# 第四章

## 1. 题目

11. **(上机题)** 已知 Hilbert 矩阵 $H_n = (h_{ij})_{n\times n}$ 的元素为

$$h_{ij} = \frac{1}{(i+j-1)}.$$

完成如下几个问题:

- 编程给出计算 $H$ 的行范数函数;
- 编程计算 $H$ 的行范数条件数. 可调用求逆函数, 比如 Mathematica 中的 `Inverse[H]`, MATLAB 中的 `inv(H)`(其它语言自行查找);
- 对 $n = 1, 2, \cdots, 20$, 计算 $H$ 的行范数条件数, 并画出 $n$ 同条件数的对数之间的关系图;
- 令 $x = (1.0, 1.0, \cdots, 1.0)^T$, $b = Hx$, 对 $n = 1, 2, \cdots, 20$, 解 $H\hat{x} = b$, 并计算

$$x - \hat{x},\ b - H\hat{x}$$

以及它们的无穷范数;

- 通过以上的数值实验, 你理解到了什么?

## 2. 程序代码

代码一：编写构建 Hilbert 矩阵、求矩阵行范数、矩阵求逆、求矩阵行范数条件数等几个函数

test_4.py

```python
import numpy as np
def Hilbert(num):
    matrix_H=np.zeros((num,num))
    for i in range(num):
        for j in range(num):
            matrix_H[i][j]=1/(i+1+j+1-1)
    return matrix_H
'''
def hangfanshu(arr):
    n=len(arr[0])
    arr_max=0
    for i in range(n):
        arr_max=arr_max+arr[0][i]
    return arr_max
'''
def hangfanshu(arr):
    n=len(arr[0])
    sum_hang=np.zeros(n)
    for i in range(n):
        sum_hang[i]=np.sum(np.abs(arr[i]))
    max_hang=np.max(sum_hang)
    return max_hang
def matrix_qiuni(arr):
    n=len(arr[0])
    A_inv = np.linalg.inv(arr)
    return A_inv

def cal_hangfanshutiaojianshu(arr):
    hangfanshutiaojianshu=hangfanshu(arr)*hangfanshu(matrix_qiuni(arr))
    return hangfanshutiaojianshu
```

代码二：用所编写的函数计算 H 的行范数条件数，并做图

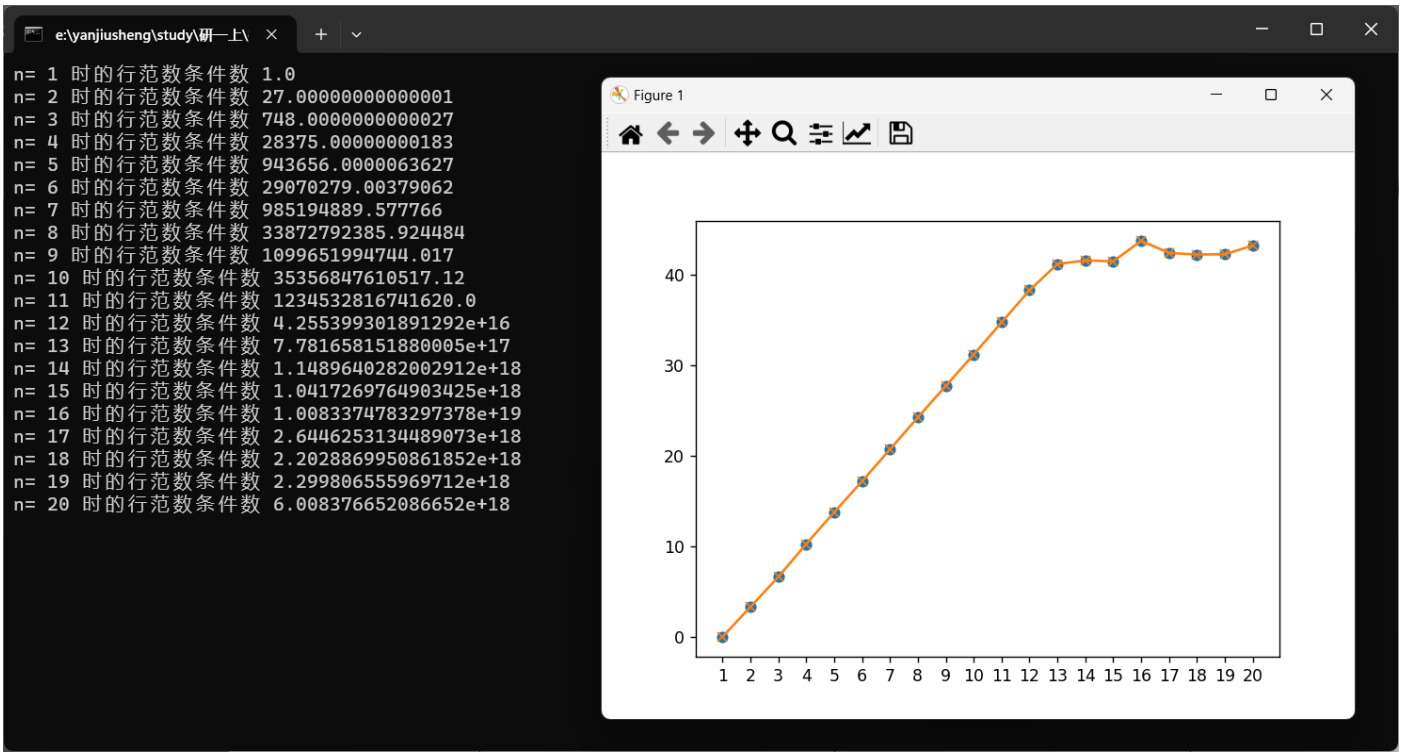
```python
from test_4 import cal_hangfanshutiaojianshu,Hilbert
import numpy as np
#import matplotlib.pyplot as plt
import matplotlib.pyplot as plt
'''
for i in range(1,21):
    print('n=',i,"时的行范数条件数",cal_hangfanshutiaojianshu(Hilbert(i)))
'''
tiaojianshu=np.zeros(20)
for i in range(20):
    tiaojianshu[i]=cal_hangfanshutiaojianshu(Hilbert(i+1))
    print('n=',(i+1),"时的行范数条件数",tiaojianshu[i])
tiaojianshu_log=np.log(tiaojianshu)
x=np.linspace(1,20,num=20)
y = np.interp(x, x, tiaojianshu_log)
plt.plot(x,y,'o')
plt.plot(x,y, '-x') #黄色的区域
plt.xticks(x)
plt.show()
```

代码三：求解 H$\hat{x}$=b，并计算 x-$\hat{x}$、b- H$\hat{x}$以及他们的无穷范数

```python
from test_4 import hangfanshu,Hilbert,matrix_qiuni
import numpy as np
#x_cal=np.zeros(20,20)
for n in range (1,21):
    H=Hilbert(n)
    #print("H\n",H)
    x_1=np.ones(len(H))
    x=x_1[:,np.newaxis]
    #print("x\n",x)
    b=np.dot(H,x)
    #print("b\n",b)
    #x_cal = np.linalg.solve(H, b)
    x_cal=np.dot(matrix_qiuni(H),b)
    print("n=",n,"时：")
    print("x-x_cal:\n",x-x_cal)
    print("x-x_cal的无穷范数:",hangfanshu(x-x_cal))
    print("b-H*x_cal\n",b-np.dot(H,x_cal))
    print("b-H*x_cal的无穷范数:",hangfanshu(b-np.dot(H,x_cal)))

```

## 3. 运行结果


e:\yanjiusheng\study\研一上\
n= 1 时的行范数条件数 1.0
n= 2 时的行范数条件数 27.000000000000001
n= 3 时的行范数条件数 748.0000000000027
n= 4 时的行范数条件数 28375.00000000183
n= 5 时的行范数条件数 943656.0000063627
n= 6 时的行范数条件数 29070279.00379062
n= 7 时的行范数条件数 985194889.577766
n= 8 时的行范数条件数 33872792385.924484
n= 9 时的行范数条件数 1099651994744.017
n= 10 时的行范数条件数 35356847610517.12
n= 11 时的行范数条件数 1234532816741620.0
n= 12 时的行范数条件数 4.255399301891292e+16
n= 13 时的行范数条件数 7.781658151880005e+17
n= 14 时的行范数条件数 1.1489640282002912e+18
n= 15 时的行范数条件数 1.0417269764903425e+18
n= 16 时的行范数条件数 1.0083374783297378e+19
n= 17 时的行范数条件数 2.6446253134489073e+18
n= 18 时的行范数条件数 2.2028869950861852e+18
n= 19 时的行范数条件数 2.299806555969712e+18
n= 20 时的行范数条件数 6.008376652086652e+18
Figure 1
0
10
20
30
40
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

```
n= 1 时:
x-x_cal:
 [[0.]]
x-x_cal的无穷范数:  0.0
b-H*x_cal
 [[0.]]
b-H*x_cal的无穷范数:  0.0
n= 2 时:
x-x_cal:
 [[-4.4408921e-16]
 [ 8.8817842e-16]]
x-x_cal的无穷范数:  4.440892098500626e-16
b-H*x_cal
 [[0.]
 [0.]]
b-H*x_cal的无穷范数:  0.0
n= 3 时:
x-x_cal:
 [[-1.13242749e-14]
 [ 3.59712260e-14]
 [ 8.54871729e-15]]
x-x_cal的无穷范数:  1.1324274851176597e-14
b-H*x_cal
 [[9.54791801e-15]
 [8.43769499e-15]
 [6.88338275e-15]]
b-H*x_cal的无穷范数:  9.547918011776346e-15
n= 4 时:
x-x_cal:
 [[ 5.68434189e-14]
 [-9.09494702e-13]
 [ 9.09494702e-13]
 [ 0.00000000e+00]]
x-x_cal的无穷范数:  5.6843418860808e-14
b-H*x_cal
 [[-9.50350909e-14]
 [-4.75175455e-14]
 [-2.65343303e-14]
 [-1.62092562e-14]]
b-H*x_cal的无穷范数:  9.50350909079134e-14
n= 5 时:
x-x_cal:
 [[-9.16888787e-12]
 [ 3.49741347e-11]
 [-7.69044828e-11]
 [ 3.40576456e-11]
 [ 3.00026670e-12]]
x-x_cal的无穷范数:  9.168887871169318e-12
```

图 1 程序结果

n 和 H 的条件数行范数结果如表 1 所示：

表 1 求解结果

| n | H 的条件数行范数 |
|---|---|
| 1 | 1.0 |
| 2 | 27.00000000000001 |
| 3 | 748.0000000000027 |
| 4 | 28375.00000000183 |
| 5 | 943656.0000063627 |
| 6 | 29070279.00379062 |
| 7 | 985194889.577766 |
| 8 | 33872792385.924484 |
| 9 | 1099651994744.017 |

| | |
|---|---|
| 10 | 35356847610517.12 |
| 11 | 1234532816741620.0 |
| 12 | 4.255399301891292e+16 |
| 13 | 7.781658151880005e+17 |
| 14 | 1.1489640282002912e+18 |
| 15 | 1.0417269764903425e+18 |
| 16 | 1.0083374783297378e+19 |
| 17 | 2.6446253134489073e+18 |
| 18 | 2.2028869950861852e+18 |
| 19 | 2.299806555969712e+18 |
| 20 | 6.008376652086652e+18 |

N 和 H 的行范数条件数的对数之间的关系图如图 2 所示：

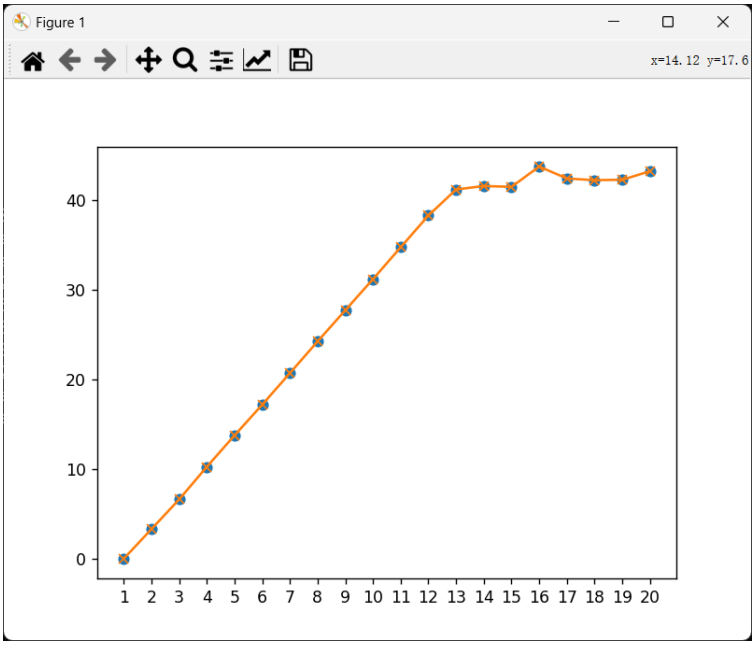



图 2 N 和行范数条件数的对数关系图

求解 H$\hat{x}$=b，并计算 x-$\hat{x}$和 b-H$\hat{x}$以及他们的无穷范数：

求解结果如图 3 所示：

```
n= 1 时:
x-x_cal:
 [[0.]]
x-x_cal的无穷范数:  0.0
b-H*x_cal
 [[0.]]
b-H*x_cal的无穷范数:  0.0
```

```
n= 2 时:
x-x_cal:
 [[-4.4408921e-16]
 [ 8.8817842e-16]]
x-x_cal的无穷范数:  4.440892098500626e-16
b-H*x_cal
 [[0.]
 [0.]]
b-H*x_cal的无穷范数:  0.0
```

```
n= 3 时：
x-x_cal：
 [[-1.13242749e-14]
 [ 3.59712260e-14]
 [ 8.54871729e-15]]
x-x_cal的无穷范数： 1.1324274851176597e-14
b-H*x_cal
 [[9.54791801e-15]
 [8.43769499e-15]
 [6.88338275e-15]]
b-H*x_cal的无穷范数： 9.547918011776346e-15
```

```
n= 4 时：
x-x_cal：
 [[ 5.68434189e-14]
 [-9.09494702e-13]
 [ 9.09494702e-13]
 [ 0.00000000e+00]]
x-x_cal的无穷范数： 5.684341886080802e-14
b-H*x_cal
 [[-9.50350909e-14]
 [-4.75175455e-14]
 [-2.65343303e-14]
 [-1.62092562e-14]]
b-H*x_cal的无穷范数： 9.50350909079134e-14
```

```
n= 5 时：
x-x_cal：
 [[-9.16888787e-12]
 [ 3.49741347e-11]
 [-7.69044828e-11]
 [ 3.40576456e-11]
 [ 3.00026670e-12]]
x-x_cal的无穷范数： 9.168887871169318e-12
b-H*x_cal
 [[-8.20232771e-12]
 [-4.84101648e-12]
 [-3.58868490e-12]
 [-2.87447843e-12]
 [-2.40052422e-12]]
b-H*x_cal的无穷范数： 8.202327705930657e-12
```

```
n= 6 时：
x-x_cal：
 [[ 1.90993887e-11]
 [-2.91038305e-10]
 [ 1.04773790e-09]
 [-4.65661287e-10]
 [ 4.65661287e-10]
 [-2.32830644e-10]]
x-x_cal的无穷范数： 1.9099388737231493e-11
b-H*x_cal
 [[1.60738090e-10]
 [1.25687682e-10]
 [1.02963416e-10]
 [8.70048478e-11]
 [7.52398144e-11]
 [6.62331301e-11]]
b-H*x_cal的无穷范数： 1.6073808950523016e-10
```

```
n= 7 时：
x-x_cal：
 [[-2.23371899e-09]
 [ 5.12227416e-09]
 [ 5.21540642e-08]
 [-1.11758709e-07]
 [ 5.21540642e-08]
 [ 1.49011612e-08]
 [-1.86264515e-09]]
x-x_cal的无穷范数： 2.2337189875543118e-09
b-H*x_cal
 [[2.42067610e-09]
 [1.86559013e-09]
 [1.44662216e-09]
 [1.18151733e-09]
 [1.00339126e-09]
 [8.75927553e-10]
 [7.79968201e-10]]
b-H*x_cal的无穷范数： 2.4206761040090896e-09
```

```
n= 8 时：
x-x_cal:
 [[ 2.27009878e-07]
 [-1.03935599e-06]
 [ 8.94069672e-07]
 [ 4.76837158e-07]
 [ 4.76837158e-07]
 [-1.90734863e-06]
 [ 9.53674316e-07]
 [-2.38418579e-07]]
x-x_cal的无穷范数： 2.270098775625229e-07
b-H*x_cal
 [[ 8.47725268e-09]
 [-1.43493899e-08]
 [-1.40592753e-08]
 [-1.26174891e-08]
 [-1.14186465e-08]
 [-1.04824781e-08]
 [-9.73194880e-09]
 [-9.10963849e-09]]
b-H*x_cal的无穷范数： 8.477252677607794e-09
```

```
n= 9 时：
x-x_cal:
 [[-1.23305451e-05]
 [ 6.49141422e-05]
 [-8.98046165e-05]
 [ 4.77007724e-05]
 [ 2.77701730e-05]
 [-5.80417889e-05]
 [ 4.38892887e-05]
 [-1.58576381e-06]
 [-2.06865165e-06]]
x-x_cal的无穷范数： 1.23305451258382e-05
b-H*x_cal
 [[3.83907774e-06]
 [4.00155317e-06]
 [3.34945524e-06]
 [2.84171699e-06]
 [2.46633748e-06]
 [2.18105589e-06]
 [1.95721346e-06]
 [1.77670550e-06]
 [1.62784401e-06]]
b-H*x_cal的无穷范数： 3.839077742817665e-06
```

```
n= 10 时：
x-x_cal：
 [[-2.54168641e-04]
 [ 2.16242671e-03]
 [-5.54656982e-03]
 [ 5.08880615e-03]
 [ 9.15527344e-04]
 [-4.02832031e-03]
 [ 1.58691406e-03]
 [ 3.66210938e-04]
 [-1.22070312e-04]
 [-4.57763672e-05]]
x-x_cal的无穷范数：  0.00025416864082217216
b-H*x_cal
 [[1.64457656e-05]
 [2.46427604e-05]
 [1.99866527e-05]
 [1.66264820e-05]
 [1.43203323e-05]
 [1.26419808e-05]
 [1.13542041e-05]
 [1.03264101e-05]
 [9.48200131e-06]
 [8.77296239e-06]]
b-H*x_cal的无穷范数：  1.6445765596895257e-05
```

```
n= 11 时：
x-x_cal：
 [[ 0.00043994]
 [-0.00579691]
 [-0.03123474]
 [ 0.1953125 ]
 [-0.28613281]
 [ 0.11132812]
 [ 0.078125  ]
 [-0.09375   ]
 [ 0.03125   ]
 [-0.0078125 ]
 [-0.00097656]]
x-x_cal的无穷范数：  0.0004399418830871582
b-H*x_cal
 [[-0.00066969]
 [-0.00056095]
 [-0.00053724]
 [-0.00052699]
 [-0.00051545]
 [-0.00050115]
 [-0.00048498]
 [-0.00046793]
 [-0.00045073]
 [-0.00043383]
 [-0.00041751]]
b-H*x_cal的无穷范数：  0.0006696850878511462
```

```
n= 12 时：
x-x_cal：
 [[ -0.54020345]
 [  3.64012146]
 [ -8.09277344]
 [  3.5390625 ]
 [ 11.        ]
 [-20.75      ]
 [ 19.        ]
 [ -9.        ]
 [  1.        ]
 [  0.        ]
 [  0.25      ]
 [ -0.0625    ]]
x-x_cal的无穷范数：  0.5402034521102905
b-H*x_cal
 [[-7.33858115e-02]
 [-1.20355054e-02]
 [-4.28313291e-03]
 [-1.95535102e-03]
 [-9.03015915e-04]
 [-3.44964781e-04]
 [-3.25590340e-05]
 [ 1.43212138e-04]
 [ 2.38818896e-04]
 [ 2.86130732e-04]
 [ 3.04019095e-04]
 [ 3.03997990e-04]]
b-H*x_cal的无穷范数：  0.07338581150572354
```

```
n= 13 时：
x-x_cal：
 [[-3.51051425e+00]
 [ 8.47091365e+00]
 [ 1.09388565e+01]
 [ 2.67924357e+01]
 [-2.81463434e+02]
 [ 6.39543912e+02]
 [-6.90865585e+02]
 [ 3.85654218e+02]
 [-1.11469086e+02]
 [ 5.74916540e+01]
 [-2.22327559e+01]
 [ 1.03151750e+00]
 [-2.55582130e-01]]
x-x_cal的无穷范数：  3.5105142474559115
b-H*x_cal
 [[2.28784714]
 [2.39466349]
 [2.14108708]
 [1.91569331]
 [1.73328331]
 [1.58455895]
 [1.46100023]
 [1.35650101]
 [1.26677219]
 [1.18874677]
 [1.12017645]
 [1.05937267]
 [1.0050394 ]]
b-H*x_cal的无穷范数：  2.2878471406666474
```

```
n= 14 时：
x-x_cal：
 [[  -3.67026141]
 [  36.33379984]
 [-150.12664032]
 [ 366.27783203]
 [-557.95703125]
 [ 535.5       ]
 [-260.5       ]
 [-105.5       ]
 [ 287.        ]
 [-167.        ]
 [ -13.        ]
 [  38.        ]
 [  -7.        ]
 [   1.1875    ]]
x-x_cal的无穷范数：  3.6702614054083824
b-H*x_cal
 [[ 7.88381534e-04]
 [ 1.59573547e-01]
 [ 1.11995222e-01]
 [ 7.55821417e-02]
 [ 5.18290816e-02]
 [ 3.60084558e-02]
 [ 2.50975198e-02]
 [ 1.73351095e-02]
 [ 1.16697128e-02]
 [ 7.44823337e-03]
 [ 4.24937907e-03]
 [ 1.79219560e-03]
 [-1.16128693e-04]
 [-1.61121987e-03]]
b-H*x_cal的无穷范数：  0.0007883815341340 75
```

```
n= 15 时：
x-x_cal：
 [[ 9.20875587e-01]
 [-3.00058117e+01]
 [ 9.53674927e+01]
 [-1.03103516e+02]
 [ 2.16953125e+02]
 [-7.32375000e+02]
 [ 1.18150000e+03]
 [-9.45000000e+02]
 [ 3.53000000e+02]
 [ 1.00000000e+00]
 [-6.30000000e+01]
 [ 1.70000000e+01]
 [-4.00000000e+00]
 [ 2.00000000e+00]
 [-3.84765625e-01]]
x-x_cal的无穷范数：  0.9208755865693092
b-H*x_cal
 [[-1.2587757 ]
 [-0.82680263]
 [-0.70263443]
 [-0.64719101]
 [-0.60997084]
 [-0.57902714]
 [-0.55121261]
 [-0.52561235]
 [-0.50190472]
 [-0.47992268]
 [-0.45953573]
 [-0.44062116]
 [-0.42305975]
 [-0.40673716]
 [-0.39154572]]
b-H*x_cal的无穷范数：  1.2587756992337327
```

```
n= 16 时：
x-x_cal：
 [[ 5.55691977e+01]
 [-8.27601318e+02]
 [ 4.10992188e+03]
 [-9.79737500e+03]
 [ 1.22530000e+04]
 [-6.94300000e+03]
 [-1.03900000e+03]
 [ 8.65000000e+02]
 [ 8.51300000e+03]
 [-1.55990000e+04]
 [ 1.43050000e+04]
 [-8.27900000e+03]
 [ 2.81700000e+03]
 [-3.83000000e+02]
 [-6.30000000e+01]
 [ 1.00000000e+00]]
x-x_cal的无穷范数：  55.56919765472412
b-H*x_cal
 [[-2.74705877]
 [-3.28285732]
 [-2.44752508]
 [-1.91737539]
 [-1.56330988]
 [-1.3105851 ]
 [-1.12223737]
 [-0.97763669]
 [-0.86407432]
 [-0.77320308]
 [-0.69930456]
 [-0.63834237]
 [-0.58740095]
 [-0.54433409]
 [-0.50753588]
 [-0.4757873 ]]
b-H*x_cal的无穷范数：  2.7470587749351325
```

```
n= 17 时：
x-x_cal：
 [[-5.01947268e+00]
 [ 2.62542559e+00]
 [ 3.25515692e+02]
 [-1.82846638e+03]
 [ 3.49761817e+03]
 [-3.65223216e+02]
 [-7.65834280e+03]
 [ 1.07291073e+04]
 [-4.66625036e+03]
 [-3.04146561e+03]
 [ 5.54866633e+03]
 [-3.53983985e+03]
 [ 1.08028805e+03]
 [-9.78402456e-01]
 [-1.81691668e+01]
 [ 1.68688504e+01]
 [-2.34124786e+00]]
x-x_cal的无穷范数：  5.019472683181327
b-H*x_cal
 [[2.97765357]
 [3.4392828 ]
 [3.35048395]
 [3.22709729]
 [3.10773799]
 [2.99614303]
 [2.89194974]
 [2.79436424]
 [2.70268216]
 [2.61633217]
 [2.53484755]
 [2.45783625]
 [2.38495954]
 [2.31591793]
 [2.25044235]
 [2.18828833]
 [2.12923221]]
b-H*x_cal的无穷范数：  2.9776535738928196
```

```
n= 18 时：
x-x_cal:
 [[ 2.21407643e+01]
 [-1.58741234e+02]
 [ 2.88009277e+02]
 [ 1.16662109e+03]
 [-7.59959375e+03]
 [ 1.72175000e+04]
 [-1.74820000e+04]
 [ 1.57500000e+03]
 [ 1.54970000e+04]
 [-1.77730000e+04]
 [ 9.69500000e+03]
 [-2.73500000e+03]
 [ 5.10625000e+01]
 [ 2.91000000e+02]
 [-9.90000000e+01]
 [-8.00000000e+00]
 [ 2.00000000e+01]
 [-1.90625000e+00]]
x-x_cal的无穷范数：  22.140764325857162
b-H*x_cal
 [[-3.7404495 ]
 [-4.81813164]
 [-4.25159541]
 [-3.73648418]
 [-3.33062661]
 [-3.0081506 ]
 [-2.74604831]
 [-2.52843693]
 [-2.34449893]
 [-2.18669829]
 [-2.04963542]
 [-1.92933684]
 [-1.82280821]
 [-1.72774568]
 [-1.64234439]
 [-1.56516792]
 [-1.49505718]
 [-1.4310654 ]]
b-H*x_cal的无穷范数：  3.7404494993911976
```

```
n= 19 时：
x-x_cal：
 [[-5.25285314e+00]
 [ 1.79251884e+02]
 [-1.00761353e+03]
 [ 1.91344922e+03]
 [-8.83437500e+01]
 [-1.98675000e+03]
 [-6.45800000e+03]
 [ 2.44670000e+04]
 [-3.37270000e+04]
 [ 2.60570000e+04]
 [-1.21120000e+04]
 [ 2.82500000e+03]
 [ 8.12812500e+01]
 [-1.39000000e+02]
 [ 8.90000000e+01]
 [-7.90000000e+01]
 [ 2.90000000e+01]
 [-1.10000000e+01]
 [ 2.25000000e+00]]
x-x_cal的无穷范数：  5.252853140234947
b-H*x_cal
 [[4.98821125]
 [3.36261525]
 [2.70795489]
 [2.34160165]
 [2.09341224]
 [1.90793673]
 [1.76119029]
 [1.64064878]
 [1.53894867]
 [1.4514061 ]
 [1.37486823]
 [1.30711812]
 [1.24654093]
 [1.1919254 ]
 [1.14234012]
 [1.09705356]
 [1.05548057]
 [1.01714563]
 [0.98165698]]
b-H*x_cal的无穷范数：  4.988211252544307
```

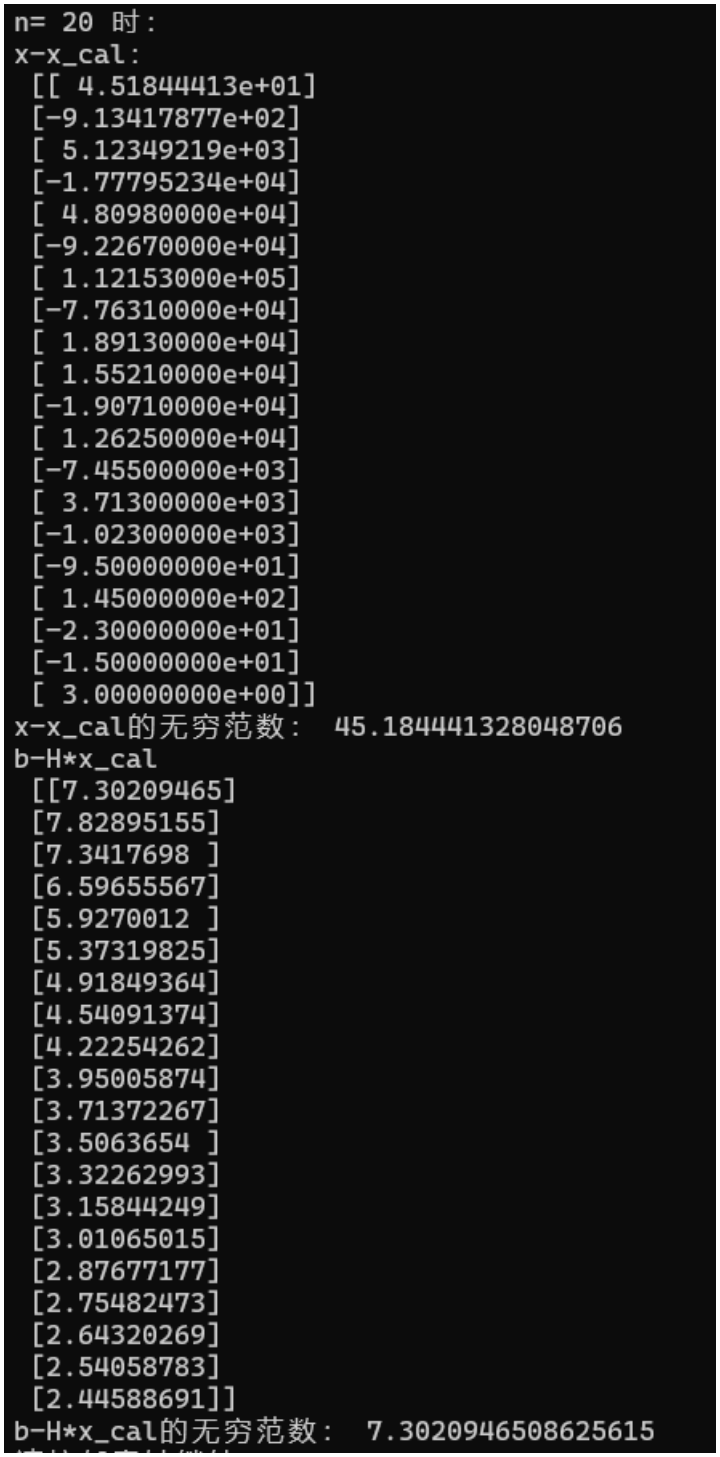



图 3 求解结果

# 4.结果分析与上机体会

4.1结果分析：

通过观察结果，可以发现随着 n 的增大，x-$\hat{x}$和 b-H$\hat{x}$的值也在增大，x-$\hat{x}$的增加程度较大，他们的无穷范数也在增大。

4.2上机体会：

本次上机实验采用了python编写，深刻理解了矩阵的行范数和行范数条件数的含义，程序运行花费了0.049695秒。