6 Equations of Curves

We have determined the equation of a line in various ways:

(i) Given the gradient, m, and the intercept on the y-axis, c, we have

$$y = mx + c.$$

(ii) Given the gradient, m, and the point (x_1, y_1) through which the line passes, we have

$$y = mx - mx_1 + y_1.$$

(iii) Given two points (x_1, y_1) and (x_2, y_2) through which the line passes, we have

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y - y_1}{x - x_1}.$$

All of these can be written in the form

$$\alpha x + \beta y + \gamma = 0,$$

where α, β, γ are constants.

For (i) $mx - y + c = 0$ ($\alpha = m$, $\beta = -1$, $\gamma = c$),

for (ii) $mx - y + y_1 - mx_1 = 0$ ($\alpha = m$, $\beta = -1$, $\gamma = y_1 - mx_1$)

and for (iii) $(y_2 - y_1)x + (x_1 - x_2)y + x_2y_1 - x_1y_2 = 0$ ($\alpha = y_2 - y_1$, $\beta = x_1 - x_2$, $\gamma = x_2y_1 - x_1y_2$).

In fact when the equation relating x and y is of the form $\alpha x + \beta y + \gamma = 0$ the geometric object is always a line. Some of the constants may be zero. For example, $x = 0$, $y - 1/2 = 0$ and $y + 2x = 0$ are equations describing lines, as is the equation $2x + 3y - 8 = 0$. For a point to lie on the line its coordinates must satisfy the equation of the line.

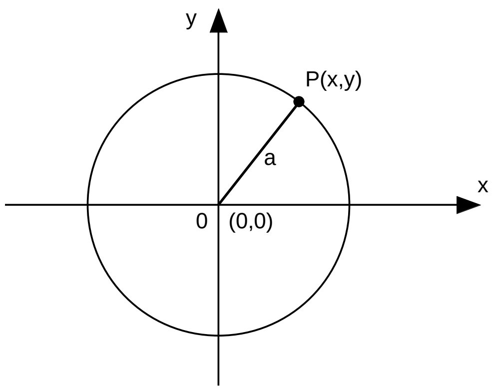
Example 12

Do the points $(3, 5)$ and $(6, 10)$ line on the line $y - 2x + 1 = 0$?

When $x = 3$, $y = 5$ we have $y - 2x + 1 = 5 - 6 + 1 = 0$, so $(3, 5)$ is on the line.

When $x = 6$, $y = 10$ we have $10 - 12 + 1 = -1 \neq 0$, so $(6, 10)$ is not on the line $y - 2x + 1 = 0$.

The equation of a line is a *first degree* curve, that is it only involves x to the power one and y to the power one. When the equation of a curve is not of first degree, in x and y, the graph will not be a straight line. We will now look at some curves of higher degree.



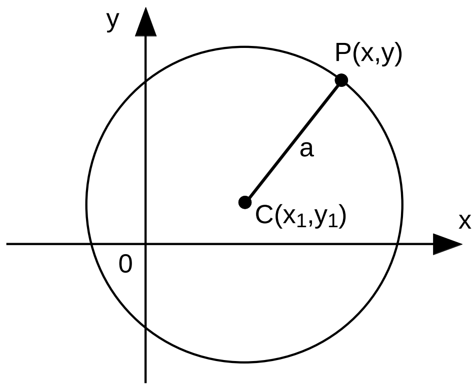
7 Circle

Consider a circle of radius a, centre at the origin.

Let $P(x, y)$ be any point on the circle. The distance between the origin and P is the radius, a. Therefore $a^2 = OP^2 = (x - 0)^2 + (y - 0)^2$ and the equation of the circle is

$$a^2 = x^2 + y^2.$$

This is the equation of the circle, it is of degree two. We can determine the values of y, for points on the circle, for any given value of x and vice versa. For any value of x(or y) there are at most two corresponding y-coordinates (or x-coordinates).



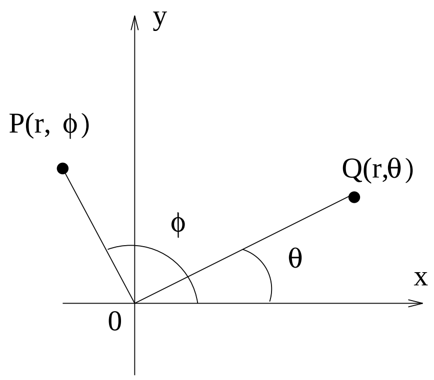
If the centre is at the point (x_1, y_1), rather than at the origin, we have $a^2 = CP^2 = (x - x_1)^2 + (y - y_1)^2$ and the equation of the circle is

$$a^2 = (x - x_1)^2 + (y - y_1)^2.$$

In fact there is a simpler equation for a circle if we use *polar coordinates* rather than Cartesian coordinates. In polar coordinates, the position of a point is given as (r, θ), where r is the distance from the origin (which is always positive) and θ is the angle between the horizontal axis and the line joining the origin to the point (θ is measured anticlockwise).

The different angle distinguishes the points $P(r, \theta)$ and $Q(r, \phi)$ which have the same radial distance, r. The relationship between polar and Cartesian coordinates is

$$x = r\cos\theta,$$



$$y = r\sin\theta.$$

So for the equation of a circle with centre at the origin, $x^2 + y^2 = a^2$, we have

$$\begin{aligned} r^2\cos^2\theta + r^2\sin^2\theta &= a^2 \\ r^2(\cos^2\theta + \sin^2\theta) &= a^2 \\ r^2 &= a^2 \\ r &= a. \end{aligned}$$

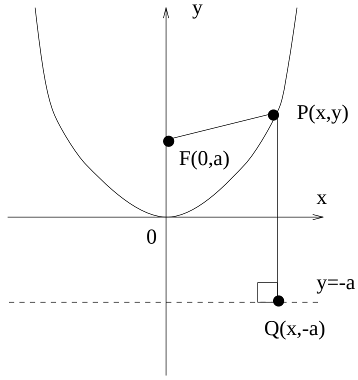
In polar coordinates the equation of a circle is $r = a$, for all values of θ.

When working with circles it is often simpler to use polar coordinates. They also have an obvious use in navigation where position is given by distance and direction, e.g. 200 miles NW of Shetland.

A *circle* is a curve on which points are all equidistant from a fixed point, that is its centre.

8 Parabola

A *parabola* is a curve on which points are equidistant from a given point (its *focus*) and a line (its *directrix).* Consider a parabola which passes through the origin, with its focus on the y-axis and its directrix parallel to the x-axis.



By definition, we require $FP = PQ$. Now

$$FP^2 = (x-0)^2 + (y-a)^2$$

and

$$PQ^2 = (x-x)^2 + (y+a)^2.$$

So

$$FP^2 = PQ^2$$

if

$$x^2 + y^2 - 2ay + a^2 = y^2 + 2ay + a^2$$

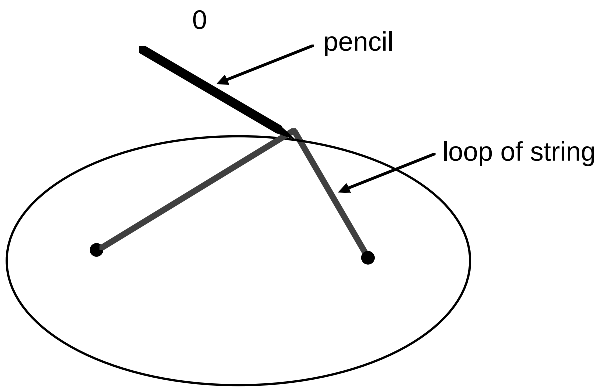
that is

$$x^2 = 4ay.$$

This is the equation of a parabola which passes through the origin and has its focus at $(0, a)$. It is symmetric about the y-axis; for any given value of y there are at most two points on the parabola, one with x-coordinate x_1, say, the other with x-coordinate $-x_1$. On the other hand, for a given value of x there is only one corresponding value for y. Like the equation of a circle it is also a second degree curve.

9 Ellipse

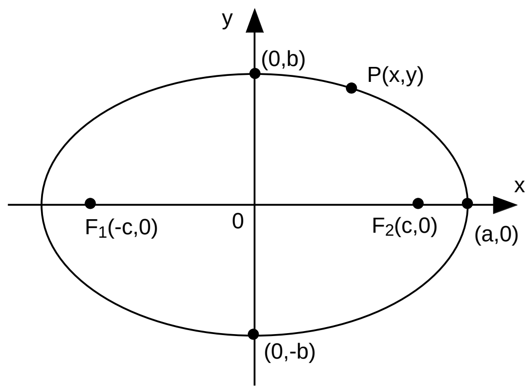
An *ellipse* is a curve on which points whose distances from two fixed points (the *foci*) have a constant sum. Think of a piece of string fixed at its ends. Place a pencil on the loop of the string and pull taut. As the pencil is moved it traces out an ellipse.



The points where the string is fixed are the foci. The length of the string is the sum of the distances from the foci.

We now determine the equation of an ellipse. Choose axes with the origin mid-way between the foci and the x-axis passing through them. Let the point where the ellipse cuts the positive x-axis be $(a, 0)$.

We have that, for an ellipse, $PF_1 + PF_2 =$ a constant. When P is the point $(a, 0)$, $PF_1 = a + c$ and $PF_2 = a - c$. Therefore $PF_1 + PF_2 = 2a$ and hence the constant is equal to $2a$. For the



general point $P(x, y)$ we have

$$\begin{aligned} PF_1 + PF_2 &= 2a \\ \sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} &= 2a \\ \sqrt{(x+c)^2 + y^2} &= 2a - \sqrt{(x-c)^2 + y^2}. \end{aligned}$$

Squaring

$$(x+c)^2 + y^2 = 4a^2 + (x-c)^2 + y^2 - 4a\sqrt{(x-c)^2 + y^2}$$

and simplifying

$$cx - a^2 = -a\sqrt{(x-c)^2 + y^2}.$$

Squaring again

$$c^2x^2 + a^4 - 2a^2cx = a^2((x-c)^2 + y^2).$$

Divide through by a^2, then

$$\begin{aligned} \frac{c^2}{a^2}x^2 + a^2 - 2cx &= x^2 - 2cx + c^2 + y^2 \\ x^2\left(\frac{c^2}{a^2} - 1\right) - y^2 &= c^2 - a^2 \\ x^2\frac{(c^2 - a^2)}{a^2} - y^2 &= c^2 - a^2 \\ \frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} &= 1 \\ \frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} &= 1. \end{aligned}$$

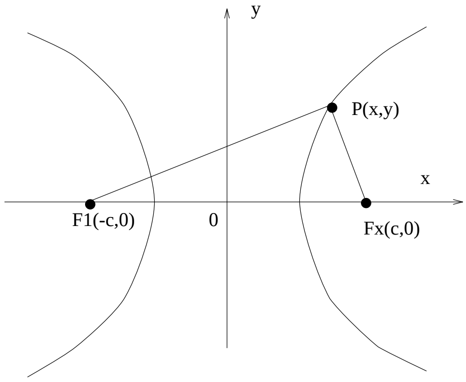
As $a > c$ then $a^2 > c^2$, so $a^2 - c^2 > 0$. We can replace $a^2 - c^2$ by b^2, which is always positive. The equation of an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

We note that when $x = 0$, $y = \pm b$, so $(0, b)$ is the point where the ellipse cuts the positive y-axis and $(0, -b)$ where it cuts the negative y-axis. The ellipse is symmetric about both axes.

10 Hyperbola

An *hyperbola* is a curve on which every point is such that its distances from two fixed points (the foci) have a constant difference.



Again we choose axes with the foci at $(c,0)$ and $(-c,0)$ with the x-axis passing through them. Let $(a,0)$ be the point where the hyperbola cuts the positive x-axis. We require $PF_1 - PF_2 =$ constant. As $P(a,0)$ is on the hyperbola we have

$$PF_1 - PF_2 = a + c - (c - a) = 2a.$$

There is another branch of the hyperbola when $PF_2 - PF_1 = 2a$. Hyperbolas have two symmetric branches. Let $P(x,y)$ be any point on the hyperbola. Then

$$\begin{aligned} PF_1 - PF_2 &= \pm 2a \\ \sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} &= \pm 2a. \end{aligned}$$

Simplifying this, as in the equation of the ellipse, gives

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1.$$

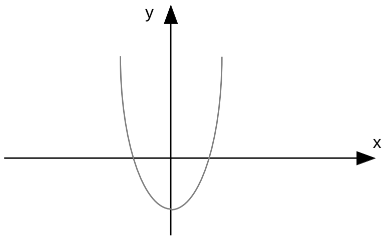
This is the same equation as that of the ellipse *but* here $a < c$ so $a^2 - c^2 < 0$ and we let $b^2 = -(a^2 - c^2) > 0$. The equation of an hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

The hyperbola is symmetric about both axes.

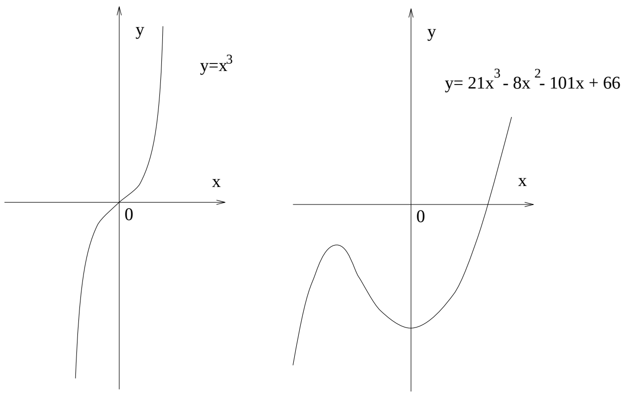
11 Other curves

We have already seen an example of a degree two curve in which y is of degree one and x is of degree two: a parabola. We can write $y = Ax^2 + Bx + C$ and often refer to y as a 'quadratic in x.' This can be thought of as a graph of y against x.



It has one turning point at most, that is one change of direction. In the diagram as x increases y decreases until $x = 0$, then y increases as x increases. The sign of the coefficient A determines the orientation; if A is positive the curve forms a 'bowl', if A is negative the 'bowl' is upside down. For every value of y there are two points on the curve, that is two values of x satisfy the equation when y is given. At the turning point these two values of x coincide.

The higher the degree of the curve, the more turning points (changes of direction) it has. In general a curve of degree n has $n - 1$ turning points. Some turning points may coincide; this is known as a point of inflexion. The following example shows two cubic (degree 3) curves.



12 Intersection of curves

A point at which two curves meet is called a point of intersection. At this point the x and y-coordinates must satisfy both equations describing the curves. The higher the degree of the curve, the more points of intersection it can have with another curve. For example, two lines can have at most one point of intersection; a line and a degree three curve can have up to three points of intersection. Some curves do not intersect at all.

Example 14

Consider the intersection of the line $y = 3x$ (line is a curve of degree one) and the second degree curve $y = x^2 + 8x + 6$. We solve the equations simultaneously to find values for x and

y that satisfy both equations

$$y = 3x \tag{12.1}$$
$$y = x^2 + 8x + 6. \tag{12.2}$$

Substitute $y = 3x$ from equation (12.1) into equation (12.2)

$$\begin{aligned} 3x &= x^2 + 8x + 6 \\ 0 &= x^2 + 5x + 6 \\ 0 &= (x+3)(x+2) \end{aligned}$$

This is satisfied when $x = -3$ or $x = -2$. When $x = -3$, $y = -9$, from (12.1) and when $x = -2$, $y = -6$. Therefore $(-3, -9)$ and $(-2, -6)$ are points of intersection of the two curves.

There cannot be more than two points where a line intersects a curve of degree two. Of course curves of higher degree can have more points of intersection with other curves. The technique for finding the points of intersection is to solve the two equations of the curves simultaneously, whatever their degree.

13 Three-dimensional coordinate geometry

So far we have considered two-dimensional geometry, that is geometry relating to points in a plane. We are now going to consider three-dimensional geometry, that is geometry of points not all of which are in the same plane. A three-dimensional Cartesian frame of reference consists of a fixed point O (the origin) and three mutually perpendicular axes Ox, Oy, Oz.

In three-dimensional geometry we fix the position of a point by projecting it onto the x-y plane, thus giving coordinates x, y as in two-dimensional axes and for the third dimension we also require its height, z, above this plane. It is possible to take the positive z-axis up or down, both directions being perpendicular to Ox and Oy. We use the convention that gives a right-handed set of axes.

A simple way to find the direction of the z-axis: Imagine turning a screwdriver from the positive x-axis towards the positive y-axis, the direction of motion of the screw being along the positive z-axis.

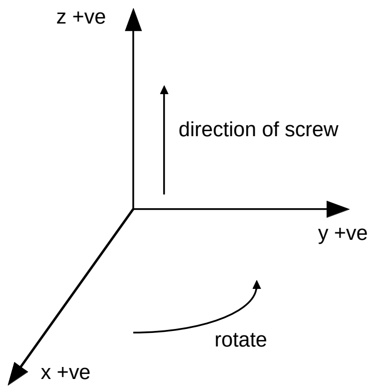
A point has coordinates (x, y, z), the order in which the coordinates are specified is important.

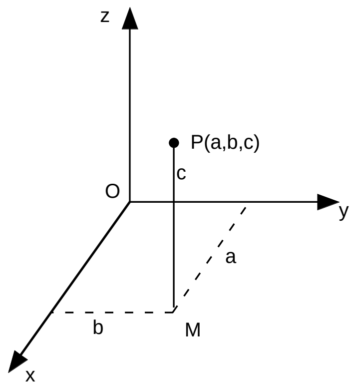
Point M is the foot of the perpendicular from the point P to the x-y plane, that is the projection of P in the x-y plane. So point M has coordinates $(a, b, 0)$. Point P is at a height c above the x-y plane, therefore $P = (a, b, c)$.

14 Distance between two points in three-dimensional space

The distance from the origin to the point P is given by

$$\begin{aligned} OP^2 &= OM^2 + MP^2 \\ &= a^2 + b^2 + c^2. \end{aligned}$$





This expression enables us to deduce an expression for the distance between two points $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$. Assume that the axes have been translated so that A is the new origin. The distance AB is the same whatever axes are used. Point $A(x_1, y_1, z_1)$ becomes $A'(0, 0, 0)$ with reference to the new axes and $B(x_2, y_2, z_2)$ becomes $B'(x_2 - x_1, y_2 - y_1, z_2 - z_1)$. So the length AB is given by the distance from the origin (A') to the point B', therefore

$$AB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2.$$

As in two dimensions this formula represents the distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) so long as due account is taken of the signs of the coordinates.

Example 15

Find the distance from the origin, O, to the point $P(4, -1, 2)$.

$$OP^2 = 16 + 1 + 4, \quad OP = \sqrt{21}.$$

Example 16

What is the distance between $A(2, 1, 3)$ and $B(1, 2, 3)$?

$$AB^2 = (1 - 2)^2 + (2 - 1)^2 + (3 - 3)^2 = 2; \quad AB = \sqrt{2}.$$

Example 17

Find the length AB if A is the point $(2, 1, 1)$ and B is $(-3, -2, -1)$.

$$\begin{aligned} AB^2 &= (-3 - 2)^2 + (-2 - 1)^2 + (-1 - 1) \\ &= 25 + 9 + 4 = 38; \quad AB = \sqrt{38}. \end{aligned}$$

15 Midpoint of a line joining two points in three-dimensional space

Rule: The midpoint of a line joining the two points $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$ is

$$M = (\tfrac{1}{2}(x_1 + x_2), \tfrac{1}{2}(y_1 + y_2), \tfrac{1}{2}(z_1 + z_2)).$$

Example 18

The midpoint of a line joining $A(-3, 1, 4)$ and $B(-2, 4, 3)$ is

$$M = \left(\frac{-5}{2}, \frac{5}{2}, \frac{7}{2}\right).$$

Example 19

If $M = (-1, 4, 2)$ is the midpoint of AB and A has coordinates $(1, 2, 3)$ find the coordinates of B.

Let $B = (x, y, z)$. Then the coordinates of the midpoint are as follows:

x-coordinate: $-1 = \frac{1}{2}(1 + x)$ which implies $x = -3$;

y-coordinate: $4 = \frac{1}{2}(2 + y)$ which implies $y = 6$;

z-coordinate: $2 = \frac{1}{2}(3 + z)$ which implies $z = 1$.

Hence the coordinates of the point B are $(-3, 6, 1)$.

16 Equations in three dimensions

In two-dimensional space equations represent curves, whereas in three-dimensional space they define surfaces. No single equation can represent a curve in three dimensions; two equations are required, the curve being the intersection of two surfaces.

A first degree equation in two dimensions such as $\alpha x + \beta y + \gamma = 0$, where α, β, γ are constants, represent a line. In three dimensions the equation $Ax + By + Cz + D = 0$, where A, B, C, D are constants, is that of a plane. If two planes intersect, they do so in a line. However, no single equation defines this line in three-dimensional space.

Example 20

The equations $2x + 3y - z + 6 = 0$ and $x - 2y + z - 4 = 0$ represent planes. We can use one of these equations to eliminate x (or y or z) from the other equation. The resulting equation is that of a line in the y-z (or x-z or x-y) plane, which is the projection of the line of intersection in that plane. So, for example, eliminating y we have $7x + z = 0$. This is the equation of a line in the $x - z$ plane; it is not the equation of the line of intersection of the two planes.

Rule: Any point which lies in two planes must lie on the line of intersection of the two planes.

Example 21

Show that $A(1,-5,-7)$ and $B(-1,1,7)$ lie on the line of intersection of the planes $2x+3y-z+6=0$ and $x-2y+z-4=0$.

We need to show that A and B lie in both planes. For the point A, $x=1$, $y=-5$ and $z=-7$. Substituting these values into the two equations we have

$$2-15+7+6=0$$

and

$$1+10-7-4=0.$$

So point A lies on the line of intersection. Similarly we can show that point B lies on the line of intersection but what about the point $C(0,1,6)$? Here $x=0$, $y=1$ and $z=6$, which when substituted into the equations of the planes gives

$$0+3-6+6=3\neq 0$$

and

$$0-2+6-4=0.$$

Therefore point C lies in the plane with equation $x-2y+z-4=0$ but not in the plane with equation $2x+3y-z+6=0$. It does not lie on the line of intersection of the two planes.

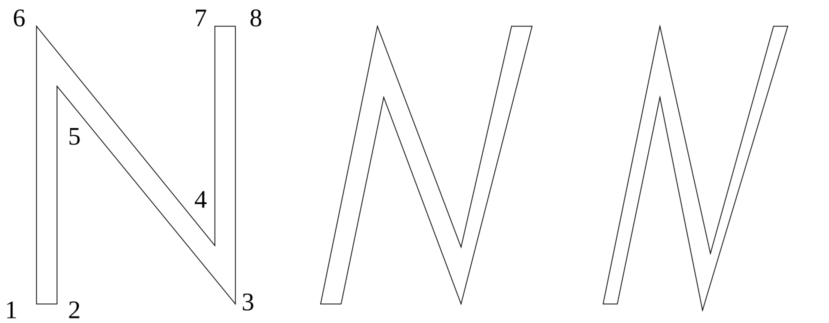
Equations of higher degree in three dimensions also represent surfaces. In general the higher the degree the more convoluted the surface. Two such equations are necessary to define a curve in three dimensions. For example, the second degree equation $x^2+y^2+z^2=a^2$ represents the surface of a sphere of radius 'a' with centre at the origin. When a plane intersects this sphere, the curve of intersection is a circle in three dimensional space – the two equations representing the sphere and the plane define the circle. Similarly $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ is the equation of the surface of an ellipsoid. The sphere and the ellipsoid are examples of closed surfaces; their counterparts in two dimensions, the circle and the ellipse, are closed curves. Of course there are many equations representing surfaces that are not closed.

17 Computer graphics

Computer graphics have widespread applications in business and industry as well as their pervasive use in computer games. As a computer screen is two-dimensional all graphical images to be displayed must be confined to a plane. Three-dimensional objects are conveyed by perspective images, that is a two-dimensional image that gives the impression of three dimensions. Coordinate geometry is fundamental in creating the image to be displayed.

Graphical images can be constructed by specifying a number of points, giving details of how points are to be connected and by providing information on colour (and shading) for any enclosed regions so formed. Often curves are approximated by line segments. The shorter the line segments the better the approximation to the actual curve. We now consider an example in which points and lines are used to describe a character which can then be transformed algebraically using shearing and scaling operations.

Example 22



Regular N Slanted N Composite transformation of N

The capital letter N in the picture is described by eight points (or vertices) connected by straight lines. In order to draw this figure graphically we need to store the coordinates of the vertices and to specify which ones are joined by a line. The image is defined by the points

$$\begin{aligned}&P_1(x_1, y_1)\\&P_2(x_2, y_2)\\&\vdots\\&P_8(x_8, y_8)\end{aligned}$$

where P_1 is the point labelled 1 in the figure, etc. and the vertices are joined in the sequence 1,2,5,3,8,7,4,6,1. The image is drawn by plotting the points and connecting P_1 to P_2, P_2 to $P_5, ..., P_6$ to P_1, using lines. For convenience choose axes with origin at P_1, x-axis passing through P_1, P_2 (and of course P_3) and y-axis passing through P_1, P_6. Then the following points could represent the letter N:

$$\begin{aligned}&P_1(0, 0)\\&P_2(0.5, 0)\\&P_3(6, 0)\\&P_4(5.5, 1.58)\\&P_5(0.5, 6.42)\\&P_6(0, 8)\\&P_7(5.5, 8)\\&P_8(6, 8)\end{aligned}$$

We may wish to make the N more like an italic N by performing a *shear* transformation on the vertices. In a shear transformation the shape is deformed in the direction of the shear, for example

The order for joining the points remains unchanged. We want to keep the same vertical height for our image so we perform a shear on the x-coordinates only. We also want the x-coordinate to move further to the right for vertices with larger y-coordinates. We choose a shear of $0.25y$ in the positive x-direction. So if the y-coordinate was zero for the original vertex the point will not be moved at all. So points P_1, P_2, P_3 will remain fixed when the

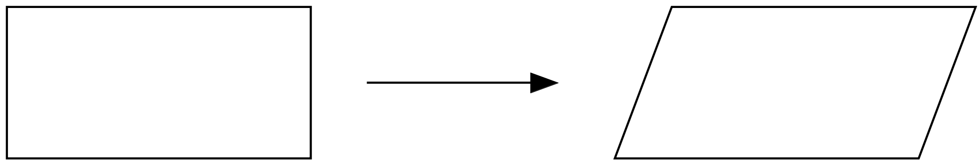


image of N undergoes the given shear transformation. That is, the base of the character remains fixed. This is usually what is required when manipulating characters. However, a point like P_6 which is on the y-axis at $(0,8)$ will move to the point $(2,8)$. The x-coordinate is increased by one quarter of the y-coordinate value.

The sheared figure is the second of the three N's pictured earlier, the 'slanted N'. The new coordinates (X,Y) are related to the original coordinates (x,y) by the equations:

$$\begin{aligned} X &= x + 0.25y \\ Y &= y. \end{aligned}$$

and the transformed vertices are:

$$\begin{array}{l} P_1'(0,0) \\ P_2'(0.5,0) \\ P_3'(6,0) \\ P_4'(5.895,1.58) \\ P_5'(2.105,6.42) \\ P_6'(2,8) \\ P_7'(7.5,8) \\ P_8'(8,8) \end{array}$$

Note that all y-coordinates remain unchanged. Once the new vertices have been calculated the image can be re-drawn by connecting appropriate vertices using straight lines. Perhaps the second N is too wide for an italic N so we wish to shrink the width. This can be done by performing a *scale* transformation. We only want to scale the x-coordinates leaving the y-coordinates unchanged. For example we could scale x by a factor of 0.75 such that the point $P_3'(6,0)$ becomes $P_3''(4.5,0)$. Similarly for other points, then the new vertices are:

$$\begin{array}{l} P_1''(0,0) \\ P_2''(0.35,0) \\ P_3''(4.5,0) \\ P_4''(4.42125,1.58) \\ P_5''(1.57875,6.42) \\ P_6''(1.5,8) \\ P_7''(5.625,8) \\ P_8''(6,8) \end{array}$$

The third N shows the result of the composite transformation, a shear followed by scaling. Mathematically these can be combined into one operation. The governing equations are

$$\begin{aligned} X &= 0.75(x + 0.25y), \\ Y &= y, \end{aligned}$$

where (x, y) are the original coordinates and (X, Y) are the sheared and scaled coordinates.

Note that in this example P_8 has returned to its start position and in fact we have retained the image within a box defined by the vertices P_1, P_3, P_6, P_8. This is precisely the type of manipulation that is required when dealing with text.