Steady State Property of LQ Systems

Notice that the LQR is a proportional state feedback control system. The controller does ...t have an integrator. This implies that the steady state of the LQR is not offset error free when the system is subjected to a constant disturbance or a constant reference input. The proportional state feedback control law is a consequence of the formulation of the standard LQ problem; recall that the plant equation did not have a disturbance term. When disturbances or nonzero reference inputs are expected, the standard LQ problem must be modified so that the controller will possess a right structure. This can be accomplished in a variety of ways.

LQI controller (Linear Quadratic Optimal Control with Integrator) for Continuous Time Systems.

The plant is described by

$$\frac{dx(t)}{dt} = Ax(t)+Bu(t), \quad y(t) = Cx(t) \tag{25-12}$$

where

x: n-dimensional state vector
u: m-dimensional input vector
y: m-dimensional output vector

We assume the following conditions:

1. Plant is controllable and observable
2. **A** is nonsingular (i.e. no open loop pole at the origin)
3. $\mathbf{CA^{-1}B}$ is nonsingular

Now consider the quadratic performance index,

$$J = \int_0^\infty [y(t)-r]^T Q_y [y(t)-r] + v^T(t) R v(t) dt \tag{25-13}$$

where $\mathbf{Q_y}$ and **R** are symmetric and positive definite, **r** is an m-dimensional constant set point vector and **v** is the derivative of **u**, i.e.

$$\mathbf{du(t)/dt = v(t)}$$

We define an (m+n)-dimensional state vector $\tilde{\mathbf{x}} = [\mathbf{e}^T, (\mathbf{dx/dt})^T]^T$ ($\mathbf{e} = \mathbf{y} - \mathbf{r}$). Then the state equation for the enlarged state vector is

$$\frac{d}{dt}\begin{bmatrix} e \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & C \\ 0 & A \end{bmatrix}\begin{bmatrix} e \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} v(t) \rightarrow \frac{d\tilde{x}(t)}{dt} = \tilde{A}\tilde{x}(t) + \tilde{B}v(t) \tag{25-14}$$

The performance index is written as

$$J = \int_0^\infty \tilde{x}^T(t)\begin{bmatrix} Q_y & 0 \\ 0 & 0 \end{bmatrix}\tilde{x}(t) + v^T(t)Rv(t)dt \tag{25-15}$$

Notice that the problem is now reformulated as a standard LQ problem.

The controllability and observability condition required for the Riccati equation to have a stationary positive definite solution must be checked for

$$(\tilde{A},\ \tilde{B}) \quad and \quad (\tilde{A},\ Q_y^{\frac{1}{2}}[I_m \quad 0_n])$$

They are satisfied under Assumptions 1, 2 and 3. You should check them yourself. The feedback control law for v(t) is given by

$$v(t) = -R^{-1}\tilde{B}^T\tilde{P}_+\tilde{x}(t) = -K_e e(t) - K_x \dot{x}(t) \tag{25-16}$$

where K_e, K_x and P_+ are given by

$$\begin{bmatrix} K_e & K_x \end{bmatrix} = \begin{bmatrix} R^{-1}B^T\tilde{P}_{xe} & R^{-1}B^T\tilde{P}_{xx} \end{bmatrix} \quad \tilde{P}_+ = \begin{bmatrix} \tilde{P}_{ee} & \tilde{P}_{ex} \\ \tilde{P}_{xe} & \tilde{P}_{xx} \end{bmatrix} > 0$$

$$\tilde{A}^T\tilde{P} + \tilde{P}\tilde{A} - \tilde{P}\tilde{B}R^{-1}\tilde{B}^T\tilde{P} + \begin{bmatrix} Q_y & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Integrating Eq. (25-16) from initial time (0) to t, we obtain

$$u(t) = K_e\int_0^t [r - y(\tau)]d\tau - K_x x(t)$$

where x(0) = 0 has been assumed. Notice that we have obtained a PI (Proportional plus Integral) control law.

Note: The main idea in this development is the use of the time derivative of u(t) in the cost functional. In the original LQ formulation, u(t) was penalized, a consequence of which is the convergence of u(t) to zero for the cost functional to remain bounded. By including the time derivative of u(t) in the cost functional, u(t) does not have to converge to zero but to any constant value.

For discrete time systems, the LQI controller can be developed in a similar manner. In particular, include $\Delta u(k)^T R \Delta u(k) = (u(k) - u(k-1))^T R(u(k) - u(k-1))$ in the cost functional instead of $u(k)^T R u(k)$. The enlarged state equation must be written for $e(k) = r - y(k)$ and $\Delta x(k) = x(k) - x(k-1)$.