Automated Termination Proving: Contributions to Graph Rewriting via Extended Weighted Type Graphs and Morphism Counting

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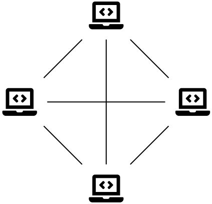
Motivation & Goal

Distributed systems:



Ensuring correctness is difficult.

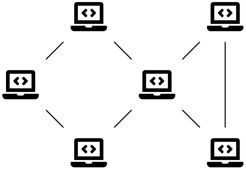
- The Needham-Schroeder protocol proved insecure 17 years after its publication.

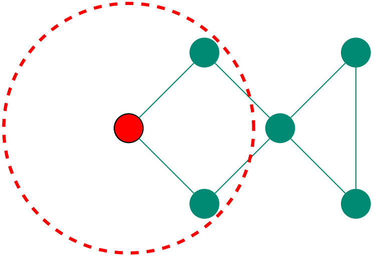
Failures can be catastrophic:

This thesis: automated verification.

- Minimal user effort
- No expertise required
- Mathematically rigorous

Graph Transformation





Modelization of distributed systems

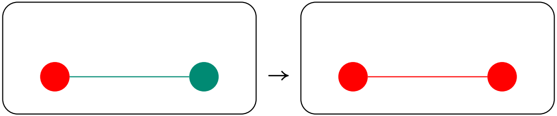
System configurations: graphs

Algorithm behaviors:

graph transformation according to local knowledge

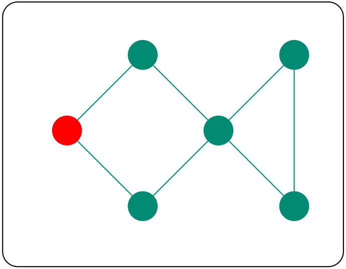
Graph Transformation

Graph transformation rule:



Replace the left-hand side by the right-hand side.

Spanning-tree construction:



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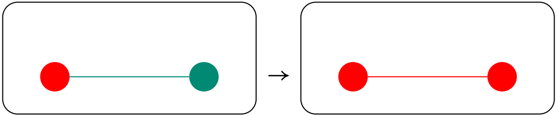

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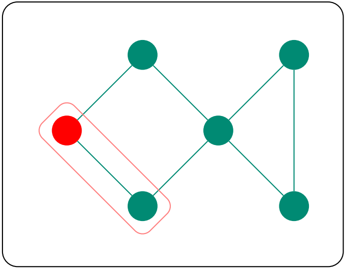

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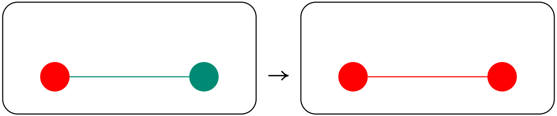

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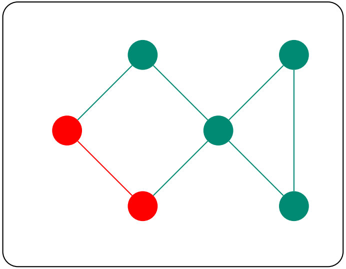

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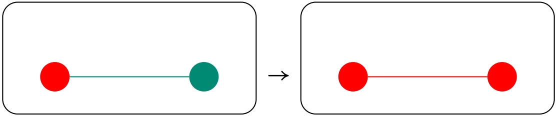

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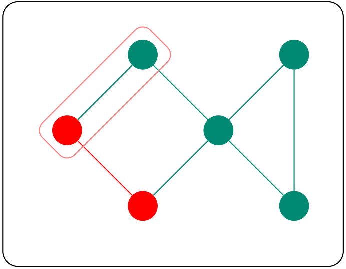

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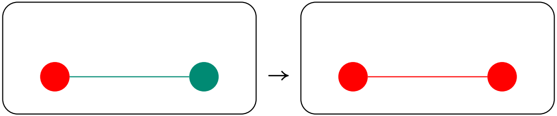

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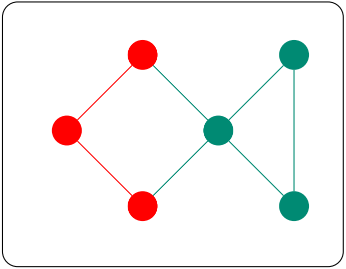

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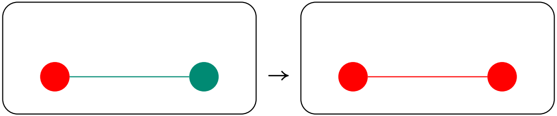

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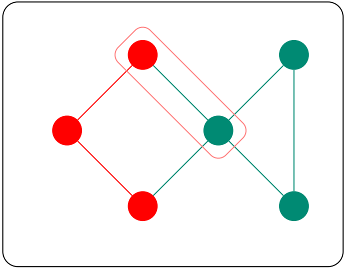

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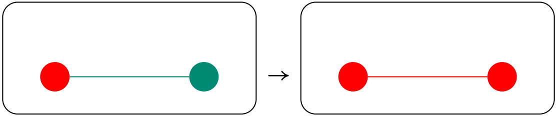

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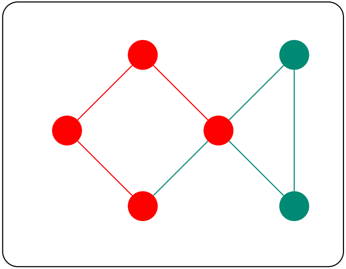

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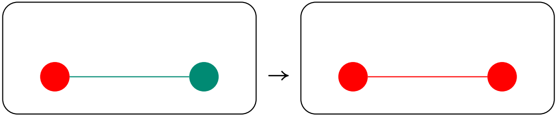

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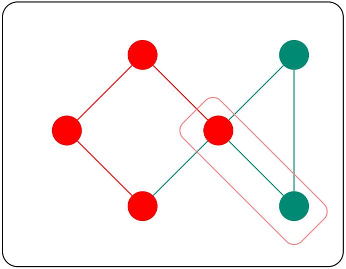

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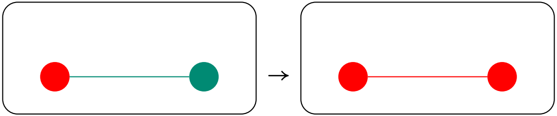

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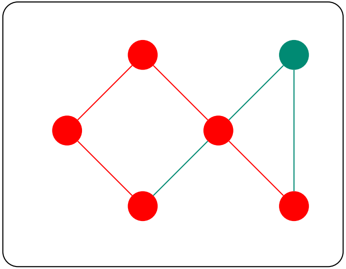

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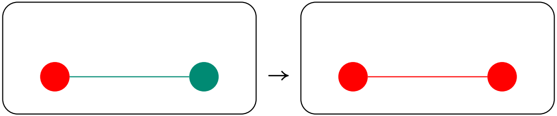

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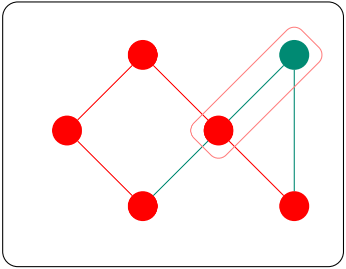

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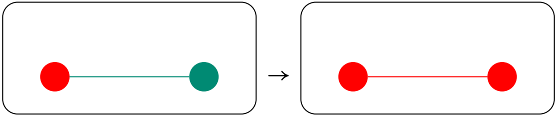

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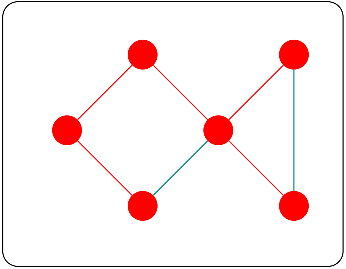

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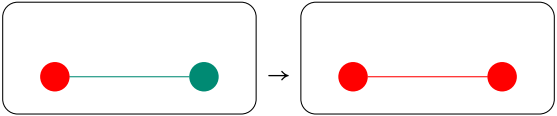

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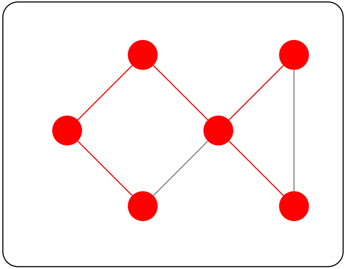

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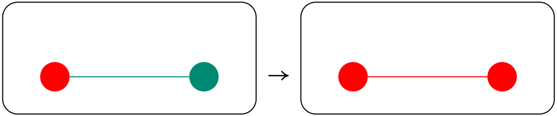

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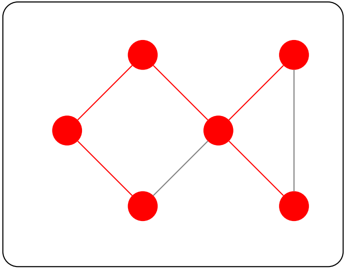


Does the transformation process terminate for any initial graph?

Termination of Graph Transformation Systems

- No graph G_0 can be transformed forever

$$G_0 \Rightarrow G_1 \Rightarrow \cdots$$

- Aligns with the notion of program termination:

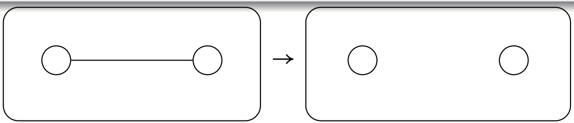
  "every execution (on any input) halts."

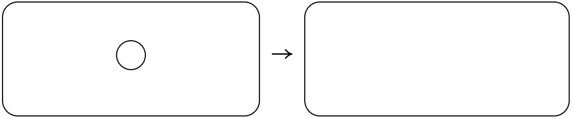
- Undecidable in general [Plu95]
  - Automated techniques for specific subclasses

Termination by interpretations

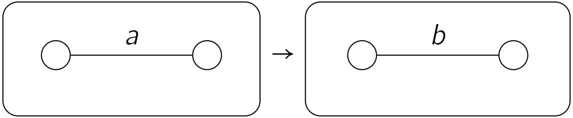
Interpret graphs as natural numbers.

Show each transformation step decreases the value.

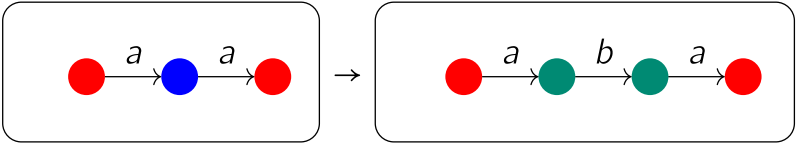
Number of edges: 

Number of nodes: 

Number of edges labeled by a:



Limitation



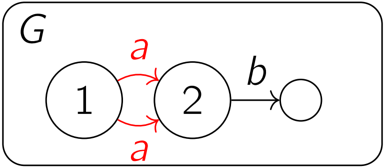
Left-hand side graph: middle has no other incident edges

Right-hand side graph: middles are fresh nodes

Can its termination be proved by interpretations?

- Weighted Type Graph Method
- Need a more powerful definition of graph transformations.

Graphs: Finite, Directed, Edge-Labeled Multigraphs

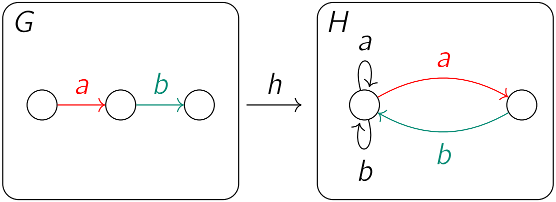


Edges with the same source, target and label are permitted.

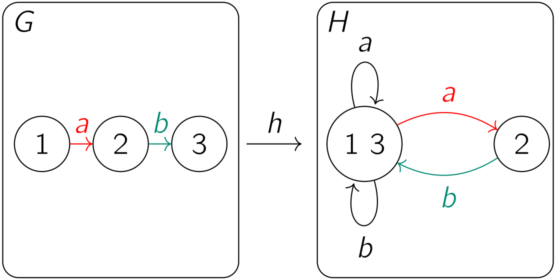
G: graph name

Numbers inside nodes: identifiers

Graph Morphisms: Structure-Preserving Functions



Colors show edge correspondence.



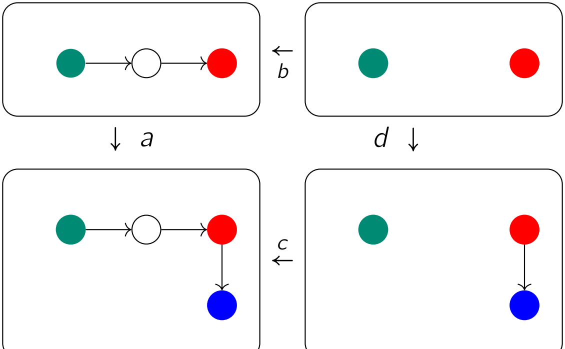
Numbers show node correspondence.

h : morphism name

Commutative Diagram

$$\begin{tikzcd} L \arrow[d, "a"] & K \arrow[l, "b"'] \arrow[d, "d"] \\ G & C \arrow[l, "c"] \end{tikzcd}$$

commutes if $a \circ b = c \circ d$.



Pushouts: Gluing Graphs Along a common part

The **pushout** of (α, β) is

$$\begin{tikzcd} & A \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ B & & C \end{tikzcd}$$

Pushouts: Gluing Graphs Along a common part

The **pushout** of (α, β) is (β', α') with

- $\square$ABDC commutes,

$$\begin{tikzcd}
 & A \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\
B \arrow[dr, "\beta'"'] \arrow[rr, phantom, "\mathrm{PO}"] & & C \arrow[dl, "\alpha'"] \\
 & D &
\end{tikzcd}$$

Pushouts: Gluing Graphs Along a common part

The **pushout** of (α, β) is $\color{red}{(\beta', \alpha')}$ with

- $\square$ABDC commutes,
- universality: for all (γ, γ'),if $\square ABEC$ commutes, then there is a unique δ such that $\triangle BDE$ and $\triangle CDE$ both commute.

$$
\begin{tikzcd}
 & A \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] \arrow[dd, phantom, "\color{red}{\mathrm{PO}}"] & \\
B \arrow[dr, red, "\beta'"'] \arrow[ddr, bend right, "\gamma'"'] & & C \arrow[dl, red, "\alpha'"] \arrow[ddl, bend left, "\gamma"] \\
 & \color{red}{D} \arrow[d, dashed, "\delta"] & \\
 & E &
\end{tikzcd}
$$

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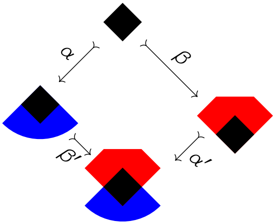

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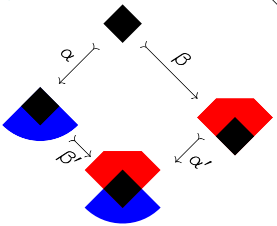
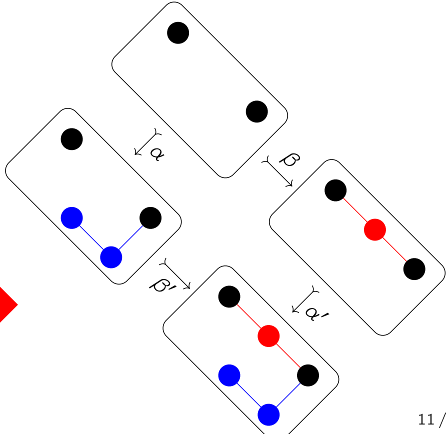
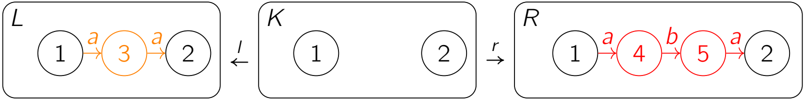




Graph Rewriting with Double-Pushout (DPO)

$$L \xleftarrow{l} K \xrightarrow{r} R$$

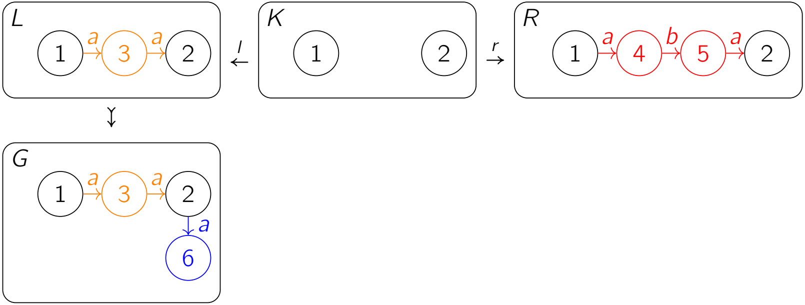
Rewriting rule with interface K



Graph Rewriting with Double-Pushout (DPO)

$$\begin{tikzcd} L \arrow[d, tail] & K \arrow[l, "l"'] \arrow[r, "r"] & R \\ G & & \end{tikzcd}$$

Rewriting rule with interface K



Graph Rewriting with Double-Pushout (DPO)

$$\begin{tikzcd} L \arrow[d, tail] \arrow[dr, phantom, "\mathrm{PO}"] & K \arrow[l, "l"'] \arrow[r, "r"] \arrow[d] & R \\ G & C \arrow[l] & \end{tikzcd}$$

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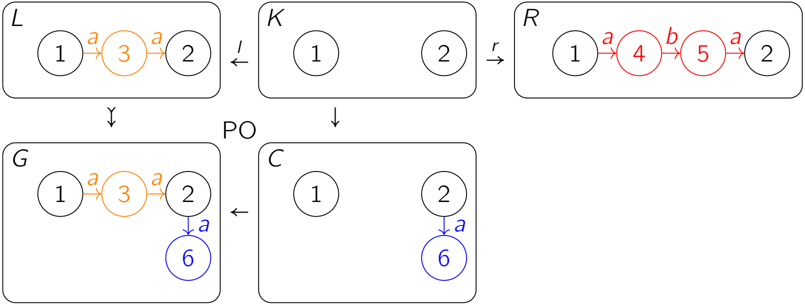

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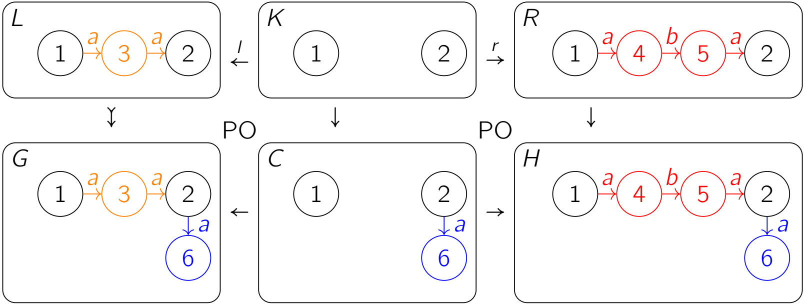
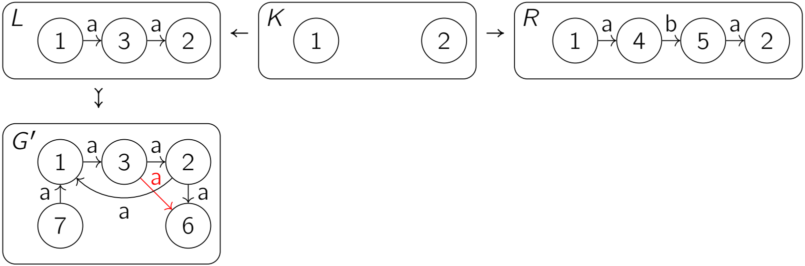
G & C \arrow[l] \arrow[r] & H
\end{tikzcd}$$

Rewriting rule with interface K

rewriting step $G \Rightarrow H$



An Invalid Rewriting Step



No implicit edge deletion by construction

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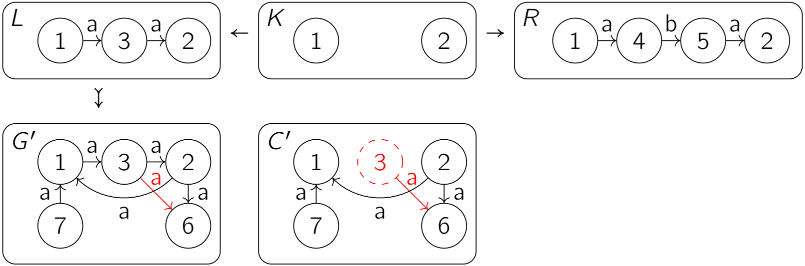


No implicit edge deletion by construction

Toward greater usability

Toward greater power

LyonParallel—A Tool for Termination of Graph Rewriting

Weighted Type Graph Method [Bru14; Bru+15; EO24b]

Termination by interpretation

Parameter: an object T in the category, called type graph

Terminology: every graph is "typed" as morphisms to T

Interpretation:

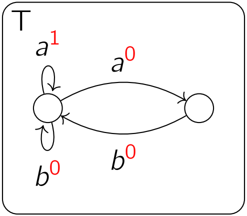
$$\begin{aligned} G &\rightsquigarrow \mathcal{F}(G, T) \\ &\rightsquigarrow \mathrm{weight}(\mathcal{F}(G, T)) \\ &\rightsquigarrow \mathrm{aggregator}(\mathrm{weight}(\mathcal{F}(G, T))) \in \mathbb{N} \end{aligned}$$

What is the morphism weight?

What is the graph weight?

Weighted Type Graph

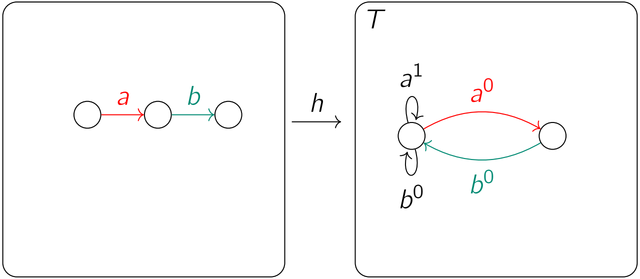
A weighted type graph is a graph with weights on edges.



Morphism Weight

The weight of a morphism $h : G \to T$ is

$$\sum_{e \in \mathsf{Edge}(\mathsf{G})} \mathrm{weight}(h(e))$$

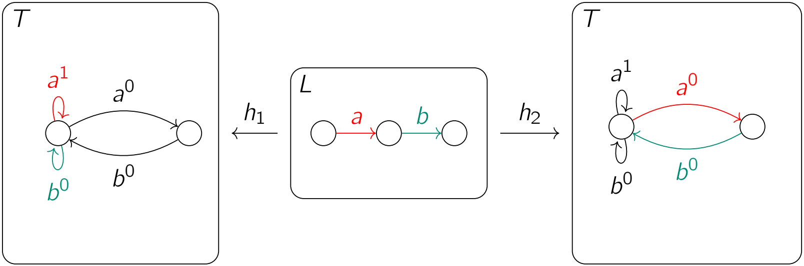


$\mathrm{weight}_T(h) = 0 + 0 = 0$

Graph Weight

The weight of a graph L is

$$\min\{h : L \to T \mid \mathrm{weight}_T(h)\}$$



$\mathrm{weight}_T(h_1) = 1 + 0 = 1$

$\mathrm{weight}_T(h_2) = 0 + 0 = 0$

$\mathrm{weight}_T(L) = \min\{1, 0\} = 0$

Termination Criterion [Bru+15]

$$\begin{tikzcd} L \arrow[dr, dashed, "t_L"'] & K \arrow[l, "l"'] \arrow[r, "r"] \arrow[d, "t_K"] & R \arrow[dl, dashed, "t_R"] \\ & T & \end{tikzcd}$$

Every rewriting step strictly decreases the weight if

- for all t_K, if there is t_L such that $\triangle KLT$ commutes, then

$$\min\{\text{weight}_T(t_L) \mid t_L.\ \triangle\ KLT \text{commutes}\}$$
$$> \min\{\text{weight}_T(t_R) \mid t_R.\ \triangle\ KRT \text{commutes}\}$$

How to find a suitable weighted type graph?

Searching for Weighted Type Graphs over $\mathbb{N}$

Restricted search space:

- no parallel edges of the same label

User-specified parameters:

- k nodes
- maximum edge weight $n \in \mathbb{N}$

The problem amounts to checking the satisfiability of an existential Presburger arithmetic theory with:

- $k^2 m$ binary variables where m is the number of labels
- $k^2 m$ integer variables

Challenge:

- $2^{k^2 m} \cdot n^{k^2 m}$ possible assignments of weights
- maximum edge weight hard to guess

Problem of the Size of the Search Space

With natural numbers as weights:

# nodes (k)	# labels (m)	# weights	# possibilities
2	2	2	$\approx 10^4$
3	3	3	$\approx 10^{21}$
3	3	10	$\approx 10^{45}$
3	3	100	$\approx 10^{87}$
4	4	4	$\approx 10^{57}$
4	4	10	$\approx 10^{95}$
4	4	100	$\approx 10^{181}$

Problems can solved by Z3 in exponential-time with respect to the number of variables $2k^2m$.

Idea

Using positive real numbers as weights

Additional constraint: there is $\delta > 0$ such that every rewriting step decreases the weight by at least δ.

Searching for Weighted Type Graphs over $\cancel{\mathbb{N}}$ $\mathbb{R}$

User-specified parameters:

- k nodes
- ~~edge weights in $\{0, 1, \ldots, n\}$~~

Assumption:

- no parallel edges of the same label

The problem amounts to checking the satisfiability of an ~~existential Presburger arithmetic theory~~ existential theory of the reals with binary variables:

- $k^2 m$ binary variables where m is the number of labels
- $k^2 m$ ~~integer~~ real variables

Challenge:

- ~~there are $2^{k^2 m} \cdot n^{k^2 m}$ possible assignments of weights~~. There are $2^{k^2 m}$ linear programs which have polynomial-time average-case complexity.

Complexity Comparison

With weights in $\mathbb{N}$:

# nodes (k)	# labels (m)	# weights	# possibilities
2	2	2	$\approx 10^4$
3	3	3	$\approx 10^{21}$
3	3	10	$\approx 10^{45}$
3	3	100	$\approx 10^{87}$
4	4	4	$\approx 10^{57}$
4	4	10	$\approx 10^{95}$
4	4	100	$\approx 10^{181}$

With weights in $\mathbb{R}$:

# nodes (k)	# labels (m)	# variables	# linear programs in $\mathbb{R}$
2	2	8	$\approx 10^2$
3	3	27	$\approx 10^8$
4	4	64	$\approx 10^{19}$

Linear programs can be solved in polynomial time with respect to the number of variables on average.

Experimental Results

	A	a	T	t	N	n
[EO24a, Example 6.3]					2.74	1.16
[EO24a, Example D.3]	2.25	1.18			2.24	1.18
[Plu95, Example 3.8]	2.95	1.90	2.94	1.87	3.49	1.87
[Plu18, Example 4]	4.26	3.19	4.24	3.13	5.82	timeout
[Plu18, Example 5]	5.54	5.55	5.53	5.50	9.11	5.62
[Bru+15, Example 4]	2.44	2.46	2.47	2.54	4.58	2.46
[Bru+15, Example 5]					7.80	timeout
[Bru+15, Example 6]					9.75	timeout
[Bru14, Example 1]	2.26	1.18			2.24	1.18
[Bru14, Example 4]	2.25	1.22	2.24	1.18	2.25	1.19
[Bru14, Example 5]	4.23	3.23	4.25	3.28	5.82	timeout

"A", "T", "N" : different configurations with weights over the natural numbers. "a", "t", "n" : corresponding configurations over the real numbers.

Implementation

LyonParallel

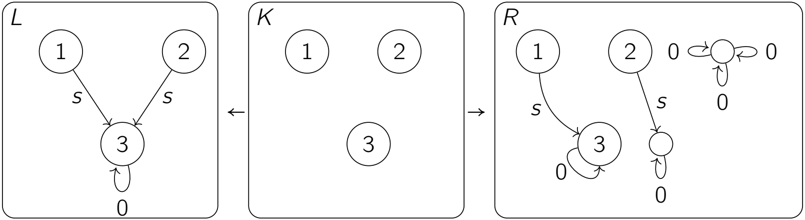
Tool in Ocaml

Relative termination

Search parallel with 6 configurations

Z3 for constraint solving

A Limitation of the Weighted Type Graph Method



All existing automated methods fail.

Intuition: the number of morphisms from $\bullet \xrightarrow{s} \bullet \xleftarrow{s} \bullet$ strictly decreases.

Toward greater usability

Toward greater power

LyonParallel—A Tool for Termination of Graph Rewriting

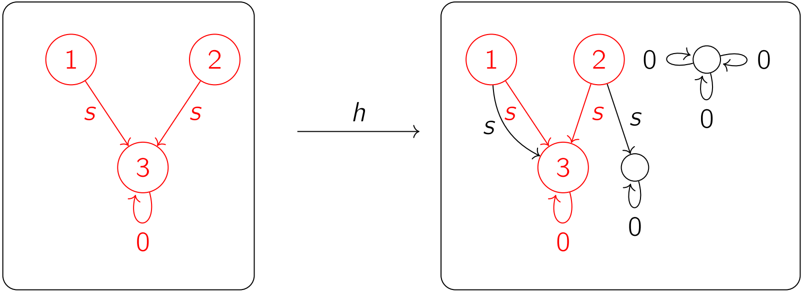
Morphism Counting

Termination by interpretation

Parameter: a graph X

Interpretation of a graph G : number of morphisms from X to G

Inclusions: morphisms h with $h(x) = x$ for all x.

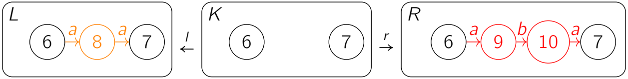


Subgraph

Graph Rewriting with Injective Rules

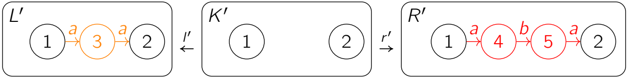
A rewriting rules consists of two inclusions.

$\varphi =$

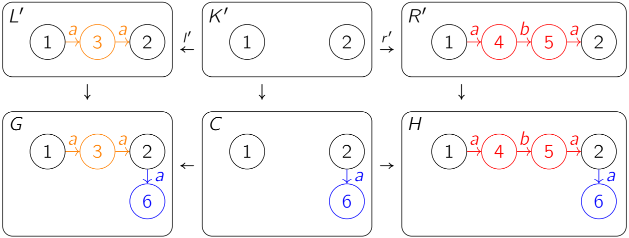


An equivalent rewriting rule expresses the same transformation.

$\varphi' =$

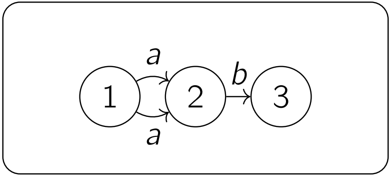


A rewriting step with φ is defined by a DPO diagram with inclusions and φ'.

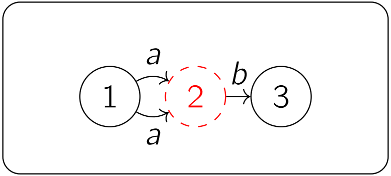


Pre-Graphs

Graph:



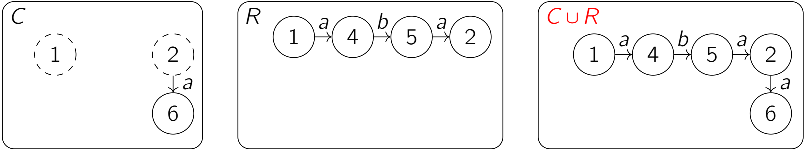
Pre-graphs obtained by removing node 2:



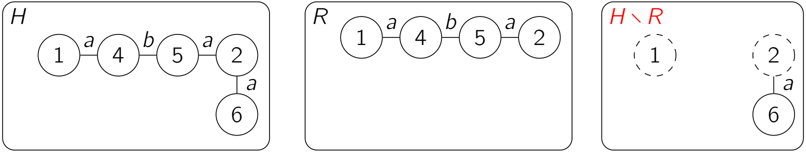
All edges are dangling.

Pre-Graph Operations

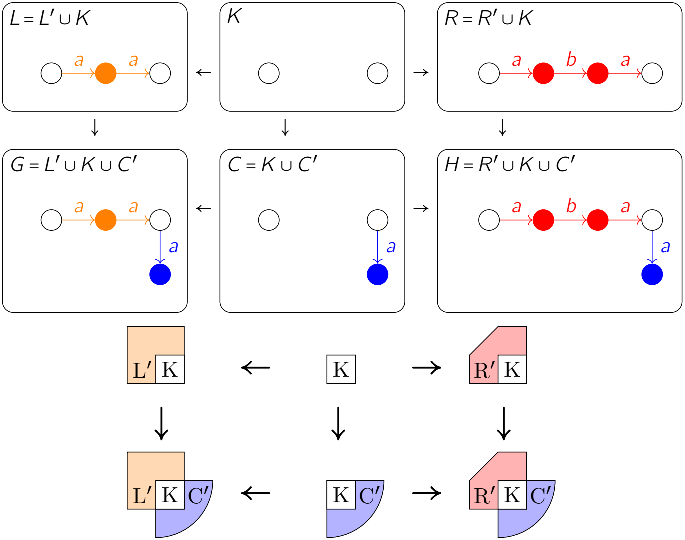
Union of two pre-graphs $C \subseteq G$ and $R \subseteq G$, denoted $C \cup R$.



Relative complement of R in H where $R \subseteq H$, denoted $H \smallsetminus R$.



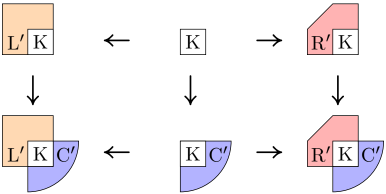
Decomposition of Graphs in Rewriting Steps



This coloring provides a classification of morphisms in rewriting steps by image node colors.

Morphisms by Image Node Colors

An X-occurrence is an injective morphism from X.

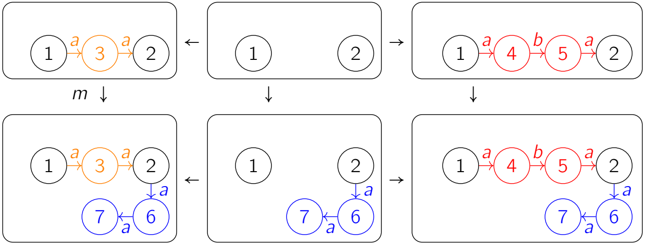


X-morphisms are classified by the colors of their image nodes:

- orange: only white and orange nodes;
- red: only white and red nodes;
- blue: only white and blue nodes;
- blue-and-orange: blue and orange nodes;
- blue-and-red: blue and red nodes.

Morphisms by Image Node Colors

Let X be the graph $\bullet \xrightarrow[a]{} \bullet \xrightarrow[a]{} \bullet$.



Blue X-morphisms: $2 \xrightarrow{a} 6 \xrightarrow{a} 7$

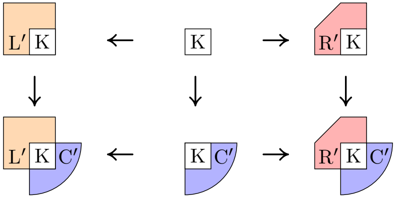
Orange X-morphisms: $1 \xrightarrow{a} 3 \xrightarrow{a} 2$

Blue-and-Orange X-morphisms: $3 \xrightarrow{a} 2 \xrightarrow{a} 6$

Red X-morphisms: none.

Blue-and-red X-morphisms: $5 \xrightarrow{a} 2 \xrightarrow{a} 6$

A Sufficient Condition for Termination

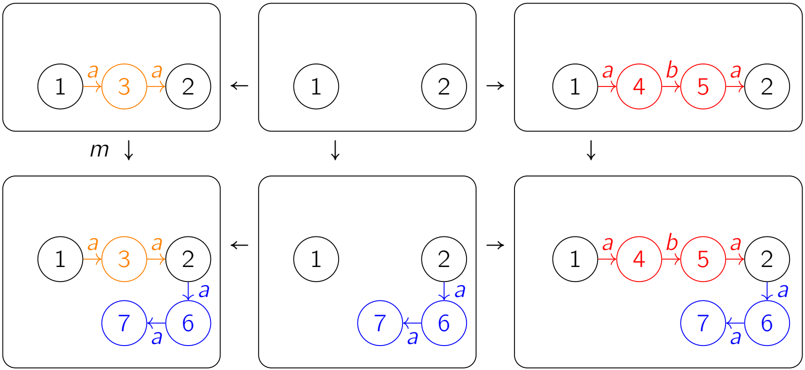


A rule terminates if, for every rewriting step, there are

- strictly more orange X-morphisms than red X-morphisms,
- more blue-and-orange X-morphisms than blue-and-red X-morphisms.

Challenge: C' is unknown before rewriting.

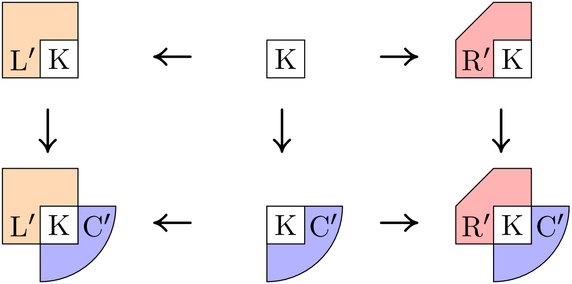
Analysis of Implicit Occurrences

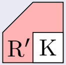


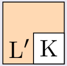
Blue-and-red X-morphisms: $5 \xrightarrow{a} 2 \xrightarrow{a} 6$

Blue-and-Orange X-morphisms: $3 \xrightarrow{a} 2 \xrightarrow{a} 6$

A Sufficient Condition for the Second Condition



There more blue-and-orange X-morphisms than blue-and-red X-morphisms, if all subgraphs of 

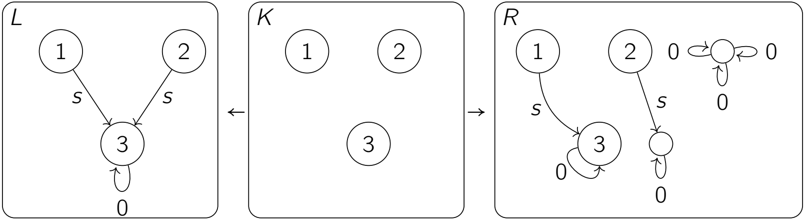
that can form an implicit X-occurrence in some rewriting step can be mapped to distinct subgraphs in 

while preserving elements in $\boxed{\mathrm{K}}$.

Imcomparable with Existing Methods

Fail in some cases where other methods succeed.

Succeed in the following case where other methods fail:



Termination proved by counting morphisms from $\bullet \xrightarrow{s} \bullet \xleftarrow{s} \bullet$.

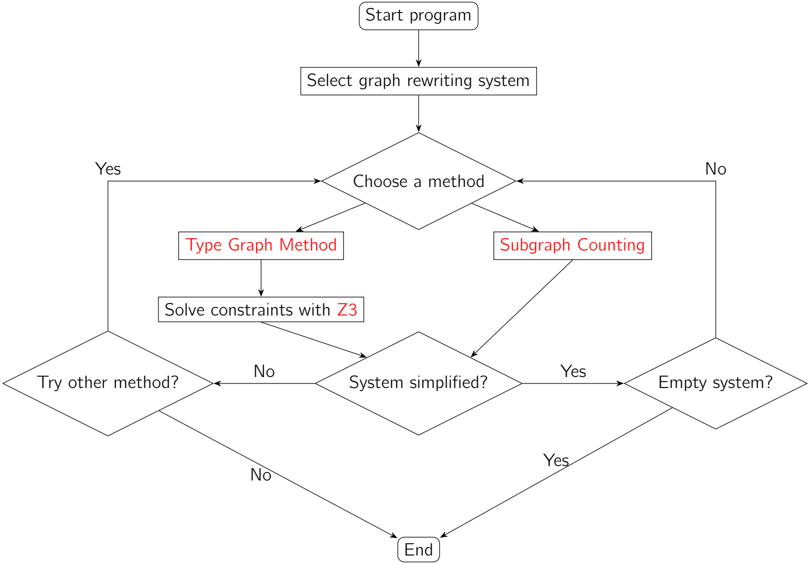
LyonParallel

Automated tool in Ocaml

Iterative elimination of graph rewriting rules

Available : `https://github.com/Qi-tchi/LyonParallel`

Process Flowchart of LyonParallel



Conclusion and Future Work

Contributions

- Extended the Weighted Type Graph Method to improve usability.
- Proposed a termination criterion applicable to new cases.
- Implemented an automated tool for termination analysis.

Future work

- Formally verify the methods.
- Generalize Morphism Counting Method to Multiple Forbidden Contexts.
- Extend the approach to other rewriting frameworks.

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[EO24a] J. Endrullis and R. Overbeek. *Generalized Weighted Type Graphs for Termination of Graph Transformation Systems*. 2024. arXiv: 2307.07601v2 [cs.LO].

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