Termination of Injective DPO Graph Rewriting Systems using Subgraph Counting

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Master's Thesis Projects

- Impact of Clause Sharing Strategies in Parallel SAT Solvers
- Formal Verification of Theorem Proving Methods

Doctoral Research: Termination of Graph Relabeling Systems

- DPO Graph Rewriting and Termination
- Termination via Translation to Term Rewriting Systems
- Termination using Weighted Type Graphs
- Termination using Subgraph Counting
- Termination using Subgraph Counting with antipatterns

Future Work

SAT solver: miniSAT
Parallelization
Single Program Multi Data (SPMD)
Message Passing Interface (MPI) / C++
A processor / a miniSAT instance / a strategy
Impact of Clause Sharing Strategies

- limit the size of the clauses to be shared
- share with different number of neighbours
- ...

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Future Work

Automated Theorem Proving by Saturation

Proving $\vdash F$ by proving $\neg F \vdash \bot$

Example:

- for proving $\vdash p \vee \neg p$
- calculate $\neg(p \vee \neg p) = \neg p \wedge p$
- we have $\{\neg p, p\} \vdash \bot$
- thus $\vdash p \vee \neg p$

Given Clause Procedure

An automated theorem prover by saturation needs to

- maintain a set of clauses S
- deduce new clauses from S
- delete redundant clauses from S

Given Clause Procedure (GC) is a set of rules for deciding

- which clauses to keep in S
- which clauses to use for deducing new clauses

Framework for Saturation Theorem Proving

Given Clause Procedure (GC)

- Process:
  - $\mathcal{N} \cup \mathcal{M} \implies_{\mathsf{GC}} \mathcal{N} \cup \mathcal{M}'$ where ...
- INFER:
  - $\mathcal{N} \cup \{(C, l)\} \implies_{\mathsf{GC}} \mathcal{N} \cup \{(C, \mathsf{active})\} \cup \mathcal{M}$ where ...

A Comprehensive Framework for Saturation Theorem Proving, Waldmann U., Tourret S., Robillard S., Blanchette J. (IJCAR 2020)

- Isabelle/HOL framework
- formalization of GC
- formalization of LGC

4 Variants employed by different Theorem Provers

1. Otter-loop(OL) : a refinement of GC
2. iProver-loop(IL) : an extension of OL
3. Discount-loop(DL) : a refinement of LGC
4. Zipperposition-loop(ZL) : an extension of DL

Correctness: Can any formula provable by a variant also be proved by GC?

Contribution

Formalization of the variants in Isabelle/HOL
Proving their correctness in Isabelle/HOL
International Conference on Automated Deduction 2023 (CADE-29)

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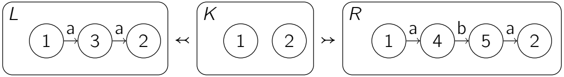
Future Work

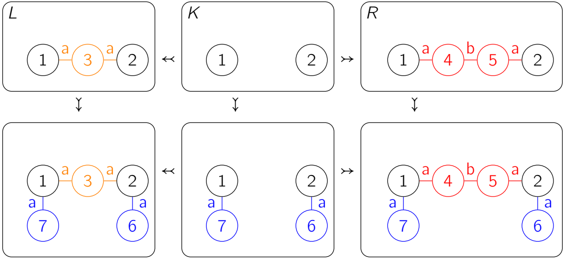
DPO Graph Rewriting and Termination

Edge-labeled directed graphs

DPO graph rewriting rule

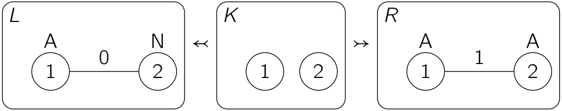


Graph transformation using DPO rewriting rule

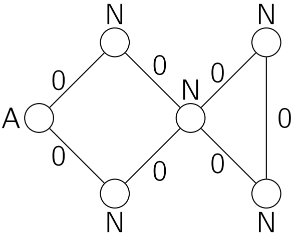


Termination of DPO graph rewriting rule

DPO Graph Rewriting System: Example



Example: the construction of a spanning tree



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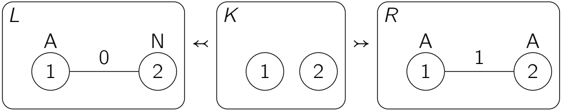

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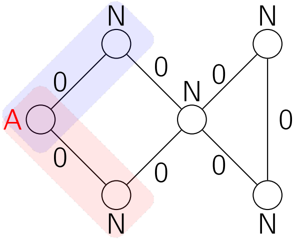

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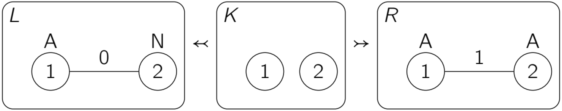

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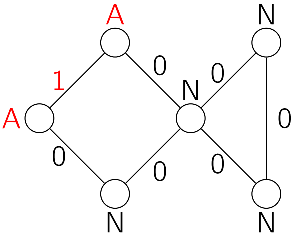

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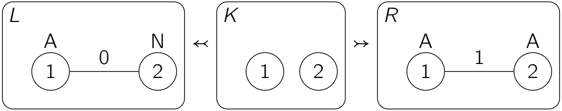

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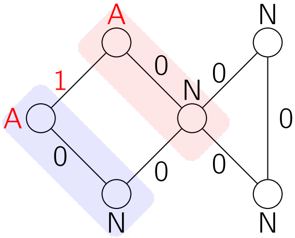

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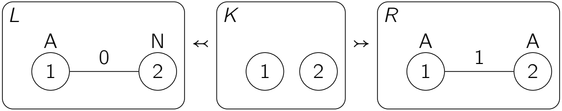

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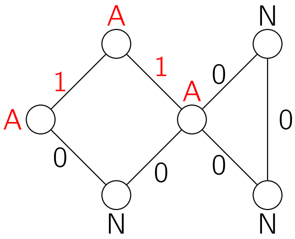

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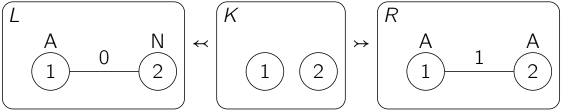

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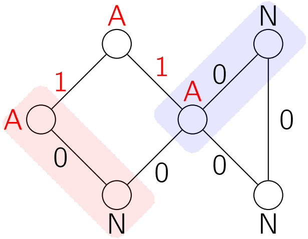

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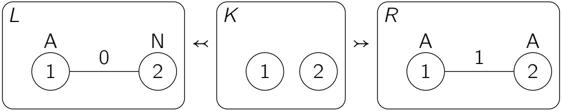

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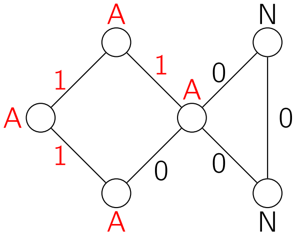

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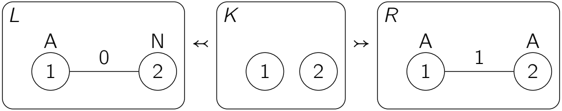

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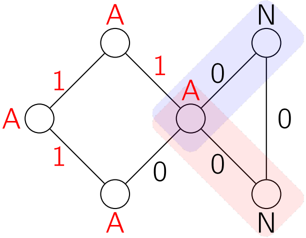

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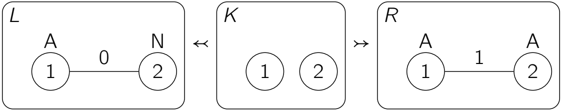

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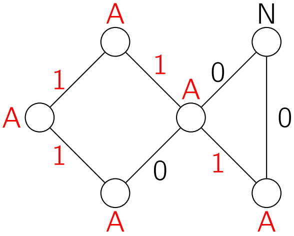

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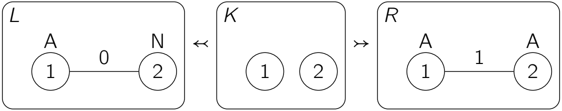

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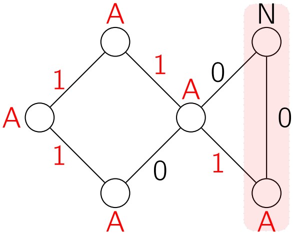

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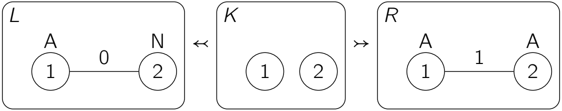

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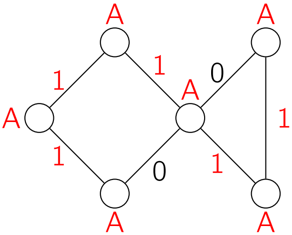

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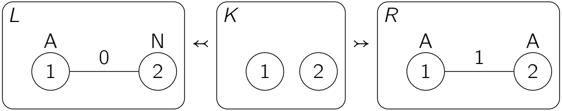

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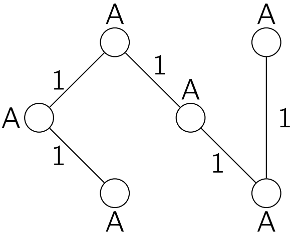

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Future Work

Term Rewriting Systems (TRS)

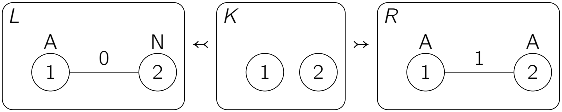
- Rule based term transformation
- Example of a TRS:

$$r_1 : f(x) \longrightarrow g(x)$$
$$r_2 : a \longrightarrow b$$

- Example of an execution:
  $f(a) \xrightarrow{r_1} g(a) \xrightarrow{r_2} g(b)$
- Can we make use existing advanced termination techniques for TRS?

Translation into TRS with AC Symbols

Example:



Translation :

$$\lambda(1, A) * \lambda(2, N) * \lambda(\{1, 2\}, 0)$$
$$\longrightarrow$$
$$\lambda(1, A) * \lambda(2, A) * \lambda(\{1, 2\}, 1)$$

Preservation of the termination property:

$$\begin{tikzcd}
\text{A terminates} & \text{GRS terminates} \arrow[l] & \text{TRS terminates} \arrow[l] \\
\text{A} \arrow[r] \arrow[u, dashed] & \text{GRS} \arrow[r] \arrow[u, dashed] & \text{TRS} \arrow[u]
\end{tikzcd}$$

Other Translations

Translation into

- Normalized Term Rewriting Systems (NTRS)
- Hierarchical Term Rewriting Systems (HTRS) with innermost strategy
- two others translations

Limitations

- Termination techniques for Term Rewriting Systems rely heavily on the tree structure of terms
- Graph has no tree structure
- New difficulties introduced by translation

Termination techniques for DPO Graph Rewriting Systems

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Future Work

Well-founded semirings

A mathematical structure $(S, \oplus, \otimes, 0, 1, \prec, \preceq)$

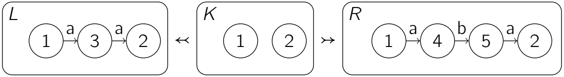
- $\otimes$, $\oplus$: binary operations
- 0, 1 : neutral elements for $\oplus$, $\otimes$
- $\prec, \preceq$: orders
- $\prec$ / $\preceq$: well-founded
- satisfies some conditions

Examples:

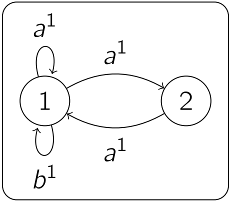
- The natural arithmetic semiring $(\mathbb{N}, +, *, 0, 1, <, \leq)$
- The natural tropical semiring: $(\mathbb{N} \cup \{+\infty\}, \mathbf{min}, +, +\infty, 0, <, \leq)$
- The natural arctic semiring: $(\mathbb{N} \cup \{-\infty\}, \mathbf{max}, +, -\infty, 0, <, \leq)$

Termination using Weighted Type Graphs

Termination of



by the weighted type graphs over the natural arithmetic semiring.



Existence of suitable weighted type graphs:

- undecidable in general

Practical Solution, its Limitation, and our Solution

Σ : finite set of labels

Searching a weighted type graph with

- $k \in \mathbb{N}$ nodes
- no parallel edges with the same label

1. Decide if $s \xrightarrow{l} t$ exists for every pair of nodes s, t and label l
2. Assign a weight to every existing edge
3. Check if the weighted type graph satisfy requirements

Complexity $O(2^{2n})$ or undecidable

- $n = k^2 \cdot |\Sigma|$

Solution: weighted type graph over real numbers

Result: $O(2^{2n}) \Rightarrow O(2^n)$ or undecidable $\Rightarrow$ decidable

Intuition

Rewriting system $\mathcal{R}$ defines binary rewriting relation $\Rightarrow_{\mathcal{R}}$

A weighted type graph over natural numbers defines

- a homomorphism $h : (\mathbf{Graph}, \Rightarrow_{\mathcal{R}}) \to (\mathbb{N}, <)$

Codomain of h can be $(\mathbb{R}, <)$ if

- there is $\delta > 0$
- for all rewriting steps $G \Rightarrow H$:
  - $h(G) \geq 0$
  - $h(G) - h(H) \geq \delta$

Non-well-founded Semirings

Non-well-founded semiring $(S, \oplus, \otimes, 0, 1, \prec, \preceq, \mu)$

- $\otimes$, $\oplus$: binary operations
- 0, 1 : neutral elements for $\oplus$, $\otimes$
- $\prec, \preceq$: orders
- homomorphism $\mu : (S, \prec) \to (\overline{\mathbb{R}}, <)$
- $\prec$, $\otimes$, $\oplus$ satisfy some conditions

Well-founded semiring $(S, \oplus, \otimes, 0, 1, \prec, \preceq)$

- $\otimes$, $\oplus$: binary operations
- 0, 1 : neutral elements for $\oplus$, $\otimes$
- $\prec, \preceq$: orders
- $\prec / \preceq$: well-founded
- $\prec, \preceq$, $\otimes$, $\oplus$ satisfy some conditions

The tropical, arctic, and arithmetic semirings are instances of non-well-founded semirings.

- The natural tropical semiring:
  $\mathfrak{T} = (\mathbb{N} \cup \{+\infty\}, \mathsf{min}, +, +\infty, 0, <, \mathsf{id}_{\mathbb{N}\cup\{+\infty\}})$
- The natural arctic semiring:
  $\mathfrak{A} = (\mathbb{N} \cup \{-\infty\}, \mathsf{max}, +, -\infty, 0, <, \mathsf{id}_{\mathbb{N}\cup\{-\infty\}})$
- The natural arithmetic semiring $\mathfrak{N} = (\mathbb{N}, +, *, 0, 1, <, \mathsf{id}_{\mathbb{N}})$

Termination result

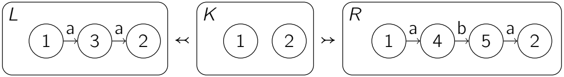
Theorem (Termination of DPO rewriting system)

Let φ be a DPO rewriting rule, T a finite weighted type graph over one of our non-well-founded semirings over real numbers such that

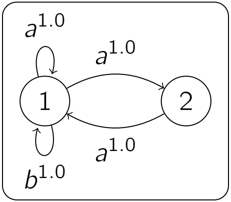
1. *edge weights are positive real numbers*
2. *the rule satisfies some conditions.*
3. *there is $\delta > 0$ such that: for all rewriting step $G \Rightarrow H$, we have $w(G) - w(H) \geq \delta$*

then φ terminates.

Example with Non-well-founded Semirings

Termination of 

by the weighted type graphs over the real arithmetic semiring.



For every rewriting step $G \Rightarrow H$, we have $w(G) - w(H) \geq 1.0$.

Contribution

Extension of the existing approach to real numbers

Automated Termination Prover: LyonParallel

- our approach
- previous approach
- parallel execution
- cooperation
- more user-friendly for non-experts than existing tools
- OCaml

Accepted for publication

- International Workshop on Graph Computation Models 2025

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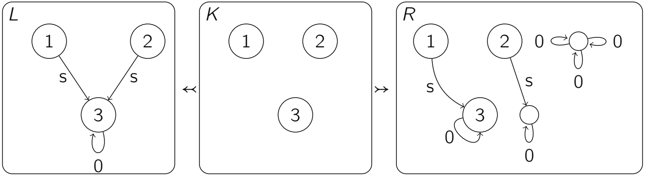
Future Work

Limitation of Existing Techniques

Injective DPO graph rewriting rule:

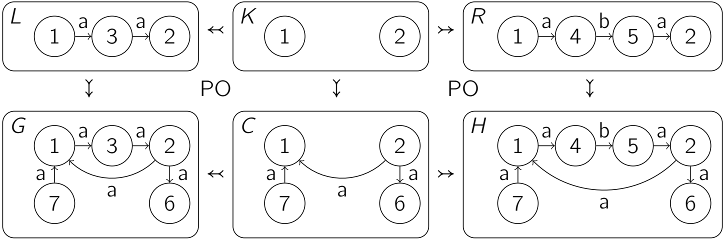


Terminating because of the strict decrease of the number of occurrences $\bullet \xrightarrow{s} \bullet \xleftarrow{s} \bullet$

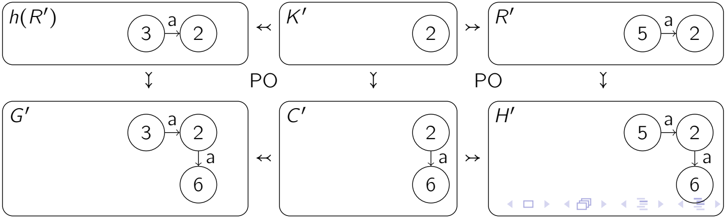
Termination cannot be proved by existing techniques

Implicit Occurrences: Occurrences depending on the context of the rewriting step

Injectively DPO graph rewriting rule:



An implicit occurrence of chain $\bullet \xrightarrow[a]{} \bullet \xrightarrow[a]{} \bullet$



X-non-increasing rule

$\varphi = L \overset{l}{\leftarrowtail} K \overset{r}{\rightarrowtail} R$: a rule
$X \subseteq R$: a graph
$D(R, X)$: subgraphs of R which can form an implicit X-occurrence in some rewriting step
φ is X-non-increasing rule if

1. For every $R_i \in D(R, X)$, there is a morphism $h_i : R_i \rightarrowtail L$ which preserves the interface elements,
2. three more conditions on h_i

Condition 1: for every implicit X-occurrence in H, there is a corresponding implicit X-occurrence in G with the same interface elements,

Other conditions: different implicit X-occurrences in H are mapped to different implicit X-occurrences in G,

for every rewriting step $G \Rightarrow H$.

Main results

$\varphi = L \twoheadleftarrow K \rightarrowtail R$: a rule
$X \subseteq R$: a graph

Lemma

For every rewriting step $G \Rightarrow H$ using φ, there are more implicit X-occurrences in G than in H if

- *φ is X-non-increasing*

Theorem (Sufficient termination condition)

φ terminates if

- *φ is X-non-increasing*
- *There are strictly more X-occurrences in L than in R*

Contribution

Machine-checkable sufficient termination condition

Termination of new classes of graph rewriting systems

Implementation in **LyonParallel**

International Conference on Graph Transformation (ICGT 2025)

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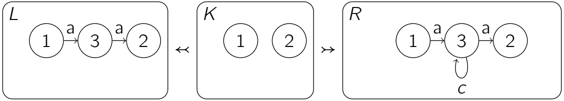
Future Work

Limitation of Subgraph Counting and Solution



Solution : Counting occurrences of L not including in an occurrence of R.

Contribution

Counting subgraphs with antipatterns

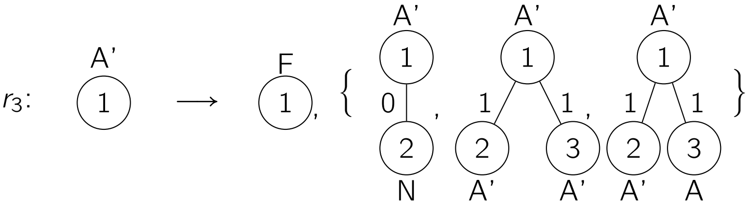
Termination of new classes of injective DPO graph rewriting systems

Tool available : **LyonParallel**

Under revision for resubmission

Future Work

Graph transformation rule with negative application conditions:



Extending the technique to DPO graph rewriting systems with negative application conditions