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On the decomposition of the *J*-integral for 3D crack problems

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Abstract. The extended *J*-integral for 3D linear elastostatic crack problems and its application to mixed mode problems is investigated. In 3D, the decomposition of the *J*-integral into its parts corresponding to the symmetric mode I and both antisymmetric modes II and III is derived explicitly. The range of validity of the decomposition method is also discussed in the framework of linear elastic fracture mechanics (LEFM). It is shown analytically that in a general mixed mode case the antisymmetric part of the *J*-integral can be split into the parts associated with mode II and mode III only in the crack near-field.

1. Introduction

The *J*-integral as a crack tip parameter for 3D crack problems was introduced by Miyamoto and Kikuchi [1], Kishimoto, Aoki and Sakata [2] and Amestoy, Bui and Labbens [3]. In the case of a single mode crack problem the path independence of these definitions enables one to calculate the stress intensity factors (SIFs) via the 'far-field' variables (Cherepanov [4]). This fact and the energy significance [3] of the *J*-integral result in robust numerical procedures for the calculation of fracture parameters.

Miyamoto and Kikuchi [1] have calculated the *J*-integral in 3D using the Finite Element Method (FEM). Later, Nikishkov and Atluri [5] proposed the 'equivalent domain integral' (EDI) method. This is an equivalent formulation to the *J*-integral but it is more convenient for calculating the *J*-integral value when using the FEM.

For mixed mode problems usually it is required to know the stress intensity factors for each mode separately. This requires the decomposition of the *J*-integral into its parts corresponding to the three modes. Ishikawa, Kitagawa and Okamura [6] derived this decomposition in 2D. Nishioka and Atluri [7] proposed the computation of mixed mode problems for dynamically propagating cracks using path independent integrals. Later on, Nikishkov and Atluri [5] applied the decomposition method to 3D crack problems.

Recently Aliabadi and Rigby [8] and independently Huber and Kuhn [9] presented the evaluation of the *J*-integral using the Boundary Element Method (BEM) in 3D. It is shown in these references, that the BEM has significant advantages over other numerical methods, which are based on domain discretisation, both in modelling and accuracy. This results from the possibility of calculating the field quantities for interior points in a precise manner using integral representation formulas for the displacement gradients (derivated forms of the Somigliana-identity) in a post-processing phase of computation. In [8] the required displacement gradients on the boundary are computed by numerical differentiation. This unstable technique could be avoided by the usage of the hypersingular identity as proposed in [9] and [10].

The accuracy of the displacement gradients inside the domain calculated by the BEM results from the fact that no domain discretization is necessary, and it is a well-known fact in the literature (see e.g. [8, 9]). Furthermore, a demonstration of highly accurate evaluation of the displacement gradients on the boundary, a necessity for J-integral calculation in 3D, for practical problems with several illustrative numerical examples are given in [10].

There are a lot of results obtained by the decomposition method available in the literature concerning various types of mixed mode problems (e.g. [5], [8], [11], [12], [13], [14]). In [5, 8, 11, 12, 13] the decomposition method is used to calculate the stress intensity factors in arbitrary mixed mode cases. In the examples included in these papers the lowest accuracy is found when evaluating the stress intensity factors for a mixed mode problem including mode II and mode III compared to other single or mixed mode cases. This fact was first realized and discussed in [11]. In the present paper, while the explicit understanding of the reasons behind these deviations is fully realized the theoretical background is also completely demonstrated. This is a necessity for proper employment of the decomposed J-integral in the 3D fracture mechanics. To the authors' knowledge it is the first time, that the decomposition of the J-integral for 3D crack problems is derived explicitly including the discussion of the path independence of the decomposed parts J_{I}, J_{II} and J_{III}. The analytical proof of why the antisymmetric part $J^{as} = J_{\mathrm{II}} + J_{\mathrm{III}}$ cannot be decomposed (when choosing a path out of the crack near field) into J_{II} and J_{III}, regarding mode II and mode III, is included.

2. The local, path independent J-integral

In elastostatics Eshelby's energy-momentum tensor [15] is defined as

$$P_{lj} = W\delta_{lj} - \sigma_{ij}u_{i,l}, \tag{1}$$

where W is the strain energy density, σ_{ij} and $u_{i,l}$ denote the components of stresses and displacement gradients respectively and δ_{ij} is the Kronecker symbol. Considering the case, when the energy density W does not depend explicitly on the system coordinates X_m $[W = W(\sigma_{ij}(u_{k,l}))]$ the divergence of P_{lj} vanishes. That is

$$P_{lj,j} = 0. \tag{2}$$

Consider an integration area consisting of a surface $A(\Gamma)$ perpendicular to the crack front. Let $A(\gamma)$ be a similar smaller surface containing the singularity. Γ and γ denote the contours of $A(\Gamma)$ and $A(\gamma)$ respectively [3] (see Fig. 1). The integral over the difference of these two surfaces which due to (2) vanishes, is

$$\int_{A(\Gamma)-A(\gamma)} P_{lj,j}\,\mathrm{d}A = 0. \tag{3}$$

Taking advantages of the superposition of the integration domains equation (3) can be rewritten as

$$\int_{A(\gamma)} P_{lj,j}\,\mathrm{d}A(\gamma) = \int_{A(\Gamma)} P_{lj,j}\,\mathrm{d}A(\Gamma). \tag{4}$$

The path independent J-integral is defined as the limit of (4) for $l = 1$ with $A(\gamma)$ tending to zero.

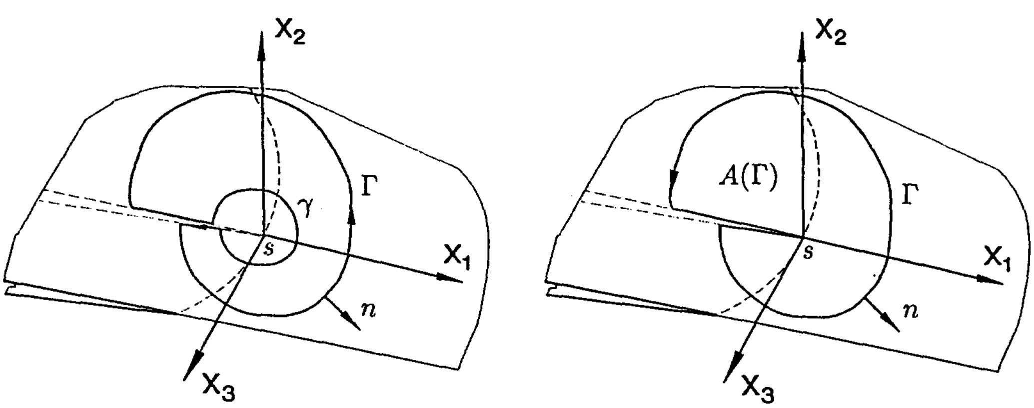

Fig. 1. Arbitrary integration path at the 3D crack front before and after the limit process.

That is

$$J^n_{x_1}(s) = \lim_{A(\gamma)\to 0} \int_{A(\gamma)} P_{1j,j}\, \mathrm{d}A(\gamma) = \int_{A(\Gamma)} P_{1j,j}\, \mathrm{d}A(\Gamma). \tag{5a}$$

(x_1, x_2, x_3) denote the local Cartesian crack front coordinate system with (x_1, x_2) perpendicular to the crack front. The parameter s characterizes the position on the crack front, where the J-integral is evaluated. The superscript n is appended to stress that the normal vector and thus the surface $A(\Gamma)$ should be perpendicular to the crack front. Using (1), the J-integral can be written as

$$J^n_{x_1}(s) = \int_{A(\Gamma)} [W_{,j}\delta_{1j} - (\sigma_{ij}u_{i,1})_{,j}]\, \mathrm{d}A. \tag{5b}$$

When applying Gauss' divergence theorem to (5b), the terms corresponding to $j = 1, 2$, can be expressed as a line integral. The term corresponding to $j = 3$ cannot be transformed to a line integral because the component associated with this direction is not coplanar with $A(\Gamma)$. Thus the final formula for the J-integral is

$$J^n_{x_1}(s) = \int_{\Gamma} (Wn_1 - \sigma_{ij}n_j u_{i,1})\, \mathrm{d}\Gamma - \int_{A(\Gamma)} (\sigma_{i3}u_{i,1})_{,3}\, \mathrm{d}A. \tag{6}$$

In the above expression n_j denotes the components of the outward normal vector to the contour Γ coplanar with $A(\Gamma)$. In addition to the classical 2D formulation of the J-integral, the surface integral including the boundary value functions at the crack surface and the singularity at the crack front have to be taken into account.

Choosing Γ as a circle with radius ε, and carrying out the limit process for $\varepsilon \to 0$, the surface integral vanishes, so that

$$\lim_{\varepsilon\to 0} J^n_{x_1}(s) = J, \tag{7}$$

as introduced by Rice [16] and Cherepanov [17].

$J^n_{x_1}(s)$ characterizes the crack front singularity at s and has the physical meaning of the energy release rate due to a translation of the crack front ds in the direction x_1 [3]. The validity of $J^n_{x_1}(s)$ as a parameter for the crack tip behaviour is limited to small strains and the deformation theory of plasticity without unloading. However, $J^n_{x_1}$ still remains significant beyond the range of small-scale yielding.

In the following, the decomposition of the J-integral which is important in the field of LEFM in order to calculate the SIFs, is discussed.

3. The decomposition of the *J*-integral

3.1. Decomposition in the crack near-field

It is well-known [18] that in the crack near-field the field quantities can be separated into mode I, mode II and mode III. The application of the J-integral to mixed mode crack problems in 3D linear elasticity requires a decomposition of the J-integral into its symmetric part J_{I} and the antisymmetric parts J_{II} and J_{III} corresponding to the modes II and III, respectively. For the decomposition of $J^n_{x_1}$ in its parts, the required field quantities u_i, $u_{i,1}$, $u_{i,13}$, σ_{ij} and $\sigma_{ij,3}$, transformed into the crack front coordinate system, should be decomposed in a sum of symmetric and antisymmetric parts regarding the crack front coordinate axes x_1 e.g.

$$u_i = u_i^I + u_i^{II} + u_i^{III}; \quad \sigma_{ij} = \sigma_{ij}^I + \sigma_{ij}^{II} + \sigma_{ij}^{III}. \tag{8}$$

Following [6], in order to get the aforementioned parts of the field quantities, one has to consider a symmetric integration path and symmetric points $P(x_1, x_2, x_3)$ and $P'(x_1, -x_2, x_3)$, as shown in Fig. 2. With the field quantities u_j^P, σ_{ij}^P and $u_j^{P'}$, $\sigma_{ij}^{P'}$, Eqn. (8) [5] becomes

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} u_1^P + u_1^{P'} \\ u_2^P - u_2^{P'} \\ u_3^P + u_3^{P'} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} u_1^P - u_1^{P'} \\ u_2^P + u_2^{P'} \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 \\ 0 \\ u_3^P - u_3^{P'} \end{bmatrix}, \tag{9a}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{22} \\ \sigma_{13} \\ \sigma_{23} \\ \sigma_{33} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \sigma_{11}^P + \sigma_{11}^{P'} \\ \sigma_{12}^P - \sigma_{12}^{P'} \\ \sigma_{22}^P + \sigma_{22}^{P'} \\ \sigma_{13}^P + \sigma_{13}^{P'} \\ \sigma_{23}^P - \sigma_{23}^{P'} \\ \sigma_{33}^P + \sigma_{33}^{P'} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \sigma_{11}^P - \sigma_{11}^{P'} \\ \sigma_{12}^P + \sigma_{12}^{P'} \\ \sigma_{22}^P - \sigma_{221}^{P'} \\ 0 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 \\ 0 \\ 0 \\ \sigma_{13}^P - \sigma_{13}^{P'} \\ \sigma_{23}^P + \sigma_{23}^{P'} \\ \sigma_{33}^P + \sigma_{33}^{P'} \end{bmatrix}. \tag{9b}$$

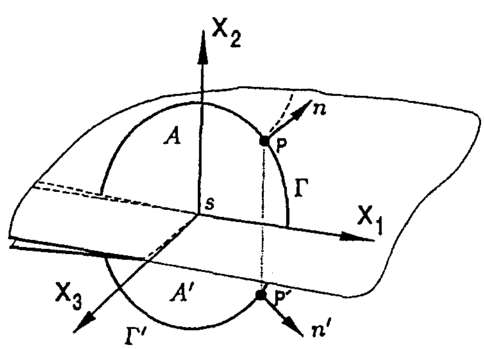

Fig. 2. Symmetric integration path.

Inserting (9) into the definition of $J^n_{x_1}$ (6) one obtains

$$
\begin{aligned}
J^n_{x_1} = \int_\Gamma \Bigg\{ & \left[\int_0^\varepsilon (\sigma^I_{ij} + \sigma^{II}_{ij} + \sigma^{III}_{ij}) d(\varepsilon^I_{ij} + \varepsilon^{II}_{ij} + \varepsilon^{III}_{ij}) \right] \mathrm{d}x_2 \\
& - (t^I_i + t^{II}_i + t^{III}_i) \frac{\partial (u^I_i + u^{II}_i + u^{III}_i)}{\partial x_1} d\Gamma \Bigg\} \\
& + \int_{A(\Gamma)} \Bigg\{ \Bigg[(\sigma^I_{i3} + \sigma^{II}_{i3} + \sigma^{III}_{i3}) \frac{\partial^2 (u^I_i + u^{II}_i + u^{III}_i)}{\partial x_1 \partial x_3} \\
& + \frac{\partial (\sigma^I_{i3} + \sigma^{II}_{i3} + \sigma^{III}_{i3})}{\partial x_3} \frac{\partial (u^I_i + u^{II}_i + u^{III}_i)}{\partial x_1} \Bigg] \mathrm{d}A \Bigg\}.
\end{aligned}
\tag{10}
$$

Multiplying and rearranging the separate terms in (10) enables the following form

$$
\begin{aligned}
J^n_{x_1} &= J^\Gamma_{\mathrm{I\,I}} + J^\Gamma_{\mathrm{I\,II}} + J^\Gamma_{\mathrm{II\,I}} + J^\Gamma_{\mathrm{II\,II}} + J^\Gamma_{\mathrm{I\,III}} \\
&\quad + J^\Gamma_{\mathrm{III\,I}} + J^\Gamma_{\mathrm{II\,III}} + J^\Gamma_{\mathrm{III\,II}} + J^\Gamma_{\mathrm{III\,III}} \\
&\quad + J^A_{\mathrm{I\,I}} + J^A_{\mathrm{I\,II}} + J^A_{\mathrm{II\,I}} + J^A_{\mathrm{II\,II}} + J^A_{\mathrm{I\,III}} \\
&\quad + J^A_{\mathrm{III\,I}} + J^A_{\mathrm{II\,III}} + J^A_{\mathrm{III\,II}} + J^A_{\mathrm{III\,III}} \\
&= J^\Gamma_{M\,N} + J^A_{M\,N} \quad (M, N = \mathrm{I, II, III})
\end{aligned}
\tag{11}
$$

with

$$
J^\Gamma_{M\,N} \int_\Gamma \left[\left(\int_0^\varepsilon \sigma^M_{ij} \, \mathrm{d}\varepsilon^N_{ij} \right) \mathrm{d}x_2 - t^M_i \frac{\partial u^N_i}{\partial x_1} \, \mathrm{d}\Gamma \right] \quad (M, N = \mathrm{I, II, III})
\tag{12}
$$

and

$$
J^A_{M\,N} = \int_{A(\Gamma)} \left(\sigma^M_{ij} \frac{\partial^2 u^N_i}{\partial x_1 \partial x_3} + \frac{\sigma^M_{i3}}{\partial x_3} \frac{\partial u^N_i}{\partial x_1} \right) \mathrm{d}A \quad (M, N = \mathrm{I, II, III}).
\tag{13}
$$

Ishikawa, Kitagawa and Okamura [6] have proved that in the 2D case the field parameters, which occur in the line integral, from P on Γ and P' on Γ' cancel out each other for $M \neq N$. This results in

$$
J^\Gamma_{M\,N} = 0 \quad \text{for} \quad M \neq N.
\tag{14}
$$

Following the same reasoning it can be shown that the mixed terms of the surface integrals also vanish.

Due to the symmetric integration path there exist relations between the field quantities at point P and the field quantities at a symmetric point P', e.g.

$$
\begin{bmatrix} u^I_1 \\ u^I_2 \\ u^I_3 \end{bmatrix}' = \begin{bmatrix} u^I_1 \\ -u^I_2 \\ u^I_3 \end{bmatrix}; \quad \begin{bmatrix} u^{II}_1 \\ u^{II}_2 \end{bmatrix}' = \begin{bmatrix} -u^{II}_1 \\ u^{II}_2 \end{bmatrix}; \quad [u^{III}_3]' = [-u^{III}_3];
\tag{15a}
$$

$$\begin{bmatrix} \sigma_{11}^{I} \\ \sigma_{12}^{I} \\ \sigma_{22}^{I} \\ \sigma_{13}^{I} \\ \sigma_{23}^{I} \\ \sigma_{33}^{I} \end{bmatrix}' = \begin{bmatrix} \sigma_{11}^{I} \\ -\sigma_{12}^{I} \\ \sigma_{22}^{I} \\ \sigma_{13}^{I} \\ -\sigma_{23}^{I} \\ \sigma_{33}^{I} \end{bmatrix}; \quad \begin{bmatrix} \sigma_{11}^{II} \\ \sigma_{12}^{II} \\ \sigma_{22}^{II} \end{bmatrix}' = \begin{bmatrix} -\sigma_{11}^{II} \\ \sigma_{12}^{II} \\ -\sigma_{22}^{II} \end{bmatrix}; \quad \begin{bmatrix} \sigma_{13}^{III} \\ \sigma_{23}^{III} \\ \sigma_{33}^{III} \end{bmatrix}' = \begin{bmatrix} -\sigma_{13}^{III} \\ \sigma_{23}^{III} \\ -\sigma_{33}^{III} \end{bmatrix}. \tag{15b}$$

These properties result in

$$\left[\sigma_{i3}^{M} \frac{\partial^2 u_i^N}{\partial x_1 \partial x_3}\right]' = \begin{cases} \sigma_{i3}^{M} \dfrac{\partial^2 u_i^N}{\partial x_1 \partial x_3} & \text{for} \quad M = N, \\ -\sigma_{i3}^{M} \dfrac{\partial^2 u_i^N}{\partial x_1 \partial x_3} & \text{for} \quad M \neq N, \end{cases} \tag{16a}$$

$$\left[\frac{\partial \sigma_{i3}^{M}}{\partial x_3} \frac{\partial u_i^N}{\partial x_1}\right]' = \begin{cases} \dfrac{\partial \sigma_{i3}^{M}}{\partial x_3} \dfrac{\partial u_i^N}{\partial x_1} & \text{for} \quad M = N, \\ -\dfrac{\partial \sigma_{i3}^{M}}{\partial x_3} \dfrac{\partial u_i^N}{\partial x_1} & \text{for} \quad M \neq N. \end{cases} \tag{16b}$$

In (13) due to relations (16a) and (16b) for $M \neq N$ the parts corresponding to A and A' are cancelling out each other, that is

$$J_{M\,N}^{A} = 0 \quad \text{for} \quad M \neq N. \tag{17}$$

Resulting from (11) with

$$\begin{aligned} J_{\mathrm{I\,I}}^{\Gamma} + J_{\mathrm{I\,I}}^{A} &= J_{\mathrm{I}}, \\ J_{\mathrm{II\,II}}^{\Gamma} + J_{\mathrm{II\,II}}^{A} &= J_{\mathrm{II}}, \\ J_{\mathrm{III\,III}}^{\Gamma} + J_{\mathrm{III\,III}}^{A} &= J_{\mathrm{III}}, \end{aligned} \tag{18}$$

we come to the desired decoupled form of the J-integral

$$J_{x_1}^{n} = J_{\mathrm{I}} + J_{\mathrm{II}} + J_{\mathrm{III}}. \tag{19}$$

Hence the J-integral and the energy release rate G

$$J_{x_1}^{n} = G = G_{\mathrm{I}} + G_{\mathrm{II}} + G_{\mathrm{III}} \tag{20}$$

are identical [3], the well-known relation [18] with the stress intensity factors

$$J_{x_1}^{n} = \frac{\kappa + 1}{8\mu}(K_{\mathrm{I}}^2 + K_{\mathrm{II}}^2) + \frac{1}{2\mu} K_{\mathrm{III}}^2 \tag{21}$$

enables after the decomposition of $J^n_{x_1}$ the separate evaluation of the SIFs

$$K_{\mathrm{I}} = \sqrt{\frac{8\mu}{\kappa+1} J_{\mathrm{I}}}, \quad K_{\mathrm{II}} = \sqrt{\frac{8\mu}{\kappa+1} J_{\mathrm{II}}}, \quad \text{and} \quad K_{\mathrm{III}} = \sqrt{2\mu J_{\mathrm{III}}}. \tag{22}$$

μ denotes the shear modulus and $\kappa = 3 - 4\nu$ or $\kappa = (3 - \nu)/(1 + \nu)$ expresses the plane strain or the plane stress condition, respectively.

A second way to get the SIFs separately as proposed in [5] is the additional use of the $J^n_{x_2}$-component of the J-vector [4] ((4) for $l = 1, 2, 3$)

$$J^n_{x_2} = -\frac{\kappa+1}{4\mu} K_{\mathrm{I}} K_{\mathrm{II}} \tag{23}$$

and the energy release rate corresponding to mode III

$$G_{\mathrm{III}} = \frac{1}{2\mu} K^2_{\mathrm{III}}. \tag{24}$$

However, the use of the $J^n_{x_2}$-integral in computations should be avoided because the contours have to terminate on the crack front, where the singularity exists. Besides, Herrmann and Herrmann [19] have demonstrated for 2D crack problems that only for special applied stress-fields, J_{x_2} becomes path independent in the same sense as J_{x_1}.

3.2. Decomposition in the far-field

In order to analyze the validity of the decomposition method in the far-field, the path independence of the separated parts of the J-integral has to be investigated. The proof of the path independence of the symmetric part $J^s = J_{\mathrm{I}}$ and the antisymmetric part $J^{as} = J_{\mathrm{II}} + J_{\mathrm{III}}$ can be given as follows. Considering an area $A(\Gamma^*)$ without any singularity, with the help of

$$W^{M\,N}_{,j} \delta_{1j} = \sigma^M_{ij} \frac{\partial \varepsilon^N_{ij}}{\partial x_1} \tag{25}$$

and

$$(\sigma^M_{ij} u^N_{i,1})_{,j} = \sigma^M_{ij,j} \frac{\partial u^N_i}{\partial x_1} + \sigma^M_{ij} \frac{\partial u^N_{i,j}}{\partial x_1}, \tag{26}$$

(5b) gives

$$J^*_{MN} = \int_{A(\Gamma^*)} \left[\sigma^M_{ij} \frac{\partial \varepsilon^N_{ij}}{\partial x_1} - \sigma^M_{ij,j} \frac{\partial u^N_i}{\partial x_1} - \sigma^M_{ij} \frac{\partial u^N_{i,j}}{\partial x_1} \right] \mathrm{d}A(\Gamma^*). \tag{27}$$

The equilibrium of forces is expressed without body forces as

$$\sigma_{ij,j} = 0. \tag{28}$$

The equilibrium of moments yields $\sigma_{ij} = \sigma_{ji}$ and therefore

$$\sigma_{ij}\frac{\partial u_{i,j}}{\partial x_1} = \sigma_{ij}\frac{\partial \varepsilon_{ij}}{\partial x_1}. \tag{29}$$

Using these equilibrium conditions in (27), J^*_{MN} would be zero, which proves the path independence of J_{MN}. However, the equilibrium equations are only valid for the entire stress state or for the symmetric part and the entire antisymmetric part. Thus the path independence is proved for $J^s = J_{\mathrm{I}}$ and J^{as}, but it is not possible to show the path independence of J_{II} and J_{III} separately.

For the stress field in the neighbourhood of a crack front, the possibility of the decomposition into parts relating to mode II and mode III is analyzed considering the general solution of a straight crack front as proposed in [20] and [21].

The description of the straining state surrounding the crack front is carried out by using the series expansions

$$\begin{pmatrix} u_r \\ u_\varphi \\ u_z \end{pmatrix} = \sum_k r^{k/2} \begin{pmatrix} f_k(\varphi, z) \\ g_k(\varphi, z) \\ h_k(\varphi, z) \end{pmatrix}, \quad k = 1, 2, 3, \ldots \tag{30}$$

r, φ, z denote the cylindrical coordinates. Inserting (30) into the Lamé-Navier-equations (in the absence of volume forces and without temperature changes) gives

$$\begin{aligned}
&\sum_k \{r^{k/2-2}[f_{k,\varphi\varphi} + (\eta + 2)(\tfrac{1}{4}k^2 - 1)f_k + (\eta(\tfrac{1}{2}k) + \tfrac{1}{2}k - \eta - 3)g_{k,\varphi}] \\
&\quad + r^{k/2-1}(\eta + 1)(\tfrac{1}{2}k)h_{k,z} + r^{k/2}f_{k,zz}\} = 0, \\
&\sum_k \{r^{k/2-2}[(\eta + 2)g_{k,\varphi\varphi} + (\tfrac{1}{4}k^2 - 1)g_k + (\eta(\tfrac{1}{2}k) + \tfrac{1}{2}k + \eta + 3)f_{k,\varphi}] \\
&\quad + r^{k/2-1}(\eta + 1)h_{k,\varphi z} + r^{k/2}g_{k,zz}\} = 0, \\
&\sum_k \{r^{k/2-2}[h_{k,\varphi\varphi} + (\tfrac{1}{4}k^2)h_k] + r^{k/2-1}[(\eta + 1)(\tfrac{1}{2}k + 1)f_{k,z} + (\eta + 1)g_{k,\varphi z}]\} = 0.
\end{aligned} \tag{31}$$

Comparing terms with equal exponents of r in the series expansion (31) yields

$$\begin{aligned}
&f_{k,\varphi\varphi} + (\eta + 2)(\tfrac{1}{4}k^2 - 1)f_k + (\eta(\tfrac{1}{2}k) + \tfrac{1}{2}k - \eta - 3)g_{k,\varphi} = -(\eta + 1)(\tfrac{1}{2}k - 1)h_{k-2,z} - f_{k-4,zz}, \\
&(\eta + 2)g_{k,\varphi\varphi} + (\tfrac{1}{4}k^2 - 1)g_k + (\eta(\tfrac{1}{2}k) + \tfrac{1}{2}k + \eta + 3)f_{k,\varphi} = -(\eta + 1)h_{k-2,\varphi z} - g_{k-4,zz}, \\
&h_{k,\varphi\varphi} + \tfrac{1}{4}k^2 h_k = -(\eta + 1)\tfrac{1}{2}k f_{k-2,z} - (\eta + 1)g_{k-2,\varphi z}.
\end{aligned} \tag{32}$$

When solving the system of equations (32) the boundary conditions at the stress-free surfaces $\sigma_{\varphi\varphi}(\varphi = \pm\pi) = \tau_{r\varphi}(\varphi = \pm\pi) = \tau_{\varphi z}(\varphi = \pm\pi) = 0$ have to be taken into account.

With the help of the relationships between deformations and strains and Hooke's law the stresses can be expressed as follows.

$$
\begin{aligned}
\sigma_{rr} &= \mu \sum_k \{ r^{(k/2)-1} [(k + \eta(\tfrac{1}{2}k) + \eta) f_k + \eta g_{k,\varphi}] + r^{k/2} \eta h_{k,z} \},\\
\sigma_{\varphi\varphi} &= \mu \sum_k \{ r^{(k/2)-1} [(2 + \eta(\tfrac{1}{2}k) + \eta) f_k + (2+\eta) g_{k,\varphi}] + r^{k/2} \eta h_{k,z} \},\\
\sigma_{zz} &= \mu \sum_k \{ r^{(k/2)-1} [\eta(1 + \tfrac{1}{2}k) f_k + \eta g_{k,\varphi}] + r^{k/2} (2+\eta) h_{k,z} \},\\
\tau_{r\varphi} &= \mu \sum_k \{ r^{(k/2)-1} [f_{k,\varphi} + (\tfrac{1}{2}k - 1) g_k] \},\\
\tau_{\varphi z} &= \mu \sum_k \{ r^{(k/2)-1} h_{k,\varphi} + r^{k/2} g_{k,z} \},\\
\tau_{zr} &= \mu \sum_k \{ r^{(k/2)-1} (\tfrac{1}{2}k) h_{k,\varphi} + r^{k/2} f_{k,z} \}.
\end{aligned}
\tag{33}
$$

μ and λ denote the Lamé constants and

$$
\eta = \frac{\lambda}{\mu} = \frac{2\nu}{1-2\nu}.
$$

The typical stress field around the crack front results from expressions with odd k and only these terms are discussed. Starting with $k = 1$ in (32) and with $r^{-(1/2)}$ in (33) respectively, taking into account the boundary conditions at the stress-free crack surfaces, one obtains the well-known singular stress field

$$
\begin{aligned}
&\begin{pmatrix} \sigma_{rr} \\ \sigma_{\varphi\varphi} \\ \tau_{r\varphi} \end{pmatrix} = \mu r^{-(1/2)} A(z) \begin{pmatrix} 5\cos\tfrac{1}{2}\varphi - \cos\tfrac{1}{2}3\varphi \\ 3\cos\tfrac{1}{2}\varphi + \cos\tfrac{1}{2}3\varphi \\ \sin\tfrac{1}{2}\varphi + \sin\tfrac{1}{2}3\varphi \end{pmatrix} + \mu r^{-(1/2)} B(z) \begin{pmatrix} -5\sin\tfrac{1}{2}\varphi + 3\sin\tfrac{1}{2}3\varphi \\ -3\sin\tfrac{1}{2}\varphi - 3\sin\tfrac{1}{2}3\varphi \\ \cos\tfrac{1}{2}\varphi + 3\cos\tfrac{1}{2}3\varphi \end{pmatrix},\\
&\sigma_{zz} = \mu(\sigma_{rr} + \sigma_{\varphi\varphi}), \quad \begin{pmatrix} \tau_{\varphi z} \\ \tau_{zr} \end{pmatrix} = \mu r^{-(1/2)} C(z) \begin{pmatrix} \cos\tfrac{1}{2}\varphi \\ \sin\tfrac{1}{2}\varphi \end{pmatrix},
\end{aligned}
\tag{34}
$$

with the decomposition into the three modes I, II and III.

For $k = 3$ (32) becomes inhomogeneous and there exists a dependence between f_3, g_3, h_1 and h_3, f_1, g_1 either in (32) or in (33). That means that the decomposition of mode II and mode III can be accomplished only for the singular stress field, which is described by the first term in the series expansions (30). The same procedure is also valid for curved crack fronts as shown in [22].

4. Conclusion

The decomposition of the J-integral, which enables the calculation of the stress intensity factors (SIFs) in mixed mode problems, is discussed in 3D elastostatics. In the range of the crack near-field the decomposition of the field quantities according to the three fracture modes is valid because of the independence of the first dominating terms in the asymptotic solution of a straight or curved crack front. Taking into account that the higher order terms of this solution have considerable influence in the far-field of a cracked body, the antisymmetric part cannot be separated in a general manner. Thus, in 3D it is only possible to decompose the far-field quantities into one symmetric part and one antisymmetric part. This is valid for the J-integral calculated via these field variables as well. In order to split the antisymmetric part into mode II and mode III the crack near-field has to be considered.

Computationally it is convenient to approximate the crack near-field with singular crack front elements (see e.g. [9], [23]). Thus, a close contour around the crack front can be chosen, in order to enable a complete decomposition of the J-integral or to calculate the SIFs directly from the approximated crack near-field by correlation and extrapolation techniques [9], [23].

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