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# The elastic energy-momentum tensor

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ABSTRACT

The application to continuum mechanics of the general methods of the classical theory of fields is advocated and illustrated by the example of the static elastic field. The non-linear theory of elasticity is set up in the most convenient form (Lagrangian coordinates and stress tensor). The appropriate energy-momentum tensor is derived, and it is shown that the integral of its normal component over a closed surface gives the force (as the term is used in the theory of solids) on defects and inhomogeneities within the surface. Other topics discussed are Günther's and related integrals, symmetrization of the energy-momentum tensor, and the Eulerian formulation. Some further extensions, existing and potential, are indicated.

RÉSUMÉ

On signale les avantages de l'application à la méchanique des milieux continus des méthodes de la théorie classique des champs. A titre d'example la théorie de l'élasticité est construite sous la forme la plus convenable à cette fin (coordonées de Lagrange, composantes de tension de Boussinesq). On déduit ensuite le tenseur d'énergie-impulsion dont l'intégrale de la composante normale étendue sur une surface fermée donne la force (comme on l'entend en théorie des solides) agissant sur les défaults dans son intérieur. On discute aussi quelques intégrales semblables, entr'elles celles de Günther, la symétrisation du tenseur d'énergie-impulsion et la formulation en coordonées d'Euler. On récapitule par conclure certains autres résultats déjà connus, tout en indiquant quelques extensions possibles.

## 1. Introduction

In that corner of the theory of solids which deals with lattice defects (dislocations, impurity and interstitial atoms, vacant lattice sites and so forth) which are capable of altering their position or configuration in a crystal, it has been found useful to introduce the concept of the force acting on a defect. If it is enough to specify simply the position of a defect the force is defined to be minus the rate of increase of the total energy of the system with respect to variation of the three position coordinates of the defect. If a larger (possibly infinite) number of parameters is required to specify the configuration of the defect, generalized forces are defined similarly in terms of the variation of the total energy with the parameters in a way familiar in analytical mechanics and thermodynamics. By total energy is meant the energy of the solid being investigated plus the potential energy of any external loading mechanism which is acting on it. The distinction between the two is to some extent arbitrary. In solid state applications the forces are equated to zero to find the equilibrium configuration, and in favourable cases the rate of approach to equilibrium can be estimated from them with the help of the Einstein relation [1] between mobility and diffusion.

In many cases an elastic model of a crystal containing defects will be adequate, or at any rate the only tractable one. The crystal becomes an elastic solid, a crack becomes a crack, a crystal dislocation one of Volterra's elastic dislocations, an oversize impurity atom an elastic sphere forced into a hole too small for it, a lattice vacancy a small region (with or without misfit) whose elastic constants are less than those of the bulk of the medium, and so on. The normal theory of elasticity recognizes nothing which corresponds with the force on a defect. (It has nothing to do with the ordinary body force, of course.) But in fact the appropriate concept has been to hand ever since the appearance of a paper by Noether [1] in 1918, in the form of the energy-momentum tensor [2] which the elastic field possesses in common with every field whose governing equations are derivable from a variational principle, and some for which they are not.

The archetypal energy-momentum tensor is Maxwell's stress tensor in electrostatics ([2], p. 84). The integral of its normal component taken over a closed surface gives the total force on all the electric charges inside that surface (cf. [2], p. 87). The writer, having looked at a book on field theory, felt that the force on a defect ought to be given by a similar expression involving the energy-momentum tensor appropriate to the elastic field. For linear elasticity it proved possible [3] to manipulate an existing expression for the force until it took on the hoped-for form. Later a simpler and less artificial demonstration was devised [4, 5] which led naturally to the desired expression and which, moreover, applied equally well to the case of the finite deformation of a solid with an arbitrary stress-strain relation.

To give a sensible result the integral for the force on a defect must be surface-independent in the sense that its value is unchanged when the surface of integration is deformed in any way, provided it continues to enclose the defect we are interested in and does not embrace any others. The fact that the divergence of the energy-momentum tensor vanishes ensures that this is so. In two dimensions the integral for the force becomes a path-independent one. If the path of integration embraces the tip of a two-dimensional crack the expression for the component of the force parallel to the crack becomes Rice's [6, 7] independently-discovered $J$-integral (cf. also [8, 9]). In what follows we shall, for brevity, use the expression "path-independent", which has become familiar in fracture mechanics, to stand for "surface-independent or path-independent as appropriate".

Apart from its connection with the theory of lattice defects the energy-momentum tensor and kindred concepts associated with elastic and other material media are of interest for their own sake, but they have received scarcely any attention from applied mathematicians, even during the intensive re-examination and extension of continuum

[1] See, for example, [5] p. 425.

2 As we shall be almost entirely concerned with static situations this term is not particularly appropriate, but the alternative, stress-energy tensor, is not much better.

mechanics which has been under way for the last couple of decades, perhaps because of the artificial separation which has grown up between applied mathematics and theoretical physics. The papers of Günther [10], Knowles and Sternberg [11] and Green [12] are exceptions.

It is hoped that the present paper may perhaps help to dispel this lack of interest. Apart from anything it may do for the theory itself a treatment of elasticity on the lines of other physical fields furnishes homely (in the British sense) illustrations of the results of the classical part of general field theory. By classical field theory we mean, roughly, the body of doctrine, based on a Lagrangian density depending on a set of field variables and their derivatives, which is often presented as a preliminary to second quantization in books and articles devoted to the quantum theory of fields [13, 14, 15]. As long as one is content with a single-particle description the wave-function of an elementary particle is a perfectly respectable classical field in the sense in which we are using the term [16, 17]. Only when recourse must be had to second quantization is this no longer true. Also relevant are some discussions of the theory of relativity, particularly those in which extra field variables appear alongside the components of the metric tensor which serve as gravitational potentials [18, 19].

As an exercise-ground for field theory, elasticity has the advantage of being non-linear, in contrast to most of the fields commonly considered. One exception is Born's non-linear electrodynamics [16, 20, 21], which in fact has a close connection with the theory of elasticity. There is an exact correspondence between time-dependent linear anti-plane elastic and plane electromagnetic fields [5] and it is not hard to show that there is the same correspondence between the two-dimensional version of Born's theory [22] and the type of non-linear anti-plane elasticity discussed by Neuber [23].

## 2. Elasticity in Lagrangian coordinates

Unless the contrary is stated our results will relate to the finite deformation of an elastic solid with an arbitrary stress-strain relation. Our coordinate system, measure of deformation and stress tensor are chosen so as to make the description of the elastic field conform with that of other physical fields, though they may not be the best choice for solving specific problems. We introduce fixed rectangular Cartesian coordinates $X_m$ and label each particle of the medium with the $X_m$ appropriate to the position it occupies in the undeformed state (Lagrangian coordinates). After deformation the particle labelled with $X_m$ is at $x_i$, referred to the same fixed coordinate system, and

$$u_i(X_m) = x_i(X_m) - X_i \tag{2.1}$$

is its displacement vector. If from each point $X_m$ we draw a vector arrow $u_i(X_m)$ with its tail at $X_m$ its tip defines the final position of the particle labelled with $X_m$. Contrast this with say, the electrostatic field. If we treat the electric field vector $E_i(X_m)$ in the same way as $u_i(X_m)$ the position of its tip has no meaning; indeed it would depend on how many metres we used to denote one volt per metre. If we wish to develop the theory of the elastic field in the same way as for other physical fields we must, in fact, look upon $u_i(X_m)$ in the same way as $E_i(X_m)$, as an unlocated vector associated with the point $X_m$, though its aspect as a displacement of the material must, of course, be taken into account in the interpretation of any results we obtain.

To match our Lagrangian[3] formulation, we choose our stress tensor so that $p_{ij}$ denotes the force parallel to the $X_i$-axis on an element of material which was of unit area and perpendicular to the $X_j$-axis before deformation. Thus $p_{ij}$ is, to use Hill's apt term, the tensor of nominal stresses; it is also known as the Lagrangian, first Piola-Kirchhoff or Boussinesq tensor. The traction on an element of surface embedded in the material is $p_{ij}dS_j$ where, in $dS_j = n_j dS$, $dS$ is the area and $n_j$ the normal of the element before deformation. (We employ the usual summation convention for repeated suffixes.) In the absence of body forces $p_{ij}$ obeys the equilibrium equations

$$\frac{\partial p_{ij}}{\partial X_j} = 0. \tag{2.2}$$

When convenient we shall use a comma followed by suffixes to denote differentiation with respect to the $X_m$, so that, for example, the last equation could be re-written $p_{ij,j} = 0$, and $u_{i,j}$, $u_{i,jk}$ denote the strain gradient tensor $\partial u_i/\partial X_j$ and its $X_k$-derivative. If $W(u_{i,j}, X_m)$ is the elastic energy density per unit undeformed volume (in an inhomogeneous medium it may depend on $X_m$) consideration of the work done in a further small deformation of a material already finitely deformed gives the relation

$$p_{ij} = \frac{\partial W}{\partial u_{i,j}}. \tag{2.3}$$

The $p_{ij}$ are not symmetric, but they obey a relation,

$$p_{li} - p_{jl} = p_{ji}u_{l,i} - p_{li}u_{j,i}, \tag{2.4}$$

which implies that the stress tensor is symmetric when referred to Cartesian final coordinates. The easiest way to derive it is to start with the relation

$$\int_S (x_l p_{ji} - x_j p_{li}) dS_i = 0,$$

which states that the total couple on the material inside $S$ is zero (the final position vector $x_l = X_l + u_l$ rather than $X_l$ is obviously the appropriate lever arm), convert to a volume integral,

$$\int [(\delta_{li} + u_{l,i})p_{ji} - (\delta_{ji} + u_{j,i})p_{li}] dv = 0,$$

and note that, since $S$ is arbitrary, the integrand must vanish.

We shall also need the relation (2.5) below. Consider two elementary cubes of material, the first with edges $\varepsilon \boldsymbol{i}_m$ parallel to the coordinate axes and the second with edges $\varepsilon \boldsymbol{i}'_m$ slightly rotated relative to those of the first, and let the displacement gradient $u'_{l,i}$ impart to the second the same strain (change of shape and size) as $u_{l,i}$ does to the first. Evidently $u'_{l,i}$ must have components $u_{l,i}$ when referred to axes parallel to the $\boldsymbol{i}'_m$, that is,

[3] In the sense of Lagrangian versus Eulerian coordinates. Also, by chance, in the sense of a Lagrangian as against a Hamiltonian formulation of mechanics: for the latter we would have to use a (complementary) energy function depending on stress rather than deformation.

$$u'_{l,i} = u_{r,s}(\delta_{rl}+\omega_{rl})(\delta_{si}+\omega_{si}),$$

where $\omega_{rl} = -\omega_{lr} \ll 1$ specifies the relative orientation of the cubes. In an anisotropic medium, a piece of wood say, the energies of the cubes will be different, for although they have undergone the same changes of shape and size, the grain runs differently in them. But if the material is isotropic the energies are equal, and we must have $\delta u_{l,i}\partial W/\partial u_{l,i} = 0$ with $\delta u_{l,i} = u'_{l,i}-u_{l,i}$, which gives, after a suitable re-naming of suffixes,

$$p_{ij}u_{i,l}-p_{il}u_{i,j}+p_{ji}u_{l,i}-p_{li}u_{j,i} = 0, \tag{2.5}$$

or, with (2.4),

$$p_{lj}-p_{jl} = p_{il}u_{i,j}-p_{ij}u_{i,l}. \tag{2.6}$$

For the linear theory with $p_{lj} = p_{jl}$ neither (2.4) nor (2.6) is true but fortunately (2.5), which then takes the form

$$p_{ij}e_{il} = p_{il}e_{ij},$$

with

$$p_{ij} = \lambda e_{mm}\delta_{ij}+2\mu e_{ij}, \qquad e_{ij} = \tfrac{1}{2}(u_{i,j}+u_{j,i}),$$

is.

What we have called elastic energy is, more precisely, internal energy under adiabatic conditions and Helmholtz free energy under isothermal conditions [24]. For most solids the difference between them is small enough for us to hope that our results will also apply in intermediate situations.

## 3. The energy-momentum tensor

In this section we show how an energy-momentum tensor can be generated for (almost) any field. The usual treatment, following Noether, is a variational one, but we shall give a more direct method which will serve our purposes. It is rather more general than would be required for the theory of elasticity as set up in the last section. The $X_i$ can be $N$-dimensional curvilinear coordinates. The number of $u_i$ need not be the same as the number of $X_i$ and they need not be the components of a vector. They could, for example, be a set of scalars, or $u_1, \ldots u_9$ could be the elements of a $3 \times 3$ matrix conventionally numbered.

We suppose that the field is governed by the equations

$$\frac{\partial}{\partial X_j}\frac{\partial W}{\partial u_{i,j}} = \frac{\partial W}{\partial u_i} \tag{3.1}$$

with

$$W = W(u_i, u_{i,j}, X_i). \tag{3.2}$$

Of course (3.1) are the Euler equations which would be found by requiring the integral of (3.2) over a region of $X$-space to be stationary, but we make no explicit use of this fact.

The $W$ of the last section does not depend on the $u_i$, but it would do so if the $X_i$ were curvilinear coordinates, for $W$ would then contain covariant derivatives of the $u_i$ which have a term linear in the $u_i$. Likewise, if the $X_i$ stand for the coordinates $x_i$ of the last section, even Cartesian ones, $W$ may depend on the $u_i$; see section 7. In any case certain manipulations seem less artificial if $W$ is supposed to contain the $u_i$ even if it does not. In order to generate the energy-momentum tensor we must first distinguish between two $X$-derivatives of $W$, namely the ordinary gradient,

$$\frac{\partial W}{\partial X_l} = (\text{grad } W)_l$$

and the derivative

$$\left(\frac{\partial W}{\partial X_l}\right)_{\text{exp.}} = \frac{\partial}{\partial X_l} W(u_i, u_{i,j}, X_m) \Bigg| \begin{array}{l} u_i, u_{i,j} \text{ const.} \\ X_m \text{ const.}, m \neq l \end{array}$$

which gives the explicit dependence on $X_l$. This rather clumsy notation, common in field theory, will be adequate since $(\partial W/\partial X_l)_{\text{exp.}}$ will not often appear in what follows. For the $u_i$ there is, of course, no distinction between the two types of derivative. We have

$$\frac{\partial W}{\partial X_l} = \frac{\partial W}{\partial u_i} u_{i,l} + \frac{\partial W}{\partial u_{i,j}} u_{i,jl} + \left(\frac{\partial W}{\partial X_l}\right)_{\text{exp.}} \tag{3.3}$$

If we use (3.1) to replace $\partial W/\partial u_i$ by $\partial(\partial W/\partial u_{i,j})/\partial X_j$ the first two terms on the right of (3.3) combine to give

$$\frac{\partial}{\partial X_j}\left(\frac{\partial W}{\partial u_{i,j}} u_{i,l}\right).$$

Whence, writing $\partial W/\partial X_l = \partial(W\delta_{lj})/\partial X_j$ and re-arranging, we have

$$\frac{\partial P_{lj}}{\partial X_j} = \left(\frac{\partial W}{\partial X_l}\right)_{\text{exp.}}, \tag{3.4}$$

where, by definition,

$$P_{lj} = W\delta_{lj} - \frac{\partial W}{\partial u_{i,j}} u_{i,l} \tag{3.5}$$

is the (canonical) energy-momentum tensor derived from $W$.

If $W = W(u_i, u_{i,j} u_{i,jk}, X_m)$ depends also on the second derivatives of $u_i$ (grade 2 material) the right-hand sides of (3.1) and (3.3) acquire additional terms

$$-\frac{\partial^2}{\partial X_k \partial X_j} \frac{\partial W}{\partial u_{i,jk}} \quad \text{and} \quad \frac{\partial W}{\partial u_{i,jk}} u_{i,jkl},$$

respectively, and a similar calculation leads to (3.4) with

$$P_{lj} = W\delta_{lj} - \left(\frac{\partial W}{\partial u_{i,j}} - \frac{\partial}{\partial X_k} \frac{\partial W}{\partial u_{i,jk}}\right) u_{i,l} - \frac{\partial W}{\partial u_{i,jk}} u_{i,kl}. \tag{3.6}$$

One can go on to derive the energy-momentum tensor for a material of grade $N$.

Though it does not play a very important role in field theory the energy-momentum tensor for a field in which the Lagrangian density depends on the first $N$ derivatives of the field variables has been repeatedly re-derived; see, for example, the papers of Podolsky and Kikuchi [25], Chang [26] and Thielheim [27]. Equations (3.5), (3.6) and their higher-order analogues agree with the results of these and other authors.

## 4. The force on a defect

For the elastic field (2.3) and (3.5) give

$$P_{lj} = W\delta_{lj} - p_{ij}u_{i,l}. \tag{4.1}$$

According to (3.4) the integral

$$F_l = \int_S P_{lj}dS_j \tag{4.2}$$

is zero when taken over a closed surface $S$ within which the material is homogeneous and free of defects so that

$$\left(\frac{\partial W}{\partial X_l}\right)_{\text{exp.}} = 0 \tag{4.3}$$

inside $S$. If a surface $S_1$ encloses a second surface $S_2$ so that (4.3) holds on $S_1$ and $S_2$ and in the volume between them, but not within $S_2$, then (4.2) is not zero, but, since the divergence of the integrand is zero it has the same value over $S_1$, $S_2$, and any intermediate surface which is inside $S_1$ and embraces $S_2$. In the region between $S_1$ and $S_2$ it is thus path-independent in the sense explained in the Introduction. Although in the present paper we do not want to over-emphasize this particular role of the energy-momentum tensor it is perhaps worth sketching the physical argument which shows that (4.2) gives the force on a defect inside $S$ in the sense mentioned in the Introduction.

The left of fig. 1 shows a loaded body containing a defect enclosed by an arbitrary surface $S$. We shall refer to it as the original system. On the right is an exact replica of it with $S$ marked out and also $S'$, the surface produced by giving $S$ a vector displacement $-\delta\xi_l$ in the *undeformed* state.

To find the energy change associated with a displacement of the singularity by $+\delta\xi_l$ we carry out the following imaginary operations.

(i) In the original system cut out the material inside $S$ and discard it. Apply suitable tractions to the surface of the resulting hole to prevent relaxation.

(ii) Cut out the piece of material inside $S'$ in the replica and apply surface tractions to prevent relaxation. The energy $E_{s'}$ inside $S'$ differs from the energy $E_s$ originally inside $S$ by the addition of the energy in the crescent-shaped region 1 and the removal of the energy in the crescent-shaped region 2, so that

$$E_{s'} - E_s = -\delta\xi_l \int_S W dS_l. \tag{4.4}$$

At this stage there is no change in the energy in the material outside the hole in the original system, or the energy of its loading mechanism.

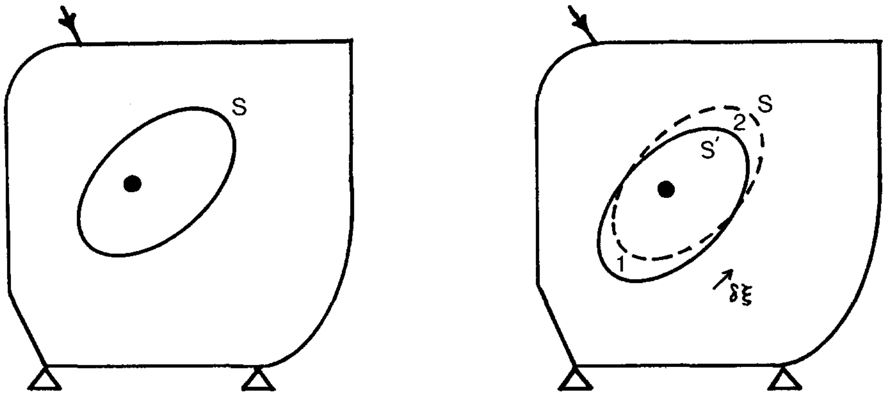


Figure 1. Calculation of the force on a defect.

(iii) We now try to fit the body bounded by $S'$ into the hole $S$. But since, by construction, $S$ and $S'$ can be made to coincide by a simple translation in the undeformed state they will not usually do so after deformation. In fact after a suitable translation to move it from replica to original the displacement on $S'$ will differ from that on $S$ by

$$\delta u_i = -\delta\xi_l u_{i,l}. \tag{4.5}$$

We therefore give a displacement (4.5) to the surface of the hole, which requires the expenditure of an amount of work

$$\delta W = -\int_S \delta u_i p_{ij} dS_j, \tag{4.6}$$

some of which goes to raise the elastic energy in the material outside $S$ and some to increase the potential energy of the loading mechanism. (The minus sign is correct if, in $dS_j = n_j dS$, $n_j$ is the outward normal to $S$).

(iv) We can now fit $S'$ into $S$ and weld across the interface. We have not yet quite reached final state, however. Though the displacement matches across the interface the tractions on either side of it will differ by a quantity of order $\delta\xi_l$, and so there is a layer of body force of order $\delta\xi_l$ spread over the interface. As we relax this unwanted force the displacement changes by a quantity of order $\delta\xi_l$ and an amount of energy of order $(\delta\xi_l)^2$ is extracted, which may be neglected in comparison with (4.4) and (4.6).

We are now left with the system as it was to begin with except that the defect has been shifted by $+\delta\xi_l$, as required. The associated change of energy is the sum of (4.4) and (4.6), and if we choose to write it as $-F_l\delta\xi_l$, the effective force $F_l$ is given by (4.2), as claimed. We add a few comments.

The material in the region where the interiors of $S$ and $S'$ overlap (see the right of the figure) plays no direct part in the energy calculation; indeed our set of operations is so devised that we do not even have to say precisely what we mean by a defect in this region. In particular a possible formal infinity in the energy of a singular defect is subtracted out in forming the difference (4.4).

As we apply the displacement $\delta u_i$ to $S$ in stage (iii) the traction changes from $p_{ij}n_j$ to

$$p_{ij}n_j + 0(\delta\xi_l), \tag{4.7}$$

where the extra term will depend on the exact properties of the medium and on the way the forces exerted by the loading mechanism vary as their points of application move. However, as the extra term in (4.7) only alters (4.6) by a quantity of order $(\delta\xi_l)^2$ our final result is independent of the precise elastic behaviour of the medium and, to use an engineering term, the degree of hardness or softness of the loading mechanism.

Our argument might suggest that in (4.2) the first term of (4.1) gives the change of energy of the material inside $S$ and the second the change in the energy of the loading mechanism and the material outside $S$, but this is not so. Although the final relaxation in stage iv only changes the total energy by a quantity of order $(\delta\xi_l)^2$ it involves an interchange of energy of order $\delta\xi_l$ across $S$ which, fortunately, we are not required to work out.

It is not hard to see that in the two-dimensional version of the above argument the tip of a crack qualifies as a defect in its own right, even though the crack passes out of the contour $S$, provided that the part of the crack which lies inside $S$ is straight and $\delta\xi_l$ is parallel to the crack. (For a more careful treatment see [9], and for an extension to three dimensions, [28].) If the crack is parallel to the $X_1$-axis $F_1$ is equal to $G$, the crack extension force or Rice's $J$-integral. It is also possible to give an interpretation to the other component, $F_2$. Suppose that the crack begins to spread in a direction inclined at an angle $\alpha$ to its original direction. Then $G$ at the new tip is a function of $\alpha$, and equation (5.1) below can be used to show that

$$F_2 = \left(\frac{dG(\alpha)}{d\alpha}\right)_{\alpha=0} \tag{4.8}$$

provided that the integral for $F_2$ is evaluated round a small circuit embracing the tip. Equation (4.8) may be of use in treating the curved path of a crack in a non-uniform stress field.

## 5. Günther's and related integrals

Günther [10] seems to have been the first to make an explicit application of Noether's theorem to elasticity. In addition to $F_l$, equation (4.2), he found the path-independent integrals

$$L_{kl} = \int (X_k P_{lj} - X_l P_{kj} + u_k p_{lj} - u_l p_{kj}) dS_j \tag{5.1}$$

and

$$M = \int (X_l P_{lj} - \tfrac{1}{2} u_l p_{lj}) dS_j. \tag{5.2}$$

The $L_{kl}$ are path-independent for a non-linear homogeneous isotropic medium, as may be verified with the help of Gauss' theorem, (2.2) and (2.5). For $L_{12}$ to be path-independent we only need transverse isotropy about the $X_3$ axis. More generally $L_{12}$ is path-independent if the material has the elastic properties of a treetrunk where, with cylindrical polars $\rho$, $\phi$, $X_3$, the functional form of $W$ depends on $\rho$ and $X_3$ but not $\phi$, provided it is everywhere referred to the local coordinate directions.

The integral $M$ is path-independent for a homogeneous linear an isotropic medium (use Gauss' theorem, (2.2) and $W = \frac{1}{2}p_{ij}u_{i,j}$) or more generally if with spherical polars $r$, $\theta$, $\phi$, $W$ depends on $\theta$, $\phi$ but not $r$. To apply (5.2) to a plane-strain situation we may integrate over a disk with its flat faces in the $X_1X_2$-plane. Gauss' theorem converts the contributions from the faces into an integral round the rim, where it cancels the second term in (5.2) to leave

$$M = \int X_l P_{lj} dS_j \tag{5.3}$$

in plane strain. For some applications of (5.3) see [29].

We may be led directly to the differential forms of (5.1) and (5.2) by introducing spherical polars $r$, $\theta$, $\phi$ and treating $\partial W/\partial\phi$ and $r\partial W/\partial r$ on the lines of $\partial W/\partial X_l$ in (3.3). For a nonlinear homogeneous anisotropic medium $r\partial W/\partial r$ leads to

$$(X_l P_{lj} + u_i \partial W/\partial u_{i,j})_{,j} = 3W + u_i \partial W/\partial u_i, \tag{5.4}$$

the differential form of a result of Green's [12]; for use in a moment we have included a term which only appears if $W$ depends on $u_i$. For the linear case (5.4) expresses the path-independence of $M$.

As conservation laws the four-dimensional analogues of $F_l$ and $L_{kl}$ have long been familiar in field theory, but though stated for the electromagnetic field in 1921 [30] the analogue of $M$ has only recently achieved prominence, in connection with high-energy physics (Dilatation or scale symmetry [31]). The $W$ appropriate to an elementary particle of mass $m$ contains a term $\frac{1}{2}m^2u_iu_i$, and so there is a term $3m^2u_iu_i$ in (5.4) and it does not express the vanishing of a four-dimensional divergence. (The 3, which stands for $\delta_{ii}$, is now a 4). However, $u_i$ is a wave-like solution of the governing equations with a frequency proportional to the energy of the particle. Thus at high enough energies the other terms in (5.4), because they contain gradients of $u_i$, swamp the awkward term and there is an approximate conservation law.

Knowles and Sternberg [11] have given an argument to show that in three dimensions $F_l$, $L_{kl}$ and $M$ are the only path-independent integrals of Noether's type, i.e. of the form

$$\int (\xi_l P_{lj} - \eta_l p_{lj}) dS_j, \tag{5.5}$$

with $\xi_l$ independent of $u_i$ and $p_{lj}$, and that nothing new appears in plane situations beyond the trivial change from (5.2) to (5.3). However, whether we regard it as three-dimensional or plane, a state of linear isotropic anti-plane strain provides an exception. For it we have $\partial u_3/\partial X_3 = 0$, $u_1 = 0$, $u_2 = 0$, $P_{12} = P_{21}$, $P_{11} + P_{22} = 0$ and Gauss' theorem shows that (5.5) is path-independent if $\eta_l$ is zero and $\xi_1$, $\xi_2$ are any pair of conjugate harmonic functions. There is an analogue of this result for plane strain, though not precisely of the form (5.5). The two may be used to simplify the derivation of some standard results in fracture mechanics [29]. Possibly further integrals may appear when other severe restrictions are placed on the form of $u_i$.

The canonical energy-momentum tensor $P_{lj}$ is not symmetric. Field theorists have studied the problem of symmetrizing it, that is, finding a symmetric tensor $P^S_{lj}$ such that the divergence of $P^S_{lj} - P_{lj}$ is identically zero. The most elegant method is that of

Rosenfeld [32]. Suppose that the $X_i$ are curvilinear with metric tensor $g_{ij}$ and that $W$ is the energy per unit coordinate mesh, a scalar density. Then

$$g^{lm}g^{jn}P^{S}_{mn} = \frac{\delta W}{\delta g_{lj}} + \frac{\delta W}{\delta g_{jl}}, \tag{5.6}$$

where

$$\frac{\delta W}{\delta g_{lj}} = \frac{\partial W}{\partial g_{lj}} - \left(\frac{\partial W}{\partial g_{lj,k}}\right)_{,k}$$

is the so-called variational derivative, and $g_{lj}, g_{jl}$ are regarded as independent. Equation (5.6) has the disadvantage that even if we are only interested in Cartesian coordinates we still have to go over to general coordinates, evaluate (5.6) and then set $g_{ij}$ equal to $\delta_{ij}$. Belinfante's [33] recipe is more direct, and leads to the same result as (5.6) [34]. For an isotropic grade one elastic material a simplified version goes as follows.

From the isotropy condition (2.5) and the equilibrium equation (2.2) we have

$$\begin{aligned} P_{lj} - P_{jl} &= p_{il}u_{i,j} - p_{ij}u_{i,l} \\ &= p_{ji}u_{l,i} - p_{li}u_{j,i} \\ &= (p_{ji}u_l - p_{li}u_j)_{,i} \\ &= H_{lji,i} \end{aligned}$$

say, with

$$H_{lji} = p_{ji}u_l - p_{li}u_j. \tag{5.7}$$

Put, tentatively,

$$H_{lji} = S_{jli} - S_{lji}. \tag{5.8}$$

Then

$$P^{S}_{lj} = P_{lj} + S_{lji,i} \tag{5.9}$$

is symmetric. We also require that the second term in (5.9) shall have no divergence, and this can be secured by making $S_{lji}$ antisymmetric in $i, j$:

$$S_{lji} = -S_{lij}. \tag{5.10}$$

Between them (5.8) and (5.10) determine the $S_{lji}$: add to (5.8) the results of applying the permutations $lji \to jil \to lij$ to it and use (5.10) to get

$$S_{jli} = \tfrac{1}{2}(H_{lji} + H_{jil} + H_{lij}).$$

With (5.7) this gives

$$P^{S}_{lj} = P_{lj} + \tfrac{1}{2}(p_{ij}u_l - p_{ji}u_l + p_{il}u_j - p_{jl}u_i + p_{li}u_j - p_{lj}u_i)_{,i}$$

or for linear isotropy $P^{S}_{lj} = P_{lj} + (p_{li}u_j - p_{lj}u_i)_{,i}$. An application of Stokes' theorem will show that $\int S_{lji,i}\,dS_j$ is equal to zero for a closed surface or to a line-integral round the boundary if it is open. Hence $\int P^{S}_{lj}\,dS_j$ is equal to (4.2) for a closed surface but not an open one.

In the elastic case there is another symmetric energy-momentum tensor which seems to have no counterpart for other fields [28]. Eqn. (2.6) states that the antisymmetric parts of $P_{lj}$ and $p_{lj}$ are the same, and so

$$P^*_{lj} = P_{lj} - p_{lj} \tag{5.11}$$

is symmetric for an isotropic medium. We may write it as

$$P^*_{lj} = W\delta_{lj} - \frac{\partial W}{\partial\left(\dfrac{\partial x_i}{\partial X_j}\right)}\frac{\partial x_i}{\partial X_l}, \tag{5.12}$$

a form we should have obtained to begin with if, following the often-quoted advice of Kirchhoff, we had taken the $x_i = u_i + X_i$ rather than the $u_i$ as the field variables on which $W$ is supposed to depend. The force on a defect, $F_l$, eqn. (4.2) will usually be the same as

$$F^*_l = \int P^*_{lj} dS_j, \tag{5.13}$$

since we commonly consider only cases where there is no uncompensated body force associated with the defect and the second term in (5.11) contributes nothing. When, rarely, it is sensible to consider body force as forming part of a defect, (5.13) rather than (4.2) is the correct expression to use [28]. The tensor $P^*_{lj}$ is not symmetric for an anisotropic material, neither is it in the linear isotropic theory where $p_{lj}$ is symmetric; this theory is a consistent field theory, but not a consistent theory of an elastic body to the second order. Nevertheless in these cases too $P^*_{lj}$ is an acceptable substitute for $P_{lj}$.

## 6. Eulerian formulation

It is possible to derive an energy-momentum tensor in terms of the rectangular final (Eulerian) coordinates $x_i$ of section 2 with the help of the general formulation of section 3, but in such coordinates it is not easy to repeat the argument of section 4 which shows that it gives the force on a defect. The simplest way is to define the appropriate tensor, $\Sigma_{lj}$ say, and the ordinary Eulerian stress $\sigma_{lj}$ by

$$\Sigma_{lj} ds_j = P_{lj} dS_j, \qquad \sigma_{lj} ds_j = p_{lj} dS_j, \tag{6.1}$$

where $ds_j$ is the directed surface element which $dS_j$ transforms into on deformation. Then the integral

$$F_l = \int_s \Sigma_{lj} ds_j$$

over the surface $s$ into which $S$ transforms will agree with (4.2). A tedious calculation gives an expression

$$\Sigma_{lj} = w\delta_{lj} - \frac{\partial w}{\partial u_{i,j}} u_{i,l} \tag{6.2}$$

precisely analogous to $P_{lj}$; $w$ is the energy per unit final volume and $u_i(x_m) = x_i - X_i(x_m)$ is now a function of the $x_m$ as opposed to our previous $u_i(X_m) = x_i(X_m) - X_i$. On the other hand the second of (6.1) gives

$$\sigma_{lj} = \Sigma_{lj} + \frac{\partial w}{\partial u_{l,j}} = w\delta_{lj} - \frac{\partial w}{\partial\left(\dfrac{\partial X_i}{\partial x_j}\right)} \frac{\partial X_i}{\partial x_l}, \tag{6.3}$$

which is analogous not to $p_{lj}$ but rather to $P^*_{lj}$, equation (5.13). It is not hard to show that, as required, it is symmetric. Chadwick [35] has independently noted the relationship between $\sigma_{lj}$ and $P^*_{lj}$.

It is interesting to see how the formalism of section 3 handles things when left to itself. With suitable changes in notation (3.4) gives

$$\frac{\partial \Sigma_{lj}}{\partial x_j} = \left(\frac{\partial w}{\partial x_l}\right)_{\text{exp.}} \tag{6.4}$$

with the $\Sigma_{lj}$ of (6.2), but in place of the expected

$$\frac{\partial \sigma_{lj}}{\partial x_j} = 0, \tag{6.5}$$

(3.1) gives

$$\frac{\partial}{\partial x_j} \frac{\partial w}{\partial u_{l,j}} = \frac{\partial w}{\partial u_l}, \tag{6.6}$$

where the quantity whose divergence appears on the left is not one of the more familiar representations of the stress tensor. The second member of (6.6) appears because we must now allow $w$ to depend on $u_i$ since in an inhomogeneous medium the form of the energy function for the particle now at $x_i$ depends on where it originally came from. However, it actually only depends on $u_i$ and $x_i$ through the combination $x_i + u_i = X_i$, which specifies its initial position, and so if we add (6.6) to (6.4) the right-hand members cancel and we are left with (6.5).

The fact that $\sigma_{lj}$ involves $\Sigma_{lj}$ has some interesting consequences in dynamical situations. Strictly speaking the form of $\Sigma_{lj}$ then needs changing but we can make the point intended without working out the details. Suppose that we are interested in the mean force exerted by a wave on an obstacle in a fluid. If the obstacle is fixed we naturally use Eulerian coordinates, and the force on it is the integral of the normal component of $\sigma_{lj}$ over its surface. If we work to a linear approximation and the process is periodic the time-average of the term $\partial w/\partial u_{l,j}$ in (6.3) is zero and the force appears to be given by the mean value of the energy-momentum tensor $\Sigma_{lj}$, or rather its dynamical generalization. This result is useful because of its suggestive analogy with radiation pressure in electromagnetism, but at the same time physically misleading.

## 7. Closure

The methods of sections 3 and 4 can be applied to continua (see, e.g. [36]) more complicated than a simple elastic solid. We can, for example take $(u_1, u_2, u_3)$ to be the displacement and $(u_4, u_5, u_6)$ a director. If $W$ is known, equation (3.5) or (3.6) or a higher-order analogue provides an energy-momentum tensor. (If there are constraints, incompressibility or constancy of the length of a director say, they need not be explicitly recognised provided that only solutions which conform with them are fed into the final formulas.) An argument on the lines of section 4 will then show that (4.2) with the appropriate $P_{lj}$ gives the force on a defect in the sense outlined in the Introduction. At least, this is what the writer believes, but see reference [37].

The formalism of section 4 can be applied to time-dependent situations by introducing an extra independent variable $X_0 = t$ ($u_i$ still has only three components) and interpreting $-W$ as the density of the appropriate Lagrangian, the kinetic energy minus the elastic energy. Some fairly interesting results may be derived for a medium with or without defects [28]. It ought to be possible to do the same for more complex continua.

Dissipative media present difficulties since the governing equations are not self-adjoint and can only be written in the form (3.1) at the expense of letting $W$ depend on additional dummy field variables. However, the case of steady viscous flow with the inertial terms ignored can be handled with the help of Goodier's [38] analogy. If in a solution of the equations of linear isotropic elasticity for an incompressible medium we interpret displacement, stress and shear modulus as velocity, stress and viscosity we have the solution of a problem in slow viscous flow. The energy density becomes half the rate of dissipation of energy per unit volume. Correspondingly the integral (4.2) gives minus half the rate of change of the total rate of dissipation with respect to the change of position of a defect, now, say, a small rigid sphere perturbing the flow. Jeffery [39] studied the motion of a rigid spheroid in a viscous flow, ignoring the inertial terms. He found that after a long time the details of the motion still depended on initial orientation, whereas experiment showed that spheroids which started off with different orientations all ultimately settled down to one common motion with no memory of their initial conditions, presumably under the influence of the neglected terms. He suggested that the final motion was given by his solution with the initial orientation chosen so as to minimize[4] the final rate of dissipation. A similar principle has been invoked in some studies of the migration of solid particles in viscous flows. Unfortunately there seems to be no respectable physical principle which justifies this procedure. It is equivalent to requiring that for the actual ultimate motion the "force" (4.2) and the "couple" (5.1) be zero for a surface enclosing the solid. Even if the principle is incorrect its consequences might still be salvaged if one could show that those motions of the solid for which $F_l$ and $L_{kl}$ are zero, but not others, are, to the first order, unaltered when the small terms which have been left out are switched on again. The writer has tried hard to devise a proof of this conjecture, but so far without success, possibly because it is not true.

[4] At constant imposed strain rate. This is the same as maximizing it at constant imposed stress [40].

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