Pi Calculus

CS242

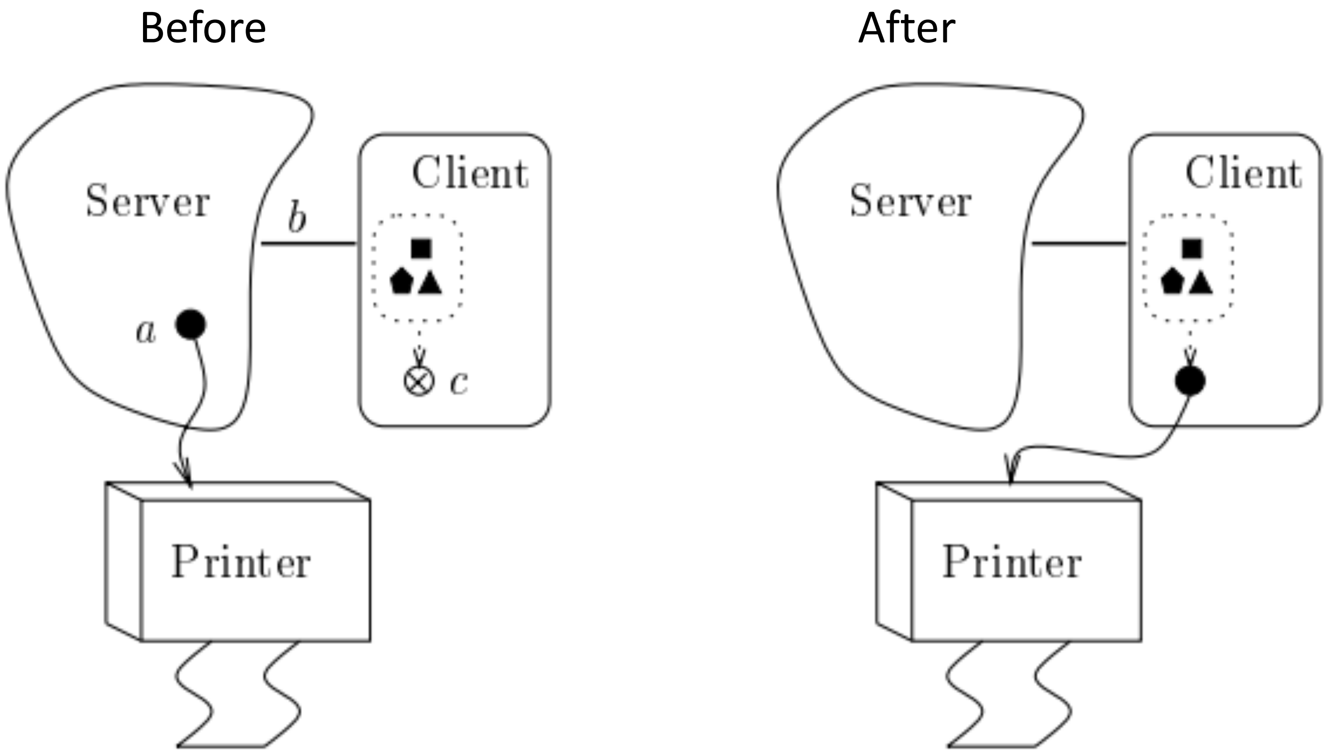
Lecture 13

Motivation

- Assume we have three entities
  - A server
  - A client
  - A resource controlled by the server, such as a printer

- How do we model the client requesting access to the printer from the server?

Picture



From “An Introduction to the Pi-Calculus”. J. Parrow

Process Calculi

- The essence of this problem is modeling the interaction of concurrent processes

- Models are known as *process calculi*

- There are many!
  - Much more diverse than the situation with, say, sequential programming
    - With functions, the only thing to observe is the output for a given input
  - Many more choices about what can be observed/modeled with concurrent systems

Process Calculi

- All process calculi share communication over *channels*

- A *prefix* can
  - Send a message *x* on channel *a*: $\bar{a}x$
  - Receive a message *x* on channel *a*: $a(x)$
- An *agent* can be
  - Empty 0
  - A prefix *p* followed by an agent *A: p.A*
- Example
  - Parallel composition of two agents: $\bar{a}x$.P | $a(y)$.Q

Pi Calculus

- The distinguishing feature of the Pi Calculus is that channels are values
  - Channels can themselves be sent as messages

- As we shall see, this gives the Pi Calculus great expressivity

- Pi Calculus is one of the most popular process calculi

Syntax

- Prefixes p = $\bar{a}x$ | $a(x)$

- Agents P = 0 empty

P = 0	empty
p.P	prefix
P + P	choice
P | P	parallel
x op y => P	match (op is = or ≠)
υx P	restriction
!P	iteration

Semantics

- Prefixes p = $\bar{a}x \mid a(x)$

- Agents P = 0 empty
  - p.P prefix
  - P + P choice
  - P | P parallel
  - x op y => P match (op is = or $\neq$)
  - υx P restriction
  - !P iteration

$$\bar{a}x.\mathrm{P} \mid a(y).\mathrm{Q} \rightarrow \mathrm{P} \mid \mathrm{Q}\{ y := x\}$$

Semantics

- Prefixes p = $\bar{a}x \mid a(x)$

- Agents

P = 0	empty
p.P	prefix
P + P	choice
P | P	parallel
x op y => P	match (op is = or ≠)
υx P	restriction
!P	iteration

Idiom: (x = y => P) + (x ≠ y ⇒ Q) → P if x = y

Semantics

- Prefixes p = $\bar{a}x \mid a(x)$

- Agents P = 0 empty

P = 0	empty
p.P	prefix
P + P	choice
P | P	parallel
x op y => P	match (op is = or $\neq$)
υx P	restriction
!P	iteration

$$!P \rightarrow P \mid !P$$

Semantics

- Prefixes p = $\bar{a}x \mid a(x)$

- Agents P = 0 empty
  - p.P prefix
  - P + P choice
  - P | P parallel
  - x op y => P match (op is = or $\neq$)
  - υx P restriction
  - !P iteration

P = 0	empty
p.P	prefix
P + P	choice
P | P	parallel
x op y => P	match (op is = or $\neq$)
υx P	restriction
!P	iteration

! x(y).0 | ! $\bar{x}z.0$ → x(y).0 | !x(y).0 | $\bar{x}z$.0| !$\bar{x}z$.0→ ! x(y).0 | !$\bar{x}z$.0

Semantics

- Prefixes p = $\bar{a}x \mid a(x)$

- Agents

P =	0	empty
	p.P	prefix
	P + P	choice
	P | P	parallel
	x op y => P	match (op is = or $\neq$)
	υx P	restriction
	!P	iteration

! x(y) | ! $\bar{x}z$ $\rightarrow$ x(y) | !x(y) | $\bar{x}z$| !$\bar{x}z$ $\rightarrow$! x(y) | !$\bar{x}z$

Not: The trailing 0's are usually elided

Restriction

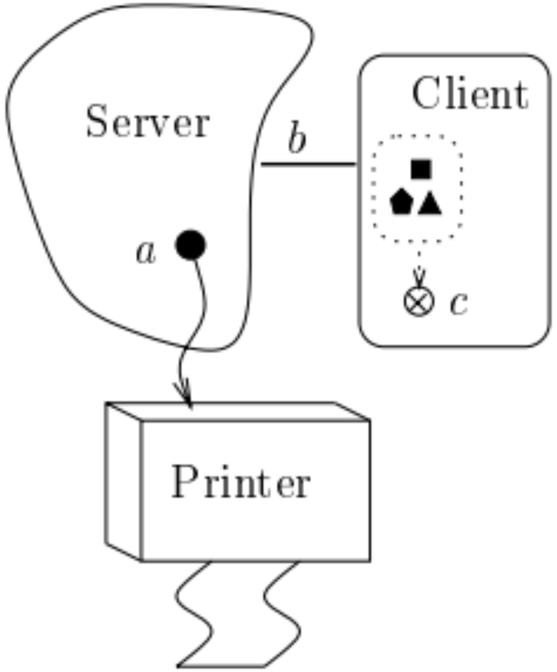
- υx P introduces a channel local to P

- Example: R | υx (P | Q)
  - P and Q share the name x and can communicate with each other over x
  - R does not have access to x
  - R | υx ($\bar{x}a$.P | $x(b)$.Q) → R υx (P | Q{b := a})

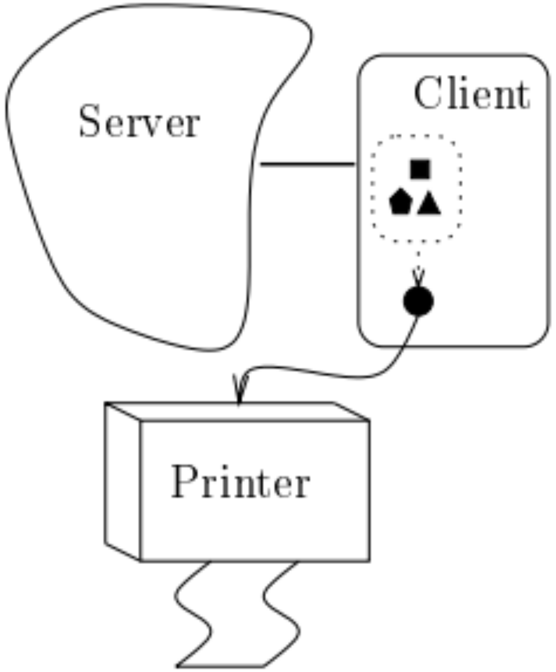
  - But x can be passed to R on a different channel that R shares with P or Q
  - Passing a restricted channel expands its scope to include the new agent
  - $a(z)$.R | υx ($\bar{a}x$.P | Q) → υx (R{z := x} | P | Q)
  - Note alpha renaming may be required to avoid capture of existing *x* in R

Back to the Example …

Before



After



$$(\nu a\,(\bar{b}a.\mathrm{S} \mid a(e).\mathrm{P})) \mid b(c).\,\bar{c}d.\,\mathrm{C}$$

From “An Introduction to the Pi-Calclus”. J. Parrow

Unbounded Computation

We can create as many fresh channel names as we like:

$!(\upsilon x\ P) \rightarrow$

$\upsilon x\ P \mid\ !(\upsilon x\ P) \rightarrow$

$\upsilon x\ P \mid \upsilon x\ P \mid\ !(\upsilon x\ P) \rightarrow$

$\upsilon x\ P \mid \upsilon y\ P\{x := y\} \mid\ !(\upsilon x\ P)$

What Can We Do With That?

- A lot!

- Example: Linked Lists

Lists

- Use two constants (unused channels) Nil and Cons
- The translation function L takes a list *l* and a channel *c* and returns an agent that provides access to a representation of *l* through *c*
- Idea: A list element communicates its kind (Nil or Cons) and then its arguments over c.

L(Nil, x) = $\bar{x}$ Nil

L(Cons H T, x) = $\upsilon y\ \upsilon z\ \bar{x}\mathrm{Cons}.\bar{x}y.\bar{x}z$ | L(H,y) | L(T,z)

Example

L(Nil, x) = $\bar{x}$ Nil

L(Cons H T, x) = υy υz $\bar{x}\mathrm{Cons}.\bar{x}y.\bar{x}z$ | L(H,y) | L(T,z)

L(Cons Nil Nil, x) =

υy υz $\bar{x}\mathrm{Cons}.\bar{x}y.\bar{x}z$ | L(Nil,y) | L(Nil,z) =

υy υz $\bar{x}\mathrm{Cons}.\bar{x}y.\bar{x}z$ | $\bar{y}$ Nil | L(Nil,z) =

υy υz $\bar{x}\mathrm{Cons}.\bar{x}y.\bar{x}z$ | $\bar{y}$ Nil | $\bar{z}$ Nil

Length of a List

Len computes the length of a list x and leaves the result on channel a.

Len(x,a) = νp νk νs Fold(k,s) | Sum(s,p,a) | $\bar{p}0$ | $\bar{k}$x

Sum(s,p,a) = !(s(*i*).p(n). i = 0 => $\bar{a}$n + i $\neq$ 0 => $\bar{p}$ n+i)

Fold(k,s) = !(k(z).z(y). y = Nil => $\bar{s}$0 + y = Cons => $\bar{s}$1.z(h).z(t).$\bar{k}$t)

Example

$(\upsilon$y υz $\bar{x}$Cons.$\bar{x}y.\,\bar{x}z$ | $\bar{y}$ Nil | $\bar{z}$ Nil)

|

Len(x,a)

Example

$(\upsilon \mathrm{y}\ \upsilon \mathrm{z}\ \bar{x}\mathrm{Cons}.\bar{x}y.\bar{x}z$ | $\bar{y}$ Nil | $\bar{z}$ Nil)

|

Fold(k,s)

|

Sum(s,p,a)

|

$\bar{p}0$

|

$\bar{k}\mathrm{x}$

(dropping top-level binders for p, k and s for brevity)

Example

$(\upsilon$y υz $\bar{x}$Cons.$\bar{x}y.\,\bar{x}z$ | $\bar{y}$ Nil | $\bar{z}$ Nil)

|

!(k(z).z(y). y = Nil => $\bar{s}0$ + y = Cons => $\bar{s}1.\,$z(h).z(t).$\bar{k}$t)

|

!(s(i).p(n). i = 0 => $\bar{a}$n + i $\neq$ 0 => $\bar{p}$ n+i)

|

$\bar{p}0$

|

$\bar{k}$x

Example

$(\upsilon$y υz $\bar{x}\mathrm{Cons}.\bar{x}y.\,\bar{x}z$ | $\bar{y}$ Nil | $\bar{z}$ Nil)

|

(k(z).z(y). y = Nil => $\bar{s}$0 + y = Cons => $\bar{s}1.\,\mathrm{z}$(h).z(t).$\bar{k}$t) | Fold(k,s)

|

!(s(*i*).p(n). i = 0 => $\bar{a}\mathrm{n}$ + i $\neq$ 0 => $\bar{p}$ n+i)

|

$\bar{p}0$

|

$\bar{k}\mathrm{x}$

Example

$(\upsilon$y υz $\bar{x}$Cons.$\bar{x}y.\bar{x}z$ | $\bar{y}$ Nil | $\bar{z}$ Nil)

|

x(y). y = Nil => $\bar{s}$0 + y = Cons => $\bar{s}1.$ x(h).x(t).$\bar{k}$t | Fold(k,s)

|

!(s(*i*).p(n). i = 0 => $\bar{a}$n + i $\neq$ 0 => $\bar{p}$ n+i)

|

$\bar{p}0$

|

0

Example

$(\upsilon$y υz $\bar{x}y.\bar{x}z$ | $\bar{y}$ Nil | $\bar{z}$ Nil)

|

Cons = Nil => $\bar{s}0$ + Cons = Cons => $\bar{s}1.\mathrm{x}$(h).x(t).$\bar{k}$t | Fold(k,s)

|

!(s(*i*).p(n). i = 0 => $\bar{a}\mathrm{n}$ + i $\neq$ 0 => $\bar{p}$ n+i)

|

$\bar{p}0$

|

0

Example

$(\upsilon$y υz $\bar{x}y.\,\bar{x}z$ | $\bar{y}$ Nil | $\bar{z}$ Nil)

|

Cons = Cons => $\bar{s}1.\,$x(h).x(t).$\bar{k}$t | Fold(k,s)

|

!(s(*i*).p(n). i = 0 => $\bar{a}$n + i $\neq$ 0 => $\bar{p}$ n+i)

|

$\bar{p}0$

|

0

Example

$(\upsilon$y υz $\bar{x}y.\bar{x}z$ | $\bar{y}$ Nil | $\bar{z}$ Nil)

|

$\bar{s}1.$ x(h).x(t).$\bar{k}$t | Fold(k,s)

|

!(s(i).p(n). i = 0 => $\bar{a}$n + i $\neq$ 0 => $\bar{p}$ n+i)

|

$\bar{p}0$

|

0

Example

$(\upsilon$y υz $\bar{x}y.\bar{x}z$ | $\bar{y}$ Nil | $\bar{z}$ Nil)

|

$\bar{s}1.$ x(h).x(t).$\bar{k}$t | Fold(k,s)

|

(s(i).p(n). i = 0 => $\bar{a}$n + i $\neq$ 0 => $\bar{p}$ n+i) | Sum(s,p,a)

|

$\bar{p}0$

|

0

Example

$(\upsilon$y υz $\bar{x}y.\,\bar{x}z$ | $\bar{y}$ Nil | $\bar{z}$ Nil)

|

x(h).x(t).$\bar{k}$t | Fold(k,s)

|

p(n). 1 = 0 => $\bar{a}$n + 1 $\neq$ 0 => $\bar{p}$ n+1 | Sum(s,p,a)

|

$\bar{p}0$

|

0

Example

$(\upsilon$y υz $\bar{x}y.\bar{x}z$ | $\bar{y}$ Nil | $\bar{z}$ Nil)

|

x(h).x(t).$\bar{k}$t | Fold(k,s)

|

1 = 0 => $\bar{a}$n + 1 $\neq$ 0 => $\bar{p}$ 0+1 | Sum(s,p,a)

|

0

|

0

Example

$(\upsilon \mathrm{y}\ \upsilon \mathrm{z}\ \bar{x} y . \bar{x} z$ | $\bar{y}$ Nil | $\bar{z}$ Nil)

|

x(h).x(t).$\bar{k}$t | Fold(k,s)

|

1 $\neq$ 0 => $\bar{p}$ 0+1 | Sum(s,p,a)

|

0

|

0

Example

$(\upsilon$y υz $\bar{x}y.\,\bar{x}z$ | $\bar{y}$ Nil | $\bar{z}$ Nil)

|

x(h).x(t).$\bar{k}$t | Fold(k,s)

|

$\bar{p}$ 0+1| Sum(s,p,a)

|

0

|

0

Example

$(\upsilon\text{y}\ \upsilon\text{z}\ \bar{x}y.\bar{x}z \mid \bar{y}\ \text{Nil} \mid \bar{z}\ \text{Nil})$

|

$\text{x(h).x(t)}.\bar{k}\text{t} \mid \text{Fold(k,s)}$

|

$\bar{p}\ 1$

|

Sum(s,p,a)

Remove 0 agents as they will not contribute further.

Example

$(\upsilon$y υz $\bar{x}z$ | $\bar{y}$ Nil | $\bar{z}$ Nil)

|

x(t).$\bar{k}$t | Fold(k,s)

|

$\bar{p}$ 1

|

Sum(s,p,a)

Example

$(\upsilon \mathrm{y}\ \upsilon \mathrm{z}$ 0 | $\bar{y}$ Nil | $\bar{z}$ Nil)

|

$\bar{k}\mathrm{z}$ | Fold(k,s)

|

$\bar{p}$ 1

|

Sum(s,p,a)

Example

$(\upsilon y\ \upsilon z\ 0 \mid \bar{y}$ Nil $\mid \bar{z}$ Nil$)$

|

$\bar{k} z$

|

Fold(k,s)

|

$\bar{p}$ 1

|

Sum(s,p,a)

Example

$(\upsilon y\ \upsilon z\ 0 \mid \bar{y}$ Nil $\mid \bar{z}$ Nil)

|

$\bar{k}$z

|

!(k(z).z(y). y = Nil => $\bar{s}0$ + y = Cons => $\bar{s}1.$ z(h).z(t).$\bar{k}$t)

|

$\bar{p}$ 1

|

!(s(*i*).p(n). i = 0 => $\bar{a}$n + i $\neq$ 0 => $\bar{p}$ n+i)

Example

$(\upsilon \mathrm{y}\ \upsilon \mathrm{z}$ 0 | $\bar{y}$ Nil | $\bar{z}$ Nil)

|

$\bar{k}\mathrm{z}$

|

(k(z).z(y). y = Nil => $\bar{s}$0 + y = Cons => $\bar{s}1.\mathrm{z}$(h).z(t).$\bar{k}$t) | Fold(k,s)

|

$\bar{p}$ 1

|

(s(*i*).p(n). i = 0 => $\bar{a}\mathrm{n}$ + i $\neq$ 0 => $\bar{p}$ n+i) | Sum(s,p,a)

Example

$(\upsilon \mathrm{y}\ \upsilon \mathrm{z}$ 0 | $\bar{y}$ Nil | $\bar{z}$ Nil)

|

0

|

(z(y). y = Nil => $\bar{s}$0 + y = Cons => $\bar{s}1.\mathrm{z}$(h).z(t).$\bar{k}$t) | Fold(k,s)

|

$\bar{p}$ 1

|

(s(*i*).p(n). i = 0 => $\bar{a}\mathrm{n}$ + i ≠ 0 => $\bar{p}$ n+i) | Sum(s,p,a)

Example

$(\upsilon\mathrm{y}\ \upsilon\mathrm{z}\ 0 \mid \bar{y}\ \mathrm{Nil} \mid 0)$

|

0

|

(Nil = Nil => $\bar{s}0$ + Nil = Cons => $\bar{s}1.\mathrm{z}$(h).z(t).$\bar{k}$t) | Fold(k,s)

|

$\bar{p}$ 1

|

(s(*i*).p(n). i = 0 => $\bar{a}\mathrm{n}$ + i $\neq$ 0 => $\bar{p}$ n+i) | Sum(s,p,a)

Example

$(\upsilon \mathrm{y}\ \upsilon \mathrm{z}$ 0 | $\bar{y}$ Nil | 0)

|

0

|

Nil = Nil => $\bar{s}$0 | Fold(k,s)

|

$\bar{p}$ 1

|

(s(i).p(n). i = 0 => $\bar{a}\mathrm{n}$ + i $\neq$ 0 => $\bar{p}$ n+i) | Sum(s,p,a)

Example

$(\upsilon\mathrm{y}\ \upsilon\mathrm{z}$ 0 | $\bar{y}$ Nil | 0)

|

0

|

$\bar{s}$0 | Fold(k,s)

|

$\bar{p}$ 1

|

(s(*i*).p(n). i = 0 => $\bar{a}\mathrm{n}$ + i $\neq$ 0 => $\bar{p}$ n+i) | Sum(s,p,a)

Example

$(\upsilon\mathrm{y}\ \upsilon\mathrm{z}$ 0 | $\bar{y}$ Nil | 0)

|

0

|

0 | Fold(k,s)

|

$\bar{p}$ 1

|

(p(n). 0 = 0 => $\bar{a}\mathrm{n}$ + 0 $\neq$ 0 => $\bar{p}$ n+0) | Sum(s,p,a)

Example

$(\upsilon \mathrm{y}\ \upsilon \mathrm{z}$ 0 | $\bar{y}$ Nil | 0)

|

0

|

0 | Fold(k,s)

|

0

|

(0 = 0 => $\bar{a}1$ + 0 $\neq$ 0 => $\bar{p}$ 1+0) | Sum(s,p,a)

Example

Fold(k,s)

|

(0 = 0 => $\bar{a}1$ + 0 $\neq$ 0 => $\bar{p}$ 1+0)

|

Sum(s,p,a)

Remove 0 and useless agents ...

Example

Fold(k,s)

|

(0 = 0 => $\bar{a}1$)

|

Sum(s,p,a)

Example

Fold(k,s)

|

$\bar{a}1$

|

Sum(s,p,a)

No further computation can be done at this point until an external agent decides to read the length from a.

SKI

- We can also encode the SKI calculus or the lambda calculus in pi calculus.

- SKI is a little easier to explain ...

Idea

- We represent combinator expressions as trees, as usual
- Each combinator takes an extra (last argument) for its parent
- There is a separate combinator for each possible number of children
  - E.g., $S_0(p)$ is S applied to no children and just the parent
  - $S_1(x, p)$ is S applied to 1 child x and the parent
  - $S_2(x, y, p)$ is S applied to 2 children x and y and the parent

One New Construct

- Add recursive definitions

$$A(x_1,\ldots,x_n) = P$$

- Now A(...) can also appear inside P

- Not necessary, can be simulated with !P
  - For definitions with single recursive calls:
  - Each invocation of P writes its result to a unique global channel p
  - Where the result of recursive call is needed P reads the value from p
  - See length of a list example ...

Collecting Children

$S_0(p) = \nu w\ \bar{p}w.(\bar{w}s_0 \mid S_0(p))$

$S_1(p,x) = \nu w\ \bar{p}w.(\bar{w}s_1.\ \bar{w}x \mid S_1(p,x))$

$S_2(p,x,y) = \nu w\ \bar{p}w.(\bar{w}s_2.\ \bar{w}x.\bar{w}y \mid S_2(p,x))$

$K_0(p) = \nu w\ \bar{p}w.(\bar{w}k_0 \mid K_0(p))$

$K_1(p,x) = \nu w\ \bar{p}w.(\bar{w}k_1.\ \bar{w}x \mid K_1(p,x))$

$I_0(p) = \nu w\ \bar{p}w.(\bar{w}i_0 \mid I_0(p))$

$I_1(p,x) = \nu w\ \bar{p}w.(\bar{w}i_1.\ \bar{w}x \mid I_1(p,x))$

Each combinator sends a series of messages to its parent: first the private channel, then the combinator's identity, then its children. Afterwards, it resets itself for another application.

Application

The application agent @(x,y,p) applies a left child x to a right child y with parent p using the rules of SKI.

$$
\begin{aligned}
@(x,y,p) = x(w).w(a).\ & \\
(\quad & a = s_0 \Rightarrow S_1(y,p) + \\
& a = s_1 \Rightarrow w(u).S_2(u,y,p) + \\
& a = s_2 \Rightarrow w(u).w(v).\ \upsilon p_1\ \upsilon p_2 \quad @(u,y,p_1)\ | \\
& \qquad\qquad\qquad\qquad\qquad\qquad @(v,y,p_2)\ | \\
& \qquad\qquad\qquad\qquad\qquad\qquad @(p_1,p_2,p)\ + \\
& a = k_0 \Rightarrow K_1(y,p) + \\
& a = k_1 \Rightarrow w(u).\ I_1(u,p) + \\
& a = i_0 \Rightarrow I_1(y,p) + \\
& a = i_1 \Rightarrow w(u).@(u,y,p)\ )
\end{aligned}
$$

Comments

Note two new tree nodes are created for the two new applications in S u v y = (u y) (v y). Here p_1 = u y and p_2 = v y.

$$
\begin{aligned}
@(x,y,p) = x(w).w(a).\\
(\quad & a = s_0 \Rightarrow S_1(y,p) + \\
& a = s_1 \Rightarrow w(u).S_2(u,y,p) + \\
& a = s_2 \Rightarrow w(u).w(v).\ \upsilon p_1\ \upsilon p_2 \quad @(u,y,p_1)\ | \\
& \qquad\qquad\qquad\qquad\qquad\qquad @(v,y,p_2)\ | \\
& \qquad\qquad\qquad\qquad\qquad\qquad @(p_1,p_2,p)\ + \\
& a = k_0 \Rightarrow K_1(y,p) + \\
& a = k_1 \Rightarrow w(u).\ I_1(u,p) + \\
& a = i_0 \Rightarrow I_1(y,p) + \\
& a = i_1 \Rightarrow w(u).@(u,y,p)\ )
\end{aligned}
$$

Comments

I x is not immediately rewritten to x. Instead we wait until it is applied to something (I x) y = x y, which allows us to use a uniform three argument application rule.

$$
\begin{array}{ll}
@(x,y,p) = x(w).w(a). & \\
\quad (\; a = s_0 \Rightarrow S_1(y,p) + & \\
\quad \; a = s_1 \Rightarrow w(u).S_2(u,y,p) + & \\
\quad \; a = s_2 \Rightarrow w(u).w(v).\ \upsilon p_1\ \upsilon p_2 & @(u,y,p_1) \mid \\
 & @(v,y,p_2) \mid \\
 & @(p_1,p_2,p) \; + \\
\quad \; a = k_0 \Rightarrow K_1(y,p) + & \\
\quad \; a = k_1 \Rightarrow w(u).\ I_1(u,p) + & \\
\quad \; \color{red}{a = i_0 \Rightarrow I_1(y,p) +} & \\
\quad \; \color{red}{a = i_1 \Rightarrow w(u).@(u,y,p)\ )} &
\end{array}
$$

Comments

$S_0(p) = \nu w\ \bar{p}w.(\bar{w}s_0 \mid S_0(p))$

$S_1(p,x) = \nu w\ \bar{p}w.(\bar{w}s_1.\ \bar{w}x \mid S_1(p,x))$

$S_2(p,x,y) = \nu w\ \bar{p}w.(\bar{w}s_2.\ \bar{w}x.\bar{w}y \mid S_2(p,x))$

$K_0(p) = \nu w\ \bar{p}w.(\bar{w}k_0 \mid K_0(p))$

$K_1(p,x) = \nu w\ \bar{p}w.(\bar{w}k_1.\ \bar{w}x \mid K_1(p,x))$

$I_0(p) = \nu w\ \bar{p}w.(\bar{w}i_0 \mid I_0(p))$

$I_1(p,x) = \nu w\ \bar{p}w.(\bar{w}i_1.\ \bar{w}x \mid I_1(p,x))$

The sharing of combinators means that the structure we build is actually a DAG not a tree – the same subterm may be shared by multiple parents. That is a much more efficient representation, as when a subterm is reduced it is shared by any other expressions that use that same subterm.

Applications of Pi Calculus

- Languages based on pi calculus are usually *modeling languages,* not programming languages

- A modeling language is used for building models
  - And proving things about them
  - But not for running the models

Applications of Pi Calculus

- Analysis of concurrent protocols
    - e.g., client/server

- Analysis of cryptographic protocols

- Systems biology
    - Building models of cell pathways

- Modeling business processes

- And more!

What is Modeled?

- Proving properties of protocols
  - That a protocol doesn’t deadlock
  - That a cryptographic protocol is secure against an attacker (another agent)
- In general
  - That a system is guaranteed to reach some state
  - That a system is guaranteed *not* to reach some state

Summary

- Process calculi
  - Model concurrent agents
  - Fundamental operation is sending a message on a channel
  - Pi calculus allows the messages to also be channels

- A very active area
  - Lots of work
  - In many different fields where concurrent agents with a fixed set of interactions is a good description of how things work