MA28010 - Note on matrices

An $m \times n$ *matrix* is an array with m rows and n columns. If $m = n$ the matrix is said to be a square matrix. Elements of the matrix (entries in the array) may be constants (numbers), variables or expressions involving variables and constants. Individual elements are referred to by specifying their row and column number, for example, if M is the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ then $M[2,3]$ is the element in row 2 and column 3, that is 6.

Matrices can be added to (or subtracted from) one another provided that both matrices have the same number of rows and columns. Corresponding elements are added, or subtracted. For example, if $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$, $B = \begin{pmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{pmatrix}$, then $A + B = \begin{pmatrix} 8 & 10 \\ 12 & 14 \\ 16 & 18 \end{pmatrix}$.

However, if $C = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$, then $A + C$, $A - C$, $B + C$, $B - C$ are not defined.

Matrices M_1 and M_2, say, can be multiplied together, provided that M_1 has the same number of columns as M_2 has rows, that is if M_1 is $m \times n$ then M_2 must be $n \times p$. Let $R = M_1 M_2$, then the elements of R are defined as $R[i,j] =$ sum of $M_1[i,k] \times M_2[k,j]$ for $k = 1$ step 1 to n. The order in which multiplication is performed is important. In general $M_1 M_2 \neq M_2 M_1$, in fact both products $M_1 M_2$ and $M_2 M_1$ only exist if M_1, M_2 are square matrices. For example, with M and C as given above, we have

$$\begin{aligned} C.M &= \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \\ &= \begin{pmatrix} 1 \times 1 + 1 \times 4 & 1 \times 2 + 1 \times 5 & 1 \times 3 + 1 \times 6 \\ 2 \times 1 + 2 \times 4 & 2 \times 2 + 2 \times 5 & 2 \times 3 + 2 \times 6 \end{pmatrix} \\ &= \begin{pmatrix} 5 & 7 & 9 \\ 10 & 14 & 18 \end{pmatrix} \end{aligned}$$

but $M.C$ is not defined.

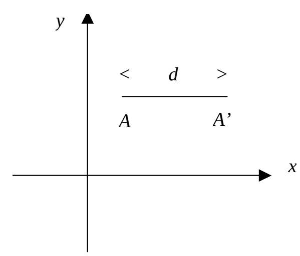
Matrices cannot be divided. If $F = G.H$, where F, G, H are matrices then $H = G^{-1}F$ (not $H = F/G$) and G^{-1} is called the *inverse* of G. The product $G.G^{-1} = G^{-1}G = I$, where I is the identity (or unit) matrix which has diagonal elements equal to one and all other elements zero. Inverse matrices exist only for square matrices. For example, if the matrix $S = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ represents a scaling of the y-coordinates by 2, with x-coordinates not scaled, then the new coordinates X, Y are given by $\begin{pmatrix} X \\ Y \end{pmatrix} = S \begin{pmatrix} x \\ y \end{pmatrix}$. To return to the original coordinates from the scaled coordinates we use the relationship

$$\begin{pmatrix} x \\ y \end{pmatrix} = S^{-1} \begin{pmatrix} X \\ Y \end{pmatrix}, \text{ where } S^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

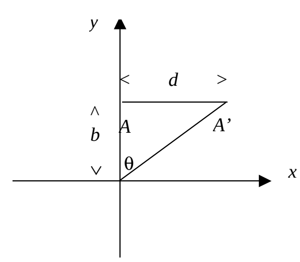
Applications to scaling and shearing

Matrices can be used to describe the transformations required when scaling or shearing a geometric object. Scaling of coordinates can be achieved by pre-multiplying the column vector containing the coordinates by $S = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$, where p is the scale factor for x and q is the scale factor for y. So $\begin{pmatrix} X \\ Y \end{pmatrix} = S \begin{pmatrix} x \\ y \end{pmatrix}$, where X, Y are the scaled coordinates. The corresponding relationships are $X = px$, $Y = qy$.

Consider a shear transformation parallel to the x-axis. The point $A(a, b)$ becomes the point $A'(a + d, b)$ after the shear.



Without loss of generality we can shift axes so that A lies on the new y-axis. New x-coordinates equal old x-coordinates minus a and y-coordinates remain unchanged.



So $d = b \tan\theta$ and the sheared x-coordinate for point A becomes $a + b\tan\theta$, where $\tan\theta$ is the shear factor. In general we have that the sheared x-coordinate equals the original x-coordinate plus the original y-coordinate times the shear factor, that is $X = x + y\times$ shear factor and $Y = y$. In matrix notation $\begin{pmatrix} X \\ Y \end{pmatrix} =$

$\begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ where f is the shear factor. Similarly a shear transformation parallel to the y-axis is equivalent to premultiplying by the matrix $\begin{pmatrix} 1 & 0 \\ f & 1 \end{pmatrix}$. For example, let $P_1(0,0)$, $P_2(0,1)$, $P_3(2,1)$ and $P_4(2,0)$ be the vertices of a rectangle. The rectangle is to be scaled by 2 in the x-direction and 5 in the y-direction, then sheared in the x-direction with a shear factor of 0.8 and finally sheared in the y-direction with a shear factor of 0.6. Matrix $S_1 = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}$ represents the scaling, $S_2 = \begin{pmatrix} 1 & 0.8 \\ 0 & 1 \end{pmatrix}$ represents the shear in the x-direction and $S_3 = \begin{pmatrix} 1 & 0 \\ 0.6 & 1 \end{pmatrix}$ represents the shear in the y-direction. Remember that the order in which the transformations are performed is important. The new (transformed) coordinates X, Y are given by $\begin{pmatrix} X \\ Y \end{pmatrix} = S_3 S_2 S_1 \begin{pmatrix} x \\ y \end{pmatrix}$. Now $S_3 S_2 S_1 = (S_3 S_2) S_1$ and $S_3 S_2 = \begin{pmatrix} 1 & 0 \\ 0.6 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0.8 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0.8 \\ 0.6 & 1.48 \end{pmatrix}$. So $S_3 S_2 S_1 = \begin{pmatrix} 1 & 0.8 \\ 0.6 & 1.48 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 1.2 & 7.4 \end{pmatrix}$. Applying the transformation to the point $P_3(2,1)$ we have $\begin{pmatrix} 2 & 4 \\ 1.2 & 7.4 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 9.8 \end{pmatrix}$. All the points can be transformed in one operation,

$$\begin{pmatrix} 2 & 4 \\ 1.2 & 7.4 \end{pmatrix} \quad \begin{array}{c} \begin{pmatrix} 0 & 0 & 2 & 2 \\ 0 & 1 & 1 & 0 \end{pmatrix} \\ \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ P_1 & P_2 & P_3 & P_4 \end{array} \end{array} \quad = \begin{pmatrix} 0 & 4 & 8 & 4 \\ 0 & 7.4 & 9.8 & 2.4 \end{pmatrix}$$