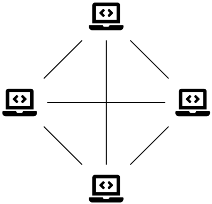
Automated Termination Proving: Contributions to Graph Rewriting via Extended Weighted Type Graphs and Morphism Counting

Qi QIU

LIRIS, UMR 5205 CNRS
Université Claude Bernard Lyon 1, France
Supervisor: Xavier URBAIN

Motivation & Goal

Distributed systems:



Failures can be catastrophic: 

Ensuring correctness is difficult.

- Needham-Schroeder protocol shown to be vulnerable 17 years after publication

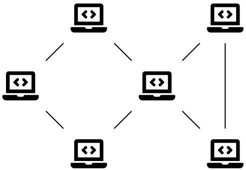
This thesis: automated verification

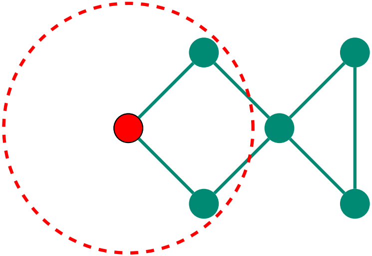
- Minimal user effort
- No expertise required
- Mathematically rigorous

Graph Transformation: Intuition

Modeling of distributed systems

System configurations: graphs



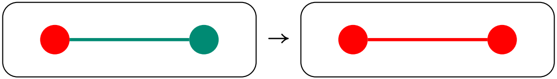


Algorithm behavior:

graph transformation based on local knowledge

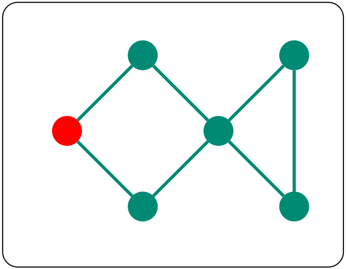
Graph Transformation: Spanning-tree Construction

Graph transformation rule:



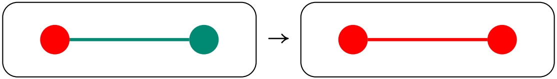
Replace the left-hand side with the right-hand side

Application of the rule while possible:



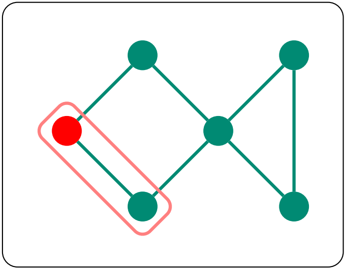
Graph Transformation: Spanning-tree Construction

Graph transformation rule:



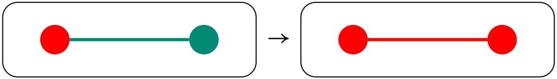
Replace the left-hand side with the right-hand side

Application of the rule while possible:



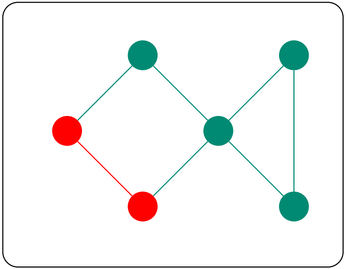
Graph Transformation: Spanning-tree Construction

Graph transformation rule:



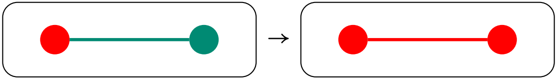
Replace the left-hand side with the right-hand side

Application of the rule while possible:



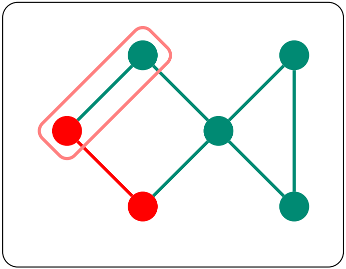
Graph Transformation: Spanning-tree Construction

Graph transformation rule:



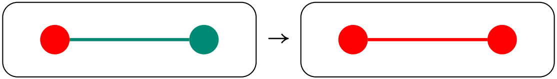
Replace the left-hand side with the right-hand side

Application of the rule while possible:



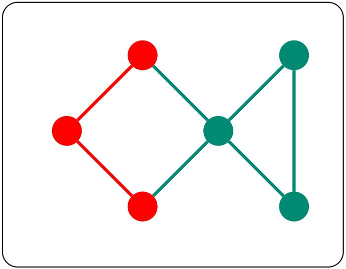
Graph Transformation: Spanning-tree Construction

Graph transformation rule:



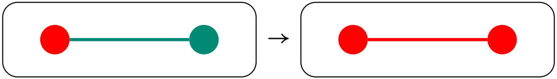
Replace the left-hand side with the right-hand side

Application of the rule while possible:



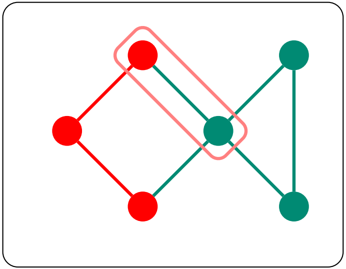
Graph Transformation: Spanning-tree Construction

Graph transformation rule:



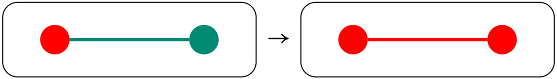
Replace the left-hand side with the right-hand side

Application of the rule while possible:



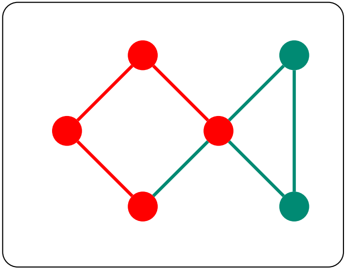
Graph Transformation: Spanning-tree Construction

Graph transformation rule:



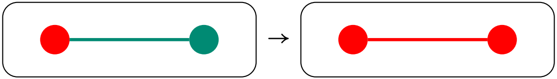
Replace the left-hand side with the right-hand side

Application of the rule while possible:



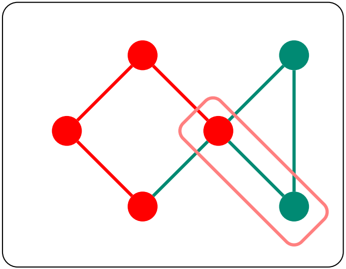
Graph Transformation: Spanning-tree Construction

Graph transformation rule:



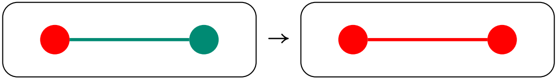
Replace the left-hand side with the right-hand side

Application of the rule while possible:



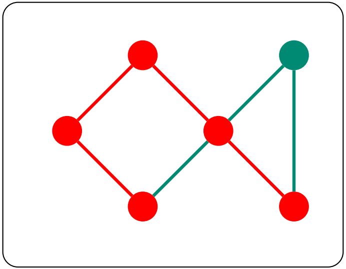
Graph Transformation: Spanning-tree Construction

Graph transformation rule:



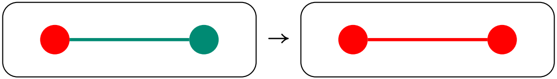
Replace the left-hand side with the right-hand side

Application of the rule while possible:



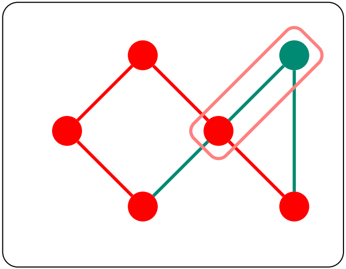
Graph Transformation: Spanning-tree Construction

Graph transformation rule:



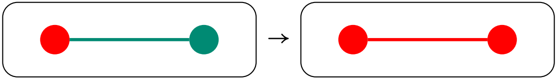
Replace the left-hand side with the right-hand side

Application of the rule while possible:



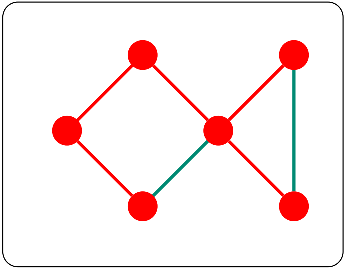
Graph Transformation: Spanning-tree Construction

Graph transformation rule:



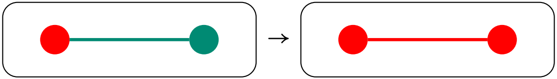
Replace the left-hand side with the right-hand side

Application of the rule while possible:



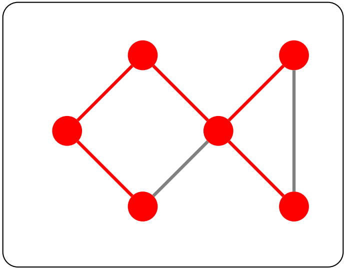
Graph Transformation: Spanning-tree Construction

Graph transformation rule:



Replace the left-hand side with the right-hand side

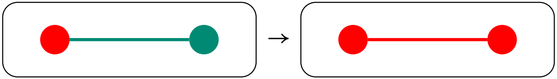
Application of the rule while possible:



The result is a spanning tree.

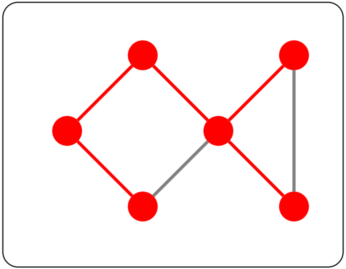
Graph Transformation: Spanning-tree Construction

Graph transformation rule:



Replace the left-hand side with the right-hand side

Application of the rule while possible:



The result is a spanning tree.

Does a result always exist?

Graph Transformation: Termination

- No graph G_0 can be transformed forever

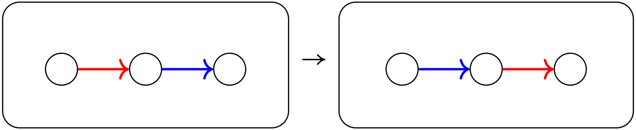
$$G_0 \Rightarrow G_1 \Rightarrow \cdots$$

- Aligns with the notion of program termination:

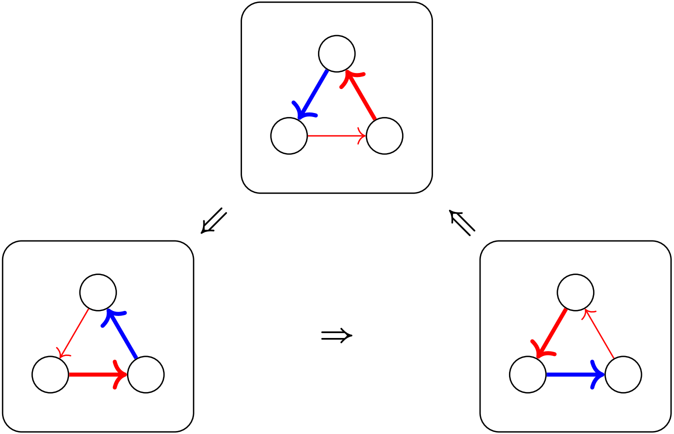
  "every execution (on any input) halts."

- Undecidable in general
  - Automated techniques
    - Power: incomplete
    - Usability: rely on user-provided parameters

Graph Transformation: A Non-Terminating Rule



Loop:

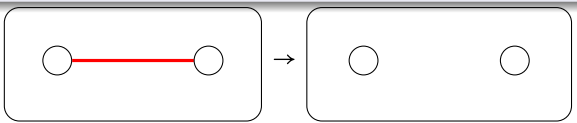


Graph Transformation: Termination by interpretations

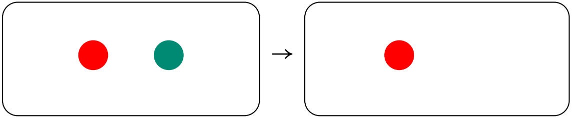
Interpret graphs as natural numbers.

Show that each transformation step strictly decreases the value.

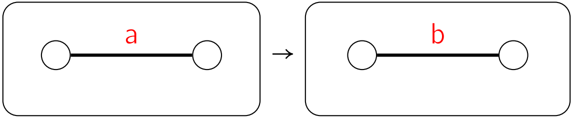
Number of edges:



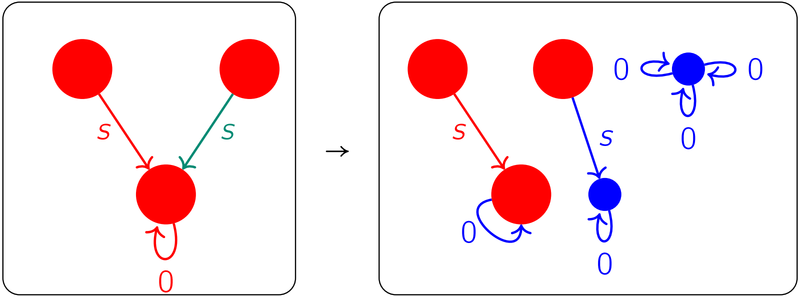
Number of nodes:



Number of edges labeled by *a*:



Limitations



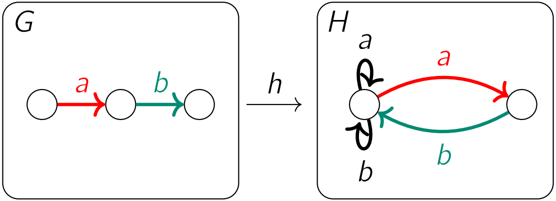
- The number of nodes/edges/labels does not decrease
- Can its termination be proved using interpretations?

Formal Definition of Graph Rewriting

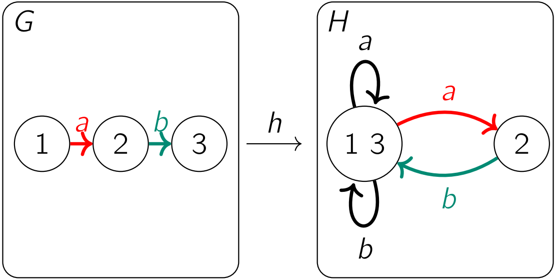
Toward Greater Usability

Toward Greater Power

Graph Morphisms: Structure-Preserving Functions

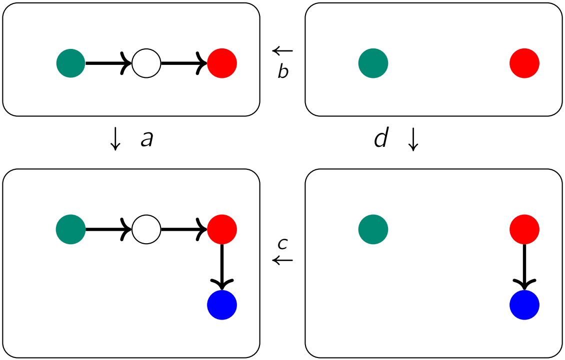


Colors indicate edge correspondence.



Numbers indicate node correspondence.

Commutative Diagram



$$
\begin{tikzcd}
L \arrow[d, "a"] & K \arrow[l, "b"'] \arrow[d, "d"] \\
G & C \arrow[l, "c"']
\end{tikzcd}
$$

commutes if $a \circ b = c \circ d$.

Pushouts: Gluing Graphs Along an Interface

The **pushout** of (α, β) is (β', α') with

- $\square$ABDC commutes,

$$\begin{tikzcd} A \arrow[r, "\beta"] \arrow[d, "\alpha"'] & C \arrow[d, "\alpha'"] \\ B \arrow[r, "\beta'"] & D \end{tikzcd}$$

D: pushout object

Pushouts: Gluing Graphs Along an Interface

The **pushout** of (α, β) is (β', α') with

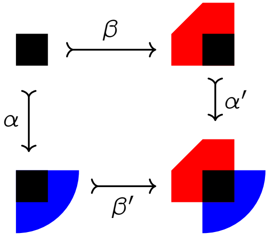
- $\square$ABDC commutes,
- universality: for all (γ, γ'),if $\square ABEC$ commutes, then there is a unique δ such that $\triangle BDE$ and $\triangle CDE$ both commute.

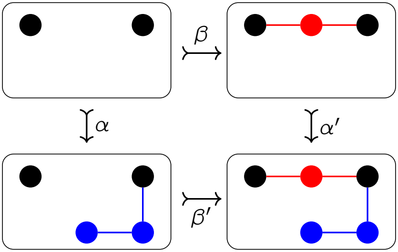
$$
\begin{tikzcd}
A \arrow[r, "\beta"] \arrow[d, "\alpha"'] & C \arrow[d, "\alpha'"'] \arrow[ddr, bend left, "\gamma'"] & \\
B \arrow[r, "\beta'"] \arrow[drr, bend right, "\gamma"'] & D \arrow[dr, dashed, "\delta"] & \\
 & & E
\end{tikzcd}
$$

D: pushout object

Pushouts: Gluing Graphs Along an Interface

$$
\begin{tikzcd}
A \arrow[r, "\beta"] \arrow[d, "\alpha"'] & C \arrow[d, "\alpha'"] \\
B \arrow[r, "\beta'"] & D
\end{tikzcd}
$$





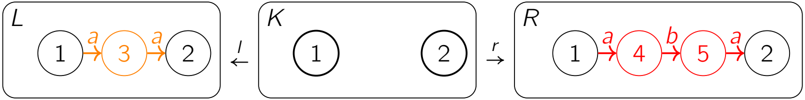
Graph Rewriting with Double-Pushout (DPO)

First algebraic approach

One of the most studied approaches

$$L \xleftarrow{l} K \xrightarrow{r} R$$

Rewriting rule with interface K



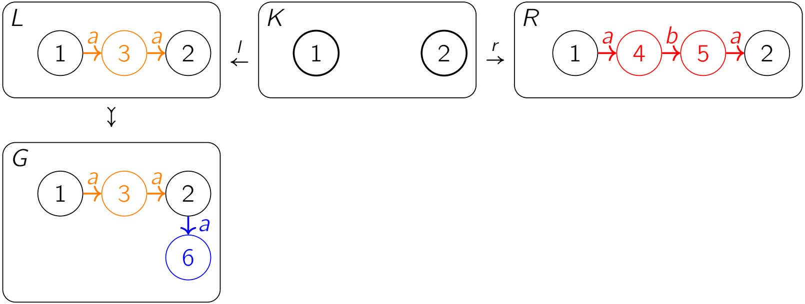
Graph Rewriting with Double-Pushout (DPO)

First algebraic approach

One of the most studied approaches

$$\begin{tikzcd} L \arrow[d, tail] & K \arrow[l, "l"'] \arrow[r, "r"] & R \\ G & & \end{tikzcd}$$

Rewriting rule with interface K



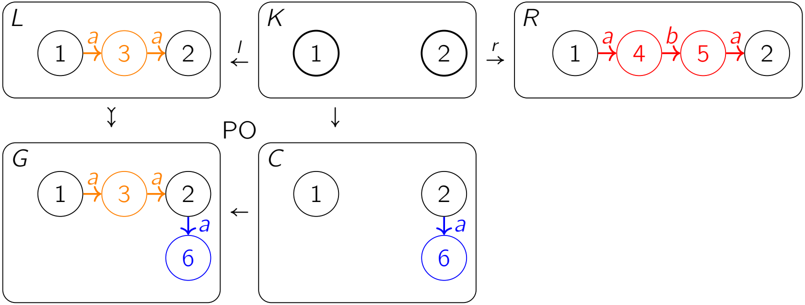
Graph Rewriting with Double-Pushout (DPO)

First algebraic approach

One of the most studied approaches

$$\begin{tikzcd}
L \arrow[d, tail] \arrow[dr, phantom, "\mathrm{PO}"] & K \arrow[l, "l"'] \arrow[r, "r"] \arrow[d] & R \\
G & C \arrow[l] &
\end{tikzcd}$$

Rewriting rule with interface K



Graph Rewriting with Double-Pushout (DPO)

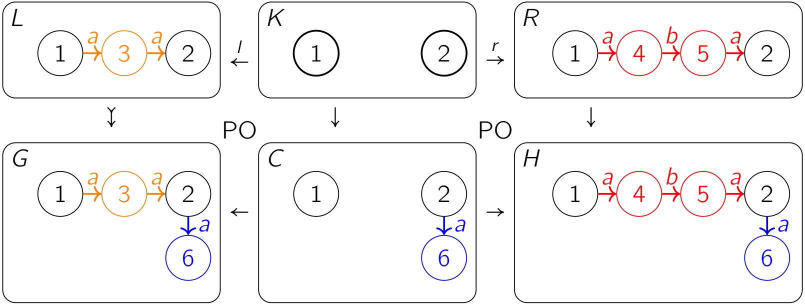
First algebraic approach

One of the most studied approaches

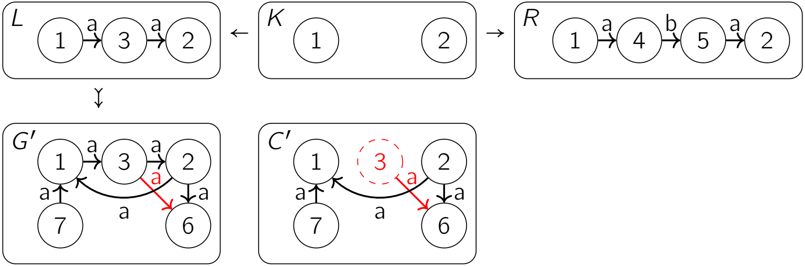
$$\begin{tikzcd} L \arrow[d, tail] \arrow[dr, phantom, "\mathrm{PO}"] & K \arrow[l, "l"'] \arrow[r, "r"] \arrow[d] \arrow[dr, phantom, "\mathrm{PO}"] & R \arrow[d] \\ G & C \arrow[l] \arrow[r] & H \end{tikzcd}$$

Rewriting rule with interface K

rewriting step $G \Rightarrow H$



An Invalid Rewriting Step



No dangling edges should be created.

Weighted Type Graph Method

Termination by interpretation

Parameter: an object T in the category, called the type graph.

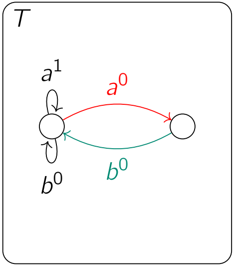
Terminology: every graph is "typed" by morphisms to T

Interpretation:

$$
\begin{aligned}
G &\rightsquigarrow \mathrm{morphisms}(G, T) \\
&\rightsquigarrow \mathrm{weight}(\mathrm{morphisms}(G, T)) \\
&\rightsquigarrow \mathrm{aggregator}(\mathrm{weight}(\mathrm{morphisms}(G, T))) \in \mathbb{N}
\end{aligned}
$$

How to choose the type graph T?
How to define the morphism weight?
How to aggregate the morphism weights?

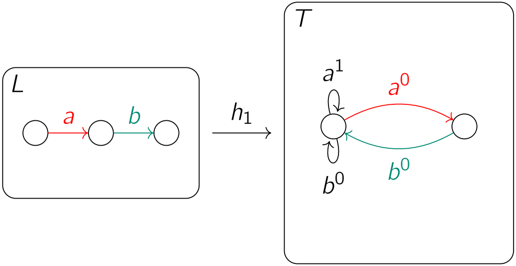
Type Graph with Weights on Edges



Morphism Weight

The weight of a morphism $h : G \to T$ is

$$\sum_{e \in \mathsf{Edge}(G)} \mathrm{weight}(h(e))$$

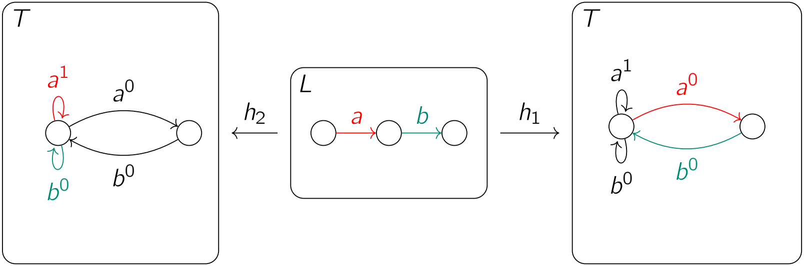


$$\mathrm{weight}_T(h_1) = 0 + 0 = 0$$

Graph Weight

The weight of a graph L is

$$\min\{h : L \to T \mid \mathrm{weight}_T(h)\}$$



$$\mathrm{weight}_T(h_2) = 1 + 0 = 1 \qquad \mathrm{weight}_T(h_1) = 0 + 0 = 0$$

$$\mathrm{weight}_T(L) = \min\{1, 0\} = 0$$

Termination Criterion [Bruggink *et al.*, 2014]

$$\begin{tikzcd} L \arrow[dr, dashed, "t_L"'] & K \arrow[l, "l"'] \arrow[r, "r"] \arrow[d, "t_K"] & R \arrow[dl, dashed, "t_R"] \\ & T & \end{tikzcd}$$

A rule terminates if there is a type graph T such that for all t_K, if there is t_L such that $\triangle KLT$ commutes, then

$$\min\{\mathrm{weight}_T(t_L) \mid t_L.\ \triangle\ KLT \text{ commutes}\}$$
$$> \min\{\mathrm{weight}_T(t_R) \mid t_R.\ \triangle\ KRT \text{ commutes}\}$$

How to find a suitable weighted type graph?

Searching for Weighted Type Graphs over $\mathbb{N}$

User-specified parameters:

- k nodes
- maximum edge weight $n \in \mathbb{N}$

The problem amounts to checking the satisfiability of an existential Presburger arithmetic theory with:

- $k^2 m$ binary variables where m is the number of labels
- $k^2 m$ integer variables

Challenges:

- Usability: difficult to guess k and n
- Search space: $2^{k^2 m} \cdot n^{k^2 m}$ possible assignments of weights

Usability Improvement

Idea: Weights in $\mathbb{R}^+$

Additional constraint: there is $\delta > 0$ such that every rewriting step decreases the weight by at least δ.

Searching for Weighted Type Graphs over $\cancel{\mathbb{N}}$ $\mathbb{R}$

User-specified parameters:

- k nodes
- ~~edge weights in $\{0, 1, \ldots, n\}$~~

The problem amounts to checking the satisfiability of an ~~existential Presburger arithmetic theory~~ existential theory of the reals with binary variables:

- $k^2 m$ binary variables where m is the number of labels
- $k^2 m$ ~~integer~~ real variables

Challenge:

- impossible to guess k and ~~maximum weight n~~
- complexity: ~~$2^{k^2 m} \cdot n^{k^2 m}$ possible assignments of weights~~ $2^{k^2 m}$ linear programs which have polynomial-time average-case complexity.

Implementation$\rightsquigarrow$Experiments

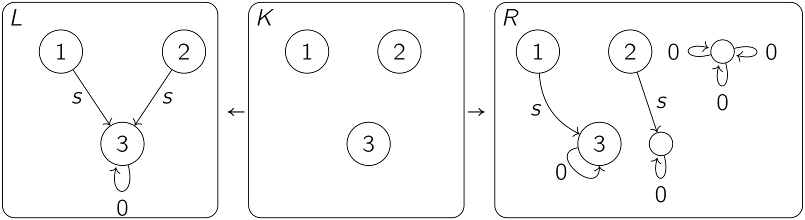
k from 1 to 4

For every k, n from 1 to 3 if over $\mathbb{N}$

	Configuration 1	Configuration 2	Configuration 3
[3, Example 6.3]			58%
[3, Example D.3]	48%		47%
[5, Example 3.8]	37%	36%	
[4, Example 4]		26%	timeout
[4, Example 5]	1%	1%	38%
[2, Example 4]	1%	1%	46%
[2, Example 5]			timeout
[2, Example 6]			timeout
[1, Example 1]	48%		47%
[1, Example 4]	46%	47%	49%
[1, Example 5]	24%	22%	timeout

Usability Improved ✓

A Limitation of the Weighted Type Graph Method



Type graph fails: existence of surjections from R to L

All existing automated methods fail.

Remark: the number of occurrences of $\bigcirc \xrightarrow{s} \bigcirc \xleftarrow{s} \bigcirc$ strictly decreases.

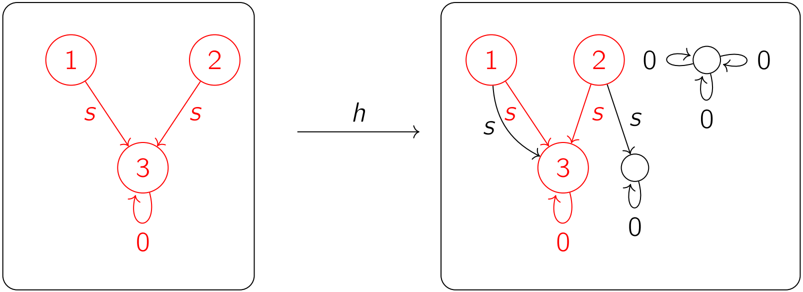
Capability Improvement: Morphism Counting

Parameter: graph X

Interpretation:

$$G \rightsquigarrow |\mathrm{morphisms}(X, G)| \in \mathbb{N}$$

Inclusions

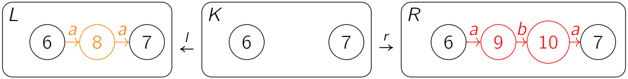


Subgraph

Graph Rewriting Systems

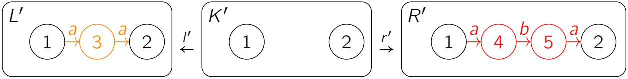
A rewriting rules consists of two inclusions.

$\varphi=$

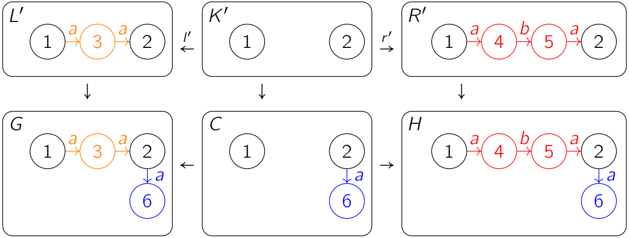


An equivalent rewriting rule expresses the same transformation.

$\varphi'=$

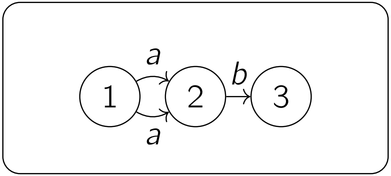


A rewriting step with φ is defined by a DPO diagram with inclusions and φ'.

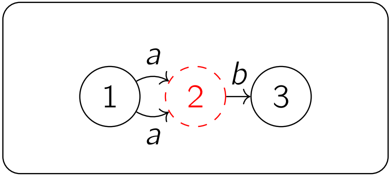


Pre-Graphs

Graph:

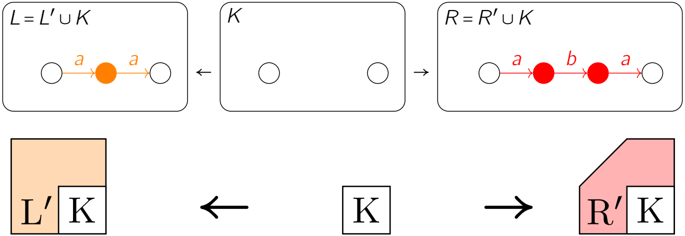


Pre-graphs obtained by removing node 2:

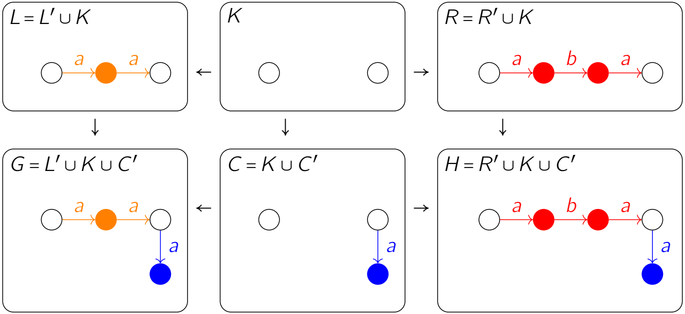


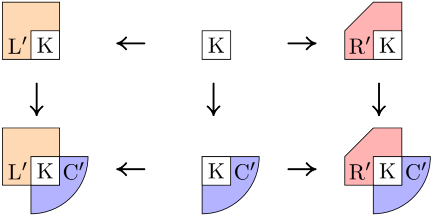
Dangling edges

Decomposition of Graphs in Rewriting Rules



Decomposition of Graphs in Rewriting Steps

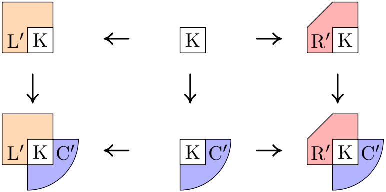




This coloring provides a classification of morphisms in rewriting steps by image node colors.

X-occurrences by Image Node Colors

> An X-occurrence is an injective morphism from X.

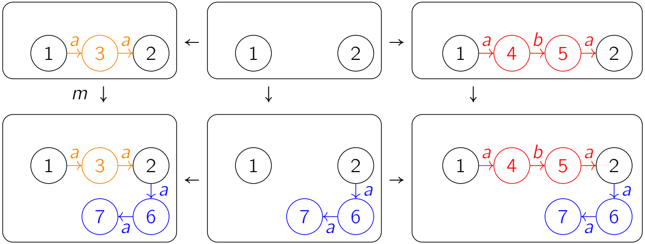


X-occurrence are classified by the colors of their image nodes:

- white: only white;
- blue: only white and at least one blue;
- blue-and-red: at least one blue and at least one red
- etc.

Morphisms by Image Node Colors

Let X be the graph $\bigcirc \xrightarrow{a} \bigcirc \xrightarrow{a} \bigcirc$.

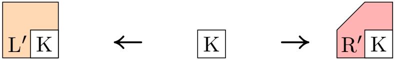


Blue X-occurrence: $2 \xrightarrow{a} 6 \xrightarrow{a} 7$

Red X-occurrences: none.

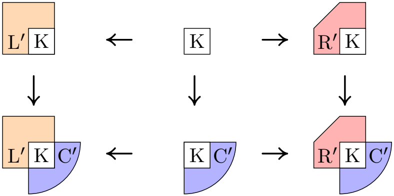
Blue-and-red X-occurrences: $5 \xrightarrow{a} 2 \xrightarrow{a} 6$

A New Sufficient Condition for Termination [Qiu]



terminates if

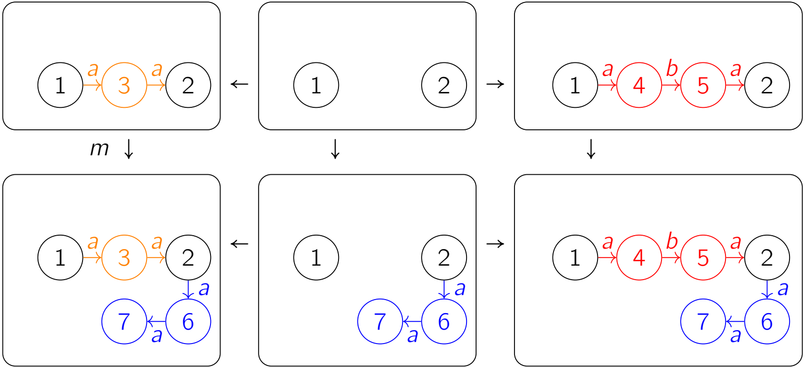
- it contains strictly more orange X-occurrences than red X-occurrences in the rule, and
- for every rewriting step:



there are more blue-and-orange X-occurrences than blue-and-red X-occurrences.

Challenge: verify the second condition for an unknown C’.

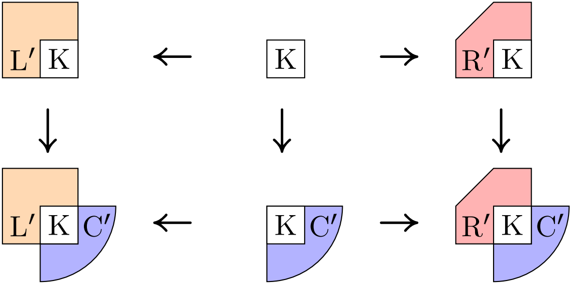
Analysis of Implicit Occurrences

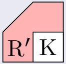


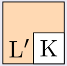
Blue-and-red X-occurrences: $5 \xrightarrow{a} 2 \xrightarrow{a} 6$

Blue-and-Orange X-occurrences: $3 \xrightarrow{a} 2 \xrightarrow{a} 6$

Sufficient Condition for the Second Condition [Qiu]

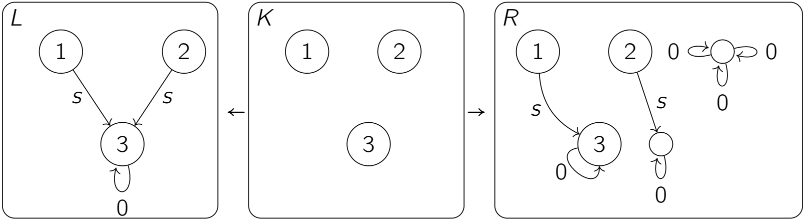


If all subgraphs of 

that can form an blue-and-red X-occurrence in any rewriting step can be mapped to distinct subgraphs in 

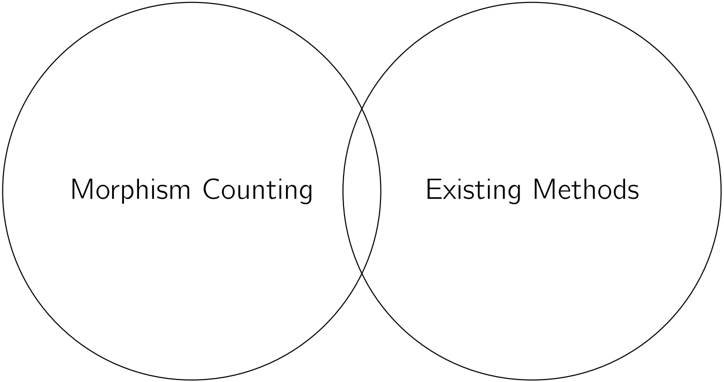
while preserving elements in $\boxed{\mathrm{K}}$, then there are more blue-and-orange X-morphisms than blue-and-red X-morphisms.

Termination of Motivating Example



Terminating by counting morphisms from $\bigcirc \xrightarrow{s} \bigcirc \xleftarrow{s} \bigcirc$ ✓

Imcomparable with Existing Methods



Succeed in some cases where all existing automated methods fail.

Fail in some cases where other methods succeed.

More power if search in parallel ✓

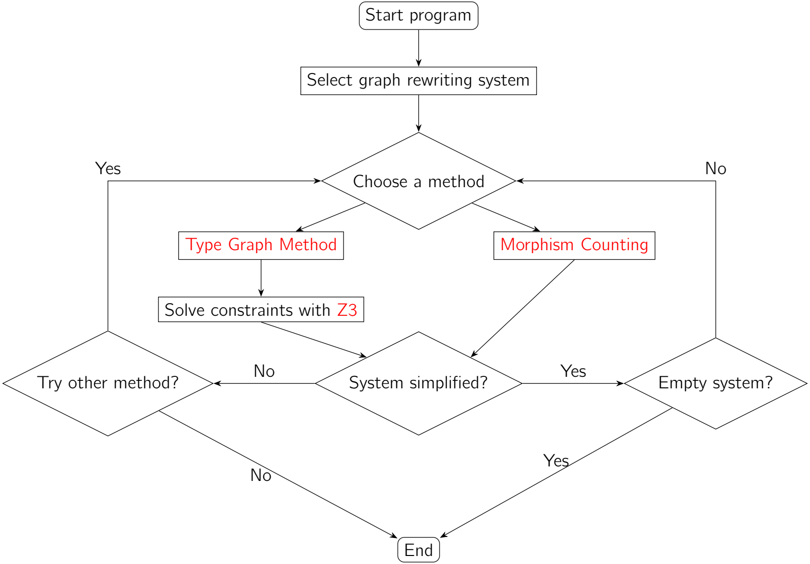
LyonParallel

Automated tool in Ocaml

Iterative elimination of graph rewriting rules

Available : `https://github.com/Qi-tchi/LyonParallel`

Process Flowchart of LyonParallel



Conclusion and Future Work

Contributions

- Improved usability of an existing method,
- Proposed a new termination criterion,
- Extended the new termination criterion (not presented here),
- Implemented an automated tool for termination analysis.

Future work

- Short term: Morphism Counting with forbidden contexts.
- Mid term: Certificate-generation mechanism.
- Long term: Extension to other graph rewriting frameworks (e.g., PBPO+)

References I

[1] H. J. Sander Bruggink. "Towards Process Mining with Graph Transformation Systems". In: *Graph Transformation*. Ed. by Holger Giese and Barbara König. Cham: Springer International Publishing, 2014, pp. 253–268. ISBN: 978-3-319-09108-2.

[2] H. J. Sander Bruggink et al. "Proving Termination of Graph Transformation Systems using Weighted Type Graphs over Semirings". In: *CoRR* abs/1505.01695 (2015). arXiv: 1505.01695.

[3] J. Endrullis and R. Overbeek. *Generalized Weighted Type Graphs for Termination of Graph Transformation Systems*. 2024. arXiv: 2307.07601v2 [cs.LO].

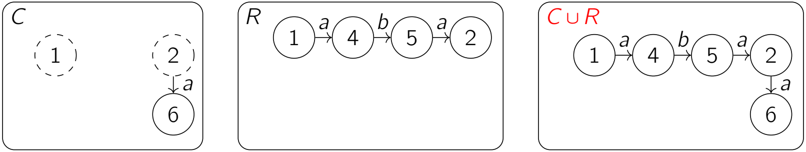
References II

[4] Detlef Plump. "Modular Termination of Graph Transformation". In: *Graph Transformation, Specifications, and Nets - In Memory of Hartmut Ehrig*. Ed. by Reiko Heckel and Gabriele Taentzer. Vol. 10800. Lecture Notes in Computer Science. Springer, 2018, pp. 231–244. DOI: 10.1007/978-3-319-75396-6_13.

[5] Detlef Plump. "On Termination of Graph Rewriting". In: *Graph-Theoretic Concepts in Computer Science, 21st International Workshop, WG '95, Aachen, Germany, June 20-22, 1995, Proceedings*. Ed. by Manfred Nagl. Vol. 1017. Lecture Notes in Computer Science. Springer, 1995, pp. 88–100. DOI: 10.1007/3-540-60618-1_68.

Pre-Graph Operations

Union of two pre-graphs $C \subseteq G$ and $R \subseteq G$, denoted $C \cup R$.



Relative complement of R in H where $R \subseteq H$, denoted $H \smallsetminus R$.