$$\bar{\boldsymbol{T}}(\mathbf{X}, t) = \boldsymbol{H}^T(t) \cdot \mathbf{N}(\mathbf{X}) \qquad \forall \, \mathbf{X} \in \partial\Omega_0 \, . \tag{68}$$

Now,

$$\begin{aligned}
\langle \boldsymbol{P} \rangle &= \frac{1}{V_0} \int_{\partial\Omega_0} \mathbf{X} \otimes \bar{\boldsymbol{T}} \, \mathrm{dA} \\
&= \frac{1}{V_0} \int_{\partial\Omega_0} \mathbf{X} \otimes (\boldsymbol{H}^T \cdot \mathbf{N} \, \mathrm{dA} \\
&= \left(\frac{1}{V_0} \int_{\partial\Omega_0} \mathbf{X} \otimes \mathbf{N} \, \mathrm{dA} \right) \cdot \boldsymbol{H} \\
&= \mathbf{1} \cdot \boldsymbol{H} = \boldsymbol{H} \, .
\end{aligned}$$

Since the surface tractions are related to the nominal stress by $\bar{B T}(\mathbf{X}, t) = \boldsymbol{P}^T(\mathbf{X}, t) \cdot \mathbf{N}(\mathbf{X})$, we must have

$$\langle \boldsymbol{P} \rangle = \boldsymbol{P} \, . \tag{69}$$

Therefore,

$$\langle \dot{\boldsymbol{F}} \cdot \boldsymbol{P} \rangle - \langle \dot{\boldsymbol{F}} \rangle \cdot \langle \boldsymbol{P} \rangle = \frac{1}{V_0} \int_{\partial\Omega_0} \dot{\mathbf{x}} \otimes \left\{ [\boldsymbol{P} - \langle \boldsymbol{P} \rangle]^T \cdot \mathbf{N} \right\} \, \mathrm{dA} = \mathbf{0}$$

or,

$$\boxed{\langle \dot{\boldsymbol{F}} \cdot \boldsymbol{P} \rangle = \langle \dot{\boldsymbol{F}} \rangle \cdot \langle \boldsymbol{P} \rangle \, .} \tag{70}$$

Similarly,

$$\boxed{\langle \boldsymbol{F} \cdot \boldsymbol{P} \rangle = \langle \boldsymbol{F} \rangle \cdot \langle \boldsymbol{P} \rangle \, .} \tag{71}$$

Hence, using the same argument as for the previous case, we have

$$\boxed{\bar{\boldsymbol{\tau}} = \langle \boldsymbol{\tau} \rangle \, .} \tag{72}$$

5 Appendix

1. Show that the balance of mass can be expressed as:

$$\dot{\rho} + \rho \, \boldsymbol{\nabla} \cdot \mathbf{v} = 0$$

where $\rho(\mathbf{x}, t)$ is the current mass density, $\dot{\rho}$ is the material time derivative of ρ, and $\mathbf{v}(\mathbf{x}, t)$ is the velocity of physical particles in the body Ω bounded by the surface $\partial\Omega$.

Recall that the general equation for the balance of a physical quantity $f(\mathbf{x}, t)$ is given by

$$\frac{d}{dt}\left[\int_\Omega f(\mathbf{x}, t) \, \mathrm{dV}\right] = \int_{\partial\Omega} f(\mathbf{x}, t)[u_n(\mathbf{x}, t) - \mathbf{v}(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}, t)] \, \mathrm{dA} + \int_{\partial\Omega} g(\mathbf{x}, t) \, \mathrm{dA} + \int_\Omega h(\mathbf{x}, t) \, \mathrm{dV} \, .$$

To derive the equation for the balance of mass, we assume that the physical quantity of interest is the mass density $\rho(\mathbf{x}, t)$. Since mass is neither created or destroyed, the surface and interior sources are zero, i.e., $g(\mathbf{x}, t) = h(\mathbf{x}, t) = 0$. Therefore, we have

$$\frac{d}{dt}\left[\int_\Omega \rho(\mathbf{x}, t) \, \mathrm{dV}\right] = \int_{\partial\Omega} \rho(\mathbf{x}, t)[u_n(\mathbf{x}, t) - \mathbf{v}(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}, t)] \, \mathrm{dA} \, .$$

Let us assume that the volume Ω is a control volume (i.e., it does not change with time). Then the surface $\partial\Omega$ has a zero velocity ($u_n = 0$) and we get

$$\int_\Omega \frac{\partial \rho}{\partial t} \, \mathrm{dV} = - \int_{\partial\Omega} \rho \, (\mathbf{v} \cdot \mathbf{n}) \, \mathrm{dA} \, .$$

Using the divergence theorem

$$\int_\Omega \boldsymbol{\nabla}\cdot\mathbf{v}\,\mathrm{dV} = \int_{\partial\Omega} \mathbf{v}\cdot\mathbf{n}\,\mathrm{dA}$$

we get

$$\int_\Omega \frac{\partial\rho}{\partial t}\,\mathrm{dV} = -\int_\Omega \boldsymbol{\nabla}\cdot(\rho\,\mathbf{v})\,\mathrm{dV}.$$

or,

$$\int_\Omega \left[\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\,\mathbf{v})\right]\,\mathrm{dV} = 0\,.$$

Since Ω is arbitrary, we must have

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\,\mathbf{v}) = 0\,.$$

Using the identity

$$\boldsymbol{\nabla}\cdot(\varphi\,\mathbf{v}) = \varphi\,\boldsymbol{\nabla}\cdot\mathbf{v} + \boldsymbol{\nabla}\varphi\cdot\mathbf{v}$$

we have

$$\frac{\partial\rho}{\partial t} + \rho\,\boldsymbol{\nabla}\cdot\mathbf{v} + \boldsymbol{\nabla}\rho\cdot\mathbf{v} = 0\,.$$

Now, the material time derivative of ρ is defined as

$$\dot{\rho} = \frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\rho\cdot\mathbf{v}\,.$$

Therefore,

$$\boxed{\dot{\rho} + \rho\,\boldsymbol{\nabla}\cdot\mathbf{v} = 0\,.}$$

2. Show that the balance of linear momentum can be expressed as:

$$\rho\,\dot{\mathbf{v}} - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} - \rho\,\mathbf{b} = 0$$

where $\rho(\mathbf{x},t)$ is the mass density, $\mathbf{v}(\mathbf{x},t)$ is the velocity, $\boldsymbol{\sigma}(\mathbf{x},t)$ is the Cauchy stress, and $\rho\,\mathbf{b}$ is the body force density.

Recall the general equation for the balance of a physical quantity

$$\frac{d}{dt}\left[\int_\Omega f(\mathbf{x},t)\,\mathrm{dV}\right] = \int_{\partial\Omega} f(\mathbf{x},t)[u_n(\mathbf{x},t) - \mathbf{v}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x},t)]\,\mathrm{dA} + \int_{\partial\Omega} g(\mathbf{x},t)\,\mathrm{dA} + \int_\Omega h(\mathbf{x},t)\,\mathrm{dV}\,.$$

In this case the physical quantity of interest is the momentum density, i.e., $f(\mathbf{x},t) = \rho(\mathbf{x},t)\,\mathbf{v}(\mathbf{x},t)$. The source of momentum flux at the surface is the surface traction, i.e., $g(\mathbf{x},t) = \mathbf{t}$. The source of momentum inside the body is the body force, i.e., $h(\mathbf{x},t) = \rho(\mathbf{x},t)\,\mathbf{b}(\mathbf{x},t)$. Therefore, we have

$$\frac{d}{dt}\left[\int_\Omega \rho\,\mathbf{v}\,\mathrm{dV}\right] = \int_{\partial\Omega} \rho\,\mathbf{v}[u_n - \mathbf{v}\cdot\mathbf{n}]\,\mathrm{dA} + \int_{\partial\Omega} \mathbf{t}\,\mathrm{dA} + \int_\Omega \rho\,\mathbf{b}\,\mathrm{dV}\,.$$

The surface tractions are related to the Cauchy stress by

$$\mathbf{t} = \boldsymbol{\sigma}\cdot\mathbf{n}\,.$$

Therefore,

$$\frac{d}{dt}\left[\int_\Omega \rho\,\mathbf{v}\,\mathrm{dV}\right] = \int_{\partial\Omega} \rho\,\mathbf{v}[u_n - \mathbf{v}\cdot\mathbf{n}]\,\mathrm{dA} + \int_{\partial\Omega} \boldsymbol{\sigma}\cdot\mathbf{n}\,\mathrm{dA} + \int_\Omega \rho\,\mathbf{b}\,\mathrm{dV}\,.$$

Let us assume that Ω is an arbitrary fixed control volume. Then,

$$\int_\Omega \frac{\partial}{\partial t}(\rho\,\mathbf{v})\,\mathrm{dV} = -\int_{\partial\Omega} \rho\,\mathbf{v}\,(\mathbf{v}\cdot\mathbf{n})\,\mathrm{dA} + \int_{\partial\Omega} \boldsymbol{\sigma}\cdot\mathbf{n}\,\mathrm{dA} + \int_\Omega \rho\,\mathbf{b}\,\mathrm{dV}\,.$$

Now, from the definition of the tensor product we have (for all vectors $\mathbf{a}$)

$$(\mathbf{u}\otimes\mathbf{v})\cdot\mathbf{a} = (\mathbf{a}\cdot\mathbf{v})\,\mathbf{u}\,.$$

Therefore,

$$\int_\Omega \frac{\partial}{\partial t}(\rho\,\mathbf{v})\,\mathrm{dV} = -\int_{\partial\Omega} \rho\,(\mathbf{v}\otimes\mathbf{v})\cdot\mathbf{n}\,\mathrm{dA} + \int_{\partial\Omega} \boldsymbol{\sigma}\cdot\mathbf{n}\,\mathrm{dA} + \int_\Omega \rho\,\mathbf{b}\,\mathrm{dV}\,.$$

Using the divergence theorem

$$\int_\Omega \boldsymbol{\nabla}\cdot\mathbf{v}\,\mathrm{dV} = \int_{\partial\Omega} \mathbf{v}\cdot\mathbf{n}\,\mathrm{dA}$$

we have

$$\int_\Omega \frac{\partial}{\partial t}(\rho\,\mathbf{v})\,\mathrm{dV} = -\int_\Omega \boldsymbol{\nabla}\cdot[\rho\,(\mathbf{v}\otimes\mathbf{v})]\,\mathrm{dV} + \int_\Omega \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}\,\mathrm{dV} + \int_\Omega \rho\,\mathbf{b}\,\mathrm{dV}$$

or,

$$\int_\Omega \left[\frac{\partial}{\partial t}(\rho\,\mathbf{v}) + \boldsymbol{\nabla}\cdot[(\rho\,\mathbf{v})\otimes\mathbf{v})] - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} - \rho\,\mathbf{b}\right]\,\mathrm{dV} = 0\,.$$

Since Ω is arbitrary, we have

$$\frac{\partial}{\partial t}(\rho\,\mathbf{v}) + \boldsymbol{\nabla}\cdot[(\rho\,\mathbf{v})\otimes\mathbf{v})] - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} - \rho\,\mathbf{b} = 0\,.$$

Using the identity

$$\boldsymbol{\nabla}\cdot(\mathbf{u}\otimes\mathbf{v}) = (\boldsymbol{\nabla}\cdot\mathbf{v})\mathbf{u} + (\boldsymbol{\nabla}\mathbf{u})\cdot\mathbf{v}$$

we get

$$\frac{\partial\rho}{\partial t}\,\mathbf{v} + \rho\,\frac{\partial\mathbf{v}}{\partial t} + (\boldsymbol{\nabla}\cdot\mathbf{v})(\rho\mathbf{v}) + \boldsymbol{\nabla}(\rho\,\mathbf{v})\cdot\mathbf{v} - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} - \rho\,\mathbf{b} = 0$$

or,

$$\left[\frac{\partial\rho}{\partial t} + \rho\,\boldsymbol{\nabla}\cdot\mathbf{v}\right]\mathbf{v} + \rho\,\frac{\partial\mathbf{v}}{\partial t} + \boldsymbol{\nabla}(\rho\,\mathbf{v})\cdot\mathbf{v} - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} - \rho\,\mathbf{b} = 0$$

Using the identity

$$\boldsymbol{\nabla}(\varphi\,\mathbf{v}) = \varphi\,\boldsymbol{\nabla}\mathbf{v} + \mathbf{v}\otimes(\boldsymbol{\nabla}\varphi)$$

we get

$$\left[\frac{\partial\rho}{\partial t} + \rho\,\boldsymbol{\nabla}\cdot\mathbf{v}\right]\mathbf{v} + \rho\,\frac{\partial\mathbf{v}}{\partial t} + [\rho\,\boldsymbol{\nabla}\mathbf{v} + \mathbf{v}\otimes(\boldsymbol{\nabla}\rho)]\cdot\mathbf{v} - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} - \rho\,\mathbf{b} = 0$$

From the definition

$$(\mathbf{u}\otimes\mathbf{v})\cdot\mathbf{a} = (\mathbf{a}\cdot\mathbf{v})\,\mathbf{u}$$

we have

$$[\mathbf{v}\otimes(\boldsymbol{\nabla}\rho)]\cdot\mathbf{v} = [\mathbf{v}\cdot(\boldsymbol{\nabla}\rho)]\,\mathbf{v}\,.$$

Hence,

$$\left[\frac{\partial\rho}{\partial t} + \rho\,\boldsymbol{\nabla}\cdot\mathbf{v}\right]\mathbf{v} + \rho\,\frac{\partial\mathbf{v}}{\partial t} + \rho\,\boldsymbol{\nabla}\mathbf{v}\cdot\mathbf{v} + [\mathbf{v}\cdot(\boldsymbol{\nabla}\rho)]\,\mathbf{v} - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} - \rho\,\mathbf{b} = 0$$

or,

$$\left[\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\rho\cdot\mathbf{v} + \rho\,\boldsymbol{\nabla}\cdot\mathbf{v}\right]\mathbf{v} + \rho\,\frac{\partial\mathbf{v}}{\partial t} + \rho\,\boldsymbol{\nabla}\mathbf{v}\cdot\mathbf{v} - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} - \rho\,\mathbf{b} = 0\,.$$

The material time derivative of ρ is defined as

$$\dot{\rho} = \frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\rho\cdot\mathbf{v}\,.$$

Therefore,

$$[\dot{\rho} + \rho\,\boldsymbol{\nabla}\cdot\mathbf{v}]\,\mathbf{v} + \rho\,\frac{\partial\mathbf{v}}{\partial t} + \rho\,\boldsymbol{\nabla}\mathbf{v}\cdot\mathbf{v} - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} - \rho\,\mathbf{b} = 0\,.$$

From the balance of mass, we have

$$\dot{\rho} + \rho\,\boldsymbol{\nabla}\cdot\mathbf{v} = 0\,.$$

Therefore,

$$\rho\,\frac{\partial\mathbf{v}}{\partial t} + \rho\,\boldsymbol{\nabla}\mathbf{v}\cdot\mathbf{v} - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} - \rho\,\mathbf{b} = 0\,.$$

The material time derivative of $\mathbf{v}$ is defined as

$$\dot{\mathbf{v}} = \frac{\partial\mathbf{v}}{\partial t} + \boldsymbol{\nabla}\mathbf{v}\cdot\mathbf{v}\,.$$

Hence,

$$\boxed{\rho\,\dot{\mathbf{v}} - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} - \rho\,\mathbf{b} = 0\,.}$$

3. Show that the balance of angular momentum can be expressed as:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$$

We assume that there are no surface couples on $\partial\Omega$ or body couples in Ω. Recall the general balance equation

$$\frac{d}{dt}\left[\int_\Omega f(\mathbf{x},t)\,\text{dV}\right] = \int_{\partial\Omega} f(\mathbf{x},t)[u_n(\mathbf{x},t) - \mathbf{v}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x},t)]\,\text{dA} + \int_{\partial\Omega} g(\mathbf{x},t)\,\text{dA} + \int_\Omega h(\mathbf{x},t)\,\text{dV}\,.$$

In this case, the physical quantity to be conserved the angular momentum density, i.e., $f = \mathbf{x}\times(\rho\,\mathbf{v})$. The angular momentum source at the surface is then $g = \mathbf{x}\times\mathbf{t}$ and the angular momentum source inside the body is $h = \mathbf{x}\times(\rho\,\mathbf{b})$. The angular momentum and moments are calculated with respect to a fixed origin. Hence we have

$$\frac{d}{dt}\left[\int_\Omega \mathbf{x}\times(\rho\,\mathbf{v})\,\text{dV}\right] = \int_{\partial\Omega}[\mathbf{x}\times(\rho\,\mathbf{v})][u_n - \mathbf{v}\cdot\mathbf{n}]\,\text{dA} + \int_{\partial\Omega}\mathbf{x}\times\mathbf{t}\,\text{dA} + \int_\Omega \mathbf{x}\times(\rho\,\mathbf{b})\,\text{dV}\,.$$

Assuming that Ω is a control volume, we have

$$\int_\Omega \mathbf{x}\times\left[\frac{\partial}{\partial t}(\rho\,\mathbf{v})\right]\,\text{dV} = -\int_{\partial\Omega}[\mathbf{x}\times(\rho\,\mathbf{v})][\mathbf{v}\cdot\mathbf{n}]\,\text{dA} + \int_{\partial\Omega}\mathbf{x}\times\mathbf{t}\,\text{dA} + \int_\Omega \mathbf{x}\times(\rho\,\mathbf{b})\,\text{dV}\,.$$

Using the definition of a tensor product we can write

$$[\mathbf{x}\times(\rho\,\mathbf{v})][\mathbf{v}\cdot\mathbf{n}] = [[\mathbf{x}\times(\rho\,\mathbf{v})]\otimes\mathbf{v}]\cdot\mathbf{n}\,.$$

Also, $\mathbf{t} = \boldsymbol{\sigma}\cdot\mathbf{n}$. Therefore we have

$$\int_\Omega \mathbf{x}\times\left[\frac{\partial}{\partial t}(\rho\,\mathbf{v})\right]\,\text{dV} = -\int_{\partial\Omega}[[\mathbf{x}\times(\rho\,\mathbf{v})]\otimes\mathbf{v}]\cdot\mathbf{n}\,\text{dA} + \int_{\partial\Omega}\mathbf{x}\times(\boldsymbol{\sigma}\cdot\mathbf{n})\,\text{dA} + \int_\Omega \mathbf{x}\times(\rho\,\mathbf{b})\,\text{dV}\,.$$

Using the divergence theorem, we get

$$\int_{\Omega} \mathbf{x} \times \left[\frac{\partial}{\partial t}(\rho\,\mathbf{v})\right] \mathrm{dV} = -\int_{\Omega} \boldsymbol{\nabla}\cdot[[\mathbf{x}\times(\rho\,\mathbf{v})]\otimes\mathbf{v}]\,\mathrm{dV} + \int_{\partial\Omega} \mathbf{x}\times(\boldsymbol{\sigma}\cdot\mathbf{n})\,\mathrm{dA} + \int_{\Omega}\mathbf{x}\times(\rho\,\mathbf{b})\,\mathrm{dV}\,.$$

To convert the surface integral in the above equation into a volume integral, it is convenient to use index notation. Thus,

$$\left[\int_{\partial\Omega}\mathbf{x}\times(\boldsymbol{\sigma}\cdot\mathbf{n})\,\mathrm{dA}\right]_i = \int_{\partial\Omega} e_{ijk}\,x_j\,\sigma_{kl}\,n_l\,\mathrm{dA} = \int_{\partial\Omega} A_{il}\,n_l\,\mathrm{dA} = \int_{\partial\Omega}\boldsymbol{A}\cdot\mathbf{n}\,\mathrm{dA}$$

where $[\,]_i$ represents the i-th component of the vector. Using the divergence theorem

$$\int_{\partial\Omega}\boldsymbol{A}\cdot\mathbf{n}\,\mathrm{dA} = \int_{\Omega}\boldsymbol{\nabla}\cdot\boldsymbol{A}\,\mathrm{dV} = \int_{\Omega}\frac{\partial A_{il}}{\partial x_l}\,\mathrm{dV} = \int_{\Omega}\frac{\partial}{\partial x_l}(e_{ijk}\,x_j\,\sigma_{kl})\,\mathrm{dV}\,.$$

Differentiating,

$$\int_{\partial\Omega}\boldsymbol{A}\cdot\mathbf{n}\,\mathrm{dA} = \int_{\Omega}\left[e_{ijk}\,\delta_{jl}\,\sigma_{kl} + e_{ijk}\,x_j\,\frac{\partial\sigma_{kl}}{\partial x_l}\right]\mathrm{dV} = \int_{\Omega}\left[e_{ijk}\,\sigma_{kj} + e_{ijk}\,x_j\,\frac{\partial\sigma_{kl}}{\partial x_l}\right]\mathrm{dV} = \int_{\Omega}\left[e_{ijk}\,\sigma_{kj} + e_{ijk}\,x_j\,[\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}]_l\right]\mathrm{dV}\,.$$

Expressed in direct tensor notation,

$$\int_{\partial\Omega}\boldsymbol{A}\cdot\mathbf{n}\,\mathrm{dA} = \int_{\Omega}\left[[\mathcal{E}:\boldsymbol{\sigma}^T]_i + [\mathbf{x}\times(\boldsymbol{\nabla}\cdot\boldsymbol{\sigma})]_i\right]\mathrm{dV}$$

where $\mathcal{E}$ is the third-order permutation tensor. Therefore,

$$\left[\int_{\partial\Omega}\mathbf{x}\times(\boldsymbol{\sigma}\cdot\mathbf{n})\,\mathrm{dA}\right]_i == \int_{\Omega}\left[[\mathcal{E}:\boldsymbol{\sigma}^T]_i + [\mathbf{x}\times(\boldsymbol{\nabla}\cdot\boldsymbol{\sigma})]_i\right]\mathrm{dV}$$

or,

$$\int_{\partial\Omega}\mathbf{x}\times(\boldsymbol{\sigma}\cdot\mathbf{n})\,\mathrm{dA} == \int_{\Omega}\left[\mathcal{E}:\boldsymbol{\sigma}^T + \mathbf{x}\times(\boldsymbol{\nabla}\cdot\boldsymbol{\sigma})\right]\mathrm{dV}\,.$$

The balance of angular momentum can then be written as

$$\int_{\Omega}\mathbf{x}\times\left[\frac{\partial}{\partial t}(\rho\,\mathbf{v})\right]\mathrm{dV} = -\int_{\Omega}\boldsymbol{\nabla}\cdot[[\mathbf{x}\times(\rho\,\mathbf{v})]\otimes\mathbf{v}]\,\mathrm{dV} + \int_{\Omega}\left[\mathcal{E}:\boldsymbol{\sigma}^T + \mathbf{x}\times(\boldsymbol{\nabla}\cdot\boldsymbol{\sigma})\right]\mathrm{dV} + \int_{\Omega}\mathbf{x}\times(\rho\,\mathbf{b})\,\mathrm{dV}\,.$$

Since Ω is an arbitrary volume, we have

$$\mathbf{x}\times\left[\frac{\partial}{\partial t}(\rho\,\mathbf{v})\right] = -\boldsymbol{\nabla}\cdot[[\mathbf{x}\times(\rho\,\mathbf{v})]\otimes\mathbf{v}] + \mathcal{E}:\boldsymbol{\sigma}^T + \mathbf{x}\times(\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}) + \mathbf{x}\times(\rho\,\mathbf{b})$$

or,

$$\mathbf{x}\times\left[\frac{\partial}{\partial t}(\rho\,\mathbf{v}) - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} - \rho\,\mathbf{b}\right] = -\boldsymbol{\nabla}\cdot[[\mathbf{x}\times(\rho\,\mathbf{v})]\otimes\mathbf{v}] + \mathcal{E}:\boldsymbol{\sigma}^T\,.$$

Using the identity,

$$\boldsymbol{\nabla}\cdot(\mathbf{u}\otimes\mathbf{v}) = (\boldsymbol{\nabla}\cdot\mathbf{v})\mathbf{u} + (\boldsymbol{\nabla}\mathbf{u})\cdot\mathbf{v}$$

we get

$$\boldsymbol{\nabla}\cdot[[\mathbf{x}\times(\rho\,\mathbf{v})]\otimes\mathbf{v}] = (\boldsymbol{\nabla}\cdot\mathbf{v})[\mathbf{x}\times(\rho\,\mathbf{v})] + (\boldsymbol{\nabla}[\mathbf{x}\times(\rho\,\mathbf{v})])\cdot\mathbf{v}\,.$$

The second term on the right can be further simplified using index notation as follows.

$$\begin{aligned}
[(\boldsymbol{\nabla}[\mathbf{x}\times(\rho\,\mathbf{v})])\cdot\mathbf{v}]_i = [(\boldsymbol{\nabla}[\rho\,(\mathbf{x}\times\mathbf{v})])\cdot\mathbf{v}]_i &= \frac{\partial}{\partial x_l}(\rho\,e_{ijk}\,x_j\,v_k)\,v_l \\
&= e_{ijk}\left[\frac{\partial\rho}{\partial x_l}\,x_j\,v_k\,v_l + \rho\,\frac{\partial x_j}{\partial x_l}\,v_k\,v_l + \rho\,x_j\,\frac{\partial v_k}{\partial x_l}\,v_l\right] \\
&= (e_{ijk}\,x_j\,v_k)\left(\frac{\partial\rho}{\partial x_l}\,v_l\right) + \rho\,(e_{ijk}\,\delta_{jl}\,v_k\,v_l) + e_{ijk}\,x_j\left(\rho\,\frac{\partial v_k}{\partial x_l}\,v_l\right) \\
&= [(\mathbf{x}\times\mathbf{v})(\boldsymbol{\nabla}\rho\cdot\mathbf{v}) + \rho\,\mathbf{v}\times\mathbf{v} + \mathbf{x}\times(\rho\,\boldsymbol{\nabla}\mathbf{v}\cdot\mathbf{v})]_i \\
&= [(\mathbf{x}\times\mathbf{v})(\boldsymbol{\nabla}\rho\cdot\mathbf{v}) + \mathbf{x}\times(\rho\,\boldsymbol{\nabla}\mathbf{v}\cdot\mathbf{v})]_i\,.
\end{aligned}$$

Therefore we can write

$$\boldsymbol{\nabla}\cdot[[\mathbf{x}\times(\rho\,\mathbf{v})]\otimes\mathbf{v}] = (\rho\,\boldsymbol{\nabla}\cdot\mathbf{v})(\mathbf{x}\times\ \mathbf{v}) + (\boldsymbol{\nabla}\rho\cdot\mathbf{v})(\mathbf{x}\times\mathbf{v}) + \mathbf{x}\times(\rho\,\boldsymbol{\nabla}\mathbf{v}\cdot\mathbf{v})]\,.$$

The balance of angular momentum then takes the form

$$\mathbf{x}\times\left[\frac{\partial}{\partial t}(\rho\,\mathbf{v}) - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} - \rho\,\mathbf{b}\right] = -(\rho\,\boldsymbol{\nabla}\cdot\mathbf{v})(\mathbf{x}\times\ \mathbf{v}) - (\boldsymbol{\nabla}\rho\cdot\mathbf{v})(\mathbf{x}\times\mathbf{v}) - \mathbf{x}\times(\rho\,\boldsymbol{\nabla}\mathbf{v}\cdot\mathbf{v}) + \mathcal{E}:\boldsymbol{\sigma}^T$$

or,

$$\mathbf{x}\times\left[\frac{\partial}{\partial t}(\rho\,\mathbf{v}) + \rho\,\boldsymbol{\nabla}\mathbf{v}\cdot\mathbf{v} - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} - \rho\,\mathbf{b}\right] = -(\rho\,\boldsymbol{\nabla}\cdot\mathbf{v})(\mathbf{x}\times\ \mathbf{v}) - (\boldsymbol{\nabla}\rho\cdot\mathbf{v})(\mathbf{x}\times\mathbf{v}) + \mathcal{E}:\boldsymbol{\sigma}^T$$

or,

$$\mathbf{x}\times\left[\rho\frac{\partial\mathbf{v}}{\partial t}+\frac{\partial\rho}{\partial t}\,\mathbf{v}+\rho\,\boldsymbol{\nabla}\mathbf{v}\cdot\mathbf{v}-\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}-\rho\,\mathbf{b}\right]=-(\rho\,\boldsymbol{\nabla}\cdot\mathbf{v})(\mathbf{x}\times\ \mathbf{v})-(\boldsymbol{\nabla}\rho\cdot\mathbf{v})(\mathbf{x}\times\mathbf{v})+\mathcal{E}:\boldsymbol{\sigma}^T$$

The material time derivative of $\mathbf{v}$ is defined as

$$\dot{\mathbf{v}}=\frac{\partial\mathbf{v}}{\partial t}+\boldsymbol{\nabla}\mathbf{v}\cdot\mathbf{v}\,.$$

Therefore,

$$\mathbf{x}\times[\rho\,\dot{\mathbf{v}}-\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}-\rho\,\mathbf{b}]=-\mathbf{x}\times\frac{\partial\rho}{\partial t}\,\mathbf{v}+-(\rho\,\boldsymbol{\nabla}\cdot\mathbf{v})(\mathbf{x}\times\ \mathbf{v})-(\boldsymbol{\nabla}\rho\cdot\mathbf{v})(\mathbf{x}\times\mathbf{v})+\mathcal{E}:\boldsymbol{\sigma}^T\,.$$

Also, from the conservation of linear momentum

$$\rho\,\dot{\mathbf{v}}-\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}-\rho\,\mathbf{b}=0\,.$$

Hence,

$$\begin{aligned}0&=\mathbf{x}\times\frac{\partial\rho}{\partial t}\,\mathbf{v}+(\rho\,\boldsymbol{\nabla}\cdot\mathbf{v})(\mathbf{x}\times\ \mathbf{v})+(\boldsymbol{\nabla}\rho\cdot\mathbf{v})(\mathbf{x}\times\mathbf{v})-\mathcal{E}:\boldsymbol{\sigma}^T\\&=\left(\frac{\partial\rho}{\partial t}+\rho\boldsymbol{\nabla}\cdot\mathbf{v}+\boldsymbol{\nabla}\rho\cdot\mathbf{v}\right)(\mathbf{x}\times\mathbf{v})-\mathcal{E}:\boldsymbol{\sigma}^T\,.\end{aligned}$$

The material time derivative of ρ is defined as

$$\dot{\rho}=\frac{\partial\rho}{\partial t}+\boldsymbol{\nabla}\rho\cdot\mathbf{v}\,.$$

Hence,

$$(\dot{\rho}+\rho\,\boldsymbol{\nabla}\cdot\mathbf{v})(\mathbf{x}\times\mathbf{v})-\mathcal{E}:\boldsymbol{\sigma}^T=0\,.$$

From the balance of mass

$$\dot{\rho}+\rho\,\boldsymbol{\nabla}\cdot\mathbf{v}=0\,.$$

Therefore,

$$\mathcal{E}:\boldsymbol{\sigma}^T=0\,.$$

In index notation,

$$e_{ijk}\,\sigma_{kj}=0\,.$$

Expanding out, we get

$$\sigma_{12}-\sigma_{21}=0\,;\;\;\sigma_{23}-\sigma_{32}=0\,;\;\;\sigma_{31}-\sigma_{13}=0\,.$$

Hence,

$$\boxed{\boldsymbol{\sigma}=\boldsymbol{\sigma}^T}$$

4. Show that the balance of energy can be expressed as:

$$\rho\,\dot{e}-\boldsymbol{\sigma}:(\boldsymbol{\nabla}\mathbf{v})+\boldsymbol{\nabla}\cdot\mathbf{q}-\rho\,s=0$$

where $\rho(\mathbf{x},t)$ is the mass density, $e(\mathbf{x},t)$ is the internal energy per unit mass, $\boldsymbol{\sigma}(\mathbf{x},t)$ is the Cauchy stress, $\mathbf{v}(\mathbf{x},t)$ is the particle velocity, $\mathbf{q}$ is the heat flux vector, and s is the rate at which energy is generated by sources inside the volume (per unit mass).

Recall the general balance equation

$$\frac{d}{dt}\left[\int_\Omega f(\mathbf{x},t)\;\text{dV}\right]=\int_{\partial\Omega}f(\mathbf{x},t)[u_n(\mathbf{x},t)-\mathbf{v}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x},t)]\;\text{dA}+\int_{\partial\Omega}g(\mathbf{x},t)\;\text{dA}+\int_\Omega h(\mathbf{x},t)\;\text{dV}\,.$$

In this case, the physical quantity to be conserved the total energy density which is the sum of the internal energy density and the kinetic energy density, i.e., $f=\rho\,e+1/2\,\rho\,|\mathbf{v}\cdot\mathbf{v}|$. The energy source at the surface is a sum of the rate of work done by the applied tractions and the rate of heat leaving the volume (per unit area), i.e, $g=\mathbf{v}\cdot\mathbf{t}-\mathbf{q}\cdot\mathbf{n}$ where $\mathbf{n}$ is the outward unit normal to the surface. The energy source inside the body is the sum of the rate of work done by the body forces and the rate of energy generated by internal sources, i.e., $h=\mathbf{v}\cdot(\rho\mathbf{b})+\rho\,s$.

Hence we have

$$\frac{d}{dt}\left[\int_\Omega\rho\,\left(e+\frac{1}{2}\,\mathbf{v}\cdot\mathbf{v}\right)\;\text{dV}\right]=\int_{\partial\Omega}\rho\,\left(e+\frac{1}{2}\,\mathbf{v}\cdot\mathbf{v}\right)(u_n-\mathbf{v}\cdot\mathbf{n})\;\text{dA}+\int_{\partial\Omega}(\mathbf{v}\cdot\mathbf{t}-\mathbf{q}\cdot\mathbf{n})\;\text{dA}+\int_\Omega\rho\,(\mathbf{v}\cdot\mathbf{b}+s)\;\text{dV}\,.$$

Let Ω be a control volume that does not change with time. Then we get

$$\int_\Omega\frac{\partial}{\partial t}\left[\rho\,\left(e+\frac{1}{2}\,\mathbf{v}\cdot\mathbf{v}\right)\right]\;\text{dV}=-\int_{\partial\Omega}\rho\,\left(e+\frac{1}{2}\,\mathbf{v}\cdot\mathbf{v}\right)(\mathbf{v}\cdot\mathbf{n})\;\text{dA}+\int_{\partial\Omega}(\mathbf{v}\cdot\mathbf{t}-\mathbf{q}\cdot\mathbf{n})\;\text{dA}+\int_\Omega\rho\,(\mathbf{v}\cdot\mathbf{b}+s)\;\text{dV}\,.$$

Using the relation $\mathbf{t}=\boldsymbol{\sigma}\cdot\mathbf{n}$, the identity $\mathbf{v}\cdot(\boldsymbol{\sigma}\cdot\mathbf{n})=(\boldsymbol{\sigma}^T\cdot\mathbf{v})\cdot\mathbf{n}$, and invoking the symmetry of the stress tensor, we get

$$\int_\Omega\frac{\partial}{\partial t}\left[\rho\,\left(e+\frac{1}{2}\,\mathbf{v}\cdot\mathbf{v}\right)\right]\;\text{dV}=-\int_{\partial\Omega}\rho\,\left(e+\frac{1}{2}\,\mathbf{v}\cdot\mathbf{v}\right)(\mathbf{v}\cdot\mathbf{n})\;\text{dA}+\int_{\partial\Omega}(\boldsymbol{\sigma}\cdot\mathbf{v}-\mathbf{q})\cdot\mathbf{n}\;\text{dA}+\int_\Omega\rho\,(\mathbf{v}\cdot\mathbf{b}+s)\;\text{dV}\,.$$

We now apply the divergence theorem to the surface integrals to get

$$\int_\Omega\frac{\partial}{\partial t}\left[\rho\,\left(e+\frac{1}{2}\,\mathbf{v}\cdot\mathbf{v}\right)\right]\;\text{dV}=-\int_\Omega\boldsymbol{\nabla}\cdot\left[\rho\,\left(e+\frac{1}{2}\,\mathbf{v}\cdot\mathbf{v}\right)\mathbf{v}\right]\;\text{dA}+\int_\Omega\boldsymbol{\nabla}\cdot(\boldsymbol{\sigma}\cdot\mathbf{v})\;\text{dA}-\int_\Omega\boldsymbol{\nabla}\cdot\mathbf{q}\;\text{dA}+\int_\Omega\rho\,(\mathbf{v}\cdot\mathbf{b}+s)\;\text{dV}\,.$$

Since Ω is arbitrary, we have

$$\frac{\partial}{\partial t}\left[\rho\,\left(e+\frac{1}{2}\,\mathbf{v}\cdot\mathbf{v}\right)\right] = -\boldsymbol{\nabla}\cdot\left[\rho\,\left(e+\frac{1}{2}\,\mathbf{v}\cdot\mathbf{v}\right)\mathbf{v}\right] + \boldsymbol{\nabla}\cdot(\boldsymbol{\sigma}\cdot\mathbf{v}) - \boldsymbol{\nabla}\cdot\mathbf{q} + \rho\,(\mathbf{v}\cdot\mathbf{b}+s)\,.$$

Expanding out the left hand side, we have

$$\begin{aligned}\frac{\partial}{\partial t}\left[\rho\,\left(e+\frac{1}{2}\,\mathbf{v}\cdot\mathbf{v}\right)\right] &= \frac{\partial\rho}{\partial t}\left(e+\frac{1}{2}\,\mathbf{v}\cdot\mathbf{v}\right)+\rho\,\left(\frac{\partial e}{\partial t}+\frac{1}{2}\,\frac{\partial}{\partial t}(\mathbf{v}\cdot\mathbf{v})\right)\\ &= \frac{\partial\rho}{\partial t}\left(e+\frac{1}{2}\,\mathbf{v}\cdot\mathbf{v}\right)+\rho\,\frac{\partial e}{\partial t}+\rho\,\frac{\partial\mathbf{v}}{\partial t}\cdot\mathbf{v}\,.\end{aligned}$$

For the first term on the right hand side, we use the identity $\boldsymbol{\nabla}\cdot(\varphi\,\mathbf{v}) = \varphi\,\boldsymbol{\nabla}\cdot\mathbf{v} + \boldsymbol{\nabla}\varphi\cdot\mathbf{v}$ to get

$$\begin{aligned}\boldsymbol{\nabla}\cdot\left[\rho\,\left(e+\frac{1}{2}\,\mathbf{v}\cdot\mathbf{v}\right)\mathbf{v}\right] &= \rho\,\left(e+\frac{1}{2}\,\mathbf{v}\cdot\mathbf{v}\right)\,\boldsymbol{\nabla}\cdot\mathbf{v} + \boldsymbol{\nabla}\left[\rho\,\left(e+\frac{1}{2}\,\mathbf{v}\cdot\mathbf{v}\right)\right]\cdot\mathbf{v}\\ &= \rho\,\left(e+\frac{1}{2}\,\mathbf{v}\cdot\mathbf{v}\right)\,\boldsymbol{\nabla}\cdot\mathbf{v} + \left(e+\frac{1}{2}\,\mathbf{v}\cdot\mathbf{v}\right)\boldsymbol{\nabla}\rho\cdot\mathbf{v} + \rho\,\boldsymbol{\nabla}\left(e+\frac{1}{2}\,\mathbf{v}\cdot\mathbf{v}\right)\cdot\mathbf{v}\\ &= \rho\,\left(e+\frac{1}{2}\,\mathbf{v}\cdot\mathbf{v}\right)\,\boldsymbol{\nabla}\cdot\mathbf{v} + \left(e+\frac{1}{2}\,\mathbf{v}\cdot\mathbf{v}\right)\boldsymbol{\nabla}\rho\cdot\mathbf{v} + \rho\,\boldsymbol{\nabla}e\cdot\mathbf{v} + \frac{1}{2}\,\rho\,\boldsymbol{\nabla}(\mathbf{v}\cdot\mathbf{v})\cdot\mathbf{v}\\ &= \rho\,\left(e+\frac{1}{2}\,\mathbf{v}\cdot\mathbf{v}\right)\,\boldsymbol{\nabla}\cdot\mathbf{v} + \left(e+\frac{1}{2}\,\mathbf{v}\cdot\mathbf{v}\right)\boldsymbol{\nabla}\rho\cdot\mathbf{v} + \rho\,\boldsymbol{\nabla}e\cdot\mathbf{v} + \rho\,(\boldsymbol{\nabla}\mathbf{v}^T\cdot\mathbf{v})\cdot\mathbf{v}\\ &= \rho\,\left(e+\frac{1}{2}\,\mathbf{v}\cdot\mathbf{v}\right)\,\boldsymbol{\nabla}\cdot\mathbf{v} + \left(e+\frac{1}{2}\,\mathbf{v}\cdot\mathbf{v}\right)\boldsymbol{\nabla}\rho\cdot\mathbf{v} + \rho\,\boldsymbol{\nabla}e\cdot\mathbf{v} + \rho\,(\boldsymbol{\nabla}\mathbf{v}\cdot\mathbf{v})\cdot\mathbf{v}\,.\end{aligned}$$

For the second term on the right we use the identity $\boldsymbol{\nabla}\cdot(\boldsymbol{S}^T\cdot\mathbf{v}) = \boldsymbol{S}:\boldsymbol{\nabla}\mathbf{v} + (\boldsymbol{\nabla}\cdot\boldsymbol{S})\cdot\mathbf{v}$ and the symmetry of the Cauchy stress tensor to get

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\sigma}\cdot\mathbf{v}) = \boldsymbol{\sigma}:\boldsymbol{\nabla}\mathbf{v} + (\boldsymbol{\nabla}\cdot\boldsymbol{\sigma})\cdot\mathbf{v}\,.$$

After collecting terms and rearranging, we get

$$\left(\frac{\partial\rho}{\partial t}+\rho\,\boldsymbol{\nabla}\cdot\mathbf{v}+\boldsymbol{\nabla}\rho\cdot\mathbf{v}\right)\left(e+\frac{1}{2}\,\mathbf{v}\cdot\mathbf{v}\right)+\left(\rho\,\frac{\partial\mathbf{v}}{\partial t}+\rho\,\boldsymbol{\nabla}\mathbf{v}\cdot\mathbf{v}-\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}-\rho\,\mathbf{b}\right)\cdot\mathbf{v}+\rho\,\left(\frac{\partial e}{\partial t}+\boldsymbol{\nabla}e\cdot\mathbf{v}\right)+-\boldsymbol{\sigma}:\boldsymbol{\nabla}\mathbf{v}+\boldsymbol{\nabla}\cdot\mathbf{q}-\rho\,s=0\,.$$

Applying the balance of mass to the first term and the balance of linear momentum to the second term, and using the material time derivative of the internal energy

$$\dot{e} = \frac{\partial e}{\partial t} + \boldsymbol{\nabla}e\cdot\mathbf{v}$$

we get the final form of the balance of energy:

$$\boxed{\rho\,\dot{e} - \boldsymbol{\sigma}:\boldsymbol{\nabla}\mathbf{v} + \boldsymbol{\nabla}\cdot\mathbf{q} - \rho\,s = 0\,.}$$

5. Show that:

$$[(\mathbf{v}\cdot\mathbf{a})(\boldsymbol{S}\cdot\mathbf{b})]\cdot\mathbf{n} = \mathbf{a}\cdot[\{\mathbf{v}\otimes(\boldsymbol{S}^T\cdot\mathbf{n})\}\cdot\mathbf{b}]\,.$$

Using the identity $\mathbf{a}\cdot(\boldsymbol{A}^T\cdot\mathbf{b}) = \mathbf{b}\cdot(\boldsymbol{A}\cdot\mathbf{a})$ we have

$$\mathbf{n}\cdot[(\mathbf{v}\cdot\mathbf{a})(\boldsymbol{S}\cdot\mathbf{b})] = \mathbf{b}\cdot[(\mathbf{v}\cdot\mathbf{a})(\boldsymbol{S}^T\cdot\mathbf{n})]\,.$$

Also, using the definition $(\mathbf{u}\otimes\mathbf{v})\cdot\mathbf{a} = (\mathbf{a}\cdot\mathbf{v})\mathbf{u}$ we have

$$(\mathbf{v}\cdot\mathbf{a})(\boldsymbol{S}^T\cdot\mathbf{n}) = [(\boldsymbol{S}^T\cdot\mathbf{n})\otimes\mathbf{v}]\cdot\mathbf{a}\,.$$

Therefore,

$$\mathbf{n}\cdot[(\mathbf{v}\cdot\mathbf{a})(\boldsymbol{S}\cdot\mathbf{b})] = \mathbf{b}\cdot[\{(\boldsymbol{S}^T\cdot\mathbf{n})\otimes\mathbf{v}\}\cdot\mathbf{a}]\,.$$

Using the identity $\mathbf{a}\cdot(\boldsymbol{A}^T\cdot\mathbf{b}) = \mathbf{b}\cdot(\boldsymbol{A}\cdot\mathbf{a})$ we have

$$\mathbf{b}\cdot[\{(\boldsymbol{S}^T\cdot\mathbf{n})\otimes\mathbf{v}\}\cdot\mathbf{a}] = \mathbf{a}\cdot[\{(\boldsymbol{S}^T\cdot\mathbf{n})\otimes\mathbf{v}\}^T\cdot\mathbf{b}]\,.$$

Finally, using the relation $(\mathbf{u}\otimes\mathbf{v})^T = \mathbf{v}\otimes\mathbf{u}$, we get

$$\mathbf{a}\cdot[\{(\boldsymbol{S}^T\cdot\mathbf{n})\otimes\mathbf{v}\}^T\cdot\mathbf{b}] = \mathbf{a}\cdot[\{\mathbf{v}\otimes(\boldsymbol{S}^T\cdot\mathbf{n})\}\cdot\mathbf{b}]\,.$$

Hence,

$$\boxed{[(\mathbf{v}\cdot\mathbf{a})(\boldsymbol{S}\cdot\mathbf{b})]\cdot\mathbf{n} = \mathbf{a}\cdot[\{\mathbf{v}\otimes(\boldsymbol{S}^T\cdot\mathbf{n})\}\cdot\mathbf{b}]} \qquad \square$$

6. Let $\mathbf{v}$ be a vector field and let $\boldsymbol{S}$ be a second-order tensor field. Let $\mathbf{a}$ and $\mathbf{b}$ be two arbitrary vectors. Show that

$$\boldsymbol{\nabla}\cdot[(\mathbf{v}\cdot\mathbf{a})(\boldsymbol{S}\cdot\mathbf{b})] = \mathbf{a}\cdot[\{\boldsymbol{\nabla}\mathbf{v}\cdot\boldsymbol{S}+\mathbf{v}\otimes(\boldsymbol{\nabla}\cdot\boldsymbol{S}^T)\}\cdot\mathbf{b}]\,.$$

Using the identity $\boldsymbol{\nabla}\cdot(\varphi\,\mathbf{u}) = \mathbf{u}\cdot\boldsymbol{\nabla}\varphi + \varphi\,\boldsymbol{\nabla}\cdot\mathbf{u}$ we have

$$\boldsymbol{\nabla}\cdot[(\mathbf{v}\cdot\mathbf{a})(\boldsymbol{S}\cdot\mathbf{b})] = (\boldsymbol{S}\cdot\mathbf{b})\cdot\boldsymbol{\nabla}(\mathbf{v}\cdot\mathbf{a}) + (\mathbf{v}\cdot\mathbf{a})\,\boldsymbol{\nabla}\cdot(\boldsymbol{S}\cdot\mathbf{b})\,.$$

From the identity $\boldsymbol{\nabla}(\mathbf{u}\cdot\mathbf{v}) = \boldsymbol{\nabla}\mathbf{u}^T\cdot\mathbf{v}+\boldsymbol{\nabla}\mathbf{v}^T\cdot\mathbf{u}$, we have $\boldsymbol{\nabla}(\mathbf{v}\cdot\mathbf{a}) = \boldsymbol{\nabla}\mathbf{v}^T\cdot\mathbf{a}+\boldsymbol{\nabla}\mathbf{a}^T\cdot\mathbf{v}$.
Since $\mathbf{a}$ is constant, $\boldsymbol{\nabla}\mathbf{a} = 0$, and we have

$$(\boldsymbol{S}\cdot\mathbf{b})\cdot\boldsymbol{\nabla}(\mathbf{v}\cdot\mathbf{a}) = (\boldsymbol{S}\cdot\mathbf{b})\cdot(\boldsymbol{\nabla}\mathbf{v}^T\cdot\mathbf{a})\,.$$

From the relation $\mathbf{a}\cdot(\boldsymbol{A}^T\cdot\mathbf{b}) = \mathbf{b}\cdot(\boldsymbol{A}\cdot\mathbf{a})$ we have

$$(\boldsymbol{S}\cdot\mathbf{b})\cdot(\boldsymbol{\nabla}\mathbf{v}^T\cdot\mathbf{a}) = \mathbf{a}\cdot[\boldsymbol{\nabla}\mathbf{v}\cdot(\boldsymbol{S}\cdot\mathbf{b})]\,.$$

Using the relation $\boldsymbol{A}\cdot(\boldsymbol{B}\cdot\mathbf{b}) = (\boldsymbol{A}\cdot\boldsymbol{B})\cdot\mathbf{b}$, we get

$$\boldsymbol{\nabla}\mathbf{v}\cdot(\boldsymbol{S}\cdot\mathbf{b}) = (\boldsymbol{\nabla}\mathbf{v}\cdot\boldsymbol{S})\cdot\mathbf{b}\,.$$

Therefore, the final form of the first term is

$$(\boldsymbol{S}\cdot\mathbf{b})\cdot\boldsymbol{\nabla}(\mathbf{v}\cdot\mathbf{a}) = \mathbf{a}\cdot[(\boldsymbol{\nabla}\mathbf{v}\cdot\boldsymbol{S})\cdot\mathbf{b}]\,.$$

For the second term, from the identity $\boldsymbol{\nabla}\cdot(\boldsymbol{S}^T\cdot\mathbf{v}) = \boldsymbol{S}:\boldsymbol{\nabla}\mathbf{v}+\mathbf{v}\cdot(\boldsymbol{\nabla}\cdot\boldsymbol{S})$ we get, $\boldsymbol{\nabla}\cdot(\boldsymbol{S}\cdot\mathbf{b}) = \boldsymbol{S}^T:\boldsymbol{\nabla}\mathbf{b}+\mathbf{b}\cdot(\boldsymbol{\nabla}\cdot\boldsymbol{S}^T)$.
Since $\mathbf{b}$ is constant, $\boldsymbol{\nabla}\mathbf{b} = 0$, and we have

$$(\mathbf{v}\cdot\mathbf{a})\,\boldsymbol{\nabla}\cdot(\boldsymbol{S}\cdot\mathbf{b}) = (\mathbf{v}\cdot\mathbf{a})\,[\mathbf{b}\cdot(\boldsymbol{\nabla}\cdot\boldsymbol{S}^T)] = \mathbf{a}\cdot[\{\mathbf{b}\cdot(\boldsymbol{\nabla}\cdot\boldsymbol{S}^T)\}\,\mathbf{v}]\,.$$

From the definition $(\mathbf{u}\otimes\mathbf{v})\cdot\mathbf{a} = (\mathbf{a}\cdot\mathbf{v})\mathbf{u}$, we get

$$[\mathbf{b}\cdot(\boldsymbol{\nabla}\cdot\boldsymbol{S}^T)]\,\mathbf{v} = [\mathbf{v}\otimes(\boldsymbol{\nabla}\cdot\boldsymbol{S}^T)]\cdot\mathbf{b}\,.$$

Therefore, the final form of the second term is

$$(\mathbf{v}\cdot\mathbf{a})\,\boldsymbol{\nabla}\cdot(\boldsymbol{S}\cdot\mathbf{b}) = \mathbf{a}\cdot[\mathbf{v}\otimes(\boldsymbol{\nabla}\cdot\boldsymbol{S}^T)]\cdot\mathbf{b}\,.$$

Adding the two terms, we get

$$\boldsymbol{\nabla}\cdot[(\mathbf{v}\cdot\mathbf{a})(\boldsymbol{S}\cdot\mathbf{b})] = \mathbf{a}\cdot[(\boldsymbol{\nabla}\mathbf{v}\cdot\boldsymbol{S})\cdot\mathbf{b}]+\mathbf{a}\cdot[\mathbf{v}\otimes(\boldsymbol{\nabla}\cdot\boldsymbol{S}^T)]\cdot\mathbf{b}\,.$$

Therefore,

$$\boxed{\boldsymbol{\nabla}\cdot[(\mathbf{v}\cdot\mathbf{a})(\boldsymbol{S}\cdot\mathbf{b})] = \mathbf{a}\cdot[\{\boldsymbol{\nabla}\mathbf{v}\cdot\boldsymbol{S}+\mathbf{v}\otimes(\boldsymbol{\nabla}\cdot\boldsymbol{S}^T)\}\cdot\mathbf{b}]} \qquad \square$$

7. Let Ω be a body and let $\partial\Omega$ be its surface. Let $\mathbf{n}$ be the normal to the surface. Let $\mathbf{v}$ be a vector field on Ω and let $\boldsymbol{S}$ be a second-order tensor field on Ω. Show that

$$\int_{\partial\Omega}\mathbf{v}\otimes(\boldsymbol{S}^T\cdot\mathbf{n})\,\text{dA} = \int_\Omega[\boldsymbol{\nabla}\mathbf{v}\cdot\boldsymbol{S}+\mathbf{v}\otimes(\boldsymbol{\nabla}\cdot\boldsymbol{S}^T)]\,\text{dV}\,.$$

Recall the relation

$$\boldsymbol{\nabla}\cdot[(\mathbf{v}\cdot\mathbf{a})(\boldsymbol{S}\cdot\mathbf{b})] = \mathbf{a}\cdot[\{\boldsymbol{\nabla}\mathbf{v}\cdot\boldsymbol{S}+\mathbf{v}\otimes(\boldsymbol{\nabla}\cdot\boldsymbol{S}^T)\}\cdot\mathbf{b}]\,.$$

Integrating over the volume, we have

$$\int_\Omega\boldsymbol{\nabla}\cdot[(\mathbf{v}\cdot\mathbf{a})(\boldsymbol{S}\cdot\mathbf{b})]\,\text{dV} = \int_\Omega\mathbf{a}\cdot[\{\boldsymbol{\nabla}\mathbf{v}\cdot\boldsymbol{S}+\mathbf{v}\otimes(\boldsymbol{\nabla}\cdot\boldsymbol{S}^T)\}\cdot\mathbf{b}\,\text{dV}\,.$$

Since $\mathbf{a}$ and $\mathbf{b}$ are constant, we have

$$\int_\Omega\boldsymbol{\nabla}\cdot[(\mathbf{v}\cdot\mathbf{a})(\boldsymbol{S}\cdot\mathbf{b})]\,\text{dV} = \mathbf{a}\cdot\left[\left\{\int_\Omega[\boldsymbol{\nabla}\mathbf{v}\cdot\boldsymbol{S}+\mathbf{v}\otimes(\boldsymbol{\nabla}\cdot\boldsymbol{S}^T)]\,\text{dV}\right\}\cdot\mathbf{b}\right]\,.$$

From the divergence theorem,

$$\int_\Omega\boldsymbol{\nabla}\cdot\mathbf{u}\,\text{dV} = \int_{\partial\Omega}\mathbf{u}\cdot\mathbf{n}\,\text{dA}$$

we get

$$\int_\Omega\boldsymbol{\nabla}\cdot[(\mathbf{v}\cdot\mathbf{a})(\boldsymbol{S}\cdot\mathbf{b})]\,\text{dV} = \int_{\partial\Omega}[(\mathbf{v}\cdot\mathbf{a})(\boldsymbol{S}\cdot\mathbf{b})]\cdot\mathbf{n}\,\text{dA}\,.$$

Using the relation

$$[(\mathbf{v}\cdot\mathbf{a})(\boldsymbol{S}\cdot\mathbf{b})]\cdot\mathbf{n} = \mathbf{a}\cdot[\{\mathbf{v}\otimes(\boldsymbol{S}^T\cdot\mathbf{n})\}\cdot\mathbf{b}]$$

we get

$$\int_\Omega\boldsymbol{\nabla}\cdot[(\mathbf{v}\cdot\mathbf{a})(\boldsymbol{S}\cdot\mathbf{b})]\,\text{dV} = \int_{\partial\Omega}\mathbf{a}\cdot[\{\mathbf{v}\otimes(\boldsymbol{S}^T\cdot\mathbf{n})\}\cdot\mathbf{b}]\,\text{dA}\,.$$

Since $\mathbf{a}$ and $\mathbf{b}$ are constant, we have

$$\int_\Omega\boldsymbol{\nabla}\cdot[(\mathbf{v}\cdot\mathbf{a})(\boldsymbol{S}\cdot\mathbf{b})]\,\text{dV} = \mathbf{a}\cdot\left[\left\{\int_{\partial\Omega}\mathbf{v}\otimes(\boldsymbol{S}^T\cdot\mathbf{n})\,\text{dA}\right\}\cdot\mathbf{b}\right]\,.$$

Therefore,

$$\mathbf{a}\cdot\left[\left\{\int_{\partial\Omega}\mathbf{v}\otimes(\boldsymbol{S}^T\cdot\mathbf{n})\,\text{dA}\right\}\cdot\mathbf{b}\right] = \mathbf{a}\cdot\left[\left\{\int_\Omega[\boldsymbol{\nabla}\mathbf{v}\cdot\boldsymbol{S}+\mathbf{v}\otimes(\boldsymbol{\nabla}\cdot\boldsymbol{S}^T)]\,\text{dV}\right\}\cdot\mathbf{b}\right]\,.$$

Since $\mathbf{a}$ and $\mathbf{b}$ are arbitrary, we have

$$\boxed{\int_{\partial\Omega}\mathbf{v}\otimes(\boldsymbol{S}^T\cdot\mathbf{n})\,\text{dA} = \int_\Omega[\boldsymbol{\nabla}\mathbf{v}\cdot\boldsymbol{S}+\mathbf{v}\otimes(\boldsymbol{\nabla}\cdot\boldsymbol{S}^T)]\,\text{dV}} \qquad \square$$

8. Let $\partial\Omega$ be a surface. Let $\mathbf{x}$ be the position vector of points on the surface and let $\mathbf{t}$ be a vector field that are defined on $\partial\Omega$. If

$$\int_{\partial\Omega} \mathbf{x} \times \mathbf{t}\ \text{dA} = \mathbf{0}$$

show that

$$\int_{\partial\Omega} \mathbf{x} \otimes \mathbf{t}\ \text{dA} = \int_{\partial\Omega} \mathbf{t} \otimes \mathbf{x}\ \text{dA}\ .$$

If we assume a Cartesian basis, we can write the given relation in index notation as

$$\int_{\partial\Omega} e_{ijk}\, x_j\, t_k\ \text{dA} = 0$$

where e_{ijk} is the Levi-Civita (permutation) symbol. Since e_{ijk} is does not depend upon the position we can write

$$e_{ijk} \int_{\partial\Omega} x_j\, t_k\ \text{dA} = 0\ .$$

Define

$$A_{jk} := \int_{\partial\Omega} x_j\, t_k\ \text{dA}\ .$$

Then,

$$e_{ijk} A_{jk} = 0\ .$$

Expanding, we get

$$\cancelto{0}{e_{i11}} A_{11} + e_{i12}A_{12} + e_{i13}A_{13} + e_{i21}A_{21} + \cancelto{0}{e_{i22}} A_{22} + e_{i23}A_{23} + e_{i31}A_{31} + e_{i32}A_{32} + \cancelto{0}{e_{i33}} A_{33} = 0 \qquad i = 1, 2, 3\ .$$

Expanding further, we get three equations

$$\begin{aligned}
\cancelto{0}{e_{112}A_{12}} + \cancelto{0}{e_{113}A_{13}} + \cancelto{0}{e_{121}A_{21}} + e_{123}A_{23} + \cancelto{0}{e_{131}A_{31}} + e_{132}A_{32} &= 0\\
\cancelto{0}{e_{212}A_{12}} + e_{213}A_{13} + \cancelto{0}{e_{221}A_{21}} + \cancelto{0}{e_{223}A_{23}} + e_{231}A_{31} + \cancelto{0}{e_{232}A_{32}} &= 0\\
e_{312}A_{12} + \cancelto{0}{e_{313}A_{13}} + e_{321}A_{21} + \cancelto{0}{e_{323}A_{23}} + \cancelto{0}{e_{331}A_{31}} + \cancelto{0}{e_{332}A_{32}} &= 0
\end{aligned}$$

or,

$$\begin{aligned}
\cancelto{1}{e_{123}} A_{23} + \cancelto{-1}{e_{132}} A_{32} &= 0\\
\cancelto{-1}{e_{213}} A_{13} + \cancelto{1}{e_{231}} A_{31} &= 0\\
\cancelto{1}{e_{312}} A_{12} + \cancelto{-1}{e_{321}} A_{21} &= 0
\end{aligned}$$

or,

$$A_{23} = A_{32}\ ;\quad A_{13} = A_{31}\ ;\quad A_{12} = A_{21}\ .$$

Therefore,

$$\int_{\partial\Omega} x_2\, t_3\ \text{dA} = \int_{\partial\Omega} x_3\, t_2\ \text{dA} = \int_{\partial\Omega} t_2\, x_3\ \text{dA}\ ;\ \int_{\partial\Omega} x_1\, t_3\ \text{dA} = \int_{\partial\Omega} x_3\, t_1\ \text{dA} = \int_{\partial\Omega} t_1\, x_3\ \text{dA}\ ;\ \int_{\partial\Omega} x_1\, t_2\ \text{dA} = \int_{\partial\Omega} x_2\, t_1\ \text{dA} = \int_{\partial\Omega} t_1\, x_2\ \text{dA}\ .$$

Also, by symmetry,

$$\int_{\partial\Omega} x_1\, t_1\ \text{dA} = \int_{\partial\Omega} t_1\, x_1\ \text{dA}\ ;\ \int_{\partial\Omega} x_2\, t_2\ \text{dA} = \int_{\partial\Omega} t_2\, x_2\ \text{dA}\ ;\ \int_{\partial\Omega} x_3\, t_3\ \text{dA} = \int_{\partial\Omega} t_3\, x_3\ \text{dA}\ .$$

Therefore we may write,

$$\int_{\partial\Omega} x_j\, t_k\ \text{dA} = \int_{\partial\Omega} t_j\, x_k\ \text{dA}\ .$$

Reverting back to direct tensor notation, we get

$$\boxed{\int_{\partial\Omega} \mathbf{x} \otimes \mathbf{t}\ \text{dA} = \int_{\partial\Omega} \mathbf{t} \otimes \mathbf{x}\ \text{dA}\ .}$$

9. Let Ω be a body and let $\partial\Omega$ be its surface. Let $\mathbf{n}$ be the normal to the surface. Let $\mathbf{v}$ be a vector field on Ω. Show that

$$\int_{\Omega} \boldsymbol{\nabla}\mathbf{v}\ \text{dV} = \int_{\partial\Omega} \mathbf{v} \otimes \mathbf{n}\ \text{dA}\ .$$

Recall that

$$\int_{\partial\Omega} \mathbf{v} \otimes (\boldsymbol{S}^T \cdot \mathbf{n})\ \text{dA} = \int_{\Omega} [\boldsymbol{\nabla}\mathbf{v} \cdot \boldsymbol{S} + \mathbf{v} \otimes (\boldsymbol{\nabla} \cdot \boldsymbol{S}^T)]\ \text{dV}$$

where $\boldsymbol{S}$ is any second-order tensor field on Ω. Let us assume that $\boldsymbol{S} = \mathbf{1}$. Then we have

$$\int_{\partial\Omega} \mathbf{v} \otimes (\mathbf{1} \cdot \mathbf{n})\ \text{dA} = \int_{\Omega} [\boldsymbol{\nabla}\mathbf{v} \cdot \mathbf{1} + \mathbf{v} \otimes (\boldsymbol{\nabla} \cdot \mathbf{1})]\ \text{dV}$$

Now,

$$\mathbf{1} \cdot \mathbf{n} = \mathbf{n} \,; \quad \boldsymbol{\nabla} \cdot \mathbf{1} = \mathbf{0} \,; \quad \boldsymbol{A} \cdot \mathbf{1} = \boldsymbol{A}$$

where $\boldsymbol{A}$ is any second-order tensor. Therefore,

$$\int_{\partial\Omega} \mathbf{v} \otimes \mathbf{n} \, \mathrm{dA} = \int_{\Omega} \boldsymbol{\nabla}\mathbf{v} \, \mathrm{dV} \,.$$

Rearranging,

$$\boxed{\int_{\Omega} \boldsymbol{\nabla}\mathbf{v} \, \mathrm{dV} = \int_{\partial\Omega} \mathbf{v} \otimes \mathbf{n} \, \mathrm{dA} \,.}$$

10. Let $\mathbf{v}$ be a vector field. Show that

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\mathbf{v}) = 0 \,.$$

For a second order tensor field $\boldsymbol{S}$, we can define the curl as

$$(\boldsymbol{\nabla} \times \boldsymbol{S}) \cdot \mathbf{a} = \boldsymbol{\nabla} \times (\boldsymbol{S}^T \cdot \mathbf{a})$$

where $\mathbf{a}$ is an arbitrary constant vector. Substituting $\boldsymbol{\nabla}\mathbf{v}$ into the definition, we have

$$[\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\mathbf{v})] \cdot \mathbf{a} = \boldsymbol{\nabla} \times (\boldsymbol{\nabla}\mathbf{v}^T \cdot \mathbf{a}) \,.$$

Since $\mathbf{a}$ is constant, we may write

$$\boldsymbol{\nabla}\mathbf{v}^T \cdot \mathbf{a} = \boldsymbol{\nabla}(\mathbf{v} \cdot \mathbf{a}) = \boldsymbol{\nabla}\varphi$$

where $\varphi = \mathbf{v} \cdot \mathbf{a}$ is a scalar. Hence,

$$[\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\mathbf{v})] \cdot \mathbf{a} = \boldsymbol{\nabla} \times (\boldsymbol{\nabla}\varphi) \,.$$

Since the curl of the gradient of a scalar field is zero (recall potential theory), we have

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\varphi) = \mathbf{0} \,.$$

Hence,

$$[\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\mathbf{v})] \cdot \mathbf{a} = \mathbf{0} \qquad \forall \ \mathbf{a} \,.$$

The arbitrary nature of $\mathbf{a}$ gives us

$$\boxed{\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\mathbf{v}) = \mathbf{0} \,.}$$

11. Let $\mathbf{v}$ be a vector field. Show that

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\mathbf{v}^T) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \times \mathbf{v}) \,.$$

The curl of a second order tensor field $\boldsymbol{S}$ is defined as

$$(\boldsymbol{\nabla} \times \boldsymbol{S}) \cdot \mathbf{a} = \boldsymbol{\nabla} \times (\boldsymbol{S}^T \cdot \mathbf{a})$$

where $\mathbf{a}$ is an arbitrary constant vector. If we write the right hand side in index notation with respect to a Cartesian basis, we have

$$[\boldsymbol{S}^T \cdot \mathbf{a}]_k = [\mathbf{b}]_k = b_k = S_{pk} \, a_p \qquad \text{and} \qquad [\boldsymbol{\nabla} \times \mathbf{b}]_i = e_{ijk} \frac{\partial b_k}{\partial x_j} = e_{ijk} \frac{\partial (S_{pk} \, a_p)}{\partial x_j} = e_{ijk} \frac{\partial S_{pk}}{\partial x_j} \, a_p = [(\boldsymbol{\nabla} \times \boldsymbol{S})]_{ip} \, a_p \,.$$

In the above a quantity $[\,]_i$ represents the i-th component of a vector, and the quantity $[\,]_{ip}$ represents the ip-th components of a second-order tensor. Therefore, in index notation, the curl of a second-order tensor $\boldsymbol{S}$ can be expressed as

$$[\boldsymbol{\nabla} \times \boldsymbol{S}]_{ip} = e_{ijk} \frac{\partial S_{pk}}{\partial x_j} \,.$$

Using the above definition, we get

$$[\boldsymbol{\nabla} \times \boldsymbol{S}^T]_{ip} = e_{ijk} \frac{\partial S_{kp}}{\partial x_j} \,.$$

If $\boldsymbol{S} = \boldsymbol{\nabla}\mathbf{v}$, we have

$$[\boldsymbol{\nabla} \times \boldsymbol{\nabla}\mathbf{v}^T]_{ip} = e_{ijk} \frac{\partial}{\partial x_j}\left(\frac{\partial v_k}{\partial x_p}\right) = \frac{\partial}{\partial x_p}\left(e_{ijk} \frac{\partial v_k}{\partial x_j}\right) = \frac{\partial}{\partial x_p}\left([\boldsymbol{\nabla} \times \mathbf{v}]_i\right) = [\boldsymbol{\nabla}(\boldsymbol{\nabla} \times \mathbf{v})]_{ip} \,.$$

Therefore,

$$\boxed{\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\mathbf{v}^T) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \times \mathbf{v}) \,.}$$

12. Let $\mathbf{u}$ be a displacement field. The displacement gradient tensor is given by $\boldsymbol{\nabla}\mathbf{u}$. Let the skew symmetric part of the displacement gradient tensor (infinitesimal rotation tensor) be

$$\boldsymbol{\omega} = \frac{1}{2}(\boldsymbol{\nabla}\mathbf{u} - \boldsymbol{\nabla}\mathbf{u}^T)\,.$$

Let $\boldsymbol{\theta}$ be the axial vector associated with the skew symmetric tensor $\boldsymbol{\omega}$. Show that

$$\boldsymbol{\theta} = \frac{1}{2}\,\boldsymbol{\nabla}\times\mathbf{u}\,.$$

The axial vector $\mathbf{w}$ of a skew-symmetric tensor $\boldsymbol{W}$ satisfies the condition

$$\boldsymbol{W}\cdot\mathbf{a} = \mathbf{w}\times\mathbf{a}$$

for all vectors $\mathbf{a}$. In index notation (with respect to a Cartesian basis), we have

$$W_{ip}\,a_p = e_{ijk}\,w_j\,a_k$$

Since $e_{ijk} = -e_{ikj}$, we can write

$$W_{ip}\,a_p = -e_{ikj}\,w_j\,a_k \equiv -e_{ipq}\,w_q\,a_p$$

or,

$$W_{ip} = -e_{ipq}\,w_q\,.$$

Therefore, the relation between the components of $\boldsymbol{\omega}$ and $\boldsymbol{\theta}$ is

$$\omega_{ij} = -e_{ijk}\,\theta_k\,.$$

Multiplying both sides by e_{pij}, we get

$$e_{pij}\,\omega_{ij} = -e_{pij}\,e_{ijk}\,\theta_k = -e_{pij}\,e_{kij}\,\theta_k\,.$$

Recall the identity

$$e_{ijk}\,e_{pqk} = \delta_{ip}\,\delta_{jq} - \delta_{iq}\,\delta_{jp}\,.$$

Therefore,

$$e_{ijk}\,e_{pjk} = \delta_{ip}\,\delta_{jj} - \delta_{ij}\,\delta_{jp} = 3\delta_{ip} - \delta_{ip} = 2\delta_{ip}$$

Using the above identity, we get

$$e_{pij}\,\omega_{ij} = -2\delta_{pk}\,\theta_k = -2\theta_p\,.$$

Rearranging,

$$\theta_p = -\frac{1}{2}\,e_{pij}\,\omega_{ij}$$

Now, the components of the tensor $\boldsymbol{\omega}$ with respect to a Cartesian basis are given by

$$\omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

Therefore, we may write

$$\theta_p = -\frac{1}{4}\,e_{pij}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

Since the curl of a vector $\mathbf{v}$ can be written in index notation as

$$\boldsymbol{\nabla}\times\mathbf{v} = e_{ijk}\,\frac{\partial u_k}{\partial x_j}\,\mathbf{e}_i$$

we have

$$e_{pij}\,\frac{\partial u_j}{\partial x_i} = [\boldsymbol{\nabla}\times\mathbf{u}]_p \qquad \text{and} \qquad e_{pij}\,\frac{\partial u_i}{\partial x_j} = -e_{pji}\,\frac{\partial u_i}{\partial x_j} = -[\boldsymbol{\nabla}\times\mathbf{u}]_p$$

where $[\,]_p$ indicates the p-th component of the vector inside the square brackets.

Hence,

$$\theta_p = -\frac{1}{4}\,(-[\boldsymbol{\nabla}\times\mathbf{u}]_p - [\boldsymbol{\nabla}\times\mathbf{u}]_p) = \frac{1}{2}\,[\boldsymbol{\nabla}\times\mathbf{u}]_p\,.$$

Therefore,

$$\boxed{\boldsymbol{\theta} = \frac{1}{2}\boldsymbol{\nabla}\times\mathbf{u}\,.}$$

13. Let $\mathbf{u}$ be a displacement field. Let $\boldsymbol{\varepsilon}$ be the strain field (infinitesimal) corresponding to the displacement field and let $\boldsymbol{\theta}$ be the corresponding infinitesimal rotation vector. Show that

$$\boldsymbol{\nabla}\times\boldsymbol{\varepsilon} = \boldsymbol{\nabla}\boldsymbol{\theta}\,.$$

The infinitesimal strain tensor is given by

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\boldsymbol{\nabla}\mathbf{u} + \boldsymbol{\nabla}\mathbf{u}^T)\,.$$

Therefore,

$$\boldsymbol{\nabla}\times\boldsymbol{\varepsilon} = \frac{1}{2}[\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\mathbf{u}) + \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\mathbf{u}^T)]\,.$$

Recall that

$$\nabla \times (\nabla \mathbf{u}) = \mathbf{0} \qquad \text{and} \qquad \nabla \times (\nabla \mathbf{u}^T) = \nabla(\nabla \times \mathbf{u}) \,.$$

Hence,

$$\nabla \times \boldsymbol{\varepsilon} = \frac{1}{2}[\nabla(\nabla \times \mathbf{u})] \,.$$

Also recall that

$$\boldsymbol{\theta} = \frac{1}{2}\nabla \times \mathbf{u} \,.$$

Therefore,

$$\boxed{\nabla \times \boldsymbol{\varepsilon} = \nabla\boldsymbol{\theta} \,.}$$

14. Show that, for a rigid body motion with infinitesimal rotations, the displacement field $\mathbf{u}(\mathbf{x})$ for can be expressed as

$$\mathbf{u}(\mathbf{x}) = \mathbf{c} + \boldsymbol{\omega} \cdot \mathbf{x}$$

where $\mathbf{c}$ is a constant vector and $\boldsymbol{\omega}$ is the infinitesimal rotation tensor.

Note that for a rigid body motion, the strain $\boldsymbol{\varepsilon}$ is zero. Since

$$\nabla \times \boldsymbol{\varepsilon} = \nabla\boldsymbol{\theta}$$

we have a $\boldsymbol{\theta} =$ constant when $\boldsymbol{\varepsilon} = 0$, i.e., the rotation is homogeneous.

For a homogeneous deformation, the displacement gradient is independent of $\mathbf{x}$, i.e.,

$$\nabla \mathbf{u} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}} = \boldsymbol{H} \qquad \leftarrow \qquad \text{constant} \,.$$

Integrating, we get

$$\mathbf{u}(\mathbf{x}) = \boldsymbol{H} \cdot \mathbf{x} + \mathbf{c} \,.$$

Now the strain and rotation tensors are given by

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T) = \frac{1}{2}(\boldsymbol{H} + \boldsymbol{H}^T) \,; \;\; \boldsymbol{\omega} = \frac{1}{2}(\nabla \mathbf{u} - \nabla \mathbf{u}^T) = \frac{1}{2}(\boldsymbol{H} - \boldsymbol{H}^T) \,.$$

For a rigid body motion, the strain $\boldsymbol{\varepsilon} = 0$. Therefore,

$$\boldsymbol{H} = -\boldsymbol{H}^T \qquad \Longrightarrow \qquad \boldsymbol{\omega} = \boldsymbol{H} \,.$$

Plugging into the expression for $\mathbf{u}$ for a homogeneous deformation, we have

$$\boxed{\mathbf{u}(\mathbf{x}) = \boldsymbol{\omega} \cdot \mathbf{x} + \mathbf{c} \,.}$$

15. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two second order tensors. Show that

$$\boldsymbol{A} : \boldsymbol{B} = (\boldsymbol{A}^T \cdot \boldsymbol{B}) : \mathbf{1} \,.$$

Using index notation,

$$\boldsymbol{A} : \boldsymbol{B} = A_{ij}\, B_{ij} = A^T_{ji}\, B_{ij} = A^T_{ji}\, B_{ik}\, \delta_{jk} = [\boldsymbol{A}^T \cdot \boldsymbol{B}]_{jk}\, \delta_{jk} = (\boldsymbol{A}^T \cdot \boldsymbol{B}) : \mathbf{1} \,.$$

Hence,

$$\boxed{\boldsymbol{A} : \boldsymbol{B} = (\boldsymbol{A}^T \cdot \boldsymbol{B}) : \mathbf{1} \,.}$$

16. Let $\boldsymbol{A}$ be a second order tensor and let $\mathbf{a}$ and $\mathbf{b}$ be two vectors. Show that

$$\boldsymbol{A} : (\mathbf{a} \otimes \mathbf{b}) = (\boldsymbol{A} \cdot \mathbf{b}) \cdot \mathbf{a} \,.$$

It is convenient to use index notation for this. We have

$$\boldsymbol{A} : (\mathbf{a} \otimes \mathbf{b}) = A_{ij}\, a_i\, b_j = (A_{ij}\, b_j)\, a_i = (\boldsymbol{A} \cdot \mathbf{b}) \cdot \mathbf{a} \,.$$

Hence,

$$\boxed{\boldsymbol{A} : (\mathbf{a} \otimes \mathbf{b}) = (\boldsymbol{A} \cdot \mathbf{b}) \cdot \mathbf{a}}$$

17. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two second order tensors and let $\mathbf{a}$ and $\mathbf{b}$ be two vectors. Show that

$$(\boldsymbol{A} \cdot \mathbf{a}) \cdot (\boldsymbol{B} \cdot \mathbf{b}) = (\boldsymbol{A}^T \cdot \boldsymbol{B}) : (\mathbf{a} \otimes \mathbf{b}) \,.$$

Using index notation,

$$(\boldsymbol{A} \cdot \mathbf{a}) \cdot (\boldsymbol{B} \cdot \mathbf{b}) = (A_{ij}\, a_j)(B_{ik}\, b_k) = (A_{ij}\, B_{ik})(a_j\, b_k) = (A^T_{ji}\, B_{ik})(a_j\, b_k) = (\boldsymbol{A}^T \cdot \boldsymbol{B}) : (\mathbf{a} \otimes \mathbf{b}) \,.$$

Hence,

$$\boxed{(\boldsymbol{A} \cdot \mathbf{a}) \cdot (\boldsymbol{B} \cdot \mathbf{b}) = (\boldsymbol{A}^T \cdot \boldsymbol{B}) : (\mathbf{a} \otimes \mathbf{b}) \,.}$$

18. Let $\boldsymbol{A}$ be a second order tensors and let $\mathbf{a}$ and $\mathbf{b}$ be two vectors. Show that

$$(\boldsymbol{A}\cdot\mathbf{a})\otimes\mathbf{b} = \boldsymbol{A}\cdot(\mathbf{a}\otimes\mathbf{b}) \qquad \text{and} \qquad \mathbf{a}\otimes(\boldsymbol{A}\cdot\mathbf{b}) = [\boldsymbol{A}\cdot(\mathbf{b}\otimes\mathbf{a})]^T = (\mathbf{a}\otimes\mathbf{b})\cdot\boldsymbol{A}^T \,.$$

For the first identity, using index notation, we have

$$[(\boldsymbol{A}\cdot\mathbf{a})\otimes\mathbf{b}]_{ik} = (A_{ij}\,a_j)\,b_k = A_{ij}\,(a_j\,b_k) = A_{ij}\,[\mathbf{a}\otimes\mathbf{b}]_{jk} = \boldsymbol{A}\cdot(\mathbf{a}\otimes\mathbf{b})\,.$$

Hence,

$$\boxed{(\boldsymbol{A}\cdot\mathbf{a})\otimes\mathbf{b} = \boldsymbol{A}\cdot(\mathbf{a}\otimes\mathbf{b})\,.}$$

For the second identity, we have

$$[\mathbf{a}\otimes(\boldsymbol{A}\cdot\mathbf{b})]_{ij} = a_i\,(A_{jk}\,b_k) = (a_i\,b_k)\,A_{jk} = (a_i\,b_k)\,A^T_{kj} = [(\mathbf{a}\otimes\mathbf{b})\cdot\boldsymbol{A}^T]_{ij}\,.$$

Therefore,

$$\mathbf{a}\otimes(\boldsymbol{A}\cdot\mathbf{b}) = (\mathbf{a}\otimes\mathbf{b})\cdot\boldsymbol{A}^T\,.$$

Now, $\mathbf{a}\otimes\mathbf{b} = [\mathbf{b}\otimes\mathbf{a}]^T$ and $(\boldsymbol{A}\cdot\boldsymbol{B})^T = \boldsymbol{B}^T\cdot\boldsymbol{A}^T$. Hence,

$$(\mathbf{a}\otimes\mathbf{b})\cdot\boldsymbol{A}^T = (\mathbf{b}\otimes\mathbf{a})^T\cdot\boldsymbol{A}^T = [\boldsymbol{A}\cdot(\mathbf{b}\otimes\mathbf{a})]^T\,.$$

Therefore,

$$\boxed{\mathbf{a}\otimes(\boldsymbol{A}\cdot\mathbf{b}) = [\boldsymbol{A}\cdot(\mathbf{b}\otimes\mathbf{a})]^T = (\mathbf{a}\otimes\mathbf{b})\cdot\boldsymbol{A}^T\,.}$$

19. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two second-order tensor fields. Let the average of any second-order tensor field ($\boldsymbol{S}$) over the region Ω (of volume V) be defined as

$$\langle\boldsymbol{S}\rangle := \frac{1}{V}\int_\Omega \boldsymbol{S}\,\text{dV}\,.$$

Show that

$$\langle\boldsymbol{A}\cdot\boldsymbol{B}\rangle - \langle\boldsymbol{A}\rangle\cdot\langle\boldsymbol{B}\rangle = \langle[\boldsymbol{A}-\langle\boldsymbol{A}\rangle]\cdot[\boldsymbol{B}-\langle\boldsymbol{B}\rangle]\rangle\,.$$

Expanding out the right hand side, we have

$$\begin{aligned}
\langle[\boldsymbol{A}-\langle\boldsymbol{A}\rangle]\cdot[\boldsymbol{B}-\langle\boldsymbol{B}\rangle]\rangle &= \frac{1}{V}\int_\Omega [\boldsymbol{A}-\langle\boldsymbol{A}\rangle]\cdot[\boldsymbol{B}-\langle\boldsymbol{B}\rangle]\,\text{dV} \\
&= \frac{1}{V}\int_\Omega [\boldsymbol{A}\cdot\boldsymbol{B} - \langle\boldsymbol{A}\rangle\cdot\boldsymbol{B} - \boldsymbol{A}\cdot\langle\boldsymbol{B}\rangle + \langle\boldsymbol{A}\rangle\cdot\langle\boldsymbol{B}\rangle]\,\text{dV} \\
&= \frac{1}{V}\int_\Omega \boldsymbol{A}\cdot\boldsymbol{B}\,\text{dV} - \frac{1}{V}\int_\Omega \langle\boldsymbol{A}\rangle\cdot\boldsymbol{B}\,\text{dV} - \frac{1}{V}\int_\Omega \boldsymbol{A}\cdot\langle\boldsymbol{B}\rangle\,\text{dV} + \frac{1}{V}\int_\Omega \langle\boldsymbol{A}\rangle\cdot\langle\boldsymbol{B}\rangle\,\text{dV}\,.
\end{aligned}$$

Now $\langle\boldsymbol{A}\rangle$ and $\langle\boldsymbol{B}\rangle$ are constants with respect to the integration. Hence,

$$\begin{aligned}
\langle[\boldsymbol{A}-\langle\boldsymbol{A}\rangle]\cdot[\boldsymbol{B}-\langle\boldsymbol{B}\rangle]\rangle &= \frac{1}{V}\int_\Omega \boldsymbol{A}\cdot\boldsymbol{B}\,\text{dV} - \langle\boldsymbol{A}\rangle\cdot\left(\frac{1}{V}\int_\Omega \boldsymbol{B}\,\text{dV}\right) - \left(\frac{1}{V}\int_\Omega \boldsymbol{A}\,\text{dV}\right)\cdot\langle\boldsymbol{B}\rangle + \langle\boldsymbol{A}\rangle\cdot\langle\boldsymbol{B}\rangle\left(\frac{1}{V}\int_\Omega \text{dV}\right) \\
&= \langle\boldsymbol{A}\cdot\boldsymbol{B}\rangle - \langle\boldsymbol{A}\rangle\cdot\langle\boldsymbol{B}\rangle - \langle\boldsymbol{A}\rangle\cdot\langle\boldsymbol{B}\rangle + \langle\boldsymbol{A}\rangle\cdot\langle\boldsymbol{B}\rangle \\
&= \langle\boldsymbol{A}\cdot\boldsymbol{B}\rangle - \langle\boldsymbol{A}\rangle\cdot\langle\boldsymbol{B}\rangle\,.
\end{aligned}$$

Therefore,

$$\boxed{\langle\boldsymbol{A}\cdot\boldsymbol{B}\rangle - \langle\boldsymbol{A}\rangle\cdot\langle\boldsymbol{B}\rangle = \langle[\boldsymbol{A}-\langle\boldsymbol{A}\rangle]\cdot[\boldsymbol{B}-\langle\boldsymbol{B}\rangle]\rangle\,.}$$

20. Let $\boldsymbol{\sigma}$ be the Cauchy stress and let $\boldsymbol{\nabla}\mathbf{v}$ be the velocity gradient in a body Ω with boundary $\partial\Omega$. Let $\mathbf{n}$ be the normal to the boundary. Let V be the volume of the body. If the skew-symmetric part of the velocity gradient is zero, i.e., $\boldsymbol{\nabla}\mathbf{v} = \boldsymbol{\nabla}\mathbf{v}^T$, or if the stress field is self equilibrated, i.e., $\langle\boldsymbol{\sigma}\rangle = \langle\boldsymbol{\sigma}\rangle^T$, show that

$$\langle\boldsymbol{\sigma}:\boldsymbol{\nabla}\mathbf{v}\rangle - \langle\boldsymbol{\sigma}\rangle:\langle\boldsymbol{\nabla}\mathbf{v}\rangle = \frac{1}{V}\int_{\partial\Omega}[\mathbf{v}-\langle\boldsymbol{\nabla}\mathbf{v}\rangle\cdot\mathbf{x}]\cdot[(\boldsymbol{\sigma}-\langle\boldsymbol{\sigma}\rangle)\cdot\mathbf{n}]\,\text{dA}\,.$$

Taking the trace of each term in the identity,

$$\langle\boldsymbol{A}\cdot\boldsymbol{B}\rangle - \langle\boldsymbol{A}\rangle\cdot\langle\boldsymbol{B}\rangle = \langle[\boldsymbol{A}-\langle\boldsymbol{A}\rangle]\cdot[\boldsymbol{B}-\langle\boldsymbol{B}\rangle]\rangle$$

the difference between the average stress power and the product of the average stress and the average velocity gradient can be written as (using either the symmetry of the stress or of the velocity gradient)

$$\begin{aligned}
\langle\boldsymbol{\sigma}:\boldsymbol{\nabla}\mathbf{v}\rangle - \langle\boldsymbol{\sigma}\rangle:\langle\boldsymbol{\nabla}\mathbf{v}\rangle &= \langle\boldsymbol{\sigma}:\boldsymbol{\nabla}\mathbf{v}\rangle - \langle\boldsymbol{\sigma}\rangle:\langle\boldsymbol{\nabla}\mathbf{v}\rangle - \langle\boldsymbol{\nabla}\mathbf{v}\rangle:\langle\boldsymbol{\sigma}\rangle + \langle\boldsymbol{\nabla}\mathbf{v}\rangle:\langle\boldsymbol{\sigma}\rangle \\
&= \langle\boldsymbol{\sigma}:\boldsymbol{\nabla}\mathbf{v}\rangle - \langle\boldsymbol{\sigma}\rangle:\langle\boldsymbol{\nabla}\mathbf{v}\rangle - \langle\boldsymbol{\nabla}\mathbf{v}\rangle:\langle\boldsymbol{\sigma}\rangle + [\langle\boldsymbol{\nabla}\mathbf{v}\rangle^T\cdot\langle\boldsymbol{\sigma}\rangle]:\mathbf{1}
\end{aligned}$$

Recall that

$$\langle\boldsymbol{\sigma}:\boldsymbol{\nabla}\mathbf{v}\rangle = \frac{1}{V}\int_{\partial\Omega}(\boldsymbol{\sigma}\cdot\mathbf{n})\cdot\mathbf{v}\,\text{dV}\,;\quad \langle\boldsymbol{\nabla}\mathbf{v}\rangle = \frac{1}{V}\int_\Omega \boldsymbol{\nabla}\mathbf{v}\,\text{dV}\,;\quad \langle\boldsymbol{\sigma}\rangle = \frac{1}{V}\int_{\partial\Omega}\mathbf{x}\otimes\bar{\mathbf{t}}\,\text{dA}\,;\quad \frac{1}{V}\int_\Omega \boldsymbol{\nabla}\mathbf{x}\,\text{dV} = \mathbf{1}\,.$$

Also, from the divergence theorem

$$\int_\Omega \boldsymbol{\nabla}\mathbf{v}\, \mathrm{dV} = \int_{\partial\Omega} \mathbf{v}\otimes\mathbf{n}\, \mathrm{dA}\,; \quad \int_\Omega \boldsymbol{\nabla}\mathbf{x}\, \mathrm{dV} = \int_{\partial\Omega} \mathbf{x}\otimes\mathbf{n}\, \mathrm{dA}\,.$$

Therefore,

$$\begin{aligned}\langle\boldsymbol{\sigma}:\boldsymbol{\nabla}\mathbf{v}\rangle - \langle\boldsymbol{\sigma}\rangle:\langle\boldsymbol{\nabla}\mathbf{v}\rangle = \frac{1}{V}\int_{\partial\Omega}(\boldsymbol{\sigma}\cdot\mathbf{n})\cdot\mathbf{v}\,\mathrm{dV} - \langle\boldsymbol{\sigma}\rangle : \left[\frac{1}{V}\int_{\partial\Omega}\mathbf{v}\otimes\mathbf{n}\,\mathrm{dA}\right] - \langle\boldsymbol{\nabla}\mathbf{v}\rangle : \left[\frac{1}{V}\int_{\partial\Omega}\mathbf{x}\otimes\bar{\mathbf{t}}\,\mathrm{dA}\right] \\ + [\langle\boldsymbol{\nabla}\mathbf{v}\rangle^T\cdot\langle\boldsymbol{\sigma}\rangle] : \left[\frac{1}{V}\int_{\partial\Omega}\mathbf{x}\otimes\mathbf{n}\,\mathrm{dA}\right]\,.\end{aligned}$$

Since $\langle\boldsymbol{\sigma}\rangle$ and $\langle\boldsymbol{\nabla}\mathbf{v}\rangle$ are independent of $\mathbf{x}$, we can take these inside the integrals to get

$$\langle\boldsymbol{\sigma}:\boldsymbol{\nabla}\mathbf{v}\rangle - \langle\boldsymbol{\sigma}\rangle:\langle\boldsymbol{\nabla}\mathbf{v}\rangle = \frac{1}{V}\int_{\partial\Omega}\Big[(\boldsymbol{\sigma}\cdot\mathbf{n})\cdot\mathbf{v} - \langle\boldsymbol{\sigma}\rangle:(\mathbf{v}\otimes\mathbf{n}) - \langle\boldsymbol{\nabla}\mathbf{v}\rangle:(\mathbf{x}\otimes\bar{\mathbf{t}}) + [\langle\boldsymbol{\nabla}\mathbf{v}\rangle^T\cdot\langle\boldsymbol{\sigma}\rangle]:(\mathbf{x}\otimes\mathbf{n})\Big]\;\mathrm{dA}$$

Using the identity

$$(\boldsymbol{A}\cdot\mathbf{a})\cdot(\boldsymbol{B}\cdot\mathbf{b}) = (\boldsymbol{A}^T\cdot\boldsymbol{B}):(\mathbf{a}\otimes\mathbf{b})$$

we get

$$[\langle\boldsymbol{\nabla}\mathbf{v}\rangle^T\cdot\langle\boldsymbol{\sigma}\rangle]:(\mathbf{x}\otimes\mathbf{n}) = [\langle\boldsymbol{\nabla}\mathbf{v}\rangle\cdot\mathbf{x}]\cdot[\langle\boldsymbol{\sigma}\rangle\cdot\mathbf{n}]\,.$$

Also, using the identity

$$\boldsymbol{A}:(\mathbf{a}\otimes\mathbf{b}) = (\boldsymbol{A}\cdot\mathbf{b})\cdot\mathbf{a}$$

we get

$$\langle\boldsymbol{\sigma}\rangle:(\mathbf{v}\otimes\mathbf{n}) = [\langle\boldsymbol{\sigma}\rangle\cdot\mathbf{n}]\cdot\mathbf{v}\,;\;\; \langle\boldsymbol{\nabla}\mathbf{v}\rangle:(\mathbf{x}\otimes\bar{\mathbf{t}}) = [\langle\boldsymbol{\nabla}\mathbf{v}\rangle\cdot\bar{\mathbf{t}}]\cdot\mathbf{x} = [\langle\boldsymbol{\nabla}\mathbf{v}\rangle^T\cdot\mathbf{x}]\cdot\bar{\mathbf{t}} = [\langle\boldsymbol{\nabla}\mathbf{v}\rangle^T\cdot\mathbf{x}]\cdot(\boldsymbol{\sigma}\cdot\mathbf{n})\,.$$

Since $\boldsymbol{\nabla}\mathbf{v}^T = \boldsymbol{\nabla}\mathbf{v}$, we have $\langle\boldsymbol{\nabla}\mathbf{v}\rangle^T = \langle\boldsymbol{\nabla}\mathbf{v}\rangle$ (we could alternatively use the symmetry of $\langle\boldsymbol{\sigma}\rangle$ to arrive at the following equation). Hence,

$$\langle\boldsymbol{\nabla}\mathbf{v}\rangle:(\mathbf{x}\otimes\bar{\mathbf{t}}) = [\langle\boldsymbol{\nabla}\mathbf{v}\rangle\cdot\mathbf{x}]\cdot(\boldsymbol{\sigma}\cdot\mathbf{n})\,.$$

Plugging these back into the original equation, we have

$$\begin{aligned}\langle\boldsymbol{\sigma}:\boldsymbol{\nabla}\mathbf{v}\rangle - \langle\boldsymbol{\sigma}\rangle:\langle\boldsymbol{\nabla}\mathbf{v}\rangle &= \frac{1}{V}\int_{\partial\Omega}\{(\boldsymbol{\sigma}\cdot\mathbf{n})\cdot\mathbf{v} - [\langle\boldsymbol{\sigma}\rangle\cdot\mathbf{n}]\cdot\mathbf{v} - [\langle\boldsymbol{\nabla}\mathbf{v}\rangle\cdot\mathbf{x}]\cdot(\boldsymbol{\sigma}\cdot\mathbf{n}) + [\langle\boldsymbol{\nabla}\mathbf{v}\rangle\cdot\mathbf{x}]\cdot[\langle\boldsymbol{\sigma}\rangle\cdot\mathbf{n}]\}\;\mathrm{dA} \\ &= \frac{1}{V}\int_{\partial\Omega}\{[(\boldsymbol{\sigma}-\langle\boldsymbol{\sigma}\rangle)\cdot\mathbf{n}]\cdot\mathbf{v} - [(\boldsymbol{\sigma}-\langle\boldsymbol{\sigma}\rangle)\cdot\mathbf{n}]\cdot(\langle\boldsymbol{\nabla}\mathbf{v}\rangle\cdot\mathbf{x})\}\;\mathrm{dA} \\ &= \frac{1}{V}\int_{\partial\Omega}\{[(\boldsymbol{\sigma}-\langle\boldsymbol{\sigma}\rangle)\cdot\mathbf{n}]\cdot[\mathbf{v} - \langle\boldsymbol{\nabla}\mathbf{v}\rangle\cdot\mathbf{x}]\}\;\mathrm{dA}\,.\end{aligned}$$

Hence

$$\boxed{\langle\boldsymbol{\sigma}:\boldsymbol{\nabla}\mathbf{v}\rangle - \langle\boldsymbol{\sigma}\rangle:\langle\boldsymbol{\nabla}\mathbf{v}\rangle = \frac{1}{V}\int_{\partial\Omega}[[\mathbf{v} - \langle\boldsymbol{\nabla}\mathbf{v}\rangle\cdot\mathbf{x}]\cdot[(\boldsymbol{\sigma}-\langle\boldsymbol{\sigma}\rangle)\cdot\mathbf{n}]]\;\mathrm{dA}\,.} \qquad \square$$

21. Let $\boldsymbol{P}$ be the first Piola-Kirchhoff stress and let $\dot{\boldsymbol{F}}$ be the time rate of the deformation gradient in a body whose reference configuration is Ω_0 with boundary $\partial\Omega_0$. Let $\mathbf{N}$ be the normal to the boundary. Let V_0 be the volume of the body. Let $\mathbf{X}$ represent the position of points in the reference configuration. Let $\dot{\mathbf{x}}$ be the material time derivative of $\mathbf{x}$. Let $\langle\boldsymbol{A}\rangle$ represent the unweighted volume average of a quantity $\boldsymbol{A}$. Show that

$$\begin{aligned}\langle\dot{\boldsymbol{F}}\cdot\boldsymbol{P}\rangle - \langle\dot{\boldsymbol{F}}\rangle\cdot\langle\boldsymbol{P}\rangle &= \frac{1}{V_0}\int_{\partial\Omega_0}[\dot{\mathbf{x}} - \langle\dot{\boldsymbol{F}}\rangle\cdot\mathbf{X}]\otimes\left\{[\boldsymbol{P}-\langle\boldsymbol{P}\rangle]^T\cdot\mathbf{N}\right\}\;\mathrm{dA} \\ &= \frac{1}{V_0}\int_{\partial\Omega_0}[\dot{\mathbf{x}} - \langle\dot{\boldsymbol{F}}\rangle\cdot\mathbf{X}]\otimes(\boldsymbol{P}^T\cdot\mathbf{N})\,\mathrm{dA} \\ &= \frac{1}{V_0}\int_{\partial\Omega_0}\dot{\mathbf{x}}\otimes\left\{[\boldsymbol{P}-\langle\boldsymbol{P}\rangle]^T\cdot\mathbf{N}\right\}\;\mathrm{dA}\,.\end{aligned}$$

Recall the identity

$$\langle\boldsymbol{A}\cdot\boldsymbol{B}\rangle - \langle\boldsymbol{A}\rangle\cdot\langle\boldsymbol{B}\rangle = \langle[\boldsymbol{A}-\langle\boldsymbol{A}\rangle]\cdot[\boldsymbol{B}-\langle\boldsymbol{B}\rangle]\rangle\,.$$

Therefore,

$$\begin{aligned}\langle\dot{\boldsymbol{F}}\cdot\boldsymbol{P}\rangle - \langle\dot{\boldsymbol{F}}\rangle\cdot\langle\boldsymbol{P}\rangle &= \langle[\dot{\boldsymbol{F}}-\langle\dot{\boldsymbol{F}}\rangle]\cdot[\boldsymbol{P}-\langle\boldsymbol{P}\rangle]\rangle \\ &= \frac{1}{V_0}\int_{\Omega_0}[\dot{\boldsymbol{F}}-\langle\dot{\boldsymbol{F}}\rangle]\cdot[\boldsymbol{P}-\langle\boldsymbol{P}\rangle]\,\mathrm{dV} \\ &= \frac{1}{V_0}\int_{\Omega_0}\dot{\boldsymbol{F}}\cdot\boldsymbol{P}\,\mathrm{dV} - \frac{1}{V_0}\int_{\Omega_0}\dot{\boldsymbol{F}}\cdot\langle\boldsymbol{P}\rangle\,\mathrm{dV} - \frac{1}{V_0}\int_{\Omega_0}\langle\dot{\boldsymbol{F}}\rangle\cdot\boldsymbol{P}\,\mathrm{dV} + \frac{1}{V_0}\int_{\Omega_0}\langle\dot{\boldsymbol{F}}\rangle\cdot\langle\boldsymbol{P}\rangle\,\mathrm{dV} \\ &= \frac{1}{V_0}\int_{\Omega_0}\dot{\boldsymbol{F}}\cdot\boldsymbol{P}\,\mathrm{dV} - \left(\frac{1}{V_0}\int_{\Omega_0}\dot{\boldsymbol{F}}\,\mathrm{dV}\right)\cdot\langle\boldsymbol{P}\rangle - \langle\dot{\boldsymbol{F}}\rangle\cdot\left(\frac{1}{V_0}\int_{\Omega_0}\boldsymbol{P}\,\mathrm{dV}\right) + \langle\dot{\boldsymbol{F}}\rangle\cdot\left(\frac{1}{V_0}\int_{\Omega_0}\mathbf{1}\,\mathrm{dV}\right)\cdot\langle\boldsymbol{P}\rangle\,.\end{aligned}$$

We want express the volume integrals above in terms of surface integrals. To do that, recall that

$$
\begin{aligned}
\int_{\Omega_0} \dot{\boldsymbol{F}} \cdot \boldsymbol{P} \, \mathrm{dV} &= \int_{\partial\Omega_0} \dot{\mathbf{x}} \otimes (\boldsymbol{P}^T \cdot \mathbf{N}) \, \mathrm{dA} \\
\int_{\Omega_0} \dot{\boldsymbol{F}} \, \mathrm{dV} &= \int_{\Omega_0} \boldsymbol{\nabla}_0 \, \dot{\mathbf{x}} \, \mathrm{dV} = \int_{\partial\Omega_0} \dot{\mathbf{x}} \otimes \mathbf{N} \, \mathrm{dA} \\
\int_{\Omega_0} \boldsymbol{P} \, \mathrm{dV} &= \int_{\partial\Omega_0} \mathbf{X} \otimes (\boldsymbol{P}^T \cdot \mathbf{N}) \, \mathrm{dA} \\
\int_{\Omega_0} \boldsymbol{1} \, \mathrm{dV} &= \int_{\Omega_0} \boldsymbol{\nabla}_0 \, \mathbf{X} \, \mathrm{dV} = \int_{\partial\Omega_0} \mathbf{X} \otimes \mathbf{N} \, \mathrm{dA} \, .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{1}{V_0} \int_{\Omega_0} \dot{\boldsymbol{F}} \cdot \boldsymbol{P} \, \mathrm{dV} &= \frac{1}{V_0} \int_{\partial\Omega_0} \dot{\mathbf{x}} \otimes (\boldsymbol{P}^T \cdot \mathbf{N}) \, \mathrm{dA} \\
\left(\frac{1}{V_0} \int_{\Omega_0} \dot{\boldsymbol{F}} \, \mathrm{dV} \right) \cdot \langle \boldsymbol{P} \rangle &= \frac{1}{V_0} \int_{\partial\Omega_0} (\dot{\mathbf{x}} \otimes \mathbf{N}) \cdot \langle \boldsymbol{P} \rangle \, \mathrm{dA} \\
&= \frac{1}{V_0} \int_{\partial\Omega_0} \dot{\mathbf{x}} \otimes [\langle \boldsymbol{P} \rangle^T \cdot \mathbf{N}] \, \mathrm{dA} \\
\langle \dot{\boldsymbol{F}} \rangle \cdot \left(\frac{1}{V_0} \int_{\Omega_0} \boldsymbol{P} \, \mathrm{dV} \right) &= \frac{1}{V_0} \int_{\partial\Omega_0} \langle \dot{\boldsymbol{F}} \rangle \cdot [\mathbf{X} \otimes (\boldsymbol{P}^T \cdot \mathbf{N})] \, \mathrm{dA} \\
&= \frac{1}{V_0} \int_{\partial\Omega_0} [\langle \dot{\boldsymbol{F}} \rangle \cdot \mathbf{X}] \otimes (\boldsymbol{P}^T \cdot \mathbf{N}) \, \mathrm{dA} \\
\langle \dot{\boldsymbol{F}} \rangle \cdot \left(\frac{1}{V_0} \int_{\Omega_0} \boldsymbol{1} \, \mathrm{dV} \right) \cdot \langle \boldsymbol{P} \rangle &= \frac{1}{V_0} \int_{\partial\Omega_0} \langle \dot{\boldsymbol{F}} \rangle \cdot (\mathbf{X} \otimes \mathbf{N}) \cdot \langle \boldsymbol{P} \rangle \, \mathrm{dA} \\
&= \frac{1}{V_0} \int_{\partial\Omega_0} [\langle \dot{\boldsymbol{F}} \rangle \cdot \mathbf{X}] \otimes [\langle \boldsymbol{P} \rangle^T \cdot \mathbf{N}] \, \mathrm{dA} \, .
\end{aligned}
$$

Collecting the terms, we have

$$
\begin{aligned}
\langle \dot{\boldsymbol{F}} \cdot \boldsymbol{P} \rangle - \langle \dot{\boldsymbol{F}} \rangle \cdot \langle \boldsymbol{P} \rangle &= \frac{1}{V_0} \int_{\partial\Omega_0} \Big\{ \dot{\mathbf{x}} \otimes (\boldsymbol{P}^T \cdot \mathbf{N}) - \dot{\mathbf{x}} \otimes [\langle \boldsymbol{P} \rangle^T \cdot \mathbf{N}] - [\langle \dot{\boldsymbol{F}} \rangle \cdot \mathbf{X}] \otimes (\boldsymbol{P}^T \cdot \mathbf{N}) + [\langle \dot{\boldsymbol{F}} \rangle \cdot \mathbf{X}] \otimes [\langle \boldsymbol{P} \rangle^T \cdot \mathbf{N}] \Big\} \, \mathrm{dA} \\
&= \frac{1}{V_0} \int_{\partial\Omega_0} \Big\{ \dot{\mathbf{x}} \otimes [\boldsymbol{P}^T \cdot \mathbf{N} - \langle \boldsymbol{P} \rangle^T \cdot \mathbf{N}] - [\langle \dot{\boldsymbol{F}} \rangle \cdot \mathbf{X}] \otimes [\boldsymbol{P}^T \cdot \mathbf{N} - \langle \boldsymbol{P} \rangle^T \cdot \mathbf{N}] \Big\} \, \mathrm{dA} \\
&= \frac{1}{V_0} \int_{\partial\Omega_0} [\dot{\mathbf{x}} - \langle \dot{\boldsymbol{F}} \rangle \cdot \mathbf{X}] \otimes [\boldsymbol{P}^T \cdot \mathbf{N} - \langle \boldsymbol{P} \rangle^T \cdot \mathbf{N}] \, \mathrm{dA} \, .
\end{aligned}
$$

Therefore,

$$
\boxed{ \langle \dot{\boldsymbol{F}} \cdot \boldsymbol{P} \rangle - \langle \dot{\boldsymbol{F}} \rangle \cdot \langle \boldsymbol{P} \rangle = \frac{1}{V_0} \int_{\partial\Omega_0} [\dot{\mathbf{x}} - \langle \dot{\boldsymbol{F}} \rangle \cdot \mathbf{X}] \otimes \Big\{ [\boldsymbol{P} - \langle \boldsymbol{P} \rangle]^T \cdot \mathbf{N} \Big\} \, \mathrm{dA} \, . }
$$

From the above, clearly

$$
\left(\frac{1}{V_0} \int_{\Omega_0} \dot{\boldsymbol{F}} \, \mathrm{dV} \right) \cdot \langle \boldsymbol{P} \rangle = \langle \dot{\boldsymbol{F}} \rangle \cdot \left(\frac{1}{V_0} \int_{\Omega_0} \boldsymbol{P} \, \mathrm{dV} \right) = \langle \dot{\boldsymbol{F}} \rangle \cdot \left(\frac{1}{V_0} \int_{\Omega_0} \boldsymbol{1} \, \mathrm{dV} \right) \cdot \langle \boldsymbol{P} \rangle \, .
$$

Therefore,

$$
\frac{1}{V_0} \int_{\partial\Omega_0} \dot{\mathbf{x}} \otimes [\langle \boldsymbol{P} \rangle^T \cdot \mathbf{N}] \, \mathrm{dA} = \frac{1}{V_0} \int_{\partial\Omega_0} [\langle \dot{\boldsymbol{F}} \rangle \cdot \mathbf{X}] \otimes (\boldsymbol{P}^T \cdot \mathbf{N}) \, \mathrm{dA} = \frac{1}{V_0} \int_{\partial\Omega_0} [\langle \dot{\boldsymbol{F}} \rangle \cdot \mathbf{X}] \otimes [\langle \boldsymbol{P} \rangle^T \cdot \mathbf{N}] \, \mathrm{dA} \, .
$$

Thus we can alternatively write the expression for the difference as

$$
\begin{aligned}
\langle \dot{\boldsymbol{F}} \cdot \boldsymbol{P} \rangle - \langle \dot{\boldsymbol{F}} \rangle \cdot \langle \boldsymbol{P} \rangle &= \frac{1}{V_0} \int_{\partial\Omega_0} \left\{ \dot{\mathbf{x}} \otimes (\boldsymbol{P}^T \cdot \mathbf{N}) - [\langle \dot{\boldsymbol{F}} \rangle \cdot \mathbf{X}] \otimes (\boldsymbol{P}^T \cdot \mathbf{N}) - \cancelto{0}{\left[\dot{\mathbf{x}} \otimes [\langle \boldsymbol{P} \rangle^T \cdot \mathbf{N}] - [\langle \dot{\boldsymbol{F}} \rangle \cdot \mathbf{X}] \otimes [\langle \boldsymbol{P} \rangle^T \cdot \mathbf{N}] \right]} \right\} \mathrm{dA} \\
&= \frac{1}{V_0} \int_{\partial\Omega_0} [\dot{\mathbf{x}} - \langle \dot{\boldsymbol{F}} \rangle \cdot \mathbf{X}] \otimes (\boldsymbol{P}^T \cdot \mathbf{N}) \, \mathrm{dA}
\end{aligned}
$$

or,

$$
\begin{aligned}
\langle \dot{\boldsymbol{F}} \cdot \boldsymbol{P} \rangle - \langle \dot{\boldsymbol{F}} \rangle \cdot \langle \boldsymbol{P} \rangle &= \frac{1}{V_0} \int_{\partial\Omega_0} \left\{ \dot{\mathbf{x}} \otimes (\boldsymbol{P}^T \cdot \mathbf{N}) - \dot{\mathbf{x}} \otimes [\langle \boldsymbol{P} \rangle^T \cdot \mathbf{N}] - \cancelto{0}{\left[\langle \dot{\boldsymbol{F}} \rangle \cdot \mathbf{X}] \otimes (\boldsymbol{P}^T \cdot \mathbf{N}) - [\langle \dot{\boldsymbol{F}} \rangle \cdot \mathbf{X}] \otimes [\langle \boldsymbol{P} \rangle^T \cdot \mathbf{N}] \right]} \right\} \mathrm{dA} \\
&= \frac{1}{V_0} \int_{\partial\Omega_0} \dot{\mathbf{x}} \otimes [\boldsymbol{P}^T \cdot \mathbf{N} - \langle \boldsymbol{P} \rangle^T \cdot \mathbf{N}] \, \mathrm{dA} \, .
\end{aligned}
$$

Hence,

$$\boxed{\begin{aligned}\langle \dot{\boldsymbol{F}} \cdot \boldsymbol{P}\rangle - \langle \dot{\boldsymbol{F}}\rangle \cdot \langle \boldsymbol{P}\rangle &= \frac{1}{V_0}\int_{\partial\Omega_0} [\dot{\mathbf{x}} - \langle \dot{\boldsymbol{F}}\rangle \cdot \mathbf{X}] \otimes \left\{[\boldsymbol{P} - \langle \boldsymbol{P}\rangle]^T \cdot \mathbf{N}\right\} \,\mathrm{dA} \\ &= \frac{1}{V_0}\int_{\partial\Omega_0} [\dot{\mathbf{x}} - \langle \dot{\boldsymbol{F}}\rangle \cdot \mathbf{X}] \otimes (\boldsymbol{P}^T \cdot \mathbf{N}) \,\mathrm{dA} \\ &= \frac{1}{V_0}\int_{\partial\Omega_0} \dot{\mathbf{x}} \otimes \left\{[\boldsymbol{P} - \langle \boldsymbol{P}\rangle]^T \cdot \mathbf{N}\right\} \,\mathrm{dA}\,.\end{aligned}}$$

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