Problem Chosen	2024 MCM/ICM Summary Sheet	Team Control Number
A		2414371

Revealing the Mysteries: Relationship between Sex Ratios and Resource Availability

Summary

The gender ratio of lamprey populations and its close relationship with local resource and environment.

For problem one, Our team abstracted the lamprey's ecosystem to include only the lamprey species, their prey layer, and the predators that feed on the lamprey. This structure forms a trophic system with three nutritional levels. We applied the **Lotka-Volterra model**, which is renowned for its interpretability and suitability in analyzing species abundance in ecosystems with **competitive relationship**. Using this model, we examined the influence of gender ratio fluctuations in the lamprey population on the ecosystem. Our analysis indicated that an increasing proportion of male individuals in the lamprey population had a negative impact on their prey, while benefiting the predators that prey on the lamprey.

For problem two, To examine how changes in the gender ratio of the lamprey population affect its advantages and disadvantages, our team developed a model based on the **Logistic** and Lotka-Volterra equations to analyze the relationship between the gender ratio and the lamprey population's size. The findings suggest that an increase in the male proportion of the lamprey population enables the species to adapt to environments with limited food resources and accumulate reserves for the upcoming population growth when conditions improve. However, there are also drawbacks to this phenomenon: **an overly high male proportion decreases the reproductive success of the population, further intensifying the lamprey's population decline.**

For problem three, In order to assess the influence of gender ratio fluctuations in the lamprey population on ecosystem stability, our team conducted an analysis by determining the **equilibrium points** of the established Lotka-Volterra model and examining the variations in these points as the gender ratio of the lamprey population changed. The findings revealed a trend of increasing stability followed by a subsequent decline as the gender ratio of the lamprey population continued to rise. Specifically, the Lotka-Volterra model exhibited optimal equilibrium when the male proportion in the lamprey population approached **67%**. However, as this proportion continued to increase, the equilibrium of the system gradually diminished.

For problem four, To comprehensively analyze the effects of changes in the gender ratio of the lamprey population on other species sharing the same ecosystem, a quantitative assessment of the changes in the populations of other species can be conducted. By refining the existing Lotka-Volterra model and incorporating a symbiotic **host-parasite relationship**, an examination of the effects of the lamprey population on the populations of other species coexisting in the ecosystem can be performed. The analysis reveals that as the male proportion within the lamprey population continues to rise, its facilitating influence on its own parasites and predators becomes increasingly prominent.

Keywords: Lotka-Volterra Model; Logistic Model; Competive Relationship; Equilibrium Points; Symbiotic Host-parasite Relationship;

Contents

1 Introduction

1.1 Background

Lamprey is a species that exhibits adaptive sexual ratio variation, adjusting its gender composition based on external environmental factors. For instance, changes in food availability can influence the growth rate of larvae, which ultimately affects the sex of the adults. These observations indicate that certain species, such as the ocean lamprey, possess the capacity to modify their sexual ratio according to resource availability. This ability confers both advantages and disadvantages to the species concerned.

Figure 1: Life diagram of lampreys

1.2 Restatement of the Problem

The research aims to investigate the gender ratio of this species and its dependence on local conditions. This includes examining how the sea lamprey adjusts its gender ratio based on the availability of resources and the advantages and disadvantages associated with this ability. To gain a deeper understanding of the interaction and impact between gender ratio variation and the ecosystem, it is necessary to develop and validate a model. The research questions to be addressed include:

➢What are the impacts on larger ecosystems when the sea lamprey population is capable of altering its gender ratio?

➢What are the advantages and disadvantages of the lamprey population itself ?

➢What are the impacts of gender ratio changes caused by lampreys on ecosystem stability ?

➢In an ecosystem, if there is a change in the sex ratio of the lamprey population, may this phenomenon provide some advantages for other organisms in the ecosystem, such as parasites?

1.3 Our Work

1. Through an analysis of the sea lamprey, a keystone species, as well as its food resources and tertiary ecological system involving its predators, a Lotka-Volterra ecological model can be developed to simulate the interactions between prey and predators. Furthermore, the effects of the sea lamprey population's ability to alter its gender ratio on the corresponding ecosystem can be explored.

2. The distinguishing feature of the sea lamprey population is its variable gender ratio, which poses problem two as an analysis of the implications of gender ratio fluctuations on the population's strengths and weaknesses. To examine the effects of gender ratio changes on the sea lamprey population's advantages and disadvantages, our team has developed a model that relates the population's gender ratio to its own size, utilizing the Logistic and Lotka-Volterra models.

3. By analyzing the original Lotka-Volterra model, we can obtain the quantities of each species in the ecosystem when it is in a stable state. This allows us to analyze the impact of changes in the gender ratio of the sea lamprey on the stability of the ecosystem.

4. By enriching the parameters of the original Lotka-Volterra model, such as introducing the parasitic rate of the sea lamprey, the disease prevalence in different seasons, and the seasonal reproductive rate, we can observe the changes in the corresponding population quantities in the improved model. This allows us to analyze whether changes in the gender ratio of the sea lamprey population can provide advantages for other species in its ecosystem.

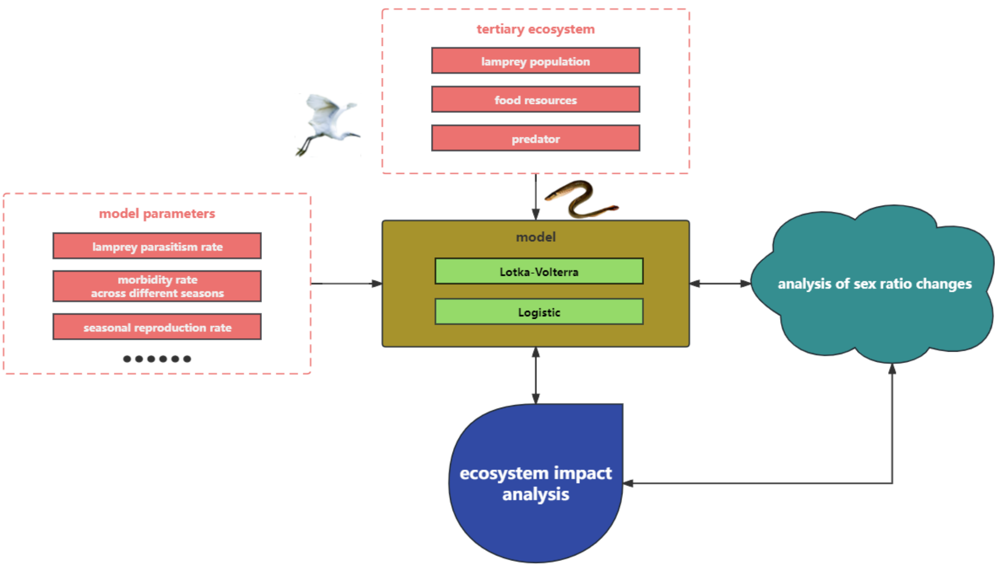

Figure 2: Our problem solving process

2 Assumptions and Justifications

- The lampreys feed on fish, and the number of these fish can affect their growth rate and sex ratio
- The changes in food resources can affect the sex ratio of lampreys, which may in turn affect reproductive rates and population structure.
- The predator of lampreys also relies on lampreys as a food source, and their quantity can be determined by the number of lampreys.

3 Notations

The primary notations used in this paper are listed in Table 1.

Table 1: Notations

Symbol	Description	Unit
u	Birth rate of immature population	%
γ	Mortality rate of immature population	%
β	Mortality rate of mature population	%
U	Adult population density of immature population	%
V	Adult population density of mature population	%
α_u	Birth rate of immature population	%
α_v	Birth rate of mature population	%
γ_u	Mortality rate during maturation of immature population	%
γ_v	Mortality rate during maturation of mature population	%
β_u	Mortality rate of immature population	%
β_v	Mortality rate of mature population	%
P	Population size	/
r	Population growth rate	%
K	Environmental carrying capacity	/

4 Lotka-Volterra Model

4.1 Problem analysis

To analyze the impact of the sea lamprey population's ability to change its gender ratio on the larger ecological environment it inhabits, we can abstract the ecological environment to consist of three main components: the sea lamprey population, the food web involving the sea lamprey's prey, and the predators that prey on the sea lamprey. We can then use the Lotka-Volterra model, commonly used to describe the population dynamics between predators and prey, to analyze the direct relationships among these three components. Specifically, by introducing the parameter of the sea lamprey population's gender ratio, we can examine its influence on the overall quantities of the three components and obtain insights into the changes in the larger ecological environment when the sea lamprey population is capable of altering its gender ratio.

4.2 Lotka-Volterra Model Establishment

In the real world, every population of organisms exists within a community and interacts with other populations, resulting in a complex web of interdependence and mutual constraints. The dynamics of predator-prey models have emerged as a crucial area of research in mathematical biology. Numerous scholars have developed models based on the interaction between two species and the unique characteristics of each species, thereby delving into the intricate dynamics of these models. As researchers have increasingly considered the impact of age structure and environmental factors, the study of biological models with time delays has gained prominence. For instance, in 1990, Aiello and Freedman constructed and analyzed a stage-structured population model with a constant maturation :

$$\begin{cases} \dfrac{du(t)}{dt} = \alpha u(t) - \gamma u(t) - \alpha e^{-\gamma\tau} u(t-\tau), \\ \dfrac{dv(t)}{dt} = \alpha e^{-\gamma\tau} v(t-\tau) - \beta v^2(t) \end{cases} \tag{1}$$

In the realm of biological models, the variables u and v represent the densities of the immature and mature populations, respectively. The parameter τ denotes the time required for maturation from birth. The constants α and γ correspond to the birth and death rates of the immature population, while β represents the death rate of the mature population.

Building upon model (1.1), literature has considered the interactions among different species and proposed a stage-structured Lotka-Volterra cooperative system. This system's dynamic behavior is examined using linearization and upper-lower solution methods. Furthermore, a delayed response diffusion model for the mutual cooperation of adult individuals of two species is established, demonstrating the existence of traveling wave solutions connecting the zero equilibrium point to the unique positive equilibrium point. Alomari and Gourley proposed a delayed Lotka-Volterra competitive model by taking into account the mutual competitive interactions between adult populations.

$$\begin{cases} \dfrac{dU(t)}{dt} = \alpha_u \displaystyle\int_0^{\infty} f_u(s) e^{-\gamma_u s} U(t-s)ds - \beta_u U^2(t) - c_1 U(t)V(t), \\ \dfrac{dV(t)}{dt} = \alpha_v \displaystyle\int_0^{\infty} f_v(s) e^{-\gamma_v s} V(t-s)ds - \beta_v V^2(t) - c_2 U(t)V(t), \\ U(t) = \varphi(t) \ge 0, V(t) = \psi(t) \ge 0, t \in (-\infty, 0] \end{cases} \tag{2}$$

In this context, U and V represent the adult population densities of the two competing species; positive constants α_u and α_v denote the birth rates of the two adult populations; $\gamma_u \gamma_v > 0$ represent the mortality rates during the maturation process; β_u, $\beta_v > 0$ indicate the mortality rates of the mature populations; $c1$ and $c2$ represent the competitive effects between the two adult populations; $fu(s)$ and $fv(s)$ are the integral kernel functions; $\phi(t)$ and $\psi(t)$ are continuous functions on $(-\infty, 0]$ with $\phi(0)$,$\psi(0)$>0.

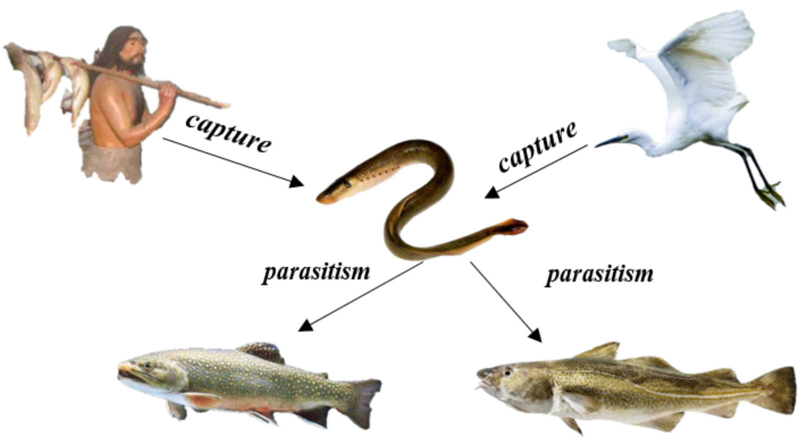

Figure 1:Lotka-Volterra Ecological model diagram

4.3 Model Solution

The task necessitates an analysis of how changes in the gender ratio of lamprey populations might affect the ecosystems they inhabit. Furthermore, it is noted that the gender ratio of marine lampreys can fluctuate based on external environmental factors – in settings with scarce food resources, growth rates are slower, leading to a male proportion of approximately 78%. In contrast, in environments where food is more readily available, the male proportion within the population ranges around 56%. Given this information, the male proportion parameter for the lamprey population in the ecosystem model constructed ranges from 56% to 78%. By conducting relevant research and considering energy transfer efficiency levels in ecosystems (approximately 10%), the initialization of the Lotka-Volterra model parameters can be as follows:

Table 2: Lotka-Volterra Model Initial parameter table

```
# Parameter initialization definition
growth_rate_of_lamprey = 0.05    # The growth rate of lampreys
carrying_capacity_of_lamprey = 2000    # The carrying capacity of lampreys
initial_population_of_lamprey = 100    # The initial population size of lampreys
growth_rate_of_prey = 0.08    # The speed of food resource regeneration
carrying_capacity_of_prey = 10000    # The carrying capacity of food resources
initial_population_of_prey = 5000    # The initial quantity of food resources
growth_rate_of_predator = 0.02    # The growth rate of predators
carrying_capacity_of_predator = 500    # The carrying capacity of predators
initial_population_of_predator = 50    # The initial population size of predators
predation_rate_of_predator = 0.0005    # The predation rate of predators
intraspecific_competition_rate_of_lamprey = 0.01 # Intraspecific rate of lampreys
```

Based on the given population growth rates, carrying capacities, and initial population sizes, calculate the growth of the lamprey population, the corresponding food resource population, and the predator population of lampreys at each time step. Then, use the Euler method to update the population numbers - estimate the population size at the next time step based on the current population size and growth rate, through discrete time steps, using the following formula:

$$Y(t+1) = Y(t) + \frac{h}{2} * f(t,y) + \frac{h}{2} * f\big(t+h, y+h*f(t,y)\big) \tag{3}$$

Where $Y(t)$ is the population size at the current time step, $Y(t+1)$ is the population size at the next time step, $f(t, Y(t))$ is the population growth rate within the current time step, and $f(t+1, Y(t+1))$ is the population growth rate within the next time step.

The final established Lotka-Volterra model shows the following changes in the population of lampreys, their corresponding food resource population, and the population of lamprey predators, corresponding to the gender ratio of the lampreys:

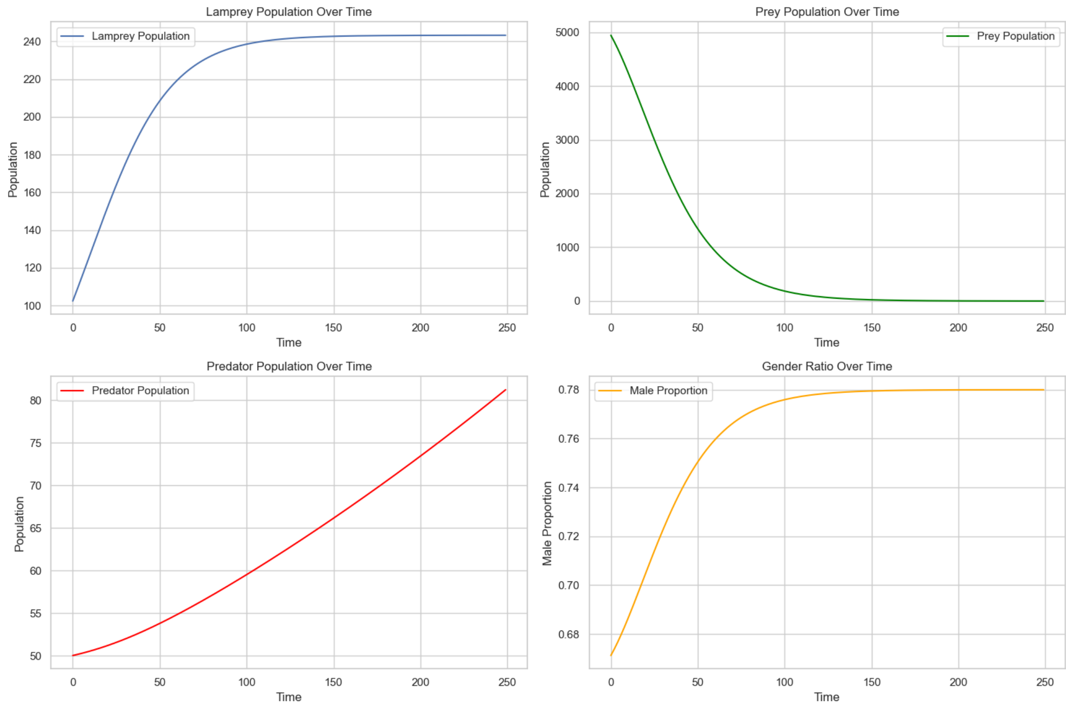

Figure 3: The variation pattern of population size at three levels with the sex ratio of lampreys

The findings suggest that as the male proportion of the lamprey population escalates, the number of lampreys in the ecosystem surges, while the population of their prey species experiences a continuous decline. concurrently, the number of lamprey predators also climbs as the male proportion in the lamprey population keeps rising. It can be inferred from this analysis that the gender ratio within the lamprey population indeed exerts a certain influence on the ecosystem they inhabit: an increase in the male proportion of the lamprey population is detrimental to their food resources but favorable to their predators.

5 Lotka-Volterra and Logistic Model

5.1 Problem Analysis

Based on the knowledge of the topic context and question one, we already know the characteristics of the eel population, which is a population that can change the sex ratio. Question two is deeper, requiring us to analyze the eel population and discuss its advantages and disadvantages. The unique feature of the eel population is its variable sex ratio, so question two also needs to combine this characteristic of the eel population for the analysis, which means that some hypothetical conditions from question one should be continued and, based on that, some adjustments and changes should be made. Furthermore, to obtain more persuasive data results for analysis, in question two, our team will add new models to simulate the entire population and even the ecosystem.

5.2 Development of a Tertiary Ecosystem Model

We understand that the population of lampreys alter their sex ratio based on the availability of resources, especially food. Consequently, for question two, we will develop a three-tiered ecosystem model encompassing the food web, which includes prey (the lamprey's food sources), intermediate predators (the lamprey population), and top predators (the indigenous species), by employing the Logistic growth model and the Lotka-Volterra model, drawing on the conceptualization of the ecosystem from question one. We will simulate the dynamics of this ecosystem by establishing initial conditions and conducting cyclical iterations to model the interactions and the dynamic equilibrium among the three levels, aiming to derive final outcomes to assess the pros and cons of the lamprey population.

To realistically simulate the ecosystem's operation, it is essential to model the growth of several populations that are assumed. The Logistic growth model is a universal framework for population or species growth, first introduced by Pierre François Verhulst in 1838 and subsequently refined to become the law of population growth. In this model, where P denotes population size and t represents time, the model is articulated through the following differential equation::

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) \tag{4}$$

Where the constant r is the population growth rate and K is the environmental carrying capacity.

In the equation, +rP leads to a growth rate with almost no resistance. The growth rate r represents the proportion of growth of population P in a unit of time. In the continuous iteration process, as the population size increases, the second term $-\frac{rP^2}{K}$ approaches the first term and tends to equalize, individuals within the population P begin to compete for some key resources (including food resources) and interfere with each other, creating an antagonistic effect, represented by the parameter K. After this competition phenomenon occurs, the overall population growth rate decreases, until P stops growing, at which point the solution of the equation is

$$P(t) = \frac{KP_0e^{rt}}{K + P_0(e^{rt} - 1)} = \frac{K}{1 + \left(\frac{K - P_0}{P_0}\right)e^{-rt}} \tag{5}$$

with

$$\lim_{t \to \infty} P(t) = K \tag{6}$$

One might say that K represents the threshold value for P, meaning the ultimate size that the population can achieve after an indefinitely long period (or effectively within a finite timeframe). It's crucial to recognize that, provided the initial value P(0) is greater than 0, no matter its amount, the population will inevitably converge to the environmental carrying capacity, even in instances where P(0) exceeds K.

In developing our previous model, we overlooked the environmental carrying capacity variable, which is subject to change over time, thus the model has been refined to

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K(t)}\right) \tag{7}$$

In certain specific situations, the carrying capacity varies over time with period T, according to the following equation:

$$K(t + T) = K(t) \tag{8}$$

For example, factors such as climate, from a macro perspective, have a typical T value of 1 year. However, in our team's modeling, the environmental carrying capacity is primarily based on food resources, therefore, it does not have a fixed periodic value, and its variation depends not only on time but also on changes in other populations. Interestingly, regardless of whether the carrying capacity varies in fixed cycles or not, in the Logistic growth model, iterating according to the model's equations, very rich behaviors can occur, such as bistability within certain parameter ranges, as well as monotonic decay, smooth exponential growth, oscillations near a stable level, among other phenomena. In the operational results of the model constructed by our team, it is not difficult to find that the final graph reflects this characteristic.

However, the above is just a pure mathematical model. Next, we need to apply this Logistic growth model to the ecosystem model, because the Logistic growth model is a universal model for the growth of population size. Therefore, for the food resource populations of the seven-gilled lamprey, the seven-gilled lamprey population itself, and the predators of the seven-gilled lamprey, all can utilize this Logistic growth model. Based on this model, we dynamically adjust the formulas and, after several iterations of the model, each iteration calculates and updates the results of the growth formulas for the three populations.

Firstly, for the derived formula of the seven-gilled lamprey population, based on the aforementioned model and considering the context and requirements of the problem, we still need to account for the effects of the population's gender ratio, reproductive success rate, and the availability of food resources on the growth of this population, while the carrying capacity term is primarily focused on the size of the seven-gilled lamprey population. Defining the reproductive success rate of the seven-gilled lamprey population as α and assuming that the gender ratio (that is, the proportion of males) is β, since the highest reproductive success occurs when the gender ratio is 1:1, we can obtain the following equation:

$$\alpha = 1 - |\beta - 0.5| \tag{9}$$

Next, in order to introduce the formula, we need to define a series of variables. Let the growth rate of the lamprey population, the regeneration rate of food resources, and the growth rate of predators be γ_1、δ_1、ε_1, respectively; the carrying capacity of the lamprey population, the carrying capacity of food resources, and the carrying capacity of predators be γ_2、δ_2、ε_2, respectively; and the total number of the lamprey population, the total amount of food resources, and the total number of predators be γ_3、δ_3、ε_3, respectively.

After obtaining the related variables, the growth formula for the seven-gilled lamprey population can be listed as follows:

$$\grave{\gamma_3} = \gamma_3 + \gamma_1\gamma_3\alpha\left(1 - \frac{\gamma_3}{\gamma_2\frac{\delta_3}{\delta_2}}\right) \tag{10}$$

Similarly, we can also establish the growth formula for food resources, the situation of being frequently preyed upon must be considered. Therefore, the growth formula needs to subtract the portion that is preyed upon. Assuming that its predation rate is θ, we can derive the formula as follows:

$$\grave{\delta}_3 = \delta_3 + \delta_1\delta_3\left(1 - \frac{\delta_3}{\delta_2}\right) - \theta\delta_3\gamma_3 \tag{11}$$

Finally, by the same analogy, the growth formula for the predator population, combined with the model, is as follows:

$$\grave{\varepsilon}_3 = \varepsilon_3 + \varepsilon_1\varepsilon_3\left(1 - \frac{\varepsilon_3}{\varepsilon_2}\right)(\theta\gamma_3) \tag{12}$$

The growth model formulas for all three populations are already set; they simply need to be implemented in the code according to these formulas. The issue pointed out that the gender ratio is impacted by food resources, with its value varying approximately from 0.56 to 0.78. This aspect will also be incorporated into the model, forming the following linear relationship:

$$\beta = \beta_{max} - (\beta_{max} - \beta_{min})\frac{\delta_3}{\delta_2} \tag{13}$$

After obtaining all the necessary functions to simulate the operation of the tri-trophic ecosystem model, we began iterating the model. In each iteration, we first calculate and update the sex ratio of the sturgeon population, then separately update the total number of sturgeon, the total food resources, and the total number of predators, and record these four data points.. The figure below shows six line graphs that display the variation of five values over time, that is, while the ecosystem model is being simulated. They describe the variation in the total number of sturgeon over time, the variation in food resources over time, the variation in the total number of predators over time, the variation in the sex ratio of the sturgeon population over time, the variation in the reproductive success rate of the sturgeon population over time, and the relationship between the total food resources and the sex ratio of the sturgeon population.

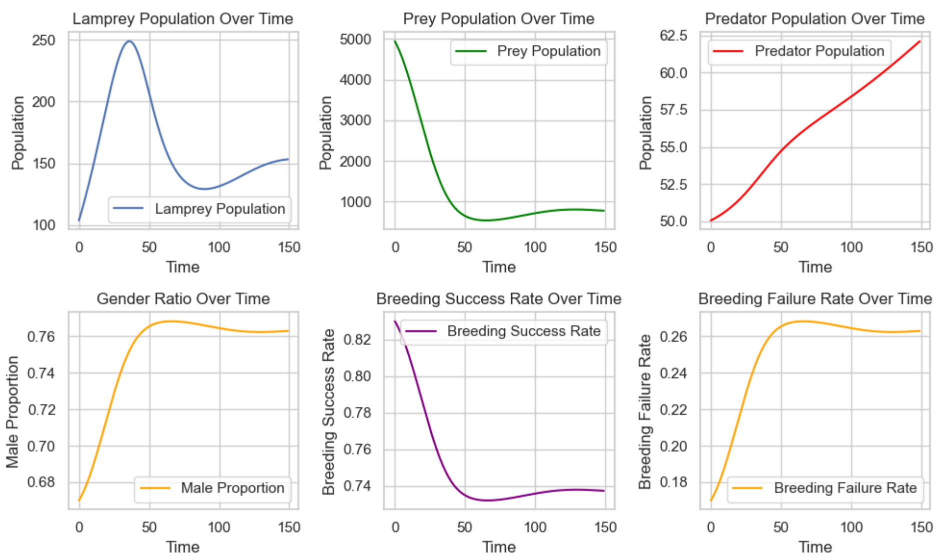

Figure 4: The impact of gender ratio on the quantity of other trophic levels in the population of lampreys

By analyzing the results of the six composite curves, we observe that over time, the total number of eels initially increases, then decreases, and finally stabilizes, while the total number of food resources decreases initially, then slightly increases, and also stabilizes, and the total number of predators consistently increases. The sexual ratio of the eel population is opposite to that of the food resources, increasing initially, then slightly decreasing, and finally stabilizing, with the reproductive success rate of the eel population resembling the changes in food resources. From this, we can understand the advantages and disadvantages of the eel population.

Advantages: Adaptive adjustments of sexual ratios enable the eel population to adapt more effectively to different environmental conditions. In the simulations, after the initial setup, due to the pre-existing condition of drastically reduced food resources, the sexual ratio of the eel population consistently increased, increasing the proportion of males to reduce the consumption of limited food resources and adapt to the environment, maintaining population stability, and demonstrating its long-term adaptability and resistance to environmental pressures.

Disadvantages: Significant changes in the sexual ratio of the eel population disrupt the internal balance between males and females, interfering with the natural balance of the ecosystem. Concurrently, when food resources are scarce, the proportion of males in the eel population increases, resulting in a reduction in the reproductive success rate of the population, which is unfavorable for long-term population development. The results also show that when the proportion of males increases to a certain point, the total number of eels begins to decline, indicating that male dominance reduces the genetic diversity of the population and increases the risk of extinction due to environmental pressures, causing negative impacts on the population.

In summary, the advantages and disadvantages of the eel population include the following points.

Advantages of Eel Populations:

- High Efficiency in Resource Utilization: The adaptive adjustment of sex ratios in eel populations allows them to adapt more broadly and effectively to different ecological conditions and ecosystems. For example, in conditions of scarce food resources, the population can adjust the proportion of males to reduce the overall demand for limited food resources.
- Dynamic Optimization of Populations: The adaptive changes in sex ratios within eel populations are favorable for their dynamic optimization, maintaining population stability, facilitating long-term adaptation, and enhancing the role the population plays in the ecosystem.
- Flexible Survival Strategies: Eel populations can flexibly adjust their sexual structure in response to changes in the environment, improving their survival capabilities and better adapting to environmental changes, thereby increasing their overall survival rates.

Disadvantages of Eel Populations:

- Disruption of Ecosystem Balance: Significant changes in the sex ratios of eel populations can disturb the natural balance of the ecosystems they inhabit, affecting the stability of the population or even the entire ecosystem.
- Reduction in Genetic Diversity: If the sex ratios within eel populations become

unbalanced, for instance, if males become overly dominant, this can reduce the genetic diversity of the population, thereby decreasing reproductive success and hindering long-term development, increasing the risk of extinction due to environmental pressures.

• Limitations in the Adaptive Change of Sex Ratios: The effectiveness of this dynamic mechanism for changing sex ratios may be limited by rapid or extreme environmental changes, which can exceed the population's ability to adjust the sex ratios, resulting in negative impacts on the population.

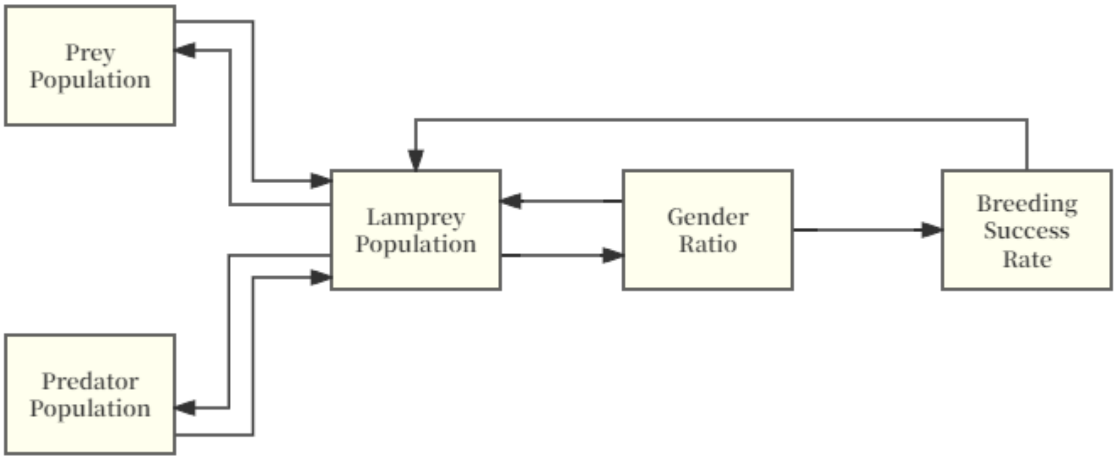

Figure 5: The relationship between the mutual influence of five factors

6 Lotka-Volterra Model's Equilibrium Points

6.1 Problem Analysis

This problem requires considering the influence of changes in the male and female sex ratio in the lamprey population on the stability of the ecosystem as a whole. The central point of this problem lies in solving the stability of the ecosystem, and the most convincing argument for the ecosystem's stability is the condition of changing the population quantity of the corresponding species over a long period of time. Furthermore, the long term means that we cannot only consider the influence of the lampreys' sex ratio on the growth rate of the current generation of this species, but we also need to consider the influence of the lampreys' sex ratio on the reproduction rate of this population. By analyzing the original Lotka-Volterra model, we can obtain the population quantity conditions of the species in the ecosystem in a stable state, and then analyze the influence of changes in the sex ratio of lampreys on the stability of the ecosystem.

6.2 Equilibrium Points In The Lotka-Volterra Model

When analyzing competition between different populations, the stability of the ecosystem as a whole is an issue that frequently attracts our attention. In the Lotka-Volterra model, it is relatively easy to locate the system's equilibrium points, also known as fixed points. If $\frac{dU}{dt}=\frac{dV}{dt}=0$, that is

$$\begin{cases} U(\alpha-\gamma V)=0 \\ V(e\gamma-\beta)=0 \end{cases} \tag{14}$$

We can solve for two equilibrium points:

$$\begin{cases} U^*=0 \\ V^*=0 \end{cases} or \begin{cases} U^*=\frac{\beta}{e\gamma} \\ V^*=\frac{\alpha}{\gamma} \end{cases} \tag{15}$$

The first equilibrium point brings the number of both populations to zero, and it will remain zero forever, a solution that is uninteresting and trivial. Furthermore, if the quantity of U undergoes any deviation, it will immediately enter exponential growth, therefore, this is an unstable equilibrium point.

The second equilibrium point, much more interesting, will be the focus of our analysis next. We will prove that, regardless of the parameter values, this equilibrium point is always stable (stable fixed point).

Suppose U and V have a slight deviation from U^* and V^*, denoted as

$$\mathrm{U(t)}=U^*+\epsilon(t), V=V^*+\delta(t) \tag{16}$$

Ignoring higher-order small terms $\epsilon\delta$, the Lotka-Volterra equation can be rewritten as

$$\begin{cases} \frac{d\epsilon}{dt}=-\frac{\beta}{e}\delta \\ \frac{d\delta}{dt}=e\alpha\epsilon \end{cases} \tag{17}$$

By combining the two equations above, we can eliminate δ and obtain an equation that contains only ϵ

$$\frac{d^2\epsilon}{dt^2}+\alpha\beta\epsilon=0 \tag{18}$$

Next, introducing the frequency $\omega=\sqrt{\alpha\beta}$, the general solution of the equation can be written as:

$$\epsilon(t)=A\cos\omega t+B\sin\omega t \tag{19}$$

Where A and B are constants determined by the initial state of the system.

By combining the equations above, we obtain the corresponding solution for δ:

$$\delta(\mathrm{t})=\mathrm{e}\sqrt{\frac{\alpha}{\beta}}(A\sin\omega t-B\cos\omega t) \tag{20}$$

These solutions indicate that, if the prey population U and the predator population V are out of the equilibrium point, future fluctuations will also just oscillate near the equilibrium point over time, exhibiting a very regular periodicity. The period of fluctuations in the population of both species is:

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\alpha\beta}} \tag{21}$$

6.3 Model Solving

After analyzing the problem, we need to consider the reproductive success of the lamprey population, it is necessary to include the reproductive success rate as a parameter in the Lotka-Volterra model . Additionally, since the reproductive success rate of the population is closely related to the sex ratio within the population, the reproductive success rate can be obtained based on the data of the sex ratio. The specific conversion formula is as follows:

$$\delta = 1 - |\alpha - \beta| \tag{22}$$

Where δ refers to the reproductive success rate of the lamprey population, α refers to the proportion of males in the lamprey population, and β refers to the proportion of males in the lamprey population in environments where food is more easily obtained, i.e., under more ideal conditions.

Additionally, since the sex ratio of lampreys is strongly related to the availability of food in their environment, we can define a sex ratio factor to adjust the sex ratio of lampreys based on the amount of food resources.

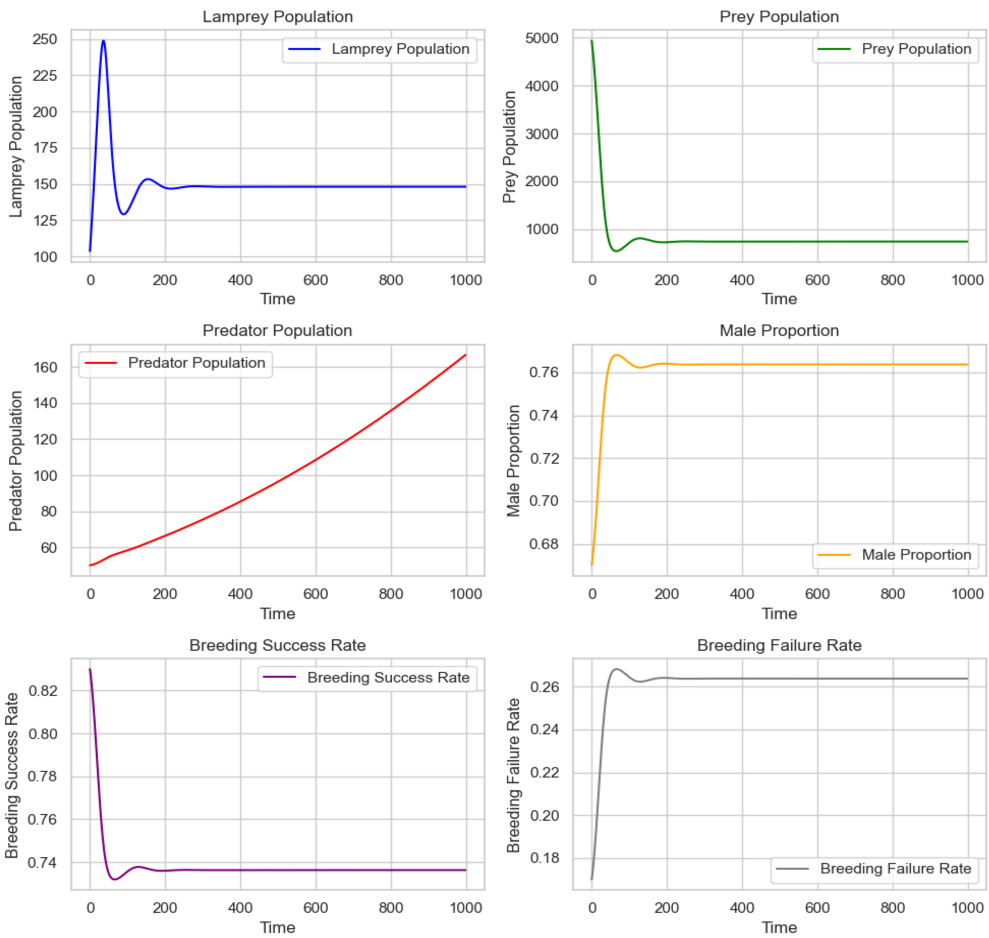

Figure 6: The impact of changes in the sex ratio of the lamprey population on the ecosystem

After analyzing the visualized image above, we can see that as the proportion of males in the lamprey population increases, the reproductive success rate of the lamprey population steadily decreases. An important observation is that at the beginning of the increase in the proportion of males in the lamprey population, the population number of lampreys was on the rise and reached a peak, then quickly declined, gradually tending towards stability over time.

This phenomenon is mainly due to the proportion of males in lampreys initially being in a more reasonable range, resulting in a higher reproductive success rate, causing the population number of lampreys to rapidly increase. Over time, the reproductive success rate of the lamprey population declined, transforming the growth trend into a decline and tending towards stability under the feedback action of the ecosystem. Additionally, we can see that the number of lamprey predators increases as the proportion of males in lampreys increases, while the food resources of lamprey predators decrease and tend towards stability over time, ultimately reaching a lower level. This partly validates the hypothesis that male lampreys are more efficient hunters than females.

We conducted sensitivity tests on the parameters of the established model to obtain sensitivity indicator data for the lamprey's reproductive success rate under different parameter conditions and visualized them. The specific results are as follows:

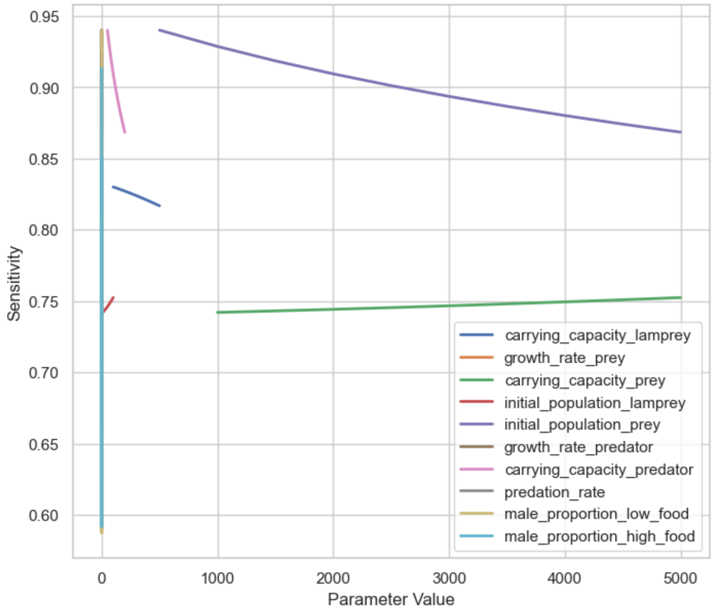

Figure 7: Lotka-Volterra model parameter sensitivity analysis

7 Problem 4: Symbiotic Host-parasite Lotka-Volterra Model

7.1 Problem analysis

Due to the fact that each biological population does not exist alone in its ecosystem, there is a certain amount of energy transfer and information exchange between it and the corresponding trophic level. As a result, considering that each biological population exists within a complex web of ecological interactions, it is crucial to understand how the gender ratio of the lamprey population influences other species in its trophic level.

For instance, some native populations consider lampreys as a dietary staple, thus making them lamprey predators. Additionally, lampreys parasitize cod and salmon to obtain nutrients, thereby positioning them as parasites of these fish. It is also essential to take into account the parasites dwelling within the lamprey population.

To examine this issue, we can enhance the traditional Lotka-Volterra model by incorporating various parameters, such as the lamprey's parasitism rate, seasonal disease prevalence, and reproductive rates. By observing the changes in population sizes within the refined model and analyzing the impacts of gender ratio changes on other species in the lamprey's ecosystem, we can determine whether the alteration offers any advantages to the ecosystem's inhabitants.

7.2 Symbiotic Host-parasite Lotka-Volterra Model Establishment

The host-parasite model is a quintessential biological mathematical model, widely explored and applied in both biology and ecology. Numerous researchers have contributed valuable insights to the study of host-parasite systems. However, these investigations predominantly focus on the assumption that an increase in parasite populations negatively impacts host survival. In reality, some parasite expansions can actually contribute to host population growth within certain ecological systems. Such models are referred to as symbiotic host-parasite models. Drawing upon the Lotka-Volterra model, a deterministic symbiotic host-parasite model is formulated as follows:

$$\begin{cases} dx_1(t) = x_1(t)[r_1 - a_{11}x_1(t) + a_{12}x_2(t)]dt, \\ dx_2(t) = x_2(t)[-r_2 + a_{21}x_1(t) - a_{22}x_2]dt \end{cases} \tag{23}$$

In this model, x_1 and x_2 represent the population sizes of the host and parasite at time t, respectively; $i = 1,2$ denotes the growth rates of x_i; $a_{ij} > 0, i, j = 1,2$, where a_{ij} is the interspecific competition coefficient of species i, and $a_{ij}, i \neq j$ represents the competitive effect of species j on species i. Through this model, it can be easily observed that the host and parasite have a mutually beneficial symbiotic relationship.

In reality, biological population models are inevitably influenced by random disturbance factors. Therefore, the establishment of the model should also consider the impact of environmental white noise. There are multiple methods to introduce environmental white noise into stochastic biological population models. Among them, a more common approach is as follows: first, define a process $X^h(kh) = (x_1^h(kh), x_2^h(kh))^T, k = 0,1,2,...$, where $X^h(0) = (x_1(0), x_2(0))^T$ is a deterministic initial value. Let $(\xi_i^h(k))_{k=0}^{\infty}$ be a sequence of random variables satisfying $\mathrm{E}\left[\xi_i^h(k)\right] = 0, \mathrm{E}\left[\xi_i^h(k)\right]^2 = \sigma_i^2 h, i = 1,2$, where σ_1^2 and σ_2^2 are constants representing the intensity of the random disturbance factors. Assuming that on the interval $[kh, (k+1)h]$ changes according to the deterministic model and is disturbed by the random term $(x_1^h(kh)\xi_1^h(k), x_2^h(k, h)\xi_1^h(k))^T$. Therefore:

$$\begin{cases} x_1^h\big((k+1)h\big) = x_1^h(kh) = +[r_1 - a_{11}x_1^h(kh) + a_{12}x_2^h(kh)]h + x_1^h(\mathrm{kh})\xi_1^h(k) \\ x_1^h\big((k+1)h\big) = \mathrm{x}_1^\mathrm{h}(\mathrm{kh}) = +\big[\mathrm{r}_1 - \mathrm{a}_{11}\mathrm{x}_1^\mathrm{h}(kh) + \mathrm{a}_{12}\mathrm{x}_2^\mathrm{h}(kh)\big]\mathrm{h} + \mathrm{x}_2^\mathrm{h}(kh)\xi_2^\mathrm{h}(\mathrm{k}), \end{cases} \tag{24}$$

As $h \to 0$, X^h weakly converges to the solution of a stochastic symbiotic host-parasite model:

$$\begin{cases} dx_1(t) = x_1(t)[r_1 - a_{11}x_1(t) + a_{12}x_2(t)]dt + \sigma_1 x_1(t)dB_1(t) \\ dx_2(t) = x_2(t)[r_1 - a_{11}x_1 + a_{12}x_2(t)]dt + \sigma_1 x_1(t)dB_1(t) \end{cases} \tag{25}$$

7.3 Model Solution

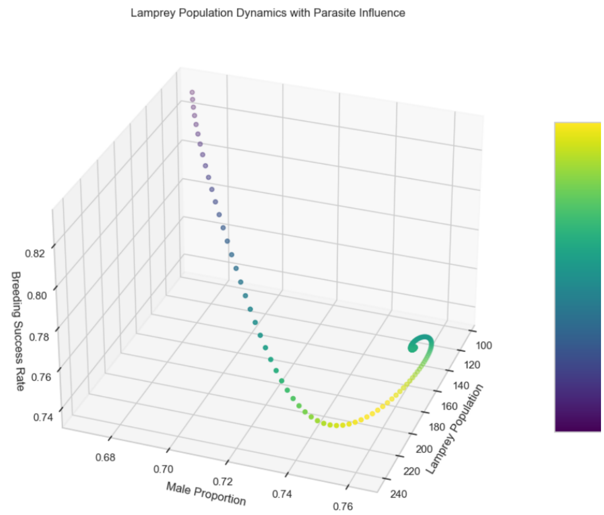

Figure 8: The number of parasites varies with the sex ratio of the lamprey population

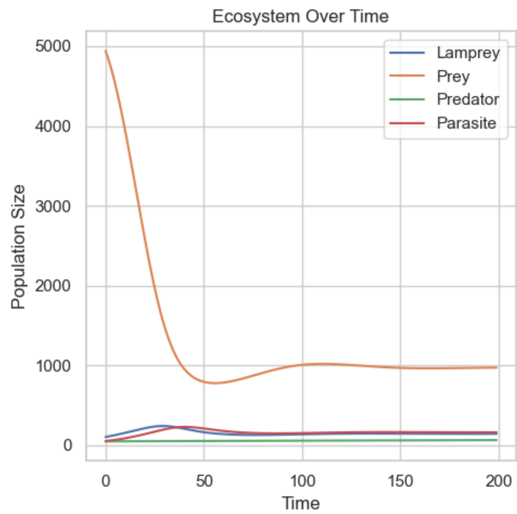

Figure 9: The number of species in the ecosystem over time

Based on the visualization results above, the change in the sex ratio of the lamprey population has a certain impact on the ecosystem it inhabits, which promotes the number of corresponding species. For example, regarding the change in the number of internal parasites in the lamprey population with the variation of the sex ratio, when the male ratio of the lamprey population reaches 0.74, the corresponding parasite population reaches its peak.

In addition to parasites, the number of predators that feed on lampreys is also slowly increasing. Since the male ratio increases, combined with the background given in the title -- in an environment with limited food supply, the male ratio in the lamprey population is relatively high -- it can be inferred that male lampreys have stronger survival abilities and hunting skills.

Therefore, as the male ratio of the lamprey population continues to increase, the number of food resources will show a clear trend of decline, which reduces the proportion of this species in the ecosystem and provides better development opportunities for competitors with the same ecological niche as the lamprey's food resources, thus promoting the species diversity of the ecosystem.

7.4 Model sensitivity analysis

Conduct a corresponding sensitivity analysis on the above model, and obtain the following results:

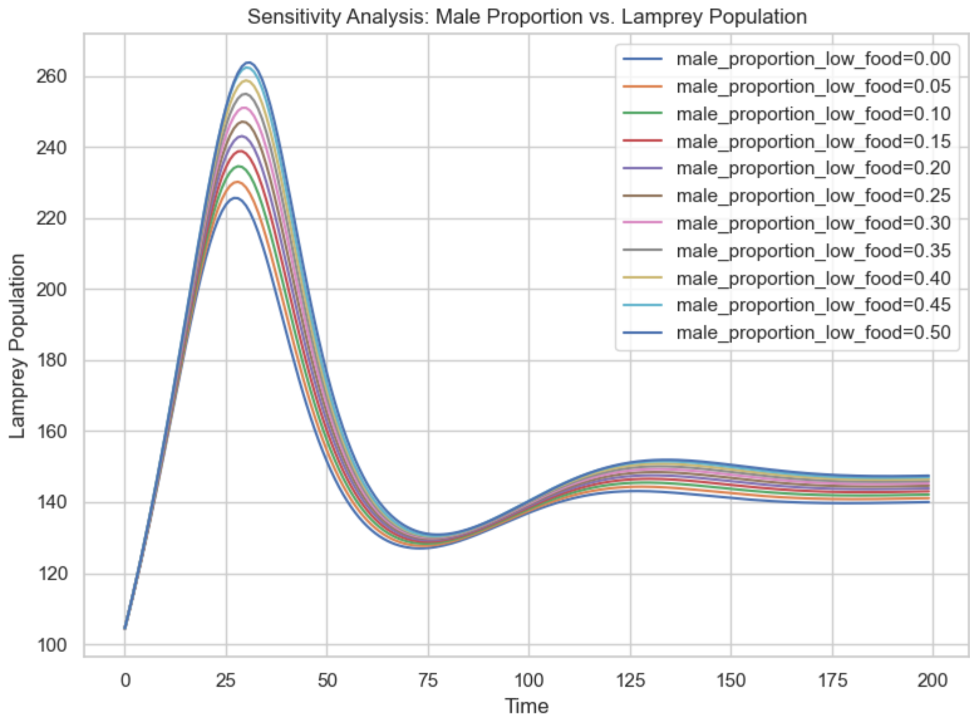

Figure 10: Sensitivity analysis of the model
after introducing species variables

From the sensitivity graph, it can be observed that as the male ratio continuously changes, the population of the lamprey exhibits a relatively reasonable range of variation. There is no significant increase or overfitting phenomenon. Therefore, it can be considered that the obtained model has good robustness and is applicable when the gender ratio of the lamprey population keeps changing.

8 Strengths and Weaknesses

8.1 Strengths

- The fully developed model utilizes the Lotka-Volterra delay model and the Lotka-Volterra host-parasite model, which can more fully reflect the fluctuations in the sex ratios of the lamprey population and their impact on the ecosystem in which they are found.
- The model is based on rigorous mathematical reasoning and is scientifically sound.
- The model took into account the necessary corresponding variables for the established Lotka-Volterra model and, based on the original, fully considered the delay effects and the relationship between the host and the parasite.

8.2 Weaknesses

- The model is based on a lot of mathematical derivation, and the definition of data initialization variables is more based on an ideal state, which may lead to discrepancies with the actual situation, so the model don't achieve a perfect fit to the actual values.

8.3 Possible Improvements

1) The established Lotka Volterra model still needs to consider factors such as but not limited to global warming and increased carbon dioxide concentration in order to be more realistic and more accurate in predicting the population size of organisms in the same ecosystem as the lamprey.

2) The Lotka Volterra model established does not fully consider the factors of human participation, and should comprehensively consider the impact of humans on the population of lampreys, rather than solely focusing on predation of lampreys. For example, the introduction of a series of restrictive measures taken by humans during the breeding period of the lamprey population will have a significant impact on the sex ratio of the lamprey population, which is still worth further investigation.

3) The prediction accuracy of the established Lotka Volterra model still needs further improvement, as the number of variables based on the established Lotka Volterra model is still relatively small, and it is easy to be affected by unilateral accidental factors that affect the overall prediction accuracy.

4) The current Lotka Volterra model is still a relatively basic model, and it should keep up with the development trend in the future, such as introducing Lotka Volterra variant models with vicious competition and other relationships to further enrich existing models.

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