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On the computation of the J-integral for three-dimensional geometries in inhomogeneous materials

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Abstract

An analytical expression of the J-integral for inhomogeneous materials is presented in a form suitable for the numerical analysis of arbitrary three-dimensional (3-D) mode-I crack configurations. This integral is given using the principle of virtual work and Eshelby's energy moment tensor. The virtual crack extension method was developed from finite element considerations and was based on the calculations of the released energy when a crack in a finite element model was extended by a small amount Δa. For numerical calculations, the 3-D analogue of the volume integral appears to be an attractive approach for obtaining pointwise values of the stress intensity factors along a crack front. The formulation is easily incorporated into a finite element program, but can also conveniently be used as part of a post-processing program, which uses stress and displacement data from a finite element analysis to calculate the stress intensity factors. Numerical examples are presented, using twenty node, isoparametric, quarter-point elements, for the compact tension specimen, homogeneous and inhomogeneous materials. Stress intensity factors are calculated for various moduli of the inclusion and distances of the crack from the interface. Numerical results are compared with results obtained by two-dimensional models.

1. Introduction

With the increasing use of composite materials and the accompanying need to better understand their fracture behaviour, there is a natural interest to study the crack propagation problems for bodies with spatially varying material properties. This composite effect is studied here by solving the problem of a homogeneous, isotropic elastic body containing an inclusion.

Since the fundamental works of Eshelby [7] and Rice [16], many papers have appeared concerning path-independent integrals and their application to fracture mechanics. In 2-D problems, the J-integral by Rice may be calculated by using analytical or numerical methods. A generalisation of this result to the 3-D case, without any restriction on the direction of crack extension, was first done by Le et al. [9]. In particular, the finite element method allows to evaluate, after determination of the displacement and stress fields in the cracked solid, the stress intensity factors, the J-integral or the energy release rate.

To solve 3-D crack problems often the finite element method is used with a great number of elements surrounding the crack front. The resolution of 3-D crack problems is in general based on two methods, the J-integral [16] and the virtual crack extension [8,15]. The virtual crack extension method was developed from a purely numerical

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standpoint and was based on the calculation of the released energy for 3-D as well as 2-D problems by Parks [15], Hellen [8] and Banks-Sills and Sherman [1,2]. An analytical expression for the energy release rate has been derived by the application of the virtual crack extension method to a consistent continuum mechanics model in 3-D crack configuration [4,5]. Nakamura and Parks [13,14] have determined the mixed-mode stress intensity factors along the crack front under remote in-plane anti-symmetrical loading. The values of these fracture-characterising parameters are obtained from 3-D conservation integrals. Li et al. [10] have developed the area J-integral and volume J-integral methods for calculating the energy released during quasi-static crack advancement.

An extension of the two-dimensional J-integral vector to mutli-phase materials was proposed by Weichert and Schulz [17]; the results have illustrated the path independency and the physical meaning of the extended J-integral. In this paper, the 3-D form of the J-integral in inhomogeneous materials using Eshelby's energy momentum tensor and material force [12] is investigated. An inhomogeneous, isotropic, linearly elastic solid subjected to quasi-static loading is considered.

2. Theoretical formulation

2.1. The J-integral in 2-D and 3-D problems

The J-integral has emerged over the last years as one of the leading parameters to characterise crack-propagation in solids. For two-dimensional, linear or nonlinear solids, the path-independent J-integral for homogeneous materials was developed by Rice [16]:

$$J = \int_{\Gamma} [w \, \mathrm{d}x_2 - T_i u_{i,1}] \, \mathrm{d}\Gamma, \tag{1}$$

where Γ is a curve surrounding the crack tip, $\boldsymbol{T}$ is the traction vector, $\boldsymbol{u}$ is the displacement vector, w is the strain energy density and $\mathrm{d}\Gamma$ is an element of arc length along Γ (Fig. 1). The J-integral can be interpreted as the energy release rate for an infinitesimal quasi-static crack advance in the plane of the crack or as the force on the crack tip in direction of the crack [7,16].

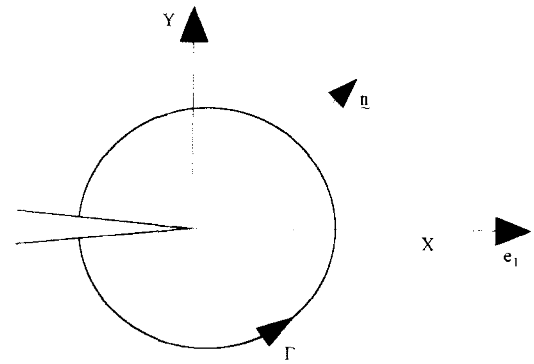

Fig. 1. Typical contour for evaluation of J-integral.

An extension of the 2-D J-integral to multiphase materials was presented by Weichert and Schulz [17]:

$$J = \int_{\Gamma} [w \, \mathrm{d}x_2 - T_i u_{i,1}] \, \mathrm{d}\Gamma - \int_{\Gamma_p} [w \, \mathrm{d}x_2 - T_i u_{i,1}] \, \mathrm{d}\Gamma, \tag{2}$$

where Γ_p is a path embracing that part of phase boundary which is enclosed by Γ.

In the 3-D case, a volume integral expression was developed by Li et al. [10] using the virtual crack extension method and Eshelby's energy momentum tensor:

$$G = \int_{V_t} \left(\sigma_{ik} \frac{\partial u_i}{\partial x_j} - w \delta_{jk} \right) \frac{\partial q_i}{\partial x_k} \, \mathrm{d}V, \tag{3}$$

where G is the total energy decrease per unit crack advance and q_j is a sufficiently smooth function in the volume enclosed by S_t and S_2. The resulting integral is identical to the one by De Lorenzi [4] derived by direct calculation of the energy difference between configurations with slightly different crack lengths.

2.2. Three-dimensional formulation for inhomogeneous materials

The purpose of the present investigation is to calculate the 3-D J-integral for inhomogeneous

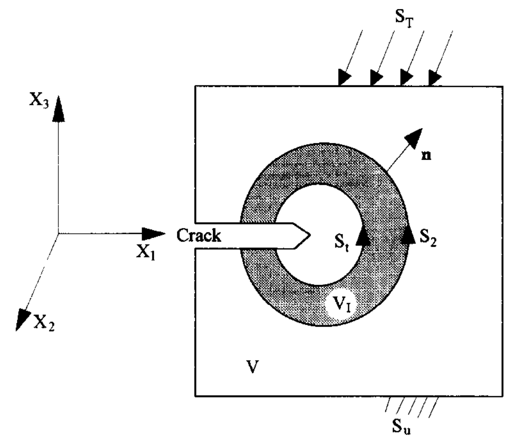

Fig. 2. Volume V_I formed by S_t and S_2.

materials. We consider an elastic solid with a traction-free portion S_t of the external surface varying with velocity v_j. The applied loads and prescribed displacements are presumed to remain fixed during this process (Fig. 2). The potential energy of the solid is given by

$$P = \int_{V(t)} w(\varepsilon, x)\, \mathrm{d}V - \int_{S_T} T_i u_i\, \mathrm{d}S_T \tag{4}$$

with the rate

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \int_{V(t)} \frac{\mathrm{d}w}{\mathrm{d}t}\, \mathrm{d}V - \int_{S_T} T_i \dot{u}_i\, \mathrm{d}S, \tag{5}$$

where T_i are the prescribed tractions on S_T. The strain energy density is denoted by w.

Multiplication of the equilibrium conditions ($\sigma_{ij,j} = 0$) by velocity $\dot{\boldsymbol{u}}$ and integration over the volume V with application of the divergence theorem gives

$$\int_V \sigma_{ij} \frac{\partial \dot{u}_i}{\partial x_j}\, \mathrm{d}V - \int_S \sigma_{ij} \dot{u}_i n_j\, \mathrm{d}S = 0, \tag{6}$$

and Eq. (5) simplifies to

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \int_{S_t} \left(w\delta_{ij} - \sigma_{ij} \frac{\partial u_i}{\partial x_k} \right) v_i n_j\, \mathrm{d}S_t \tag{7}$$

(see also Refs. [6,10]).

The velocity can be expressed as $v_i = \mathrm{d}a/\mathrm{d}t l_i$, where l_i can be chosen in the direction of the normal to S_t at each point and to be a function of x_1 and x_2 (Fig. 2). $\mathrm{d}a/\mathrm{d}t$ is independent of S_t, and using the equality $G = -\mathrm{d}P/\mathrm{d}a|_T$ we obtain

$$G = -\int_{S_t} \left(w\delta_{ij} - \sigma_{ij} \frac{\partial u_i}{\partial x_k} \right) l_i n_j\, \mathrm{d}S_t, \tag{8}$$

where G is the total energy by unit notch advance. In order to develop a volume integral expression for the energy released, we introduce the functions q_j defined by

$$q_j = \begin{cases} l_j & \text{on } S_t, \\ 0 & \text{on } S_2, \end{cases} \tag{9}$$

q_j sufficiently smooth in the volume enclosed by S_t and S_2. Using Eq. (9), the energy released can be written as

$$G = \int_{S_t - S_2} \left(\sigma_{ik} \frac{\partial u_i}{\partial x_j} - w\delta_{jk} \right) q_j n_k\, \mathrm{d}S$$

$$= -\int_{S_t - S_2} b_{jk} q_j n_k\, \mathrm{d}S, \tag{10}$$

where b is Eshelby's energy momentum tensor. The path of integration is taken along some surface surrounding the notch front and not enclosing any other notch, such as the path $S = S_2 + S_3 - S_t + S_4$. When $h \to 0$, the notch becomes a sharp crack, the traction vanishes on the crack faces S_3 and S_4 (Fig. 3).

Applying the divergence theorem, Eq. (10) becomes

$$G = \int_{V_I} \left(\sigma_{ik} \frac{\partial u_i}{\partial x_j} - w\delta_{jk} \right) \frac{\partial q_j}{\partial x_k}\, \mathrm{d}V - \int_{V_I} \frac{\partial b_{jk}}{\partial x_k} q_j\, \mathrm{d}V \tag{11}$$

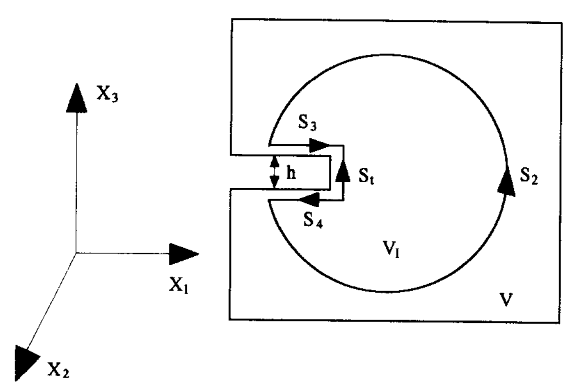

Fig. 3. Thickness h of a notch in the (X_1, X_3) plane.

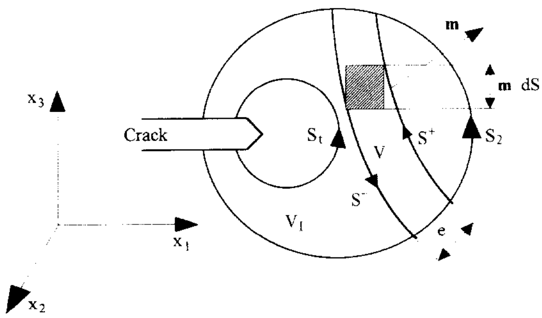

Fig. 4. Interface of dissimilar replaced by a thin layer.

The second term is an additional term describing the influence of the inhomogeneities and can be interpreted as a so-called material force [12]. To evaluate this term, we consider a cracked body with the interface, which is replaced by a thin layer of thickness e (Fig. 4).

Only within this layer and on its boundaries the material properties are assumed to vary in a steady manner, everywhere else the material is homogeneous. Then, we can evaluate the second term of Eq. (11) as the integral over a volume V of the layer and consider it for the limiting case $e \to 0$, by which the interface is described. Therefore, the energy release rate G is reduced to the J_1-integral for the case in which the crack is extended in the direction of its plane. Then the J_1-integral in the three-dimensional case for inhomogeneous materials can be established:

$$J_1 = \int_{V_I} \left(\sigma_{ik} \frac{\partial u_i}{\partial x_1} - w\delta_{1k} \right) \frac{\partial q_1}{\partial x_k} \, \mathrm{d}V - \int_S \left(w\delta_{1k} - \sigma_{ik} \frac{\partial u_i}{\partial x_1} \right) m_1 n_k \, \mathrm{d}S. \tag{12}$$

This expression represents the J_1-integral in inhomogeneous materials. The first term of the right hand side is the volume J_1-integral along the crack front, and the second term is the surface integral enclosing any inhomogeneities.

3. Finite element formulation

In the following, the two integrals of the rhs of Eq. (12) are treated separately.

3.1. Volume integral

For the chosen isoparametric element (20 nodes) $(-1 \le \xi_i \le 1,\ i = 1, 2, 3)$, the coordinates and the displacements of an arbitrary point referred to a global reference system are given by

$$x_i = \sum_{k=1}^{20} N_k X_{ik} \qquad (i = 1, 2, 3), \tag{13a}$$

$$u_i = \sum_{k=1}^{20} N_k U_{ik} \qquad (i = 1, 2, 3), \tag{13b}$$

where the X_{ik} are the nodal coordinates and the U_{ik} the nodal displacements.

The expression of q_j is

$$q_j = \sum_{I=1}^{20} N_I Q_{jI}, \tag{14}$$

where Q_{jI} are the nodal values for the Ith node. $Q_{jI} = 1$ if the Ith node is on S_t and $Q_{jI} = 0$ if the Ith node is on S_2 (Fig. 2). For nodes lying within V_I, Q_{jI} is given by interpolation between the associated boundary nodes on S_t and S_2. Using Eqs. (13) and (14), the spatial derivatives of q_j are given by

$$\frac{\partial q_i}{\partial x_j} = \sum_{I=1}^{20} \sum_{J=1}^{3} \left(\frac{\partial N_I}{\partial \eta_J} \frac{\partial \eta_J}{\partial x_j} \right) Q_{iI}. \tag{15}$$

$\partial \eta_J / \partial x_j$ is the inverse Jacobian matrix of the transformation Eq. (13).

Employing $2 \times 2 \times 2$ Gaussian integration, the discretised form for the volume integral becomes

$$J_V = \sum_{n=1}^{N} \sum_{m=1}^{M} \sum_{p=1}^{P} \sum_{r=1}^{R} \left[\left(\sigma_{ik} \frac{\partial u_i}{\partial x_1} - w\delta_{1k} \right) \times \frac{\partial q_1}{\partial x_k} |J| \right]_{\xi_m, \eta_p, \zeta_r} w_m w_p w_r, \tag{16}$$

where N is the number of elements in V_I. M, P and R are the numbers of Gaussian points per element in the x, y and z directions, respectively.

3.2. Surface integration

The surface integral J_s is evaluated along a surface embracing the inhomogeneities (Fig. 5),

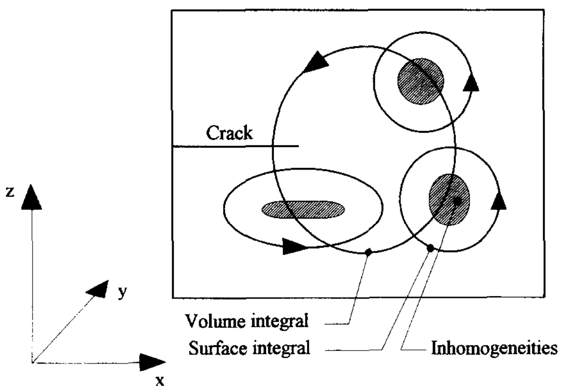

Fig. 5. Volume integral along crack front and surface integral embracing inhomogeneities.

along $(\xi, \eta) = \text{const.}$ and along $(\eta, \zeta) = \text{const.}$ (Fig. 6) given by

$$J_s = \int_S \left(w\delta_{1k} - \sigma_{ik}\frac{\partial u_i}{\partial x_1} \right) m_1 n_k \, \mathrm{d}S. \tag{17}$$

The surface integration in a three-dimensional space is performed using the Gaussian integration. In detail, the necessary transformations are the following.

(i) the components of normal vector $\boldsymbol{n}$ are

$$n_1 = \left(\frac{\partial y}{\partial \xi}\frac{\partial z}{\partial \eta} - \frac{\partial z}{\partial \xi}\frac{\partial y}{\partial \eta} \right) \Big/ |J|, \tag{18a}$$

$$n_2 = \left(\frac{\partial z}{\partial \xi}\frac{\partial x}{\partial \eta} - \frac{\partial x}{\partial \xi}\frac{\partial z}{\partial \eta} \right) \Big/ |J|, \tag{18b}$$

$$n_3 = \left(\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi}\frac{\partial x}{\partial \eta} \right) \Big/ |J|, \tag{18c}$$

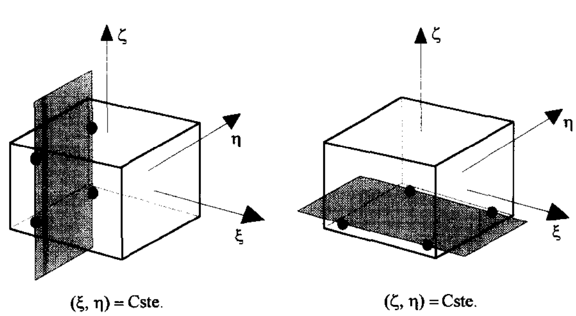

Fig. 6. Surface integration.

where

$$|J| = \left[\left(\frac{\partial y}{\partial \xi}\frac{\partial z}{\partial \eta} - \frac{\partial z}{\partial \xi}\frac{\partial y}{\partial \eta} \right)^2 + \left(\frac{\partial z}{\partial \xi}\frac{\partial x}{\partial \eta} - \frac{\partial x}{\partial \xi}\frac{\partial z}{\partial \eta} \right)^2 + \left(\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi}\frac{\partial x}{\partial \eta} \right)^2 \right]^{1/2}$$

(ii) the differential area $\mathrm{d}S$ is

$$\mathrm{d}S = |J| \, \mathrm{d}\xi \, \mathrm{d}\eta, \tag{19}$$

(iii) the strain energy density can be written as

$$w = \frac{1}{2}\left(\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \sigma_{zz}\varepsilon_{zz} + \sigma_{xy}\varepsilon_{xy} + \sigma_{yz}\varepsilon_{yz} + \sigma_{xz}\varepsilon_{xz} \right). \tag{20}$$

Using above Eqs. (18)–(20) we can express J_s as

$$J_s = \sum_{n=1}^{N} \sum_{i=1}^{I} \sum_{j=1}^{J} \left[\left(\sigma_{ik}\frac{\partial u_i}{\partial x_1} - w\delta_{1k} \right) m_1 n_k |J| \right]_{\xi_i, \eta_j} \times w_i w_j, \tag{21}$$

where I, J are the number of Gaussian points in the x and y directions, respectively. The expression in square brackets is evaluated at the Gaussian points i and j.

After the calculation of the stresses and displacements using the finite element code ANSYS, a subroutine is called which computes the J_1-integral using a numerical integration process through Eqs. (16) and (21).

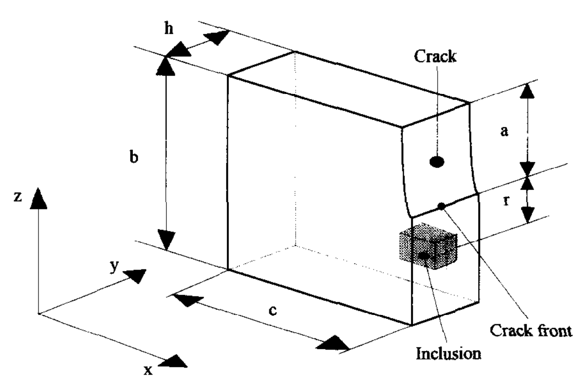

Fig. 7. Schematic of half-model containing a crack and an inclusion.

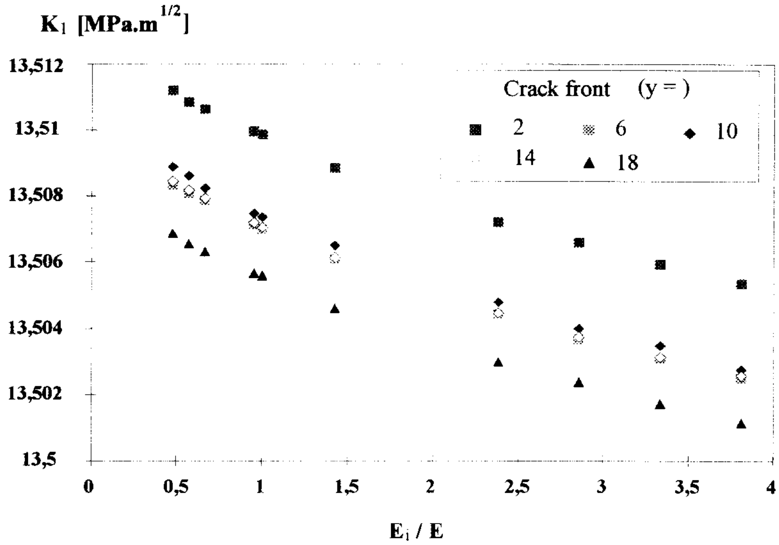

Fig. 8. Variation of k_1 for differences E_i of inclusion.

4. Numerical examples

The influence of an inclusion on the stress intensity factors (SIF) values and the J-integral in inhomogeneous materials are analysed. Due to the symmetry, only half of the body is modelled. We consider a 3-D body containing a crack with geometry defined by $a/b = 0.5$, $c/b = 1$ and $h/b = 0.2$. The body is subjected to the uniform stress of 18 MPa in the direction x infinitely far from the crack (Fig. 7).

4.1. Influence of the inclusion on the crack front

The stress intensity factors K_1 was calculated from the J-integral using the formula $K_1 = \sqrt{EJ}$.

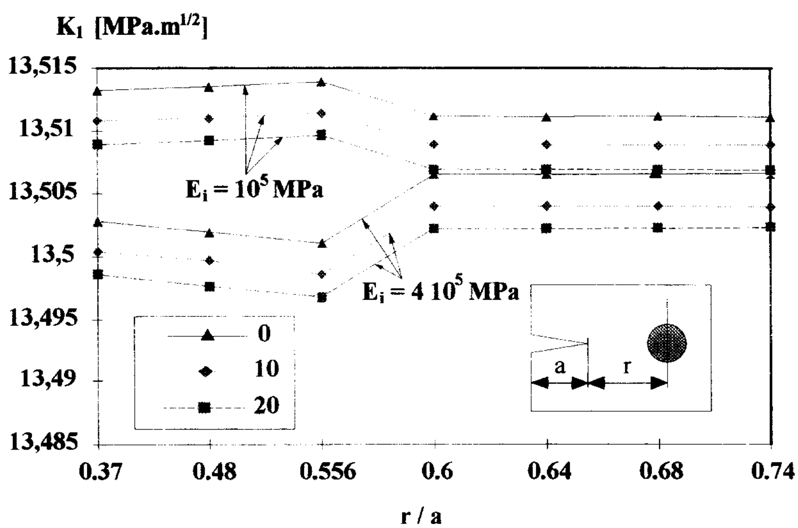

Fig. 9. Variation of K_1 for various distances r between a crack and an inclusion.

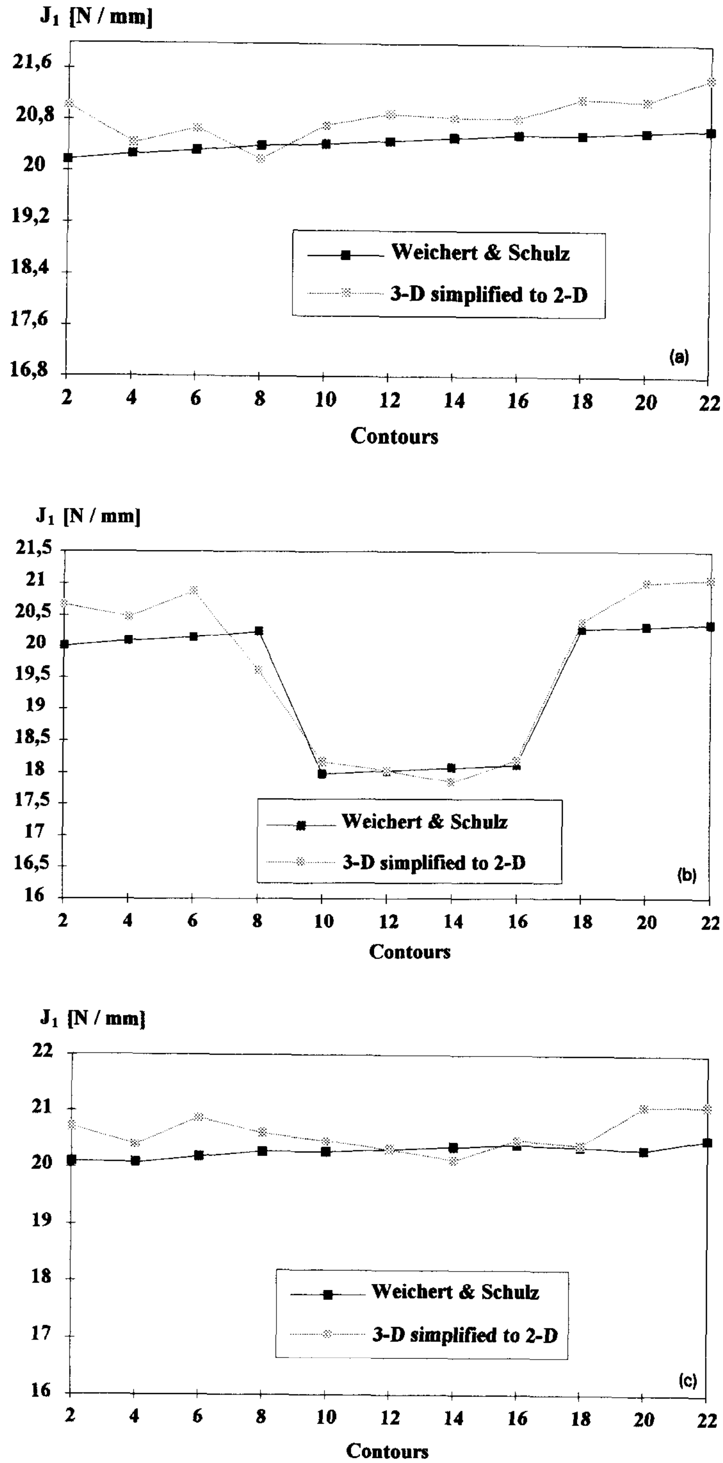

Fig. 10. Comparison between the J_1 values in the present analysis and those from Ref. [17].

We study the influence of Young's moduli of the inclusion (inclusion is in the middle of the crack front) and its distance r (distance between the inclusion and crack front) on the stress intensity factor.

(a) Influence of Young's moduli E_i of the inclusion

The stress intensity factor is evaluated for different Young's moduli of the inclusion and for various position along the crack front. The Young's moduli E_i vary between ($1–2 \times 10^5$ MPa) for a soft inclusion and between ($2.8–7 \times 10^5$ MPa) for a stiff inclusion. The stress intensity factor decreases when the inclusion becomes stiff, the crack is stabilised (Fig. 8).

(b) Influence of the distance r between crack and inclusion

As shown in (Fig. 9), the inclusion has a considerable influence on the SIF of a crack:
– for soft inclusions with the ratio $E_i/E < 1$, the obtained solutions show that the SIF increases when the inclusion approaches the crack front; an instability of the crack can be the result.
– for stiff inclusions with the ratio $E_i/E > 1$, the SIF decreases when the distances between inclusion and crack front decreases and the crack may stabilise. The obtained curves converge to the K_I value for homogeneous materials.

The results obtained by Haddi [6] in the 2-D case, treated as particular case of 3-D analysis, are in good agreement with those obtainted by Li and Chudnovsky [11].

4.2. Comparison of results for J: a genuinely 2D model versus the 3D model simplified to 2D

The J-integral is calculated by evaluating the integrand at several Gaussian points per element. The total value of the J-integral is given by summing the contributions of all elements forming the integral path. We compare the results for the J-integral obtained by Weichert and Schulz [17] with a genuinely 2-D model and the proposed 3-D model, simplified to 2-D. At first, the J-integral is evaluated for different contours for homogeneous materials (Fig. 10a). For inhomogeneous materials, the J-integral is calculated neglecting the second terms of Eqs. (2) and (12) (Fig. 10b). We see that the J-integral depends in this case on the contours. By taking the additional terms into account, the results obtained shown in Fig. 10c demonstrate the path-independence of the extended J-integral.

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