Automated Termination Proving: Contributions to Graph Rewriting via Extended Weighted Type Graphs and Morphism Counting

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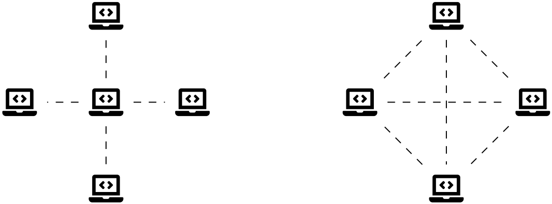
Motivation & Goal

Centralized systems and distributed systems:



Failures can be catastrophic: 

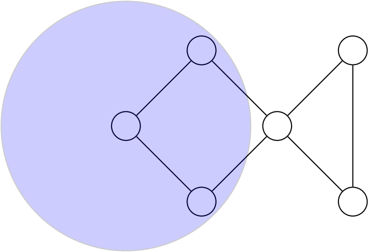
Ensuring correctness is hard.

- distributed algorithms are complex
- The Needham-Schroeder protocol flaw (1978-1995)

This thesis: automated verification.

- mathematically rigorous
- Minimal user effort

Graph Transformation

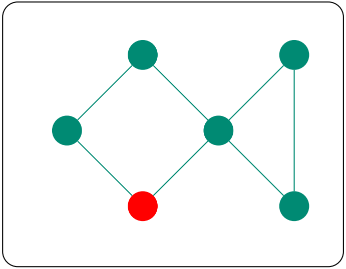


An intuitive modelization of distributed systems:

- computational units → nodes
- communication channels → edges
- system states → graphs
- algorithm behaviors
  → graph transformation rules according to local knowledge

Graph Transformation

Configuration of a distributed system:



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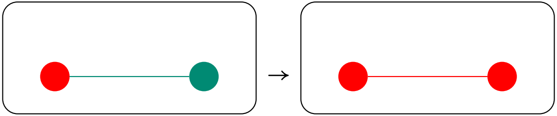

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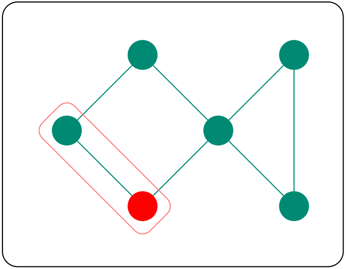

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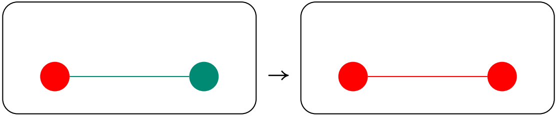

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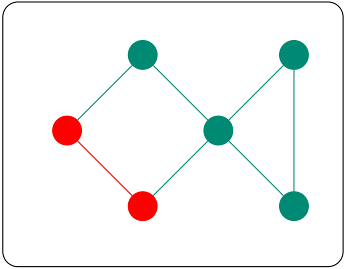

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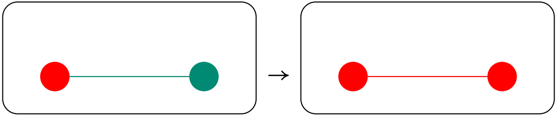

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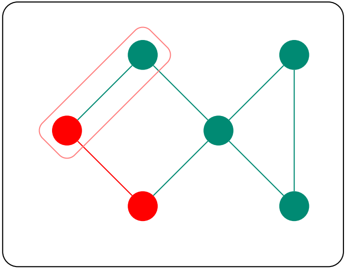

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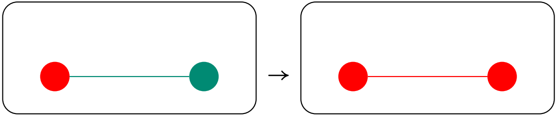

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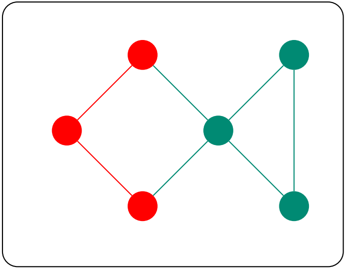

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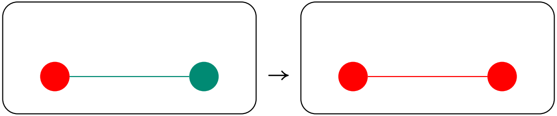

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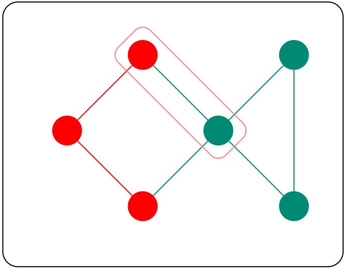

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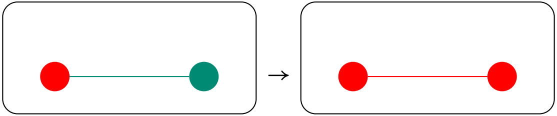

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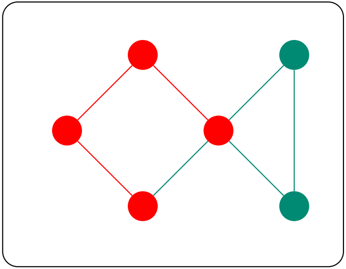

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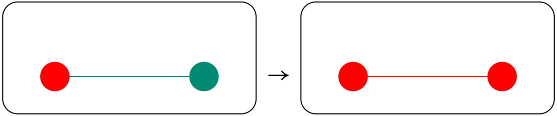

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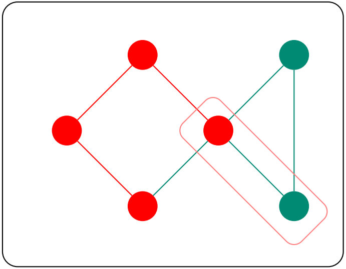

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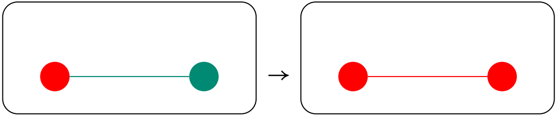

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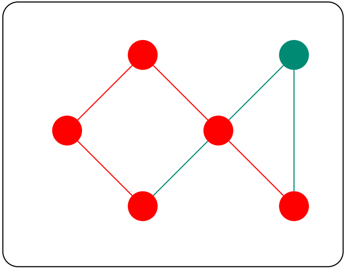

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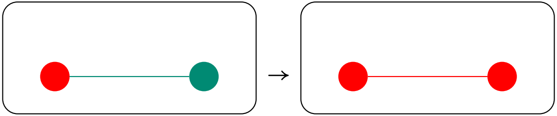

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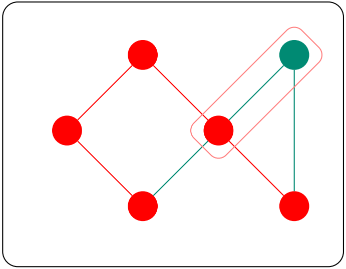

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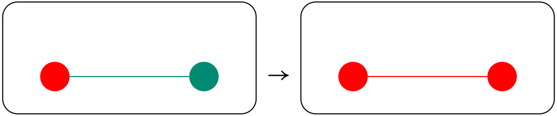

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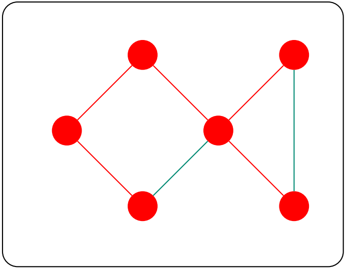

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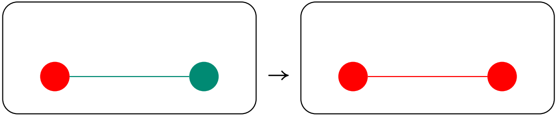
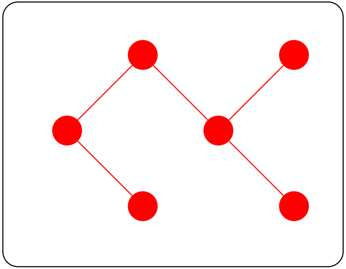


Graph Transformation

A spanning tree is obtained when the rule cannot be applied:



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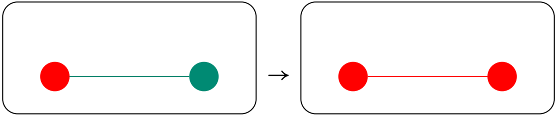

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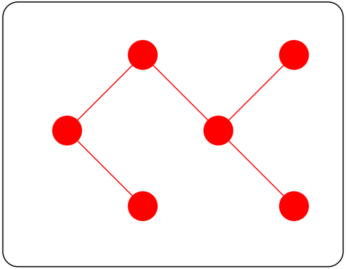

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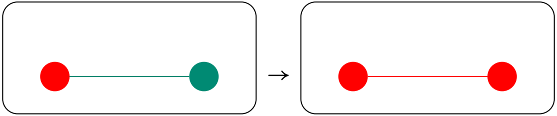


Can a given set of rules transform a given initial graph forever?

Termination

- No graph G_0 can be transformed forever

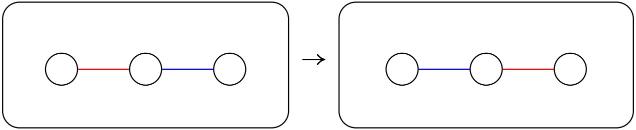
$$G_0 \Rightarrow G_1 \Rightarrow \cdots$$

- Aligns with the notion of program termination:

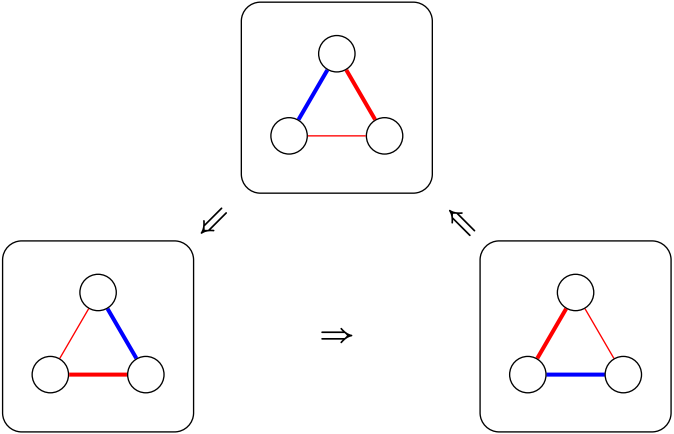
  "every execution (on any input) halts."

- Undecidable in general
- How to prove termination automatically?

A non-termination case



Loop:

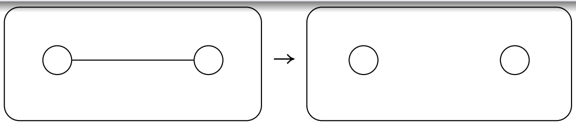


Automated Termination Proofs

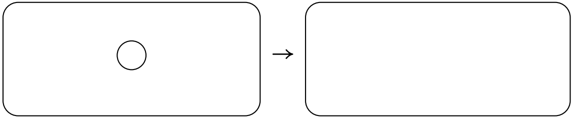
Termination by interpretations:

- interpret graphs as natural numbers;
- show that each transformation step decreases the value.

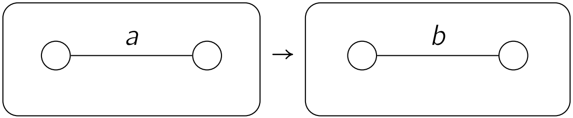
Number of edges:



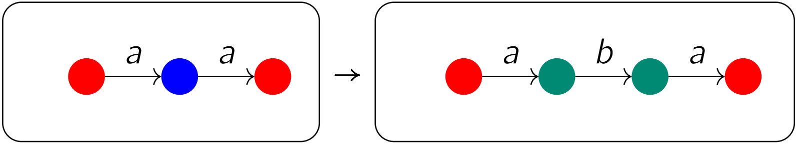
Number of nodes:



Number of edges labeled by a:



Limitation of Trivial Interpretations



Left-hand side graph: middle has no other incident edges

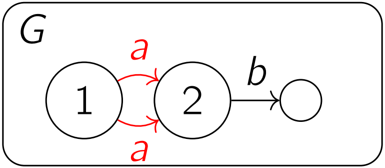
Right-hand side graph: middles are fresh nodes

Termination: number of $\circ \xrightarrow{a} \circ \xrightarrow{a} \circ$ decreases

Can its termination be proved by interpretations?

- Weighted Type Graph Method
- Need a more powerful definition of graph transformations.

Graphs: Finite, Directed, Edge-Labeled Multigraphs

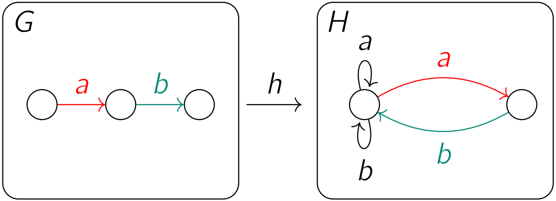


Edges with the same source, target and label are permitted.

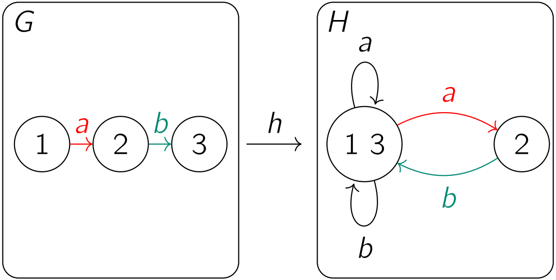
G: graph name

Numbers inside nodes: identifiers

Graph Morphisms: Structure-Preserving Functions



Colors show edge correspondence.



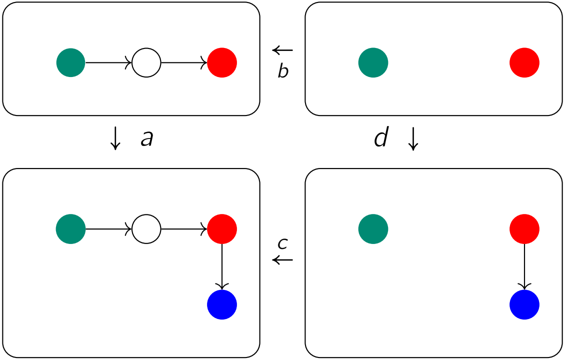
Numbers show node correspondence.

h : morphism name

Commutative Diagram

$$\begin{tikzcd} L & K \arrow[l, "b"'] \arrow[d, "d"'] \\ G & C \arrow[l, "c"] \arrow[from=1-1, to=2-1, "a"] \end{tikzcd}$$

commutes if $a \circ b = c \circ d$.



Pushouts: Gluing Graphs Along a common part

The **pushout** of (α, β) is

$$\begin{tikzcd}[column sep=small] & A \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ B & & C \end{tikzcd}$$

Pushouts: Gluing Graphs Along a common part

The **pushout** of (α, β) is (β', α') with

- $\square$ABDC commutative,

$$\begin{tikzcd} & A \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ B \arrow[dr, "\beta'"'] & \text{PO} & C \arrow[dl, "\alpha'"] \\ & D & \end{tikzcd}$$

Pushouts: Gluing Graphs Along a common part

The **pushout** of (α, β) is (β', α') with

- $\square$ABDC commutative,
- universality: for all (γ, γ'),if $\square ABEC$ commutes, then there is a unique δ such that $\triangle BDE$ and $\triangle CDE$ both commute.

$$
\begin{tikzcd}
& A \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\
B \arrow[dr, "\beta'"'] \arrow[ddr, bend right, "\gamma'"'] & \text{PO} & C \arrow[dl, "\alpha'"] \arrow[ddl, bend left, "\gamma"] \\
& D \arrow[d, dashed, "\delta"] & \\
& E &
\end{tikzcd}
$$

Pushouts: Gluing Graphs Along a common part

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- universality: for all (γ,γ'),if $\square ABEC$ commutes, then there is a unique δ such that $\triangle BDE$ and $\triangle CDE$ both commute.

$$\begin{tikzcd}
& A \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] \arrow[dd, phantom, "PO"] & \\
B \arrow[dr, "\beta'"] \arrow[ddr, bend right, "\gamma'"'] & & C \arrow[dl, "\alpha'"'] \arrow[ddl, bend left, "\gamma"] \\
& D \arrow[d, dashed, "\delta"] & \\
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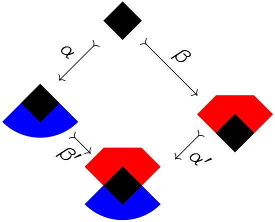

Pushouts: Gluing Graphs Along a common part

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- $\square ABDC$ commutative,
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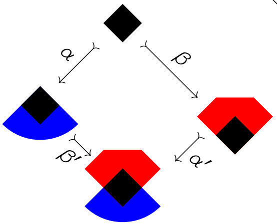
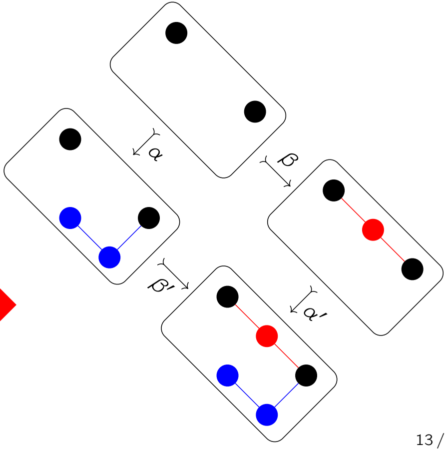
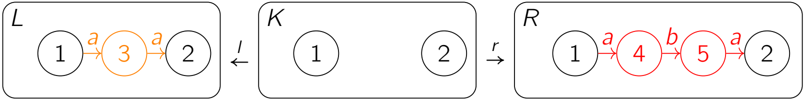




Graph Rewriting with Double-Pushout (DPO)

$$L \xleftarrow{l} K \xrightarrow{r} R$$

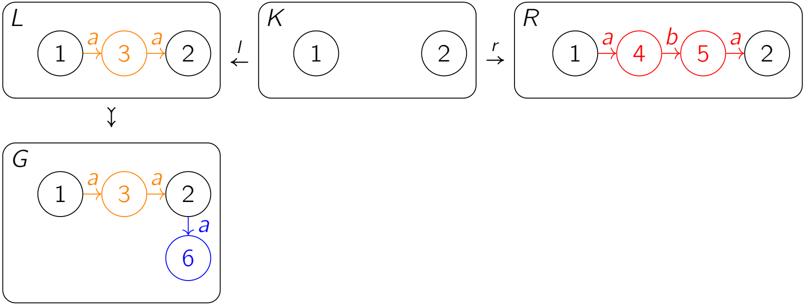
Rewriting rule with interface K



Graph Rewriting with Double-Pushout (DPO)

$$\begin{tikzcd} L \arrow[d, tail] & K \arrow[l, "l"'] \arrow[r, "r"] & R \\ G & & \end{tikzcd}$$

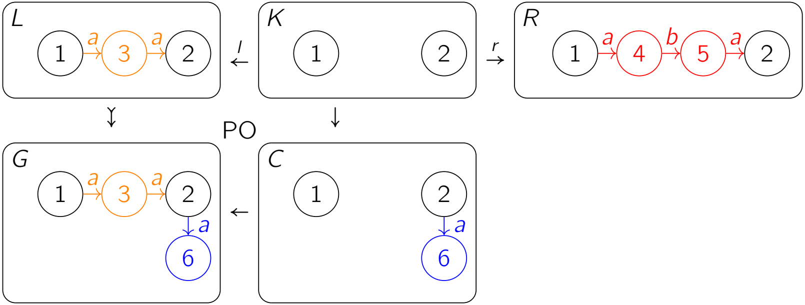
Rewriting rule with interface K



Graph Rewriting with Double-Pushout (DPO)

$$\begin{tikzcd} L \arrow[d, tail] \arrow[dr, phantom, "\mathrm{PO}"] & K \arrow[l, "l"'] \arrow[r, "r"] \arrow[d] & R \\ G & C \arrow[l] & \end{tikzcd}$$

Rewriting rule with interface K



Graph Rewriting with Double-Pushout (DPO)

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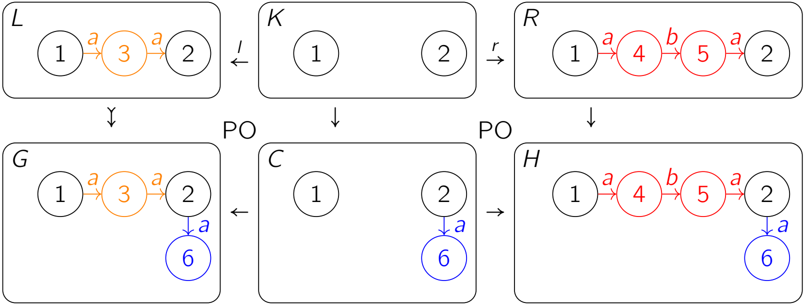
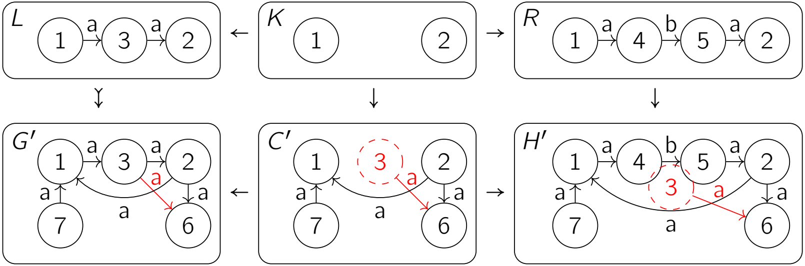
G & C \arrow[l] \arrow[r] & H
\end{tikzcd}
$$

Rewriting rule with interface K

rewriting step $G \Rightarrow H$



An Invalid Rewriting Step



No implicit edge deletion by construction

Weighted Type Graph Method [Bru14; Bru+15; EO24b]

Termination by interpretation

Parameter: an object T in the category, called type graph

Terminology: every graph is "typed" as morphisms to T

Interpretation:

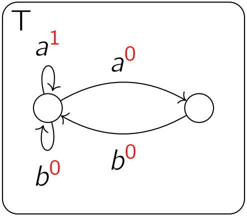
$$\begin{aligned} G &\rightsquigarrow \mathcal{F}(G, T) \\ &\rightsquigarrow \mathrm{weight}(\mathcal{F}(G, T)) \\ &\rightsquigarrow \mathrm{aggregator}(\mathrm{weight}(\mathcal{F}(G, T))) \in \mathbb{N} \end{aligned}$$

What is the morphism weight?

What is the graph weight?

Weighted Type Graph

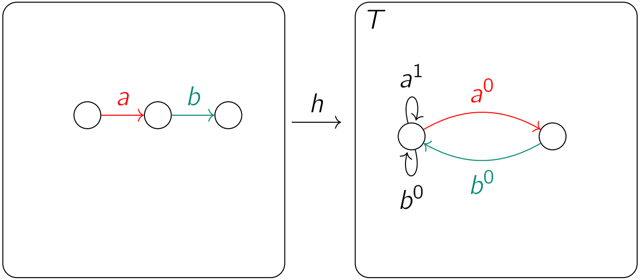
A weighted type graph is a graph with weights on edges.



Morphism Weight

The weight of a morphism $h : G \to T$ is

$$\sum_{e \in \mathsf{Edge}(\mathsf{G})} \mathrm{weight}(h(e))$$

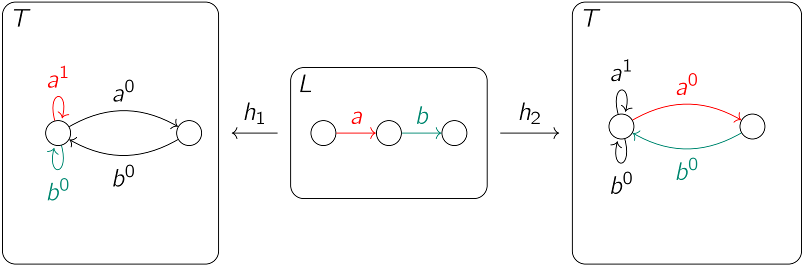


$\mathrm{weight}_T(h) = 0 + 0 = 0$

Graph Weight

The weight of a graph L is

$$\min\{h : L \to T \mid \mathrm{weight}_T(h)\}$$



$\mathrm{weight}_T(h_1) = 1 + 0 = 1$

$\mathrm{weight}_T(h_2) = 0 + 0 = 0$

$\mathrm{weight}_T(L) = \min\{1, 0\} = 0$

Termination Criterion [Bru+15]

$$\begin{tikzcd}
L & & K \arrow[ll, "l"'] \arrow[rr, "r"] \arrow[d, "t_K"] & & R \arrow[dll, dashed, "t_R"] \\
& & T \arrow[from=1-1, dashed, "t_L"'] & &
\end{tikzcd}$$

Every rewriting step strictly decreases the weight if

- for all t_K, if there is t_L such that $\triangle KLT$ commutes, then

$$\begin{aligned}
&\min\{\text{weight}_T(t_L) \mid t_L.\ \triangle\, KLT\,\text{commutes}\}\\
>&\min\{\text{weight}_T(t_R) \mid t_R.\ \triangle\, KRT\,\text{commutes}\}
\end{aligned}$$

How to find a suitable weighted type graph?

Searching for Weighted Type Graphs over $\mathbb{N}$

User-specified parameters:

- k nodes
- maximum edge weight $n \in \mathbb{N}$

Assumption:

- no parallel edges of the same label

The problem amounts to checking the satisfiability of an existential Presburger arithmetic theory with:

- $k^2 m$ binary variables where m is the number of labels
- $k^2 m$ integer variables

Challenge:

- $2^{k^2 m} \cdot n^{k^2 m}$ possible assignments of weights
- maximum edge weight hard to guess

Problem of the Size of the Search Space

With natural numbers as weights:

# nodes (k)	# labels (m)	# weights	# possibilities
2	2	2	$\approx 10^{4}$
3	3	3	$\approx 10^{21}$
3	3	10	$\approx 10^{45}$
3	3	100	$\approx 10^{87}$
4	4	4	$\approx 10^{57}$
4	4	10	$\approx 10^{95}$
4	4	100	$\approx 10^{181}$

Problems can solved by Z3 in exponential-time with respect to the number of variables $2k^2m$.

Idea

Using positive real numbers as weights

Additional constraint: there is $\delta > 0$ such that every rewriting step decreases the weight by at least δ.

Searching for Weighted Type Graphs over $\cancel{\mathbb{N}}$ $\mathbb{R}$

User-specified parameters:

- k nodes
- ~~edge weights in $\{0, 1, \ldots, n\}$~~

Assumption:

- no parallel edges of the same label

The problem amounts to checking the satisfiability of an ~~existential Presburger arithmetic theory~~ existential theory of the reals with binary variables:

- $k^2 m$ binary variables where m is the number of labels
- $k^2 m$ ~~integer~~ real variables

Challenge:

- ~~there are $2^{k^2 m} \cdot n^{k^2 m}$ possible assignments of weights~~. There are $2^{k^2 m}$ linear programs which have polynomial-time average-case complexity.

Complexity Comparison

With weights in $\mathbb{N}$:

# nodes (k)	# labels (m)	# weights	# possibilities
2	2	2	$\approx 10^4$
3	3	3	$\approx 10^{21}$
3	3	10	$\approx 10^{45}$
3	3	100	$\approx 10^{87}$
4	4	4	$\approx 10^{57}$
4	4	10	$\approx 10^{95}$
4	4	100	$\approx 10^{181}$

With weights in $\mathbb{R}$:

# nodes (k)	# labels (m)	# variables	# linear programs in $\mathbb{R}$
2	2	8	$\approx 10^2$
3	3	27	$\approx 10^8$
4	4	64	$\approx 10^{19}$

Linear programs can be solved in polynomial time with respect to the number of variables on average.

Experimental Results

	A	a	T	t	N	n
[EO24a, Example 6.3]					2.74	1.16
[EO24a, Example D.3]	2.25	1.18			2.24	1.18
[Plu95, Example 3.8]	2.95	1.90	2.94	1.87	3.49	1.87
[Plu18, Example 4]	4.26	3.19	4.24	3.13	5.82	timeout
[Plu18, Example 5]	5.54	5.55	5.53	5.50	9.11	5.62
[Bru+15, Example 4]	2.44	2.46	2.47	2.54	4.58	2.46
[Bru+15, Example 5]					7.80	timeout
[Bru+15, Example 6]					9.75	timeout
[Bru14, Example 1]	2.26	1.18			2.24	1.18
[Bru14, Example 4]	2.25	1.22	2.24	1.18	2.25	1.19
[Bru14, Example 5]	4.23	3.23	4.25	3.28	5.82	timeout

"A", "T", "N" : different configurations with weights over the natural numbers. "a", "t", "n" : corresponding configurations over the real numbers.

Implementation

LyonParallel

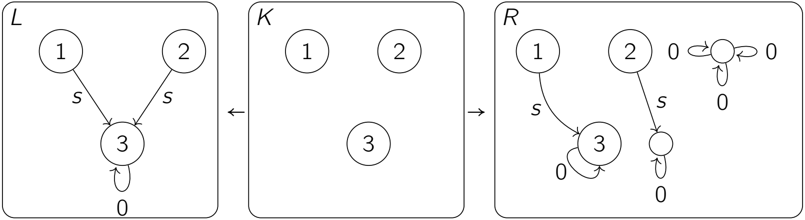
Tool in Ocaml

Relative termination

Search parallel with 6 configurations

Z3 for constraint solving

A Limitation of the Weighted Type Graph Method



All existing automated methods fail.

Intuition: the number of morphisms from $\bullet \xrightarrow{s} \bullet \xleftarrow{s} \bullet$ strictly decreases.

Preliminaries

Toward greater usability

Toward greater power

LyonParallel—A Tool for Termination of Graph Rewriting

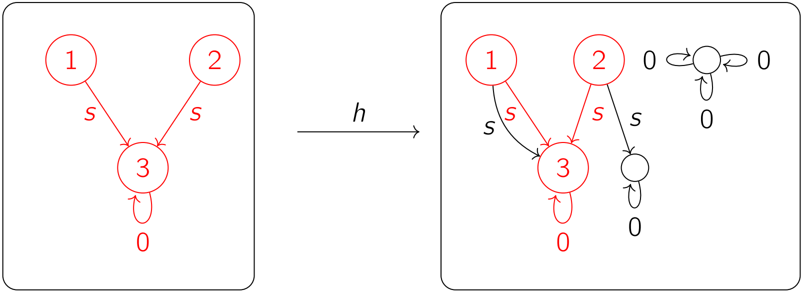
Morphism Counting

Termination by interpretation

Parameter: a graph X

Interpretation of a graph G : number of morphisms from X to G

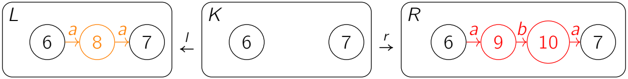
Inclusions: morphisms h with $h(x) = x$ for all x.



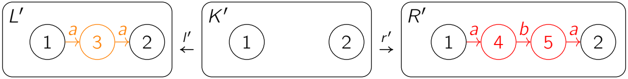
Subgraph

Graph Rewriting with Injective Rules

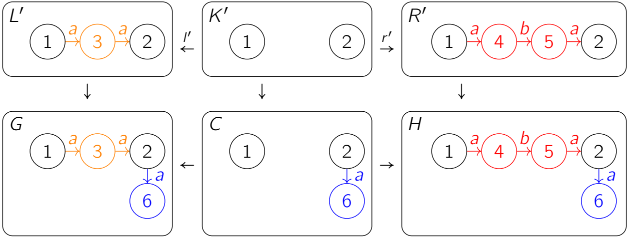
A rewriting rules consists of two inclusions.

$\varphi =$ 

An equivalent rewriting rule expresses the same transformation.

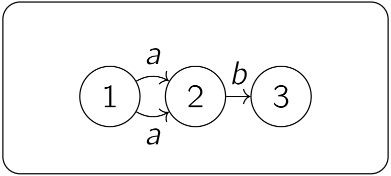
$\varphi' =$ 

A rewriting step with φ is defined by a DPO diagram with inclusions and φ'.

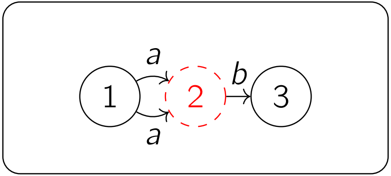


Pre-Graphs

Graph:



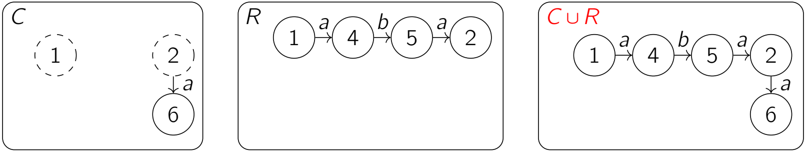
Pre-graphs obtained by removing node 2:



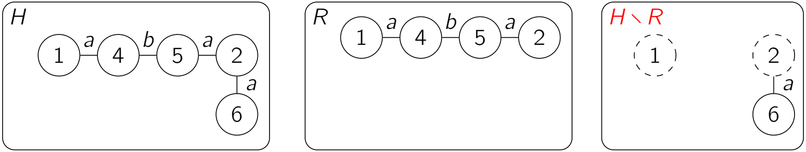
All edges are dangling.

Pre-Graph Operations

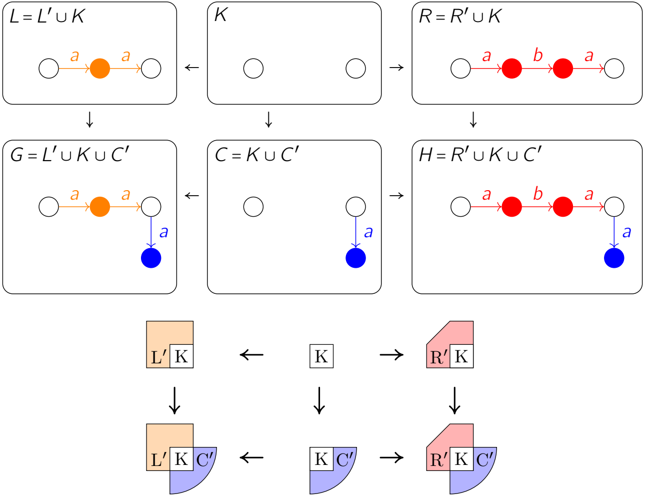
Union of two pre-graphs $C \subseteq G$ and $R \subseteq G$, denoted $C \cup R$.



Relative complement of R in H where $R \subseteq H$, denoted $H \setminus R$.



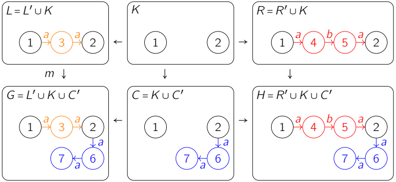
Decomposition of Graphs in Rewriting Steps

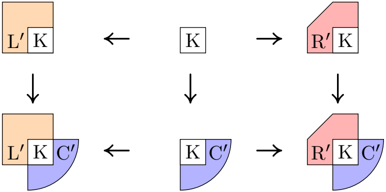


This coloring provides a classification of morphisms in rewriting steps by image node colors.

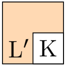
Morphisms by Image Node Colors

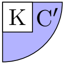
An *X*-occurrence is an injective morphism from *X*.

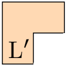


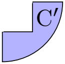


An X-occurrence is

- explicit if $\mathbf{Im}(x)$ is included in 

- shared if $\mathbf{Im}(x)$ is included in 

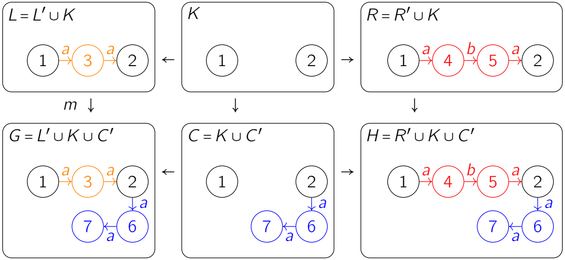
- implicit if $\mathbf{Im}(x)$ has elements in both 

 and 

Similarly, in H.

Implicit, Explicit and Shared Morphisms

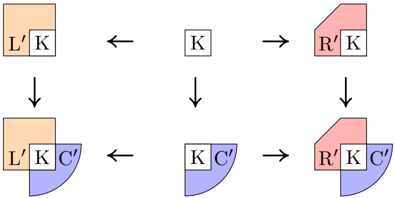
Let X be the graph $\bullet \xrightarrow[a]{} \bullet \xrightarrow[a]{} \bullet$.

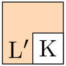


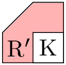
The morphisms from X has their images:

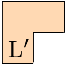
- in G and L: $1 \xrightarrow{a} 3 \xrightarrow{a} 2$
- in H and R: None
- shared by G and H: $2 \xrightarrow{a} 6 \xrightarrow{a} 7$
- formed by subgraphs of L and C: $3 \xrightarrow{a} 2 \xrightarrow{a} 6$
- formed by subgraphs of R and C: $5 \xrightarrow{a} 2 \xrightarrow{a} 6$

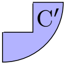
A Sufficient Condition for Termination

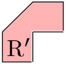


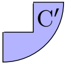
Suppose that there are strictly more X-morphisms with image in 

then in 

, φ terminates if, in every rewriting step, there are more X-morphisms whose image has elements in both 

and 

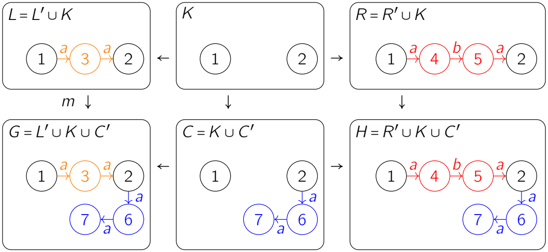
then X-morphisms whose image has elements in both 

and 

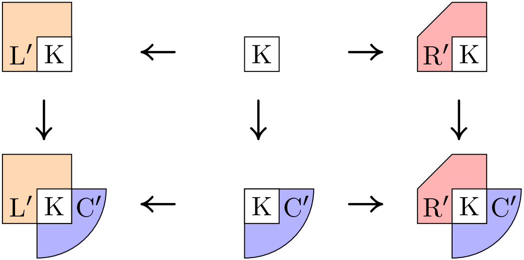
.

Challenge: the pregraph C' is unknown.

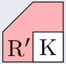
Analysis of Implicit Occurrences

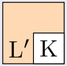


Analysis of Implicit Occurrences



There are more implicit X-occurrences before rewriting, if

- all subgraphs of 

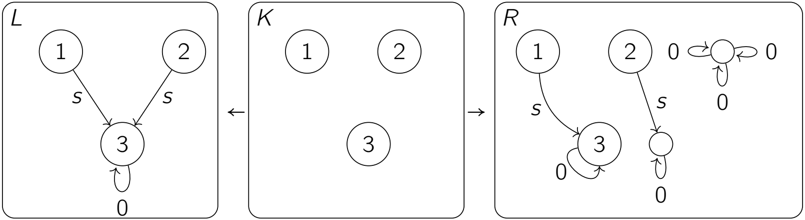
that can form an implicit X-occurrence in some rewriting step can be mapped to distinct subgraphs in 

while preserving the interface elements.

Imcomparable with Existing Methods

Fail in some cases where other methods succeed.

Succeed in the following case where other methods fail.



Termination proved by counting morphisms from $\bullet \xrightarrow{s} \bullet \xleftarrow{s} \bullet$.

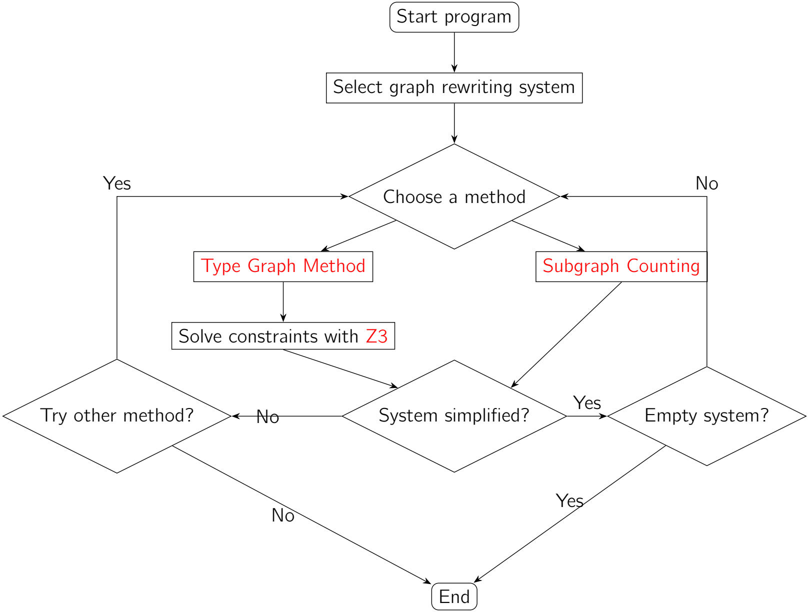
LyonParallel

Automated tool in Ocaml

Iterative elimination of graph rewriting rules

Available : `https://github.com/Qi-tchi/LyonParallel`

Process Flowchart of LyonParallel



Conclusion and Future Work

Contributions

- Extended the Weighted Type Graph Method to improve usability.
- Proposed a termination criterion applicable to new cases.
- Implemented an automated tool for termination analysis.

Future work

- Formally verify the methods.
- Generalize Morphism Counting Method to Multiple Forbidden Contexts.
- Extend the approach to other rewriting frameworks.

References

[Bru+15] H. J. Sander Bruggink et al. "Proving Termination of Graph Transformation Systems using Weighted Type Graphs over Semirings". In: *CoRR* abs/1505.01695 (2015). arXiv: 1505.01695.

[Bru14] H. J. Sander Bruggink. "Towards Process Mining with Graph Transformation Systems". In: *Graph Transformation*. Ed. by Holger Giese and Barbara König. Cham: Springer International Publishing, 2014, pp. 253–268. ISBN: 978-3-319-09108-2.

[EO24a] J. Endrullis and R. Overbeek. *Generalized Weighted Type Graphs for Termination of Graph Transformation Systems*. 2024. arXiv: 2307.07601v2 [cs.LO].

[EO24b] Jorg Endrullis and Roy Overbeek. "Generalized Weighted Type Graphs for Termination of Graph Transformation Systems". In: *Graph Transformation - 17th International Conference, ICGT 2024, Held as*