Automated Termination Proving: Contributions to Graph Rewriting via Extended Weighted Type Graphs and Morphism Counting

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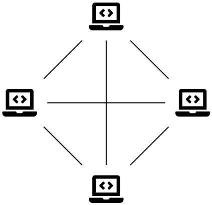
Motivation & Goal

Distributed systems:



Failures can be catastrophic: 🏥 🚆 ✈ 🚀

Ensuring correctness is difficult.

- The Needham-Schroeder protocol proved insecure 17 years after its publication.

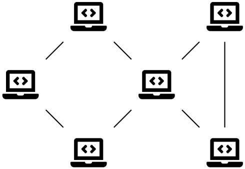
This thesis: automated verification.

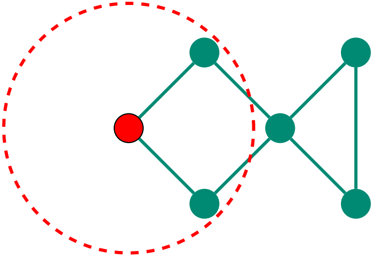
- Minimal user effort
- No expertise required
- Mathematically rigorous

Graph Transformation

Modelization of distributed systems

System configurations: graphs



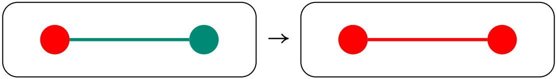


Algorithm behaviors:

graph transformation according to local knowledge

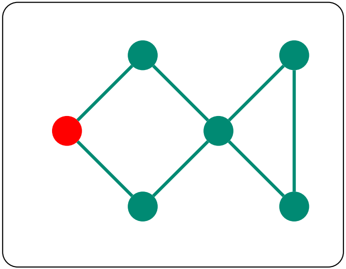
Graph Transformation

Graph transformation rule:



Replace the left-hand side by the right-hand side.

Spanning-tree construction:



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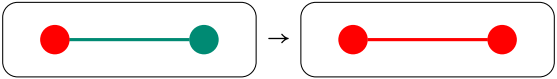

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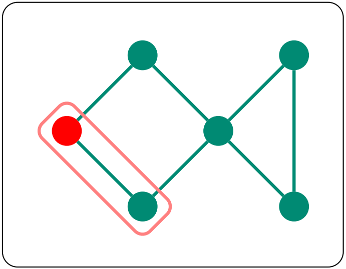

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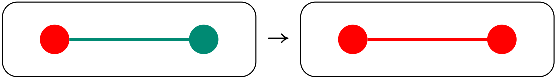

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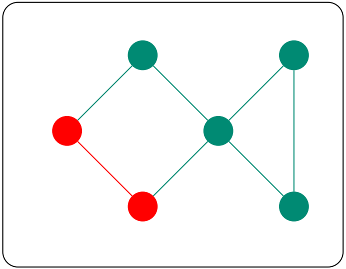

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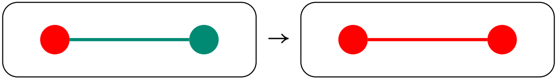

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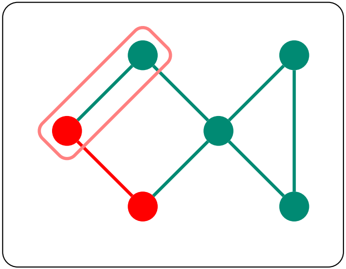

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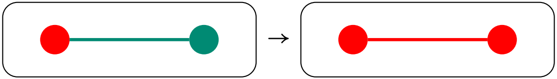

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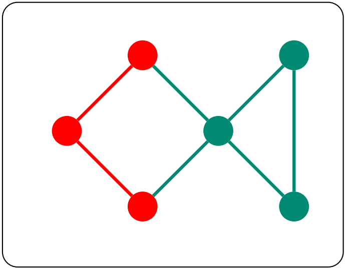

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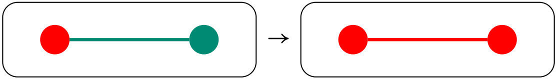

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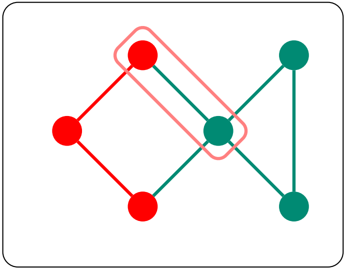

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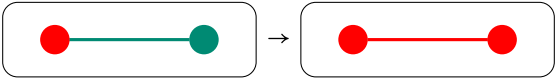

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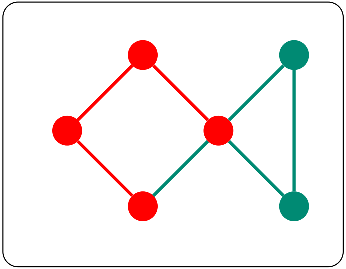

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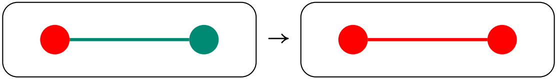

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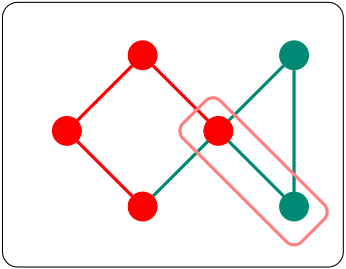

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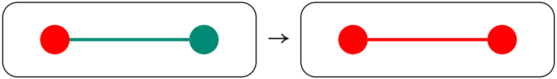

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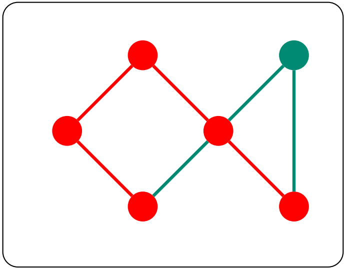

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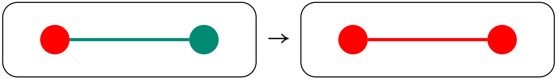

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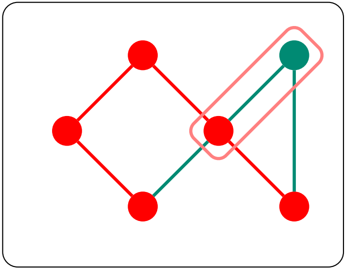

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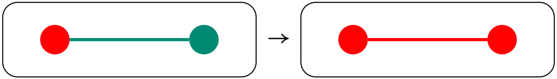

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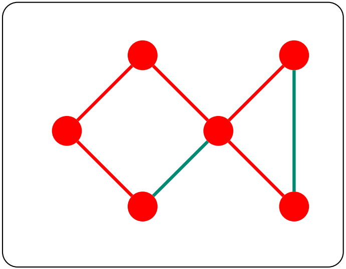

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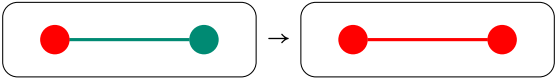

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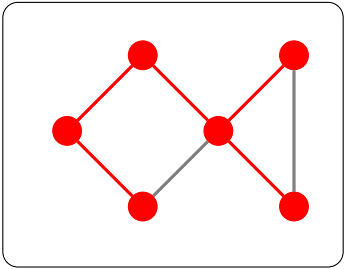
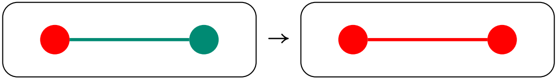


A spanning tree is obtained when the rule cannot be applied.

Graph Transformation

Graph transformation rule:



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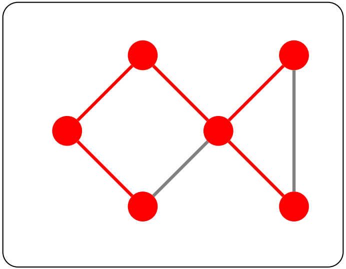


A spanning tree is obtained when the rule cannot be applied.

Does the transformation process terminate for any initial graph?

Termination of Graph Transformation Systems

- No graph G_0 can be transformed forever

$$G_0 \Rightarrow G_1 \Rightarrow \cdots$$

- Aligns with the notion of program termination:

  "every execution (on any input) halts."

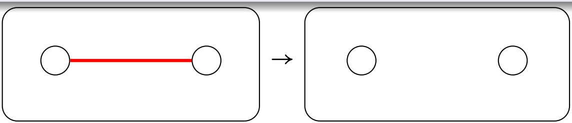
- Undecidable in general [9]
  - Automated techniques for specific subclasses

Termination by interpretations [11, 1]

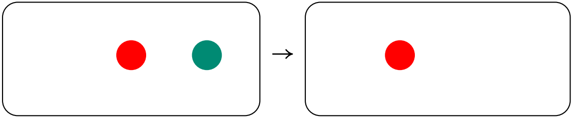
Interpret graphs as natural numbers.

Show each transformation step strictlydecreases the value.

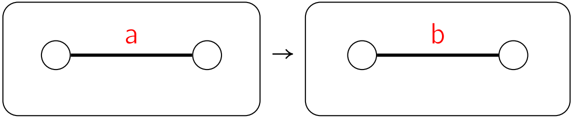
Number of edges:



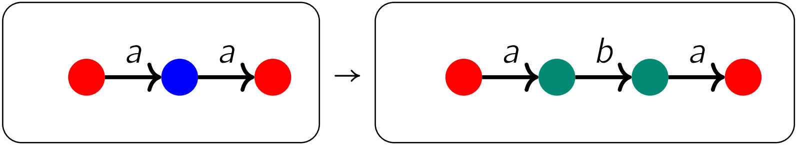
Number of nodes:



Number of edges labeled by *a*:



Limitation



Number of nodes/edges/labels do not decrease.

Can its termination be proved by interpretations?

- Need a formal definition of graph transformations.

Structure of the Remainder

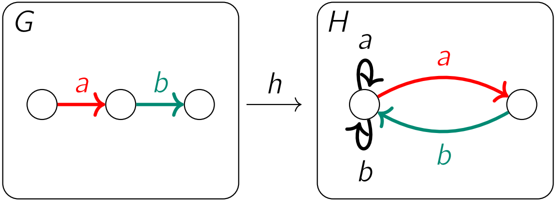
Graph Rewriting with Double-Pushout (DPO)

Toward Greater Usability

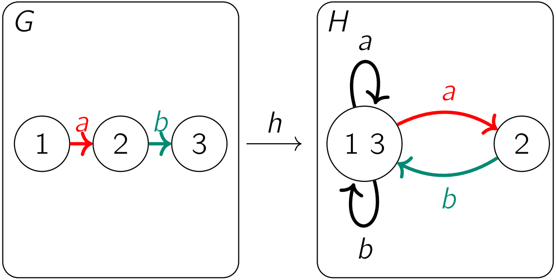
Toward Greater Power

LyonParallel—A Tool for Termination of Graph Rewriting

Graph Morphisms: Structure-Preserving Functions



Colors show edge correspondence.

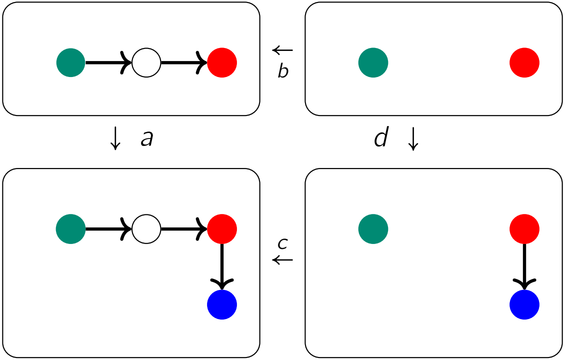


Numbers show node correspondence.

Commutative Diagram

$$\begin{tikzcd}
L \arrow[d, "a"] & K \arrow[l, "b"'] \arrow[d, "d"'] \\
G & C \arrow[l, "c"]
\end{tikzcd}$$

commutes if $a \circ b = c \circ d$.



Pushouts: Gluing Graphs Along an Interface

The **pushout** of (α, β) is (β', α') with

- □ABDC commutes,

$$\begin{tikzcd}
A \arrow[r, "\beta"] \arrow[d, "\alpha"'] & C \arrow[d, "\alpha'"'] \\
B \arrow[r, "\beta'"] & D
\end{tikzcd}$$

□ABDC: pushout square

D: pushout object

Pushouts: Gluing Graphs Along an Interface

The **pushout** of (α, β) is (β', α') with

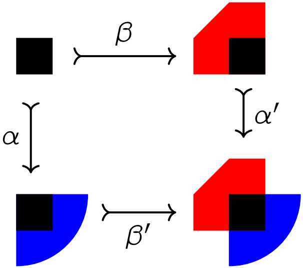
- □ABDC commutes,
- universality: for all (γ, γ'),if $\square ABEC$ commutes, then there is a unique δ such that $\triangle BDE$ and $\triangle CDE$ both commute.

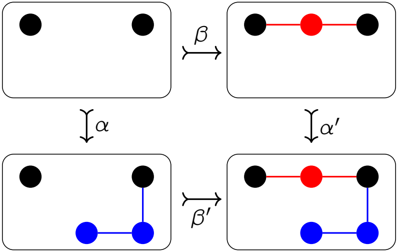
$$\begin{tikzcd}
A \arrow[r, "\beta"] \arrow[d, "\alpha"'] & C \arrow[d, "\alpha'"'] \arrow[ddr, bend left, "\gamma'"] & \\
B \arrow[r, "\beta'"] \arrow[drr, bend right, "\gamma"] & D \arrow[dr, dashed, "\delta"] & \\
& & E
\end{tikzcd}$$

□ABDC: pushout square

D: pushout object

Pushouts: Gluing Graphs Along an Interface





Graph Rewriting with Double-Pushout (DPO)

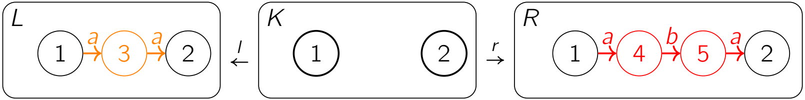
The first algebraic approach to graph rewriting [4]

One of the most studied approaches to graph rewriting [5]

Graph Rewriting with Double-Pushout (DPO) [4]

$$\begin{tikzcd} L & K \arrow[l, "l"'] \arrow[r, "r"] & R \end{tikzcd}$$

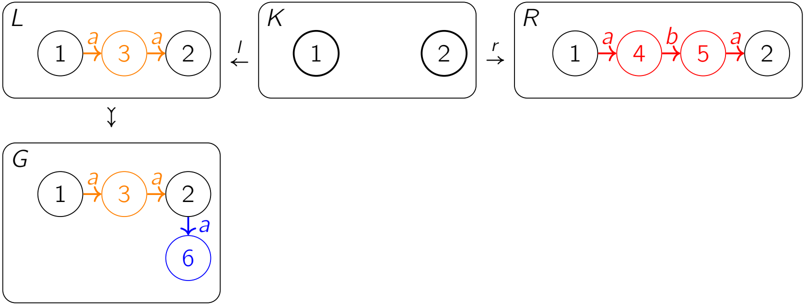
Rewriting rule with interface K



Graph Rewriting with Double-Pushout (DPO) [4]

$$\begin{tikzcd}
L \arrow[d, tail] & K \arrow[l, "l"'] \arrow[r, "r"] & R \\
G & &
\end{tikzcd}$$

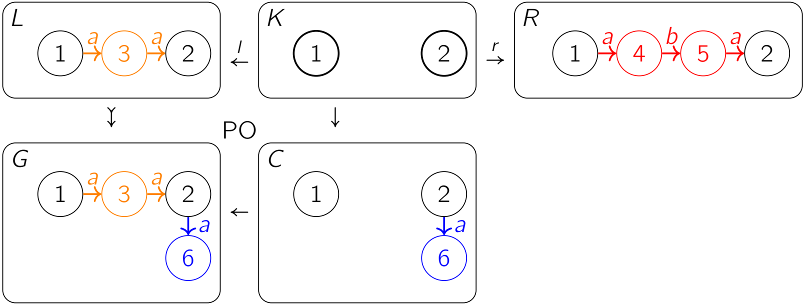
Rewriting rule with interface K



Graph Rewriting with Double-Pushout (DPO) [4]

$$\begin{tikzcd}
L \arrow[d, tail] \arrow[dr, phantom, "\mathrm{PO}"] & K \arrow[l, "l"'] \arrow[r, "r"] \arrow[d] & R \\
G & C \arrow[l]
\end{tikzcd}$$

Rewriting rule with interface K

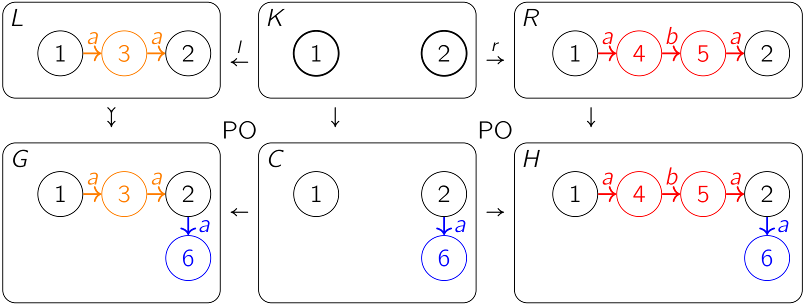


Graph Rewriting with Double-Pushout (DPO) [4]

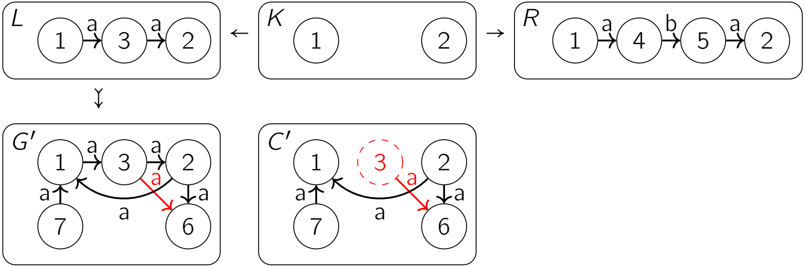
$$\begin{tikzcd} L \arrow[d, tail] \arrow[dr, phantom, "\text{PO}"] & K \arrow[l, "l"'] \arrow[r, "r"] \arrow[d] \arrow[dr, phantom, "\text{PO}"] & R \arrow[d] \\ G & C \arrow[l] \arrow[r] & H \end{tikzcd}$$

Rewriting rule with interface K

rewriting step $G \Rightarrow H$



An Invalid Rewriting Step



Weighted Type Graph Method [2, 3, 7]

Termination by interpretation

Parameter: an object T in the category, called type graph

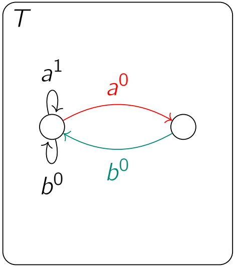
Terminology: every graph is "typed" as morphisms to T

Interpretation:

$$\begin{aligned} G &\rightsquigarrow \mathcal{M}(G,T) \\ &\rightsquigarrow \mathrm{weight}(\mathcal{M}(G,T)) \\ &\rightsquigarrow \mathrm{aggregator}(\mathrm{weight}(\mathcal{M}(G,T))) \in \mathbb{N} \end{aligned}$$

How to choose the type graph T?
What is the morphism weight?
What is the graph weight?

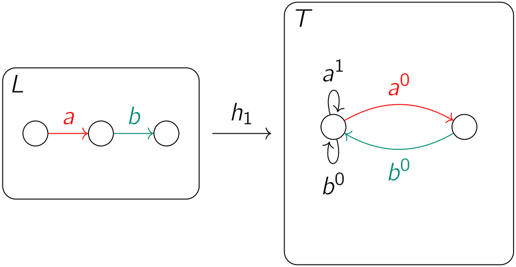
Type Graph with Weights on Edges



Morphism Weight

The weight of a morphism $h : G \to T$ is

$$\sum_{e \in \mathsf{Edge}(G)} \mathrm{weight}(h(e))$$

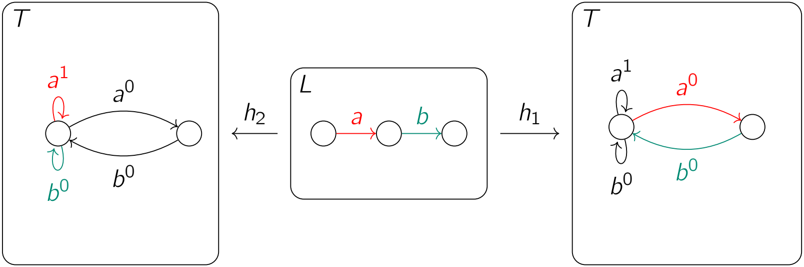


$$\mathrm{weight}_T(h_1) = 0 + 0 = 0$$

Graph Weight

The weight of a graph L is

$$\min\{h : L \to T \mid \mathrm{weight}_T(h)\}$$



$\mathrm{weight}_T(h_2) = 1 + 0 = 1$

$\mathrm{weight}_T(h_1) = 0 + 0 = 0$

$\mathrm{weight}_T(L) = \min\{1, 0\} = 0$

Termination Criterion [3]

$$\begin{tikzcd}
L \arrow[dr, dashed, "t_L"'] & K \arrow[l, "l"'] \arrow[r, "r"] \arrow[d, "t_K"] & R \arrow[dl, dashed, "t_R"] \\
& T &
\end{tikzcd}$$

A rule terminates if there is T such that for all t_K, if there is t_L such that $\triangle KLT$ commutes, then

$$\begin{aligned}
&\min\{\mathrm{weight}_T(t_L) \mid t_L.\, \triangle\, KLT\,\text{commutes}\}\\
>&\min\{\mathrm{weight}_T(t_R) \mid t_R.\, \triangle\, KRT\,\text{commutes}\}
\end{aligned}$$

How to find a suitable weighted type graph?

Searching for Weighted Type Graphs over $\mathbb{N}$ [11, 3]

User-specified parameters:

- k nodes
- maximum edge weight $n \in \mathbb{N}$

The problem amounts to checking the satisfiability of an existential Presburger arithmetic theory with:

- $k^2 m$ binary variables where m is the number of labels
- $k^2 m$ integer variables

Challenge:

- expertise: impossible to guess k and maximum weight n
- complexity: $2^{k^2 m} \cdot n^{k^2 m}$ possible assignments of weights

Problem of the Size of the Search Space

With natural numbers as weights:

# nodes (k)	# labels (m)	# weights	# possibilities
2	2	2	$\approx 10^4$
3	3	3	$\approx 10^{21}$
4	4	4	$\approx 10^{57}$
5	5	5	$\approx 10^{125}$

Problems can solved by Z3 in exponential-time with respect to the number of variables $2k^2m$.

Usability Improvement: Using Real Numbers as Weights

Using positive real numbers as weights

Additional constraint: there is $\delta > 0$ such that every rewriting step decreases the weight by at least δ.

Searching for Weighted Type Graphs over ~~$\mathbb{N}$~~ $\mathbb{R}$

User-specified parameters:

- k nodes
- ~~edge weights in $\{0, 1, \ldots, n\}$~~

The problem amounts to checking the satisfiability of an ~~existential Presburger arithmetic theory~~ existential theory of the reals with binary variables:

- k^2m binary variables where m is the number of labels
- k^2m ~~integer~~ real variables

Challenge:

- impossible to guess k and ~~maximum weight n~~
- complexity: ~~$2^{k^2m} \cdot n^{k^2m}$ possible assignments of weights~~ there are 2^{k^2m} linear programs which have polynomial-time average-case complexity.

Complexity Comparison

With weights in $\mathbb{N}$:

# nodes (k)	# labels (m)	# weights	# possibilities
2	2	2	$\approx 10^4$
3	3	3	$\approx 10^{21}$
4	4	4	$\approx 10^{57}$
5	5	5	$\approx 10^{125}$

With weights in $\mathbb{R}$:

# nodes (k)	# labels (m)	# variables	# linear programs in $\mathbb{R}$
2	2	8	$\approx 10^2$
3	3	27	$\approx 10^8$
4	4	64	$\approx 10^{19}$
5	5	125	$\approx 10^{32}$

Linear programs can be solved in polynomial time with respect to the number of variables on average by Z3.

Experimental Results

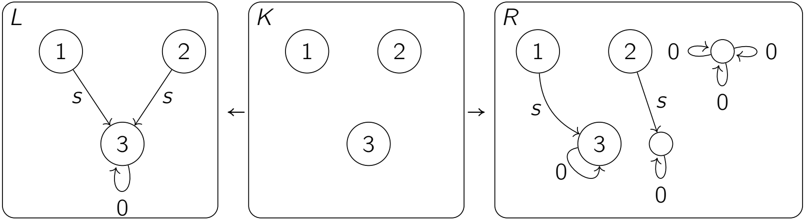
	$A_{\mathbb{N}}$	$a_{\mathbb{R}}$	$T_{\mathbb{N}}$	$t_{\mathbb{R}}$	$N_{\mathbb{N}}$	$n_{\mathbb{R}}$
[6, Example 6.3]					2.74	1.16
[6, Example D.3]	2.25	1.18			2.24	1.18
[9, Example 3.8]	2.95	1.90	2.94	1.87	3.49	1.87
[8, Example 4]	4.26	3.19	4.24	3.13	5.82	timeout
[8, Example 5]	5.54	5.55	5.53	5.50	9.11	5.62
[3, Example 4]	2.44	2.46	2.47	2.54	4.58	2.46
[3, Example 5]					7.80	timeout
[3, Example 6]					9.75	timeout
[2, Example 1]	2.26	1.18			2.24	1.18
[2, Example 4]	2.25	1.22	2.24	1.18	2.25	1.19
[2, Example 5]	4.23	3.23	4.25	3.28	5.82	timeout

Implementation

LyonParallel

Search parallel with 6 configurations

A Limitation of the Weighted Type Graph Method



Type graph fails: existence of surjections from R to L.

All existing automated methods fail.

Remark: the number of occurrences of L strictly decreases.

Capability Improvement: Morphism Counting

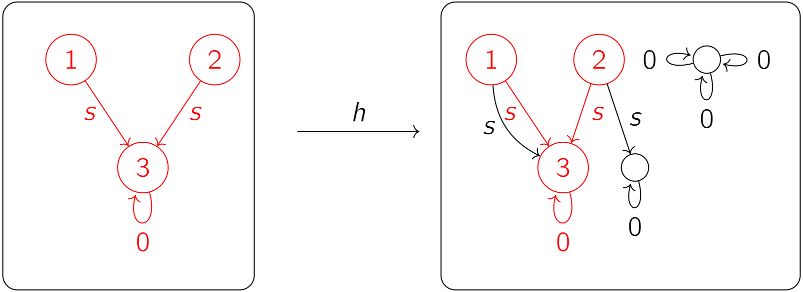
Termination by interpretation

Parameter: graph X

Interpretation:

$$G \rightsquigarrow |\mathcal{M}(X, G)| \in \mathbb{N}$$

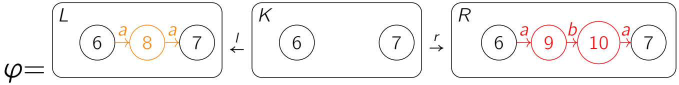
Inclusions: morphisms h with $h(x) = x$ for all x.



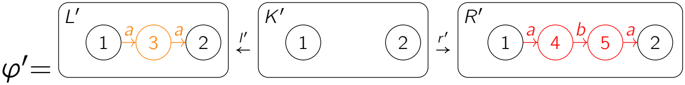
Subgraph

Graph Rewriting with Injective Rules

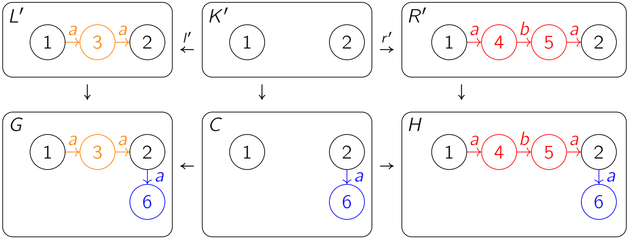
A rewriting rules consists of two inclusions.



An equivalent rewriting rule expresses the same transformation.

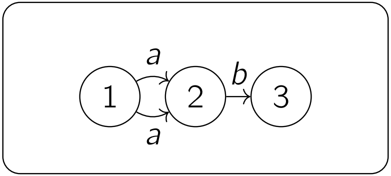


A rewriting step with φ is defined by a DPO diagram with inclusions and φ'.

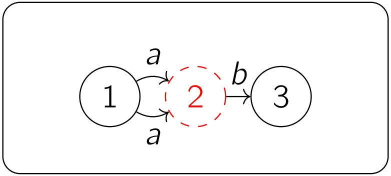


Pre-Graphs

Graph:

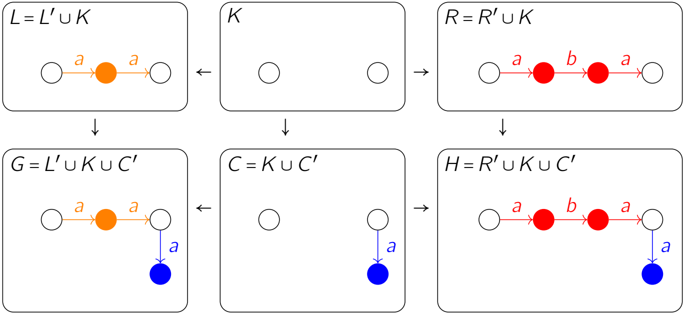


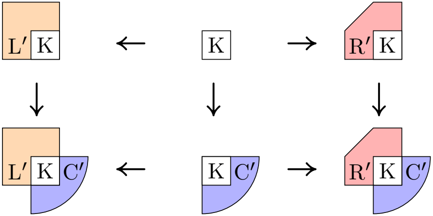
Pre-graphs obtained by removing node 2:



All edges are dangling.

Decomposition of Graphs in Rewriting Steps

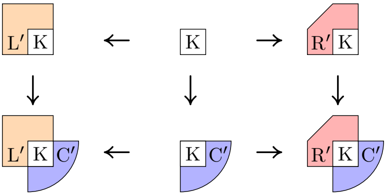




This coloring provides a classification of morphisms in rewriting steps by image node colors.

X-occurrences by Image Node Colors

An X-occurrence is an injective morphism from X.

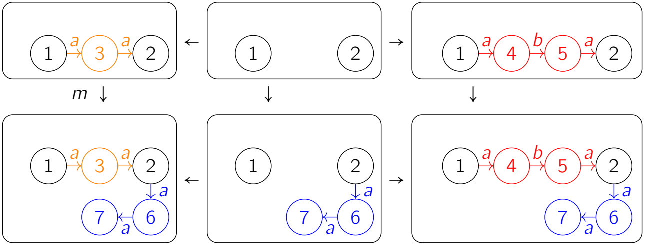


X-occurrence are classified by the colors of their image nodes:

- white: only white;
- orange: only white and at least one orange;
- blue: only white and at least one blue;
- red: only white and at least one red;
- blue-and-orange: at least one blue and at least one orange;
- blue-and-red: at least one blue and at least one red

Morphisms by Image Node Colors

Let X be the graph $\bullet \xrightarrow[a]{} \bullet \xrightarrow[a]{} \bullet$.

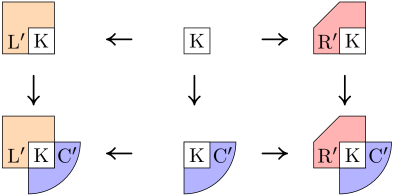


Blue X-occurrences: $2 \xrightarrow{a} 6 \xrightarrow{a} 7$

Red X-occurrences: none.

Blue-and-red X-occurrences: $5 \xrightarrow{a} 2 \xrightarrow{a} 6$

A New Sufficient Condition for Termination [10]

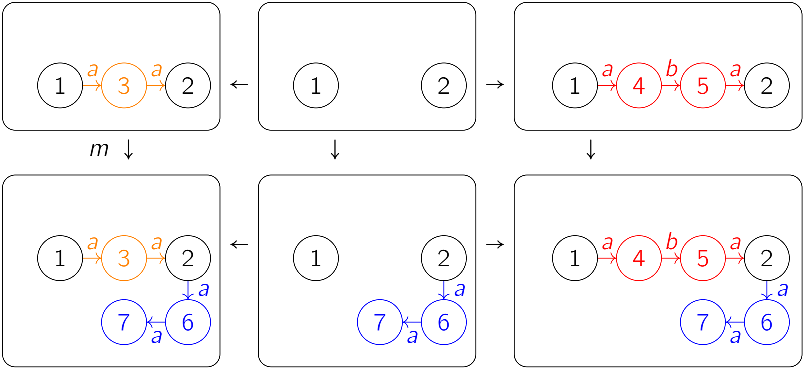


A rule terminates if, for every rewriting step, there are

- strictly more orange X-occurrences than red X-occurrences,
- more blue-and-orange X-occurrences than blue-and-red X-occurrences.

Challenge: C’ is unknown

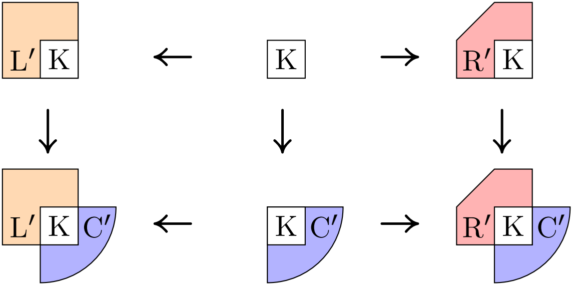
Analysis of Implicit Occurrences

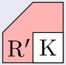


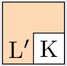
Blue-and-red X-occurrences: $5 \xrightarrow{a} 2 \xrightarrow{a} 6$

Blue-and-Orange X-occurrences: $3 \xrightarrow{a} 2 \xrightarrow{a} 6$

Sufficient Condition for the Second Condition [10]

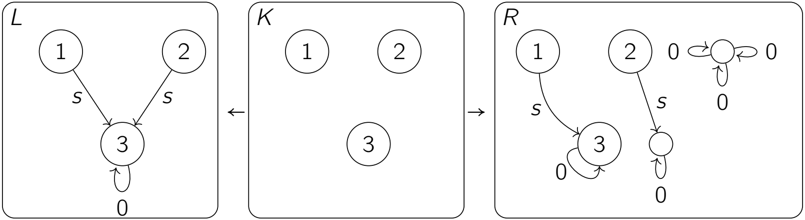


There more blue-and-orange X-morphisms than blue-and-red X-morphisms, if all subgraphs of 

that can form an blue-and-red *X*-occurrence in any rewriting step can be mapped to distinct subgraphs in 

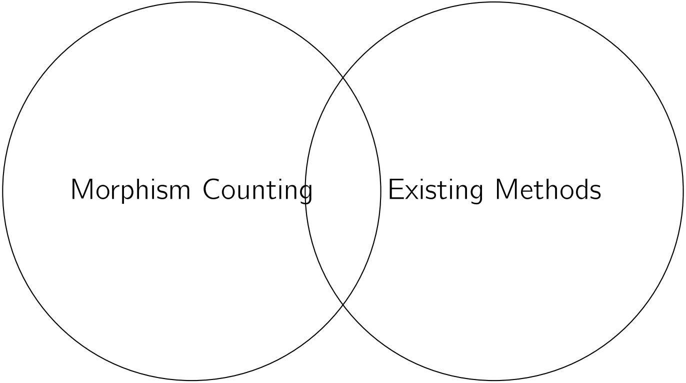
while preserving elements in $\boxed{\mathrm{K}}$.

Termination of Motivating Example



Termination proved by counting morphisms from $\bullet \xrightarrow{s} \bullet \xleftarrow{s} \bullet$.

Imcomparable with Existing Methods



Existing automated methods fail.

Fail in some cases where other methods succeed.

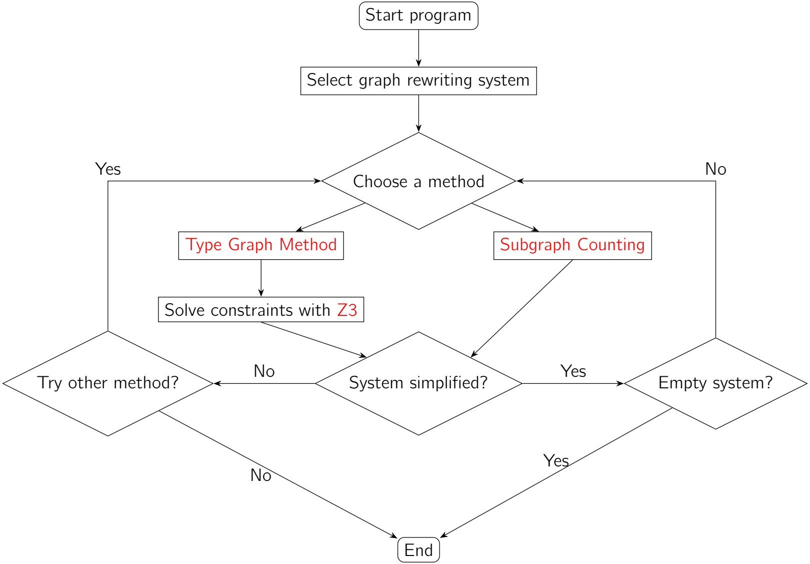
LyonParallel

Automated tool in Ocaml

Iterative elimination of graph rewriting rules

Available : `https://github.com/Qi-tchi/LyonParallel`

Process Flowchart of LyonParallel



Conclusion and Future Work

Contributions

- Extended the Weighted Type Graph Method to improve usability.
- Proposed a termination criterion applicable to new cases.
- Extended Morphism Counting to count morphisms with a forbidden context.
- Implemented an automated tool for termination analysis.

Future work

- Short term:
  - Extend Morphism Counting to count morphisms with multiple forbidden contexts.
  - Parallelizing Type Graph/Subgraph Counting in the tool.
- Mid term: Equip our tool with a certificate-generation mechanism.
- Long term: Extend to other graph rewriting frameworks (e.g., PBPO+)

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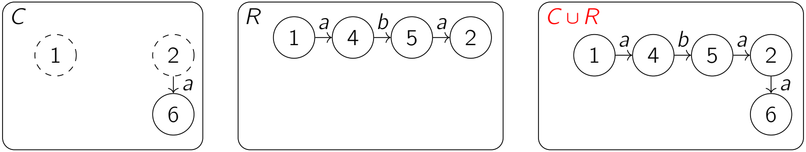
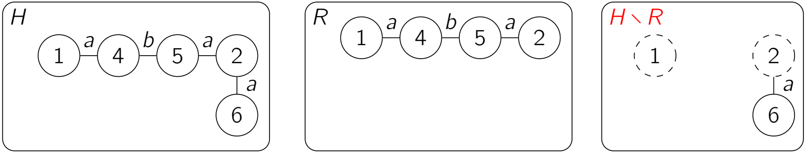
Pre-Graph Operations

Union of two pre-graphs $C \subseteq G$ and $R \subseteq G$, denoted $C \cup R$.



Relative complement of R in H where $R \subseteq H$, denoted $H \smallsetminus R$.


H
1
a
4
b
5
a
2
a
6
R
1
a
4
b
5
a
2
$H \smallsetminus R$
1
2
a
6