Module

Def.

Let R be a ring, A (left-) R module is an abelian group M with a map

$$R \times M \longrightarrow M, \quad (r, m) \longmapsto r \cdot m$$

s.t. $\forall\, r, s \in R$, $m, n \in M$

(1) $1_R \cdot m = m$

(2) $(r+s) \cdot m = r \cdot m + s \cdot m$

(3) $r \cdot (m+n) = r \cdot m + r \cdot n$

(4) $r \cdot (s \cdot m) = (rs) m$

Rmk.

① If R is field, R-Mod is linear space, which we familiar with.

② we can view module as a "ring action"

③ The most simple / important example of module is abelian group

Let G is abelian group, $\mathbb{Z}$ act naturally on G as:

$$\mathbb{Z} \times G \longrightarrow G, \quad (n, g) \longmapsto \overbrace{g + \cdots + g}^{n \text{ items}}$$

Thus with this map, G is $\mathbb{Z}$-module.

Inversely, any $\mathbb{Z}$-module $(M, \cdot)$, $\cdot$ is the natural action of $\mathbb{Z}$ on abelian group M.

So the category $\mathbb{Z}$-module and abelian group are equivalent.

We can define some basic concepts by analogy with the case of linear spaces.

Such as: submodule, quotient module, direct sum module-homomorphism, bases, etc.

However, it's important to note there are some differences:

① Module may has torsion elements, i.e. may be exist $r \in R \setminus \{0\}$, $x \in M$ s.t. $rx = 0$. For example take finite cyclic group $\mathbb{Z}/n\mathbb{Z}$

② Module may hasn't basis. For example, also take $\mathbb{Z}/n\mathbb{Z}$. (We say M is free if M has a basis)

③ Suppose R is commutative, M, N are R-module. $\mathrm{Hom}(M, N)$ can be viewed as R-module naturally.

But note that if R is noncommutative, the multiplication may not well-defined.

$\forall s, r \in R$, $\varphi \in \mathrm{Hom}(M, N)$, $x \in M$

$s\varphi(rx) = s \cdot \varphi(r \cdot x) = (s \cdot r)\varphi(x)$

$\neq (r\cdot s)\varphi(x) = r\cdot(s\varphi(x))$ $\lrcorner$

In this class, we will only consider modules over commutative rings (most of time, over PIDs)

Def (Free module generated by Set)

Let S be a set, R is Ring. we define

$$RS := \{\phi : S \to R \mid \{x : \phi(x) \neq 0\} \text{ is finite}\}$$

RS has natural R-module structure.

Rmk: Note that $\forall \sigma \in S$, $\chi_\sigma(x) := \begin{cases} 0, & x \neq \sigma \\ 1, & x = \sigma \end{cases}$ form basis of RS. We always denote χ_σ by σ. thus, $RS = \{\sum_{\sigma\in S} r_\sigma \cdot \sigma \mid \{r_\sigma\}_{\sigma \in S}$ has finite non-zero$\}$

Prop (universal property of free module)

$$\begin{tikzcd} \sigma \in S \arrow[r, "\varphi"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & M \\ RS \arrow[ur, dashed, "\exists ! \tilde{\varphi}"'] & \end{tikzcd}$$

$\sigma \mapsto \sigma$

M is R-module, φ is map.

A Glimpse of Category.

Def.

A Category $\mathcal{C}$ is following data

「1. A Set $Obj(\mathcal{C})$, whose elements are called the obefect of $\mathcal{C}$.

2. For each ordered pair (A, B) of objects with $A, B \in Obj(\mathcal{C})$, a set $Hom(A, B)$, whose elements are called the morphism from A to B.

3. For each object A, a elements $id_A \in Hom(A,A)$, called the identity morphism on X.

4. For any $X, Y, Z \in Obj(\mathcal{C})$, a Composition map

$$\circ : Hom(Y, Z) \times Hom(X, Y) \longrightarrow Hom(X, Z)$$
$$(f, g) \longmapsto f \circ g$$

which is often write simply as fg.

which satisfy.

(a) $(f\circ g)\circ h = f\circ(g\circ h)$

(b) $id_A\circ f = f\circ id_A = f$ $\lrcorner$

We have encountered several example.

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The most naive category is $\mathcal{C} = \mathrm{Set}$ with

$\mathrm{Obj}(\mathrm{Set})$ is collection of sets

$\mathrm{Hom}(A,B)$ is collection of map from A to B

$\circ$ is composition of map

② Category of group / ring / module : $\mathrm{Grp}/\mathrm{Ring}/\mathrm{Mod}_R$

In fact, there are many interesting categories whose morphism isn't map.

③ Poset $(I, \leq)$

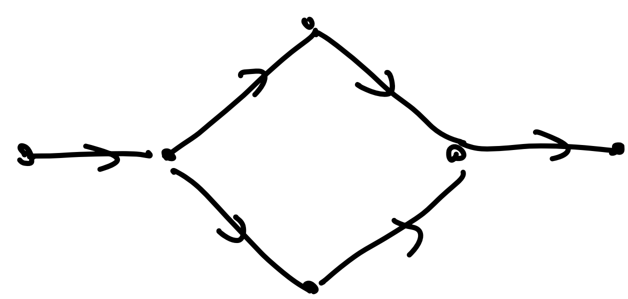
A poset is a set with partial order.

we can construct a Category $\mathcal{C}$ as

$Obj(\mathcal{C}) = I$

$\forall i, j \in Obj(\mathcal{C})$, $Hom(i,j) := \{(i,j)\}$

We can visualize morphism as the arrow from i to j.



Visualization of Poset

What connect different categories is functor.

Def (Functor)

Let $\mathcal{C}$ and $\mathcal{D}$ are category, A functor F

from $\mathcal{C}$ to $\mathcal{D}$ is following data

(1) A map (also denote as F)

$$F : Obj(\mathcal{C}) \longrightarrow Obj(\mathcal{D}),$$

(2) For each pair of object (A, B) in $\mathcal{C}$, with a map

$$F : Hom(A, B) \longrightarrow Hom(F(A), F(B))$$

they need satisfies

(a) $F(id_A) = id_{F(A)}$

(b) $F(f \circ g) = F(f) \circ F(g)$

Example:

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$$\begin{tikzcd} \{\text{Category of pointed manifold}\} \arrow[d, "\text{tangent}"] \\ \text{Vect} \end{tikzcd}$$

② {Category of pointed topological space}

$$\begin{tikzcd} \{\text{Category of pointed topological space}\} \arrow[d, "\pi_1(-)"] \\ \mathrm{Grp} \end{tikzcd}$$