# Termination of Graph Rewriting using Weighted Type Graphs over Non-well-founded Semirings

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# Plan (for myself)

- Introduction
  - Graphs and Graph Morphisms
  - Double Pushout Diagram (DPO)
  - Graph rewriting using DPO
    - Example of DPO graph rewriting : grsaa
  - Termination of DPO graph rewriting systems
    - Example: looping and terminating examples
- Type graph method in theory
- Type graph method in practice and the introduction of the main problem
- Morphism weight (by a example)
- Graph weight (by a example)
- A condition that weighted type graphs must satisfy
- Rule $\varphi$
- Example with rule $\varphi$
- Sufficient condition for termination using weighted type graph
- Sufficient condition simplified
- Example

# Outline

Introduction

Termination of Injective DPO Graph Rewriting using Morphism Counting

Etending the Type Graph Method to Non-well-founded Semirings

Termination of Injective DPO Graph Rewriting using Morphism Counting with antipatterns

Introduction
Graphs and Graph Morphisms
Termination of DPO Graph Rewriting Systems

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# Graphs

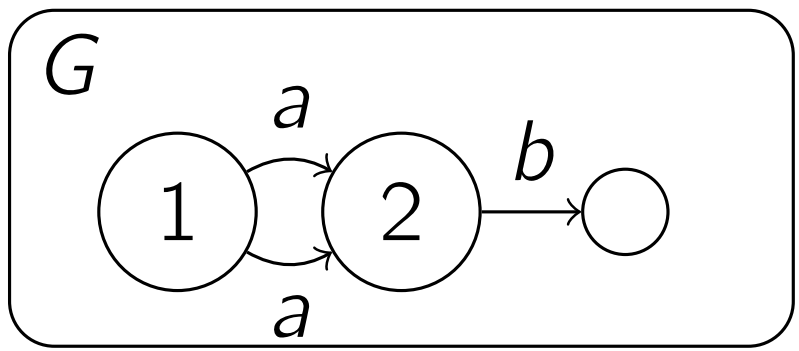
Graphs are finite and directed with:

- Parallel edges,
- Labeled edges,
- Finite labels.

Example:

- Graph $G$ with 3 nodes.
- 2 edges from node 1 to node 2 labeled by $a$.
- 1 edge from node 2 to node 3 labeled by $b$.

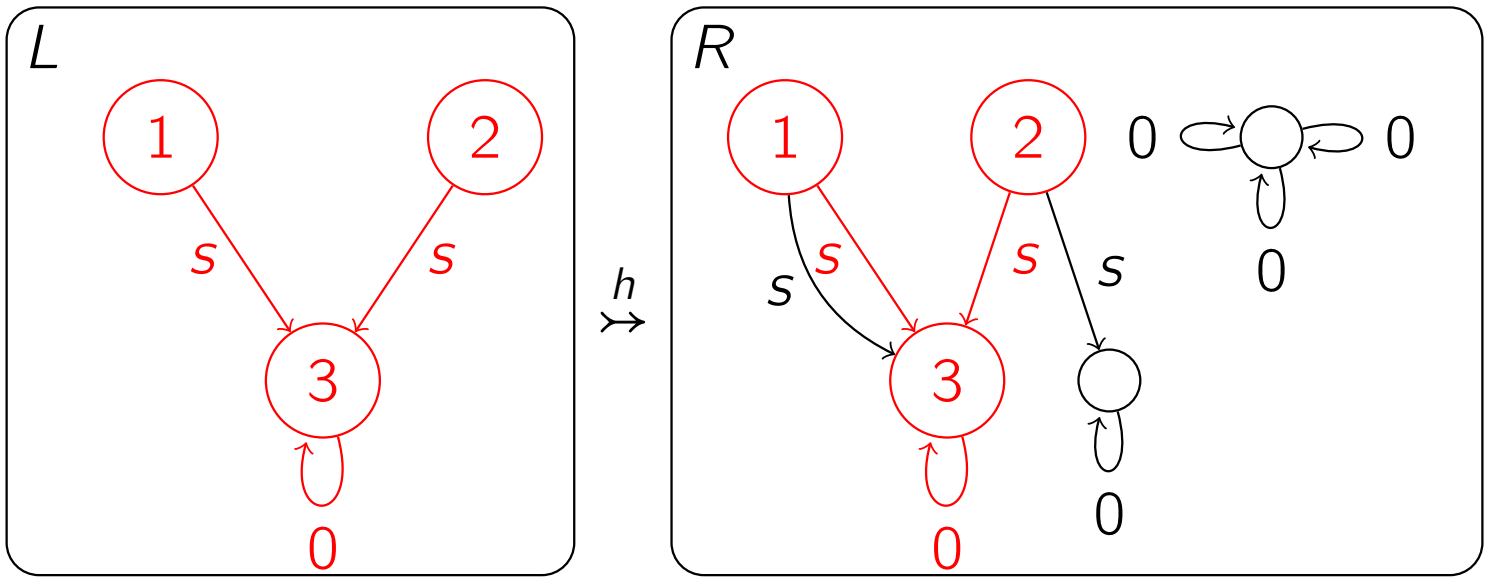
Notation:




- Graphs are contained in boxes
- Graph name $G$ is placed in the top left corner
- Numbers in nodes are identifiers, ommitted when not relevant

# Inclusion (morphism)

Morphisms: structure-preserving functions between graphs
Isomorphisms: bijective morphisms
**Inclusions**: morphisms that are indentity functions
Example in a viasual notion:

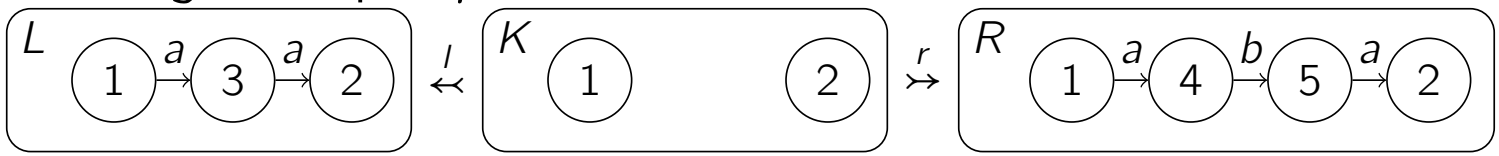



- $\overset{h}{\rightarrowtail}$ between the boxes marques an inclusion named $h$.

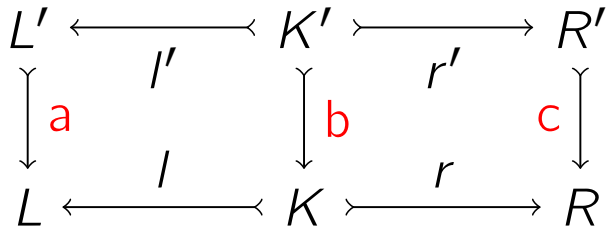
# Graph rewriting rule

**Rules** $\varphi = (L \overset{l}{\hookleftarrow} K \overset{r}{\hookrightarrow} R)$ consist of inclusions $l$ and $r$.

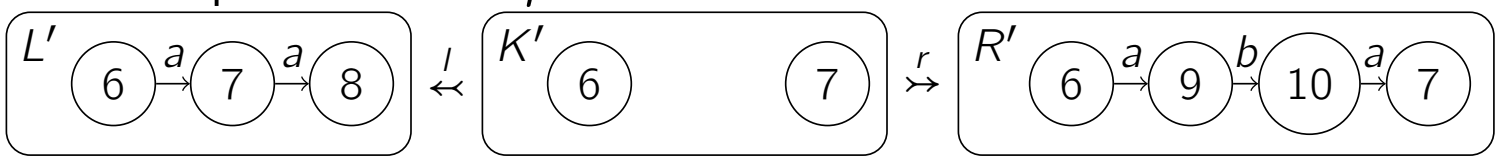
- Running example $\varphi$:




Rule $\varphi' = (L' \overset{l'}{\hookleftarrow} K' \overset{r'}{\hookrightarrow} R')$ and $\varphi$ are equivalent if there are isomorphisms $a, b, c$ such that $a \circ l' = l \circ b$ and $c \circ r' = r \circ b$:

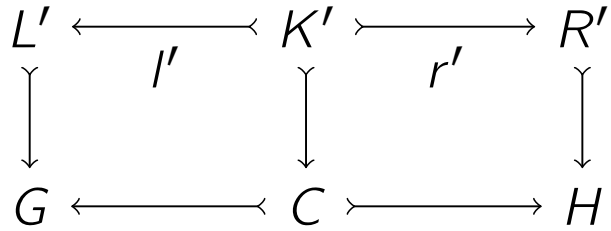



A rule equivalent to $\varphi$:

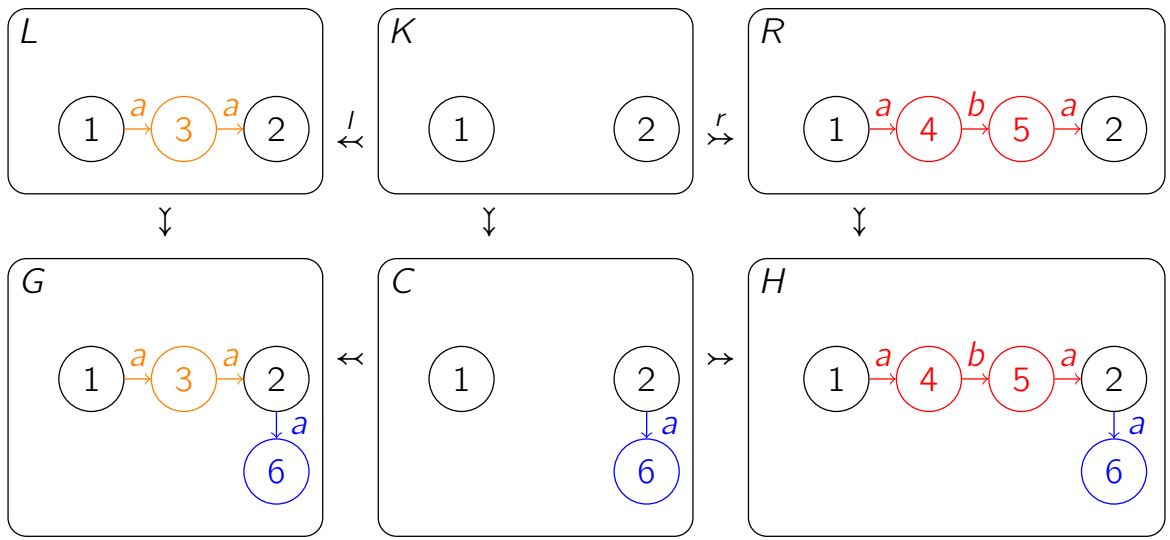


# Graph Rewriting

Rewriting steps $G \Rightarrow_\varphi H$ using rule $\varphi$ are commutative diagrams with an equivalent rule $L' \overset{l'}{\hookleftarrow} K' \overset{r'}{\hookrightarrow} R'$ where all morphisms are inclusions:

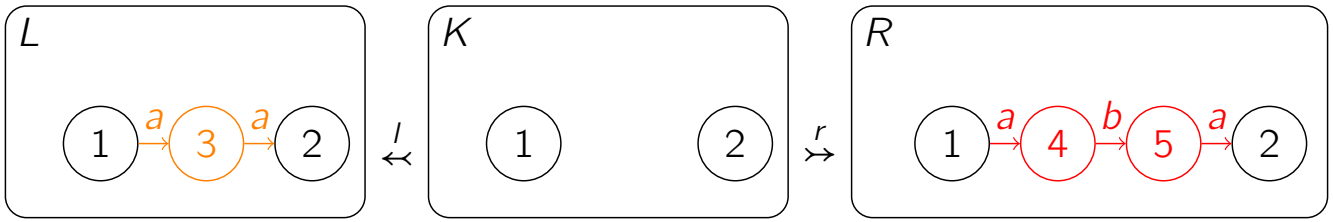



A rewriting step using $\varphi$:

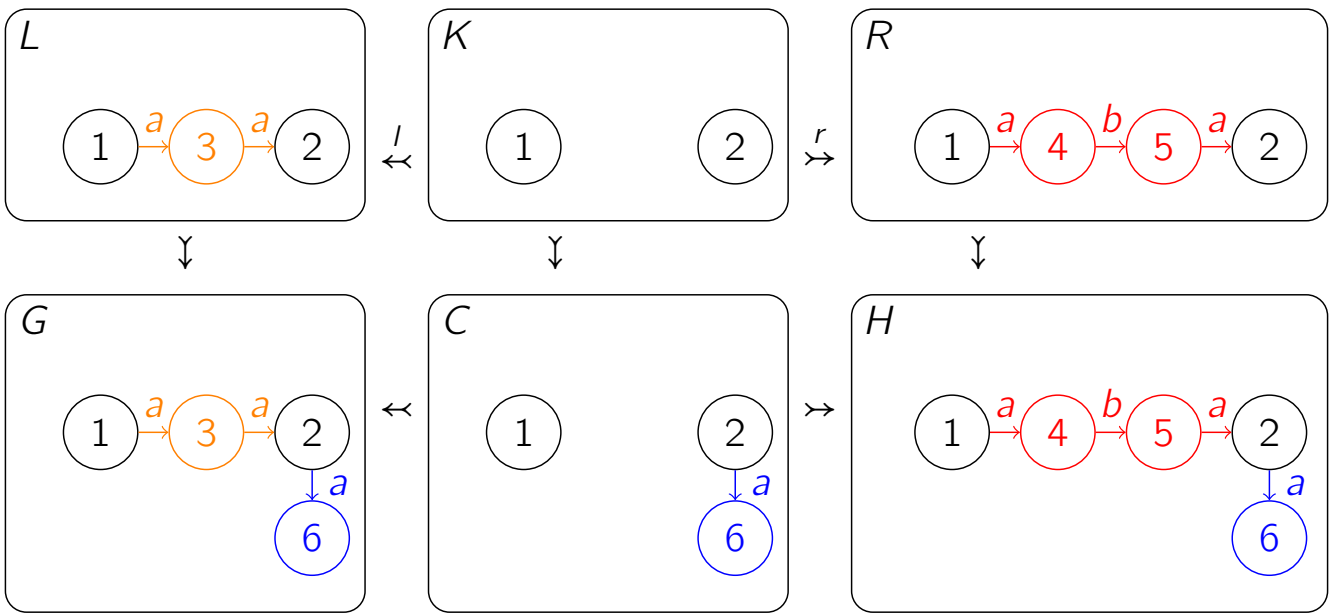


# A rewriting step of the running rule (to do : delete)

Rewriting rule $\varphi$:




A rewriting step $G \Rightarrow_\varphi H$:



# Termination of DPO Graph Rewriting Systems

- $\mathcal{R}$ : a set of rules
- No graph $G_0$ can be rewritten forever:

$$G_0 \underset{\mathcal{R}}{\Rightarrow} G_1 \underset{\mathcal{R}}{\Rightarrow} G_2 \underset{\mathcal{R}}{\Rightarrow} \cdots$$

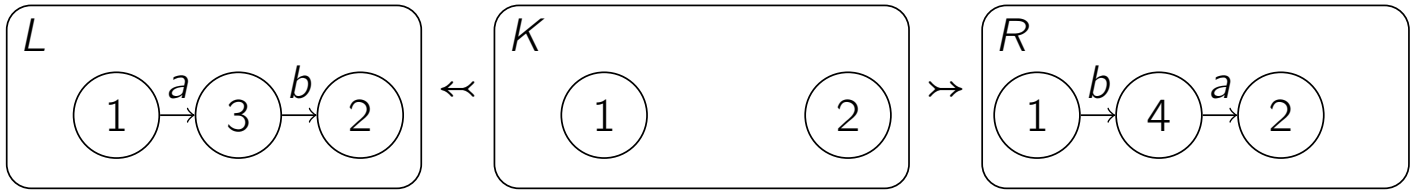
  when using the non-deterministic strategy

  "apply rules as long as possible"
- Aligns with the standard notion of program termination:

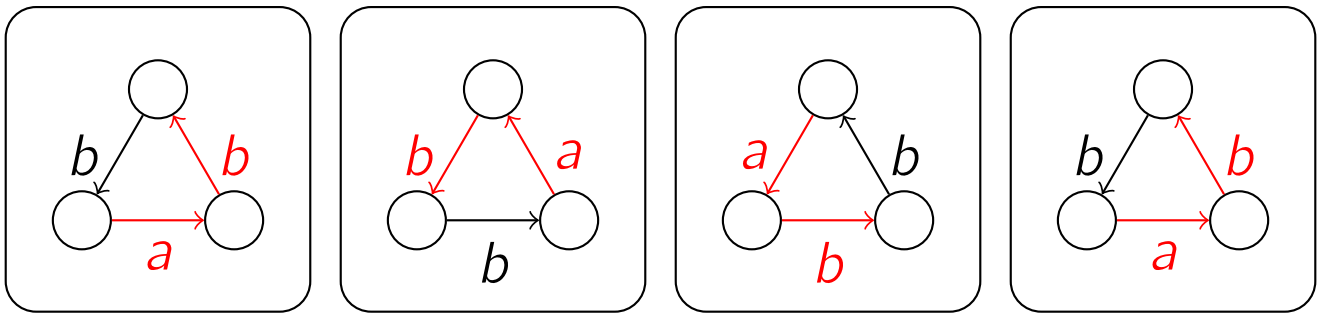
  "every execution (on any input) eventually halts."
- Undecidable in general

# One-rule examples

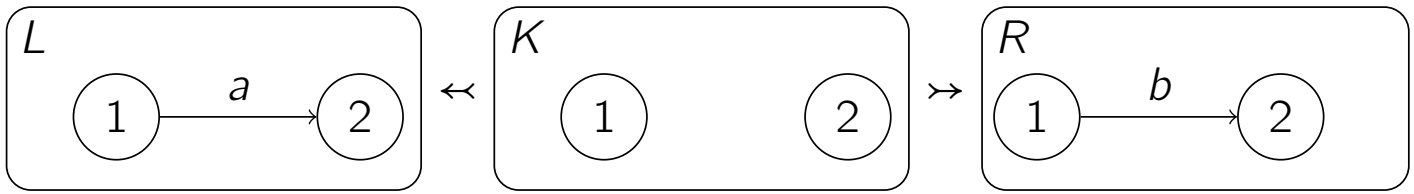
Rule $a$ :



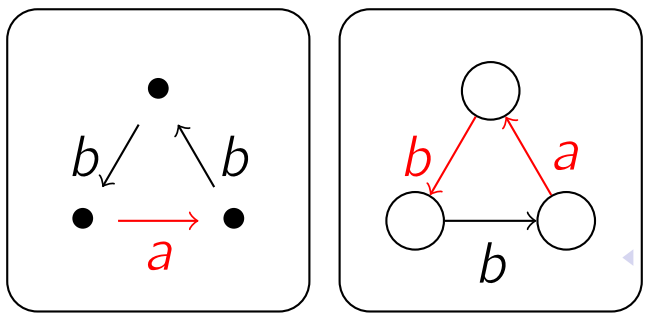

Looping:




Rule $b$:




Terminating: the number of edges labeled by "a" strictly decreases.



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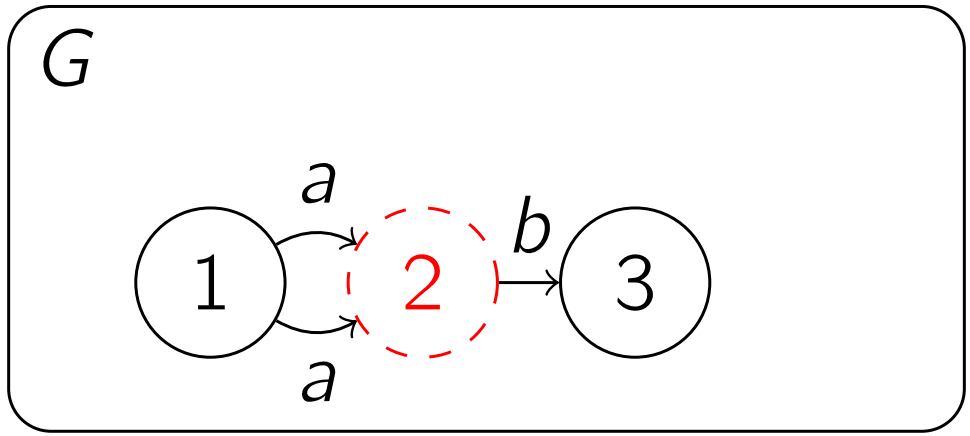
# Pre-graphs

Pre-graphs are graphs with missing nodes and dangling edges.
Example:

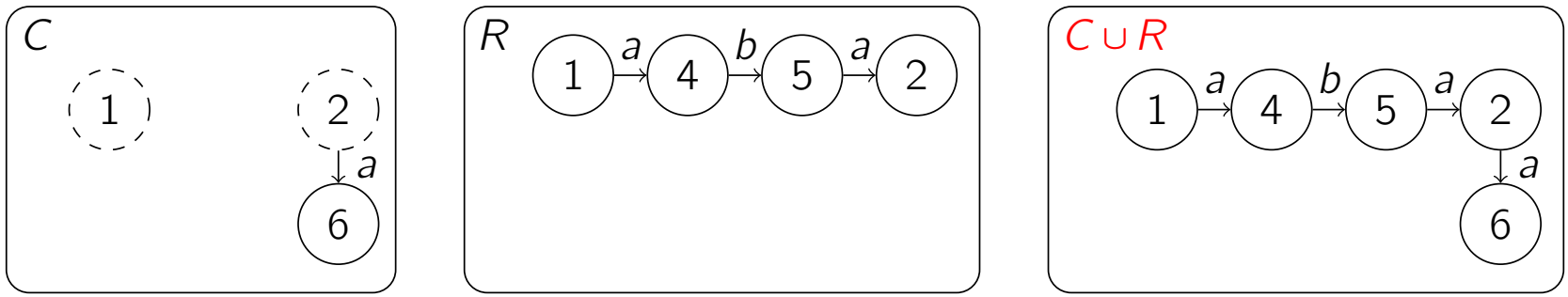



The pregraph $G$ has

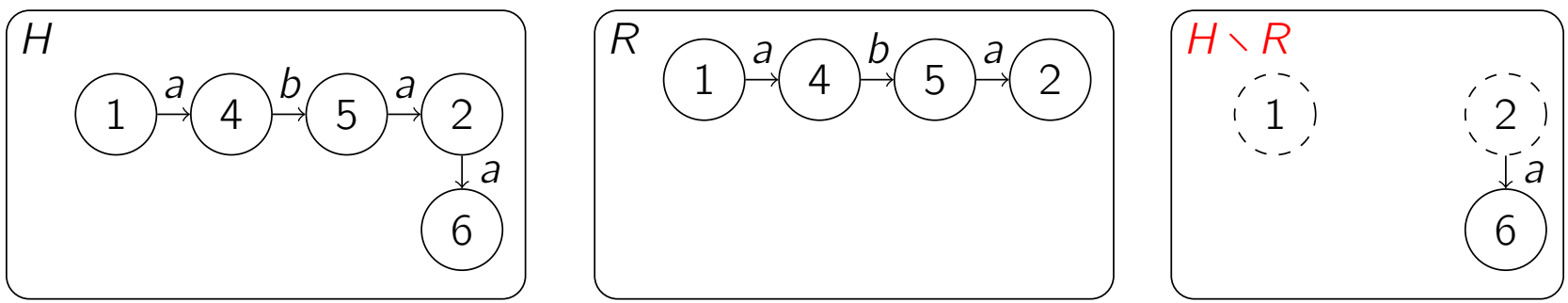
- 2 existing nodes,
- 1 missing node,
- 3 dangling edges.

# Pre-graph operations

Union of two pre-graphs $C \subseteq G$ and $R \subseteq G$, denoted $C \cup R$:




Relative complement of $R$ in $H$ where $R \subseteq H$, denoted $H \setminus R$:



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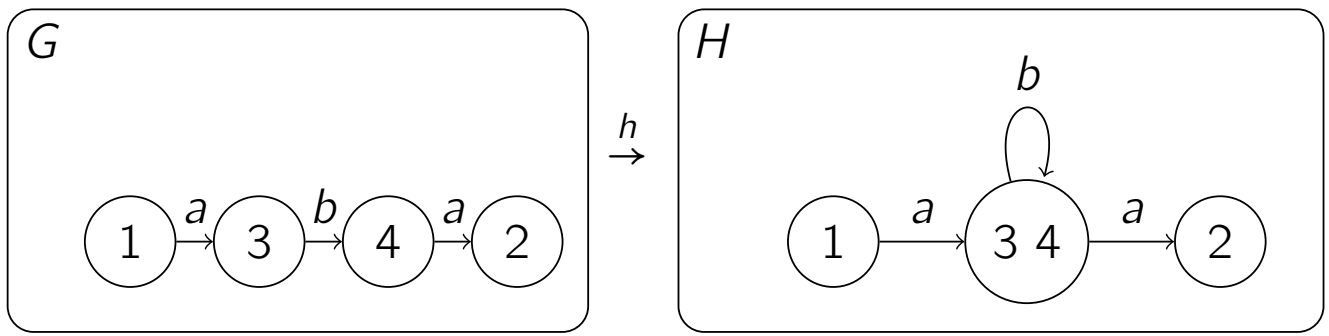
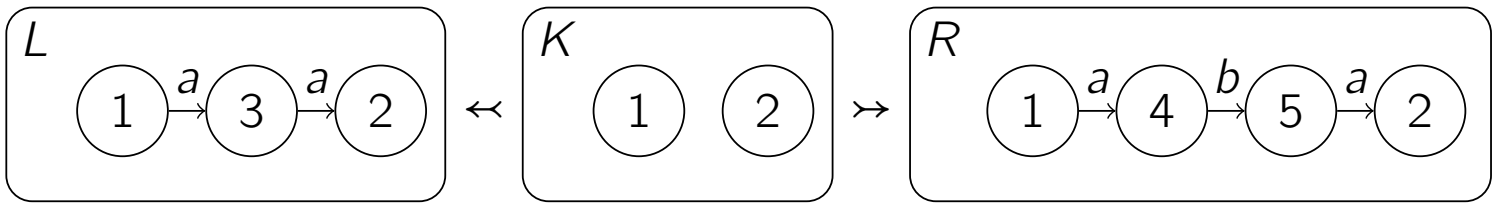
# Graphs and Graph Morphisms

- **Graph morphisms**: structure-preserving mappings.
  Ex:

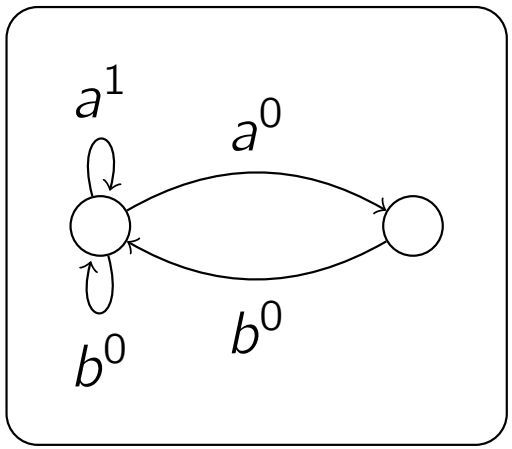


# Type graph method with weighted type graphs over natural numbers with an example

- Rule $\alpha$:

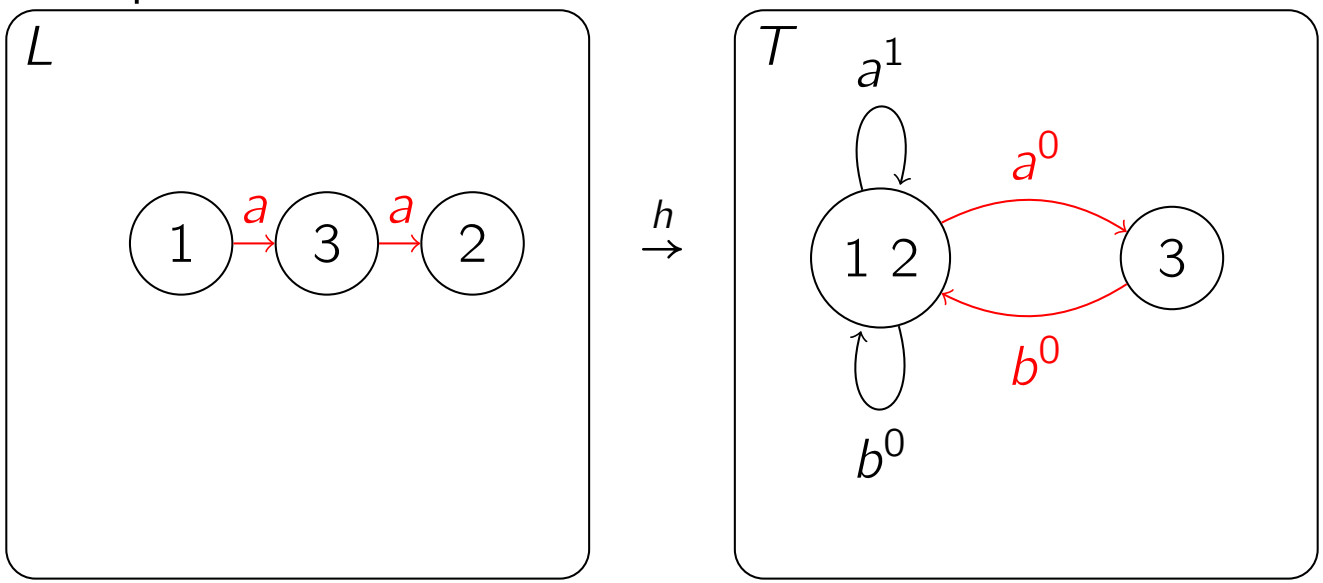



- Weighted type graph $T$ over natural numbers:



# Morphism Weight

- The weight of $h : L \to T$ is the sum of weights of all edges in $\mathsf{Im}(h)$.
- Example:

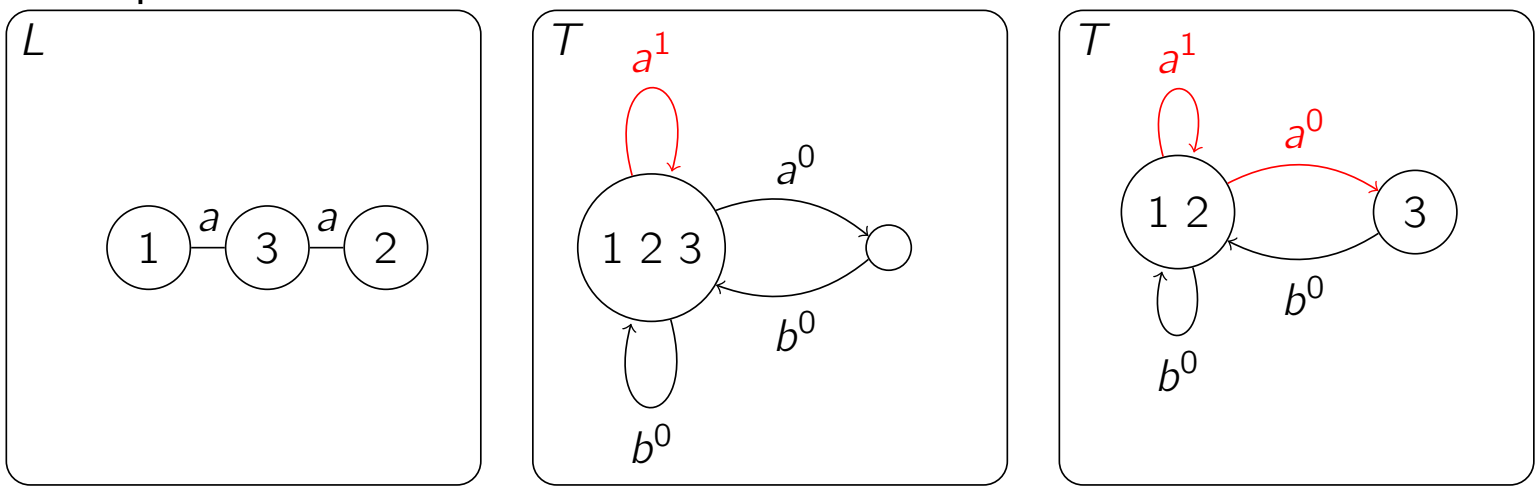



$w_T(h) = 0 + 0 = 0$

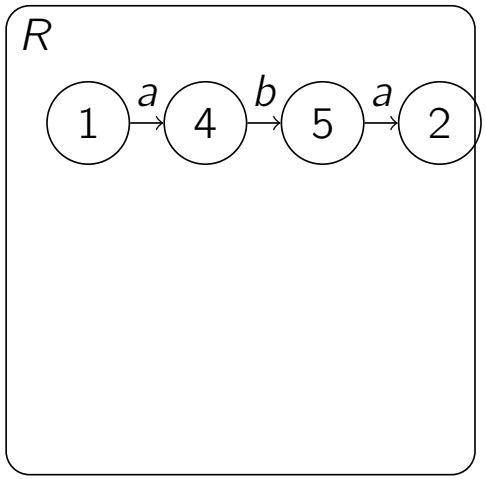
# Graph Weight

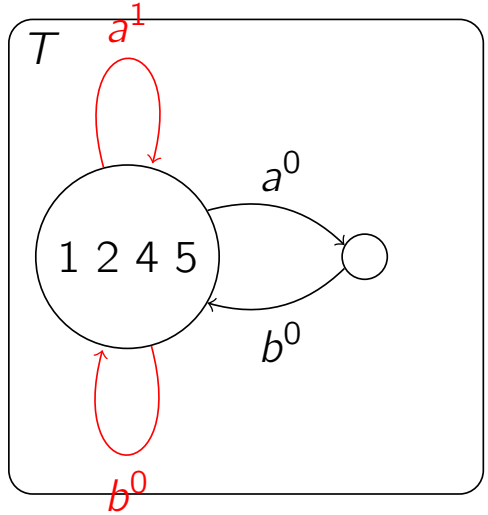
$w_T(L)$: the minimum weight $w_T(h)$ of all morphisms $h : L \to T$

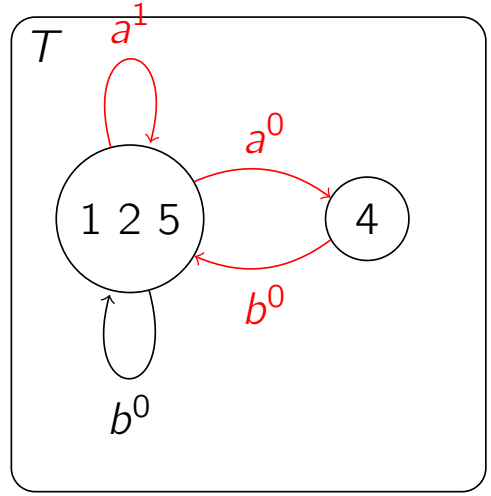
Example :

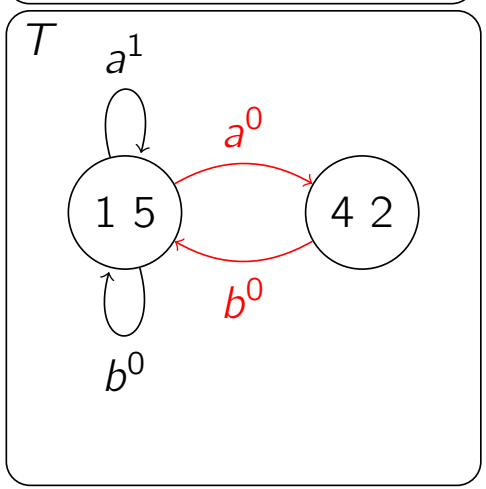



$w_T(L) = \min\{1 + 1, 1 + 0\} = 1$



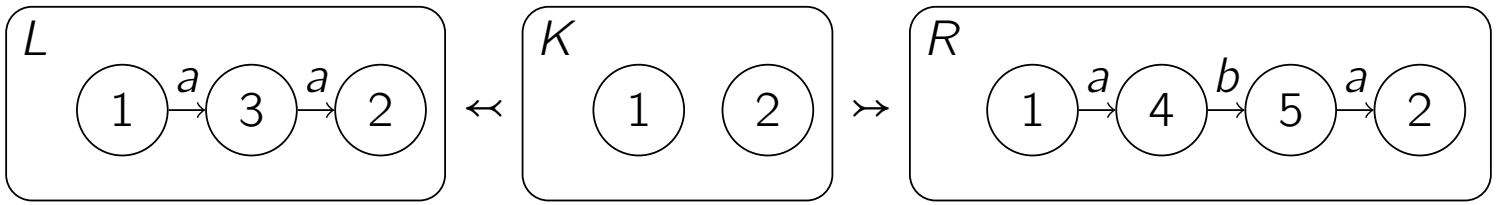




The weight of $R$ is

$$\min\{1+0+1, 0+0+1, 0+0+0\} = 0$$

Since for all $G \Rightarrow_\alpha H$, we have $w_T(G) \in \mathbb{N}$, it remains to show that every rewriting step strictly decreases the weight.
Not every weighted graph can be a weighted type graph It guarantees: for all $G \Rightarrow_\alpha H$, the weight of $G$ is defined and $w_T(G) \in \mathbb{N}$

# Termination Condition




For every morphism $t_K : K \to T$, we define

- $S(t_k, L)$ : the sum of the weights of the morphisms $t_L$ that extend $t_K$
- $S(t_k, R)$ : the sum of the weights of the morphisms $t_R$ that extend $t_K$

By [**endrullis2024generalized˙arxiv˙v2**], every rewriting step strictly decreases the weight if

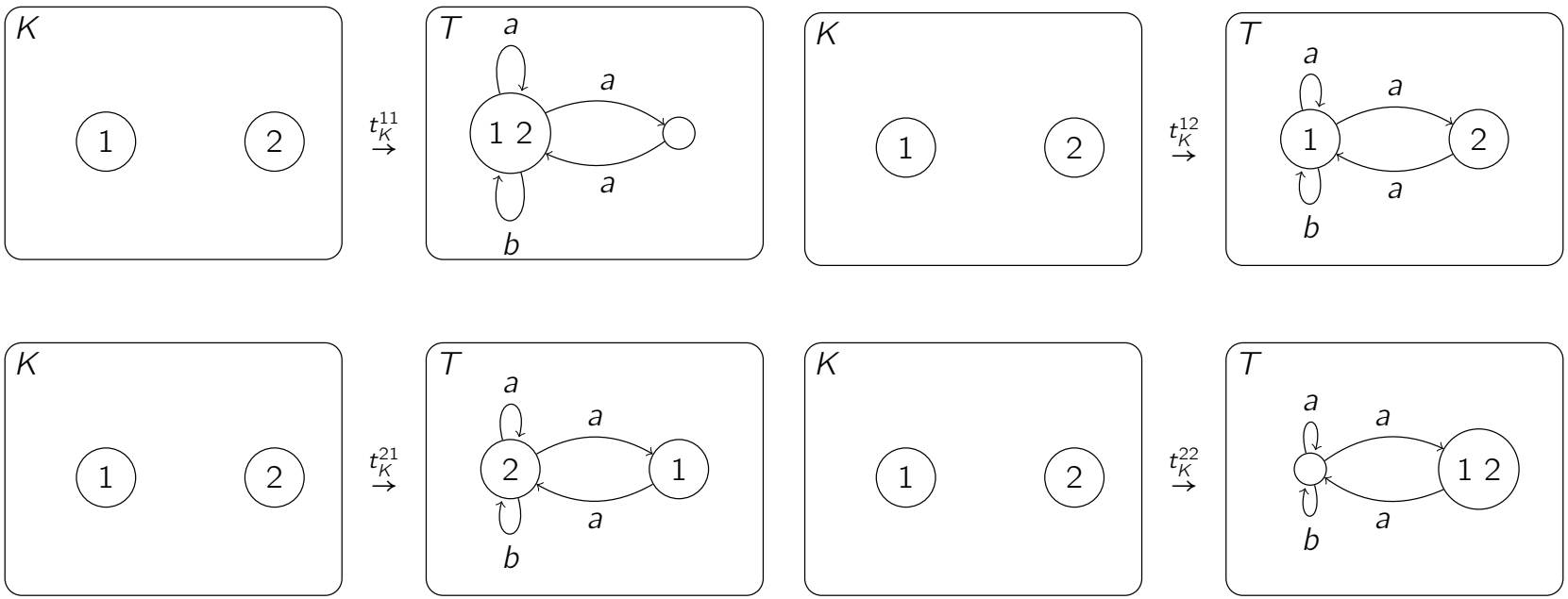
- for all $t_K : K \to T$,

$$S(t_K, L) > S(t_K, R)$$

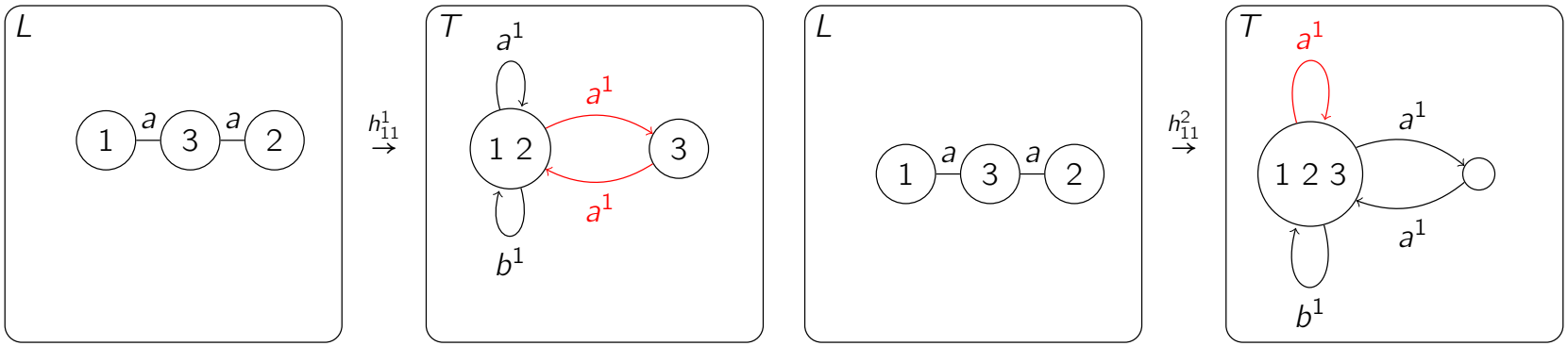
.

# Example

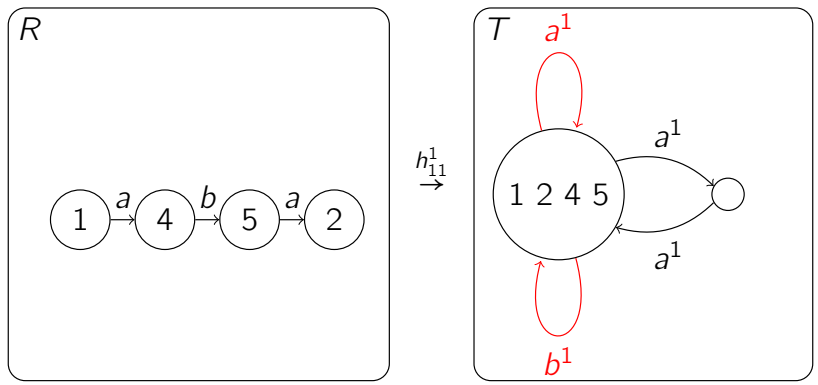
There are $t_K^{11}, t_K^{12}, t_K^{21}, t_K^{22} : K \to T$ as depicted below:



# Detail for $t_K^{11}$



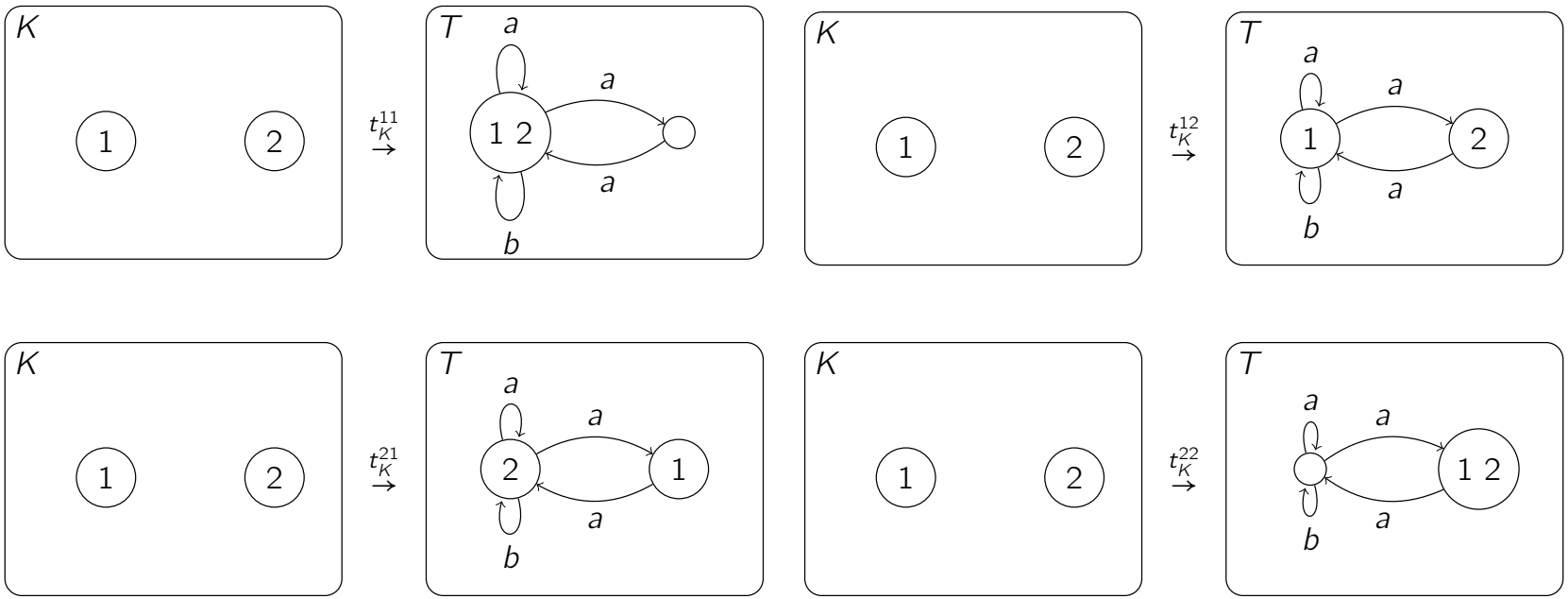

We have $S(t_k^{11}, L) = w_{\mathcal{T}}(h_{11}^1) + w_{\mathcal{T}}(h_{11}^2) = (1 * 1) + (1 * 1) = 2$




we have $S(t_k^{11}, R) = w_{\mathcal{T}}(h_{11}^3) = 1 * 1 * 1 = 1$.
Therefore, for $t_K^{11}$, we have: $S(t_k^{11}, L) = 2 > 1 = S(t_k^{11}, R)$.

# Example

There are $t_K^{11}, t_K^{12}, t_K^{21}, t_K^{22} : K \to T$ as depicted below:




- $t_K^{11} : 2 > 1$
- $t_K^{12} : 1 \geq 1$
- $t_K^{21} : 1 \geq 1$
- $t_K^{22} : 1 \geq 1$

Therefore, our Running Example is terminating by the weighted type graph $T$ over natural numbers.

# Type Graph Method with Weighted Type Graphs over Well-founded Semirings

The existence of suitable weighted type graphs is undecidable in general.

# Searching for Weighted Type Graphs over Natural Numbers

User-specified parameters:

- $k$ nodes
- edge weights in $\{0, 1, \ldots, n\}$

Assumption:

- no parallel edges of the same label

The problem amounts to checking the satisfiability of an existential Presburger arithmetic formula:

- $k^2|\Sigma|$ binary variables
- $k^2|\Sigma|$ integer variables

Challenge:

- there are $2^{k^2|\Sigma|} \cdot n^{k^2|\Sigma|}$ possible assignments of weights
- $k$ and $n$ unknown à priori

# Solution: using real numbers instead of natural numbers.

Every rewriting step strictly decreases the weight if

- for every $t_K : K \to T$,

$$S(t_K, L) > S(t_K, R)$$

.

- there is $\delta > 0$ such that for some $t_K : K \to T$,

$$S(t_K, L) > S(t_K, R) + \delta$$

.

# Searching for Weighted Type Graphs over Real Numbers

User-specified parameters:

- $k$ nodes
- ~~edge weights in $\{0, 1, \ldots, n\}$~~

Assumption:

- no parallel edges of the same label

The problem amounts to checking the satisfiability of an existential Presburger arithmetic formula:

- $k^2|\Sigma|$ binary variables
- $k^2|\Sigma|$ real variables

Challenge:

- ~~there are $2^{k^2|\Sigma|} \cdot n^{k^2|\Sigma|}$ possible assignments of weights~~
- there are $2^{k^2|\Sigma|}$ linear programs which have polynomial-time average-case complexity

# Implementation and Experimental Results

Implemented in the tool LyonParallel

- supports natural and real numbers
- searches with natural and real numbers in parallel and cooperates between them

Tested on examples from previous work:

- no need of user-specified upper bound on weights
- Acceleration

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