Bayesian Updating

Incorporates evidence one piece at a time: If the patient appears at the dental surgery with toothache

$$P(Cavity|Toothache) = P(Cavity)\frac{P(Toothache|Cavity)}{P(Toothache)}$$

When $Catch$ is observed we seek to find a way of applying Bayes rule for this single new piece of evidence. First applying the product rule as before gives:

$$P(Cavity|Toothache \wedge Catch) = \frac{P(Toothache \wedge Catch \wedge Cavity)}{P(Toothache \wedge Catch)} \tag{1}$$

Looking first at the denominator. Again by the product rule:

$$P(Toothache \wedge Catch) = P(Catch|Toothache)P(Tootache)$$

Looking now at the numerator (also by the product rule and remembering that "$\wedge$" is commutative):

$$P(Toothache \wedge Catch \wedge Cavity) = P(Catch|Toothache \wedge Cavity)P(Toothache \wedge Cavity)$$

But: $P(Toothache \wedge Cavity) = P(Cavity|Toothache)P(Toothache)$ *[funnily enough this is also by the product rule.]*
Substituting back into equation 1 gives:

$$P(Cavity|Toothache \wedge Catch) = \frac{P(Catch|Toothache \wedge Cavity)P(Cavity|Toothache)\cancel{P(Toothache)}}{P(Catch|Toothache)\cancel{P(Toothache)}}$$

In this case we know the equation for $PCavity|Toothache)$ having used it already when $Toothache$ was the only piece of evidence. So the above becomes:

$$P(Cavity|Toothache \wedge Catch) = P(Cavity)\frac{P(Toothache|Cavity)}{P(Toothache)}\frac{P(Catch|Toothache \wedge Cavity)}{P(Catch|Toothache)}$$

This equation matches the one in Russel & Norvig. So we have filled in the gaps of their presentatin for Bayesian updating.

Conditional Independence

Conditional independence expresses the relationship that pertains once we know the direct causes of some evidence. For example, once it is $Cavity$, the presence of a $Catch$ is not going to change the probability that the $Cavity$ is causing the $Toothache$. Similarly, the presence of a $Toothache$ is not going to change the probability that the $Cavity$ is causing the $Catch$.
This can be written as:

$$P(Catch|Cavity \wedge Toothache) = P(Catch|Cavity)$$
$$P(Toothache|Cavity \wedge Catch) = P(Toothache|Cavity)$$

So we can simplify the updating equation as follows:

$$P(Cavity|Toothache \wedge Catch) = P(Cavity)\frac{P(Toothache|Cavity)}{P(Toothache)}\frac{P(Catch|Cavity)}{P(Catch|Toothache)}$$

The product of the denominators of this equations can be written as: $P(Tootheache \wedge Catch)$ (from the product rule) and can be eliminated by means of normalization as follows: *(this bit is not in R & N - they leave it for you to work out).*

$$P(Cavity|Toothache \wedge Catch) = \frac{P(Cavity)P(Toothache|Cavity)P(Catch|Cavity)}{P(Toothache \wedge Catch)}$$

$$P(\tilde{}Cavity|Toothache \wedge Catch) = \frac{P(\tilde{}Cavity)P(Toothache|\tilde{}Cavity)P(Catch|\tilde{}Cavity)}{P(Toothache \wedge Catch)}$$

Remembering that these two equations must sum to 1, we can construct the sum and multiply both sides by $P(Toothache \wedge Catch)$, giving:

$$P(Toothache \wedge Catch) = P(Cavity)P(Toothache|Cavity)P(Catch|Cavity) + P(\tilde{}Cavity)P(Toothache|\tilde{}Cavity)P(Catch|\tilde{}Cavity)$$

Therefore the final form of the Bayesian updating equation is:

$$P(Cavity|Toothache \wedge Catch) = \frac{P(Cavity)P(Toothache|Cavity)P(Catch|Cavity)}{P(Cavity)P(Toothache|Cavity)P(Catch|Cavity) + P(\tilde{}Cavity)P(Toothache|\tilde{}Cavity)P(Catch|\tilde{}Cavity)}$$