Determination of Transfer Functions of 2PIBC for PI Control

Mathematical Modelling

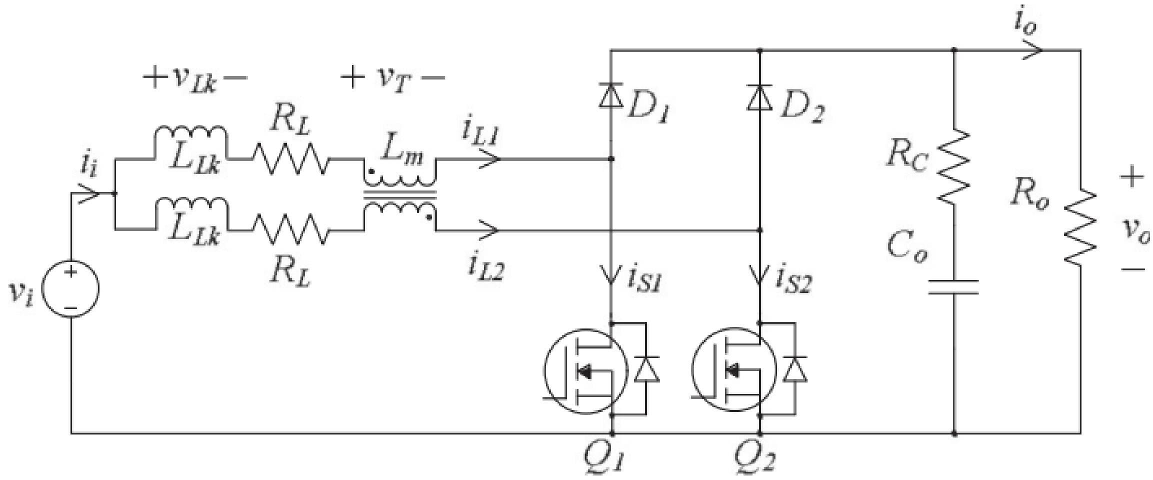

Figure 1: Circuit diagram for state modelling [1]

Assumptions:

$$\begin{aligned} i_{L1} &= i_{L2} \\ L_{Lk1} &= L_{Lk1} = L_{Lk} \\ d_{c1} &= d_{c2} = d_c \\ d_{\text{off1}} &= d_{\text{off2}} = d_{\text{off}} \end{aligned} \tag{1}$$

where L_{Lkx}, i_{Lx}, d_{cx}, and $d_{\text{off}x}$ are the leakage inductance, phase current, duty cycle (or switch conduction time), and diode conduction time of phase x, (1 or 2).

State Equations

In a coupled inductor boost converter, when the switch is closed, the input and output of the converter are disconnected from each other. The leakage inductance voltage v_{Lk} in this state is expressed as

$$v_{Lk} = L_{Lk}\frac{di_{L1}}{dt} = v_i - v_T - i_{L1}R_L \tag{2}$$

where v_i is the converter input voltage, v_T is the voltage drop across the magnetizing inductance, i_{L1} is the inductor current of phase 1, and R_L is the CL per-phase ESR. When the switch is open and the diode conducts, the input and output are directly connected to each other, and the inductor voltage v_{Lk} in this state is expressed as

$$v_{Lk} = L_{Lk}\frac{di_{L1}}{dt} = v_i - v_o - v_T - i_{L1}R_L \tag{3}$$

where v_o is the converter output voltage. By averaging (2) with (3), the dynamic equation of the inductor voltage of the CL boost converter are found as

$$L_{Lk}\frac{di_{L1}}{dt} = d_c v_i + d_{\text{off}}(v_i - v_o) - v_T - i_{L1}R_L = v_i(d_c + d_{\text{off}}) - d_{\text{off}}v_o - (d_c + d_{\text{off}})v_T - (d_c + d_{\text{off}})i_{L1}R_L. \quad (4)$$

Similarly, the output capacitor current i_{C_o} of the converter during the full cycle is

$$i_{C_o} = C_o\frac{dv_c}{dt} = i_{L1} + i_{L2} - i_{S1} - i_{S2} - \frac{v_o}{R_o} \quad (5)$$

where C_o is the output capacitance, R_o is the output load resistance, i_{Sx} is the DC switch current flowing through phase x, and v_c is the voltage drop across the capacitance. By applying Kirchhoff's current law to the output filter of the converter, and finding the capacitor voltage in terms that are known, the capacitor voltage can be expressed as

$$v_c = \left(1 + \frac{R_C}{R_o}\right)v_o - 2R_C(i_{L1} - i_{S1}) \quad (6)$$

where R_C is the output capacitor ESR. By inserting (6) into (5) and applying the assumptions given in (1), the dynamic equation of the output capacitor of the CL boost converter is

$$C_{\text{eq}}\frac{dv_o}{dt} = 2i_{L1} - 2i_{S1} + 2C_oR_C\left(\frac{di_{L1}}{dt} - \frac{di_{S1}}{dt}\right) - \frac{v_o}{R_o} \quad (7)$$

where

$$C_{\text{eq}} = C_o\left(1 + \frac{R_C}{R_o}\right). \quad (8)$$

The equations presented in (4) and (7) can be used to describe the CL boost converter when operating in CCM and in any DCM.

Small Signal Model

The magnetizing inductance voltage drop v_T is dependent on the magnetizing current circulating through the inductor

$$v_T = L_m\frac{di_m}{dt}. \quad (9)$$

However, as can be seen from Fig. 5, the average change of the magnetizing current over one full cycle is zero

$$(d_c + d_{\text{off}})v_T = 0. \quad (10)$$

The switch current i_{S1} is found as the DC inductor current of the converter averaged over the duty cycle

$$i_{S1} = d_c i_{L1}. \quad (11)$$

By inserting (10) into (4) and (11) into (7), the dynamic equations of the CCM CL boost converter are found as

$$L_{Lk}\frac{di_{L1}}{dt} = v_i - v_o - d_c v_o - i_{L1}R_L \quad (12)$$

$$C_{\text{eq}}\frac{dv_o}{dt} = 2i_{L1}(1 - d_c) + 2C_oR_C\frac{di_{L1}}{dt}\left(1 - \frac{d_c}{dt}\right) - \frac{v_o}{R_o}. \quad (13)$$

The equations presented in (12) and (13) must now be linearized before the small-signal model is found.

When linearizing, if

$$\frac{dx}{dt} = f(x, y)$$

then

$$\frac{d\tilde{x}}{dt} = \frac{\partial f(x, y)}{\partial X}\tilde{x} + \frac{\partial f(x, y)}{\partial Y}\tilde{y}$$

where (X, Y) is the steady-state operating point plus a small AC perturbation $(\tilde{x}, \tilde{y})$.

The states of the converter is linearized and transform into:

$$
\begin{aligned}
v_i &= V_I + \tilde{v}_i(t) \\
v_o &= V_O + \tilde{v}_o(t) \\
i_{L1} &= I_{L1} + \tilde{i}_{L1}(t) \\
d_c &= D_C + \tilde{d}_c(t)
\end{aligned}
$$

For example, linearizing the inductor voltage equation presented in (12) yields

$$\frac{\partial}{\partial}\left[L_{Lk}\frac{di_{L1}(t)}{dt}\right]\tilde{i}_{L1}(t) = L_{Lk}\frac{d\tilde{i}_{L1}(t)}{dt} \tag{14}$$

$$\frac{\partial}{\partial v_i}\left[L_{Lk}\frac{di_{L1}(t)}{dt}\right]\tilde{v}_i(t) = \tilde{v}_i(t) \tag{15}$$

$$\frac{\partial}{\partial v_o}\left[L_{Lk}\frac{di_{L1}(t)}{dt}\right]\tilde{v}_o(t) = -(1-D_C)\tilde{v}_o(t) \tag{16}$$

$$\frac{\partial}{\partial d_c}\left[L_{Lk}\frac{di_{L1}(t)}{dt}\right]\tilde{d}_c(t) = V_O\tilde{d}_c(t) \tag{17}$$

$$\frac{\partial}{\partial i_{L1}}\left[L_{Lk}\frac{di_{L1}(t)}{dt}\right]\tilde{i}_{L1}(t) = R_L. \tag{18}$$

Hence, the small-signal model of the inductor voltage is found to be

$$L_{Lk}\frac{d\tilde{i}_{L1}(t)}{dt} = \tilde{v}_i(t) - (1-D_C)\tilde{v}_o(t) + V_O\tilde{d}_c(t) - R_L\tilde{i}_{L1}. \tag{19}$$

By linearizing (13) in the same manner the small-signal model of the output capacitor current is found as

$$C_{\text{eq}}\frac{d\tilde{v}_o}{dt} = 2(1-D_C)\tilde{v}_i(t) - 2I_{L1}\tilde{d}_c(t) + 2C_oR_C(1-D_C)\frac{d\tilde{i}_{L1}(t)}{dt} - 2C_oR_CI_{L1}\frac{d\tilde{d}_c(t)}{dt} - \frac{\tilde{v}_o(t)}{R_o}. \tag{20}$$

With the equations linearized, they are next transformed into the Laplace domain.

$$sL_{Lk}\tilde{i}_{L1}(s) = \tilde{v}_i(s) - (1-D_C)\tilde{v}_o(s) + V_O\tilde{d}_c(s) - R_L\tilde{i}_{L1}(s) \tag{21}$$

$$sC_{\text{eq}}\tilde{v}_o(s) = 2(1-D_C)\tilde{i}_{L1}(s) - 2I_{L1}\tilde{d}_c(s) + 2sC_oR_C(1-D_C)\tilde{i}_{L1}(s) - 2sC_oR_CI_{L1}\tilde{d}_c(s) - \frac{\tilde{v}_o(s)}{R_o}. \tag{22}$$

In order to simplify the expressions, (21) and (22) can be rewritten into what are termed the unified small-signal models, i.e.,

$$sL_{Lk}\tilde{i}_{L1}(t) = \alpha_1\tilde{v}_i(s) + \beta_1\tilde{v}_o(s) + \gamma_1\tilde{d}_c(s) + \delta_1\tilde{i}_{L1}(s) \tag{23}$$

$$sC_{\text{eq}}\tilde{v}_o(s) = \alpha_2\tilde{v}_i(s) + \beta_2\tilde{v}_o(s) + \gamma_2\tilde{d}_c(s) + \delta_2\tilde{i}_{L1}(s). \tag{24}$$

where the α, β, γ, and δ coefficients are what are termed the unified model coefficients, which for CCM are

$$
\begin{aligned}
\alpha_1 &= 1 \\
\beta_1 &= -(1-D_C) \\
\gamma_1 &= V_O \\
\delta_1 &= -R_L \\
\alpha_2 &= 0 \\
\beta_2 &= -\frac{1}{R_o} \\
\gamma_2 &= -2I_{L1}(1+C_oR_Cs) \\
\delta_2 &= 2(1+C_oR_Cs)(1-D_C).
\end{aligned} \tag{25}
$$

Equations (23) and (24) are now coupled with the coefficients provided in (25) and the small-signal model of the CL converter operating in CCM is now complete.

Transfer Function

Using the unified transfer function given by [1]

A. Input Voltage-to-Output Voltage Small-Signal Model

The first transfer function to be derived is $G_{v_i v_o}(s)$, the input voltage-to-output voltage transfer function. Initially, inductor current in (23) is isolated

$$\tilde{i}_{L1}(s) = \frac{\alpha_1 \tilde{v}_i(s) + \beta_1 \tilde{v}_o(s) + \gamma_1 \tilde{d}_c(s)}{sL_{Lk} - \delta_1}. \tag{26}$$

This expression is then substituted into (24)

$$\alpha_2 \tilde{v}_i(s) + \beta_2 \tilde{v}_o(s) + \gamma_2 \tilde{d}_c(s) + sC_{\mathrm{eq}} \tilde{v}_o(s) = \delta_2 \frac{\alpha_1 \tilde{v}_i(s) + \beta_1 \tilde{v}_o(s) + \gamma_1 \tilde{d}_c(s)}{sL_{Lk} - \delta_1}. \tag{27}$$

By gathering all the terms of the dynamic coefficients together, it is found that

$$\tilde{v}_o(s) = \frac{(sL_{Lk}\alpha_2 - \delta_1\alpha_2 + \delta_2\alpha_1)\tilde{v}_i(s) + (sL_{Lk}\gamma_2 - \delta_1\gamma_2 + \delta_2\gamma_1)\tilde{d}_c(s)}{s^2 L_{Lk} C_{\mathrm{eq}} - s(L_{Lk}\beta_2 + C_{\mathrm{eq}}\delta_1) + (\delta_1\beta_2 - \delta_2\beta_1)}. \tag{28}$$

By letting $\tilde{d}_c(s) = 0$, the input voltage-to-output voltage transfer function is found as

$$G_{v_i v_o}(s) = \frac{\tilde{v}_o(s)}{\tilde{v}_i(s)} = \frac{sL_{Lk}\alpha_2 - \delta_1\alpha_2 + \delta_2\alpha_1}{s^2 L_{Lk} C_{\mathrm{eq}} - s(L_{Lk}\beta_2 + C_{\mathrm{eq}}\delta_1) + (\delta_1\beta_2 - \delta_2\beta_1)}. \tag{29}$$

B. Duty Cycle-to-Output Voltage Small-Signal Model

The next transfer function, the duty cycle-to-output voltage transfer function $G_{v_o d_c}(s)$, is found by letting $\tilde{v}_i(s)$ equal to zero in (28). Hence

$$G_{vd}(s) = \frac{\tilde{v}_o(s)}{\tilde{d}_c(s)} = \frac{sLL_k\gamma_2 - \delta_1\gamma_2 + \delta_2\gamma_1}{s^2 LL_k C_{eq} - s(LL_k\beta_2 + C_{eq}\delta_1) + (\delta_1\beta_2 - \delta_2\beta_1)}. \tag{30}$$

C. Duty Cycle-to-Inductor Current Small-Signal Model

The next transfer function to be derived is the duty cycle-to-inductor current transfer function $G_{id}(s)$. First, the output voltage in (24) is isolated

$$\tilde{v}_o(s) = \alpha_2 \tilde{v}_i(s) + \gamma_2 \tilde{d}_c(s) + \delta_2 \tilde{i}_{L1}(s) \cdot \frac{1}{sC_{eq} - \beta_2}. \tag{31}$$

This expression for the output voltage is then inserted into (23)

$$\begin{aligned} sLL_k \tilde{i}_{L1}(s) = {} & \alpha_1 \tilde{v}_i(s) + \gamma_1 \tilde{d}_c(s) + \delta_1 \tilde{i}_{L1}(s) \\ & + \beta_1 \frac{\alpha_2 \tilde{v}_i(s) + \gamma_2 \tilde{d}_c(s) + \delta_2 \tilde{i}_{L1}(s)}{sC_{eq} - \beta_2}. \end{aligned} \tag{32}$$

By gathering all the terms of the dynamic coefficients together, it is found that

$$\begin{aligned} \tilde{i}_{L1}(s) = {} & \frac{(sC_{eq}\alpha_1 - \beta_2\alpha_1 + \beta_1\alpha_2)\tilde{v}_i(s)}{s^2 LL_k C_{eq} - s(LL_k\beta_2 + C_{eq}\delta_1) + (\delta_1\beta_2 - \delta_2\beta_1)} \\ & + \frac{(sC_{eq}\gamma_1 - \beta_2\gamma_1 + \beta_1\gamma_2)\tilde{d}_c(s)}{s^2 LL_k C_{eq} - s(LL_k\beta_2 + C_{eq}\delta_1) + (\delta_1\beta_2 - \delta_2\beta_1)}. \end{aligned} \tag{33}$$

Finally, $\tilde{v}_i(s)$ is set equal to zero and the duty cycle-to-inductor current transfer function is found to be

$$G_{id}(s) = \frac{\tilde{i}_{L1}(s)}{\tilde{d}_c(s)} = \frac{(sC_{eq}\gamma_1 - \beta_2\gamma_1 + \beta_1\gamma_2)}{s^2 LL_k C_{eq} - s(LL_k\beta_2 + C_{eq}\delta_1) + (\delta_1\beta_2 - \delta_2\beta_1)}. \tag{34}$$

D. Inductor Current-to-Output Voltage Small-Signal Model

The final transfer function to be derived is $G_{vi}(s)$, the inductor current-to-output voltage transfer function. In order to find $G_{vi}(s)$, the duty cycle in (24) is isolated

$$\tilde{d}_c(s) = \frac{sC_{eq}\tilde{v}_o(s) - \alpha_2\tilde{v}_i(s) - \beta_2\tilde{v}_o(s) - \delta_2\tilde{i}_{L1}(s)}{\gamma_2}. \tag{35}$$

This expression for the duty cycle is then inserted into (23) and the dynamic coefficients gathered

$$\tilde{v}_o(s) = \frac{(s\gamma_2 LL_k + \delta_2\gamma_1 - \delta_1\gamma_2)\tilde{i}_{L1}(s) + (\alpha_2\gamma_1 - \alpha_1\gamma_2)\tilde{v}_i(s)}{s\gamma_1 C_{eq} + \beta_1\gamma_2 - \beta_2\gamma_1}. \tag{36}$$

Finally, the inductor current-to-output voltage transfer function $G_{vi}(s)$, $\tilde{v}_i(s)$ is set to zero in (36). Hence

$$G_{vi}(s) = \frac{\tilde{v}_o(s)}{\tilde{i}_{L1}(s)} = \frac{(s\gamma_2 LL_k + \delta_2\gamma_1 - \delta_1\gamma_2)}{(s\gamma_1 C_{eq} + \beta_1\gamma_2 - \beta_2\gamma_1)}. \tag{37}$$

The transfer functions represented in (29), (30), (34), and (37) can be applied to the 1L, 2L, or CL boost or buck converter.

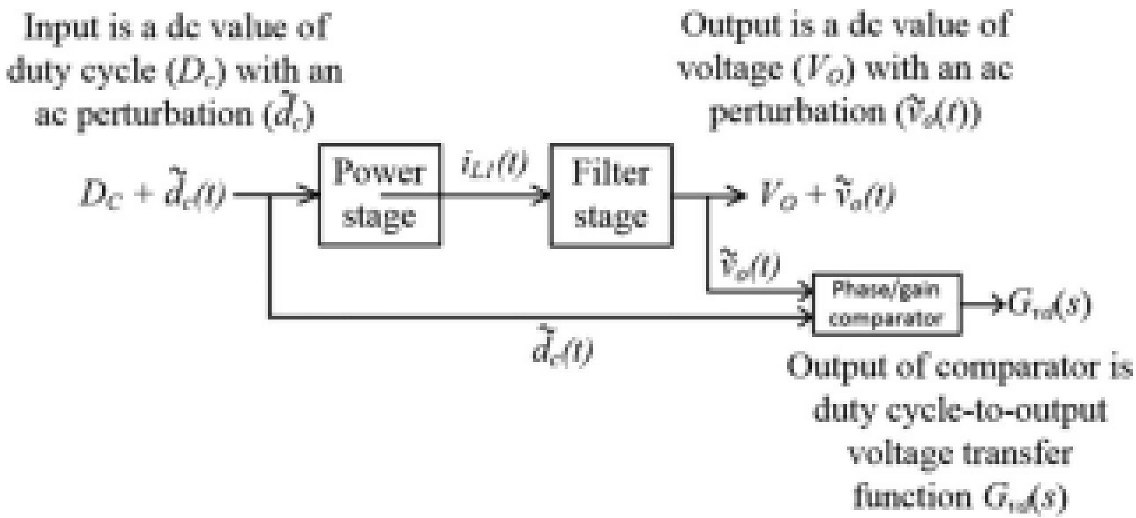

Figure 2: Block diagram of experimental frequency sweep from duty cycle to output voltage. [1]

substituting all the unified co efficients, we have

$$G_{id}(s) = \frac{sC_{eq}V_O + \frac{V_O}{R_O} + 2I_{L1}(1-D_C)(1+C_OR_Cs)}{s^2LL_kC_{eq} + s\left(\frac{LL_k}{R_O} + C_{eq}R_L\right) + \frac{R_L}{R_O} + 2(1+C_OR_Cs)(1-D_C)^2} \tag{38}$$

$$G_{vd}(s) = \frac{2(1+C_oR_Cs)\left[V_O(1-D_C) - I_{L1}\left(sLL_k + R_L\right)\right]}{s^2LL_kC_{eq} + s\left(\frac{LL_k}{R_O} + R_LC_{eq}\right) + \frac{R_L}{R_O} + 2(1+C_oR_Cs)(1-D_C)^2}. \tag{39}$$

now substituting the converter parameters: $C_{eq} = 1\times10^{-4}$ $V_O = 48v$ $R_O = 4.608v$ $I_{L1} = 17.9A$ $D_C = 0.7$ $C_O = 1\times10^{-4}v$ $R_C = 10\times10^{-3}v$ $LL_k = 32\times10^{-6}H$ $R_L = 3.43\times10^{-3}$

$$G_{id}(s) = \frac{4.81074\times10^{-3}s + 21.15667}{3.2\times10^{-9}s^2 + 7.46744\times10^{-6}s + 0.180744} \tag{40}$$

$$G_{vd}(s) = \frac{-1.1456\times10^{-9}s^2 - 0.00111792s + 28.67721}{3.2\times10^{-9}s^2 + 7.46744\times10^{-6}s + 0.180744} \tag{41}$$

References

[1] B. C. Barry, J. G. Hayes, M. S. Rylko, R. Stala, A. Penczek, A. Mondzik, and R. T. Ryan, “Small-Signal Model of the Two-Phase Interleaved Coupled-Inductor Boost Converter,” IEEE Transactions on Power Electronics, vol. 33, no. 9, pp. 8052–8064, Sep. 2018. [Online]. Available: https://ieeexplore.ieee.org/document/8080253/