# On the Global Convergence of Policy Optimization in Deep Reinforcement Learning

Zhaoran Wang
(Joint Work with Qi Cai, Jason D. Lee, Boyi Liu, Zhuoran Yang)

Department of IEMS, Northwestern University

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# Deep Reinforcement Learning: Success & Failure

- success: DL (representation) + RL (decision) = human-level AI
  - play: Atari, Texas hold'em, Go, Dota, Starcraft, Doom, ...
  - control: grasp, move, walk, swim, drive, ...
  - interact: recommend, personalize, chat, ...
  - explore: 3D scene, maze, ...
- failure: convergence, generalization, sample efficiency, reproducibility
  - "deep reinforcement learning that matters" (Henderson et al.)
  - "deep reinforcement learning doesn't work yet" (Irpan)
  - "RL never worked, and 'deep' only helped a bit" (Sahni)
  - "policy gradient is nothing more than random search" (Recht)
  - "are deep policy gradient algorithms truly policy gradient algorithms" (Iyas et al.)
- this talk: convergence & generalization of policy optimization

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# Background on RL & DL

- Markov decision process $(\mathcal{S}, \mathcal{A}, \mathcal{P}, r, \gamma)$, where $s' \sim \mathcal{P}(\cdot \,|\, s, a)$
  - actor: policy $a \sim \pi(\cdot \,|\, s)$
  - critic: action-value function

$$Q^\pi(s,a) := (1-\gamma)\mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t r(s_t, a_t) \,\middle|\, s_0 = s,\ a_0 = a,\ a_t \sim \pi(\cdot \,|\, s_t)\right]$$

  - goal: $\max_\pi J(\pi) := \mathbb{E}_{s\sim\nu}[V^\pi(s)] = \mathbb{E}_{s\sim\nu}[\langle Q^\pi(s,\cdot), \pi(\cdot \,|\, s)\rangle]$

- two-layer neural networks $(b, \alpha, \sigma, m)$ serving as actor & critic

$$u_\alpha(s,a) = \frac{1}{\sqrt{m}} \sum_{i=1}^{m} b_i \sigma([\alpha]_i^\top (s,a))$$

which is randomly initialized & "overparametrized"

- question: global convergence $J(\pi^*) - J(\pi^k) \to 0$?

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- trust region/proximal policy optimization (Schulman et al.)

$$\theta_{k+1} \leftarrow \arg\max_{\theta} \widehat{\mathbb{E}}_{s\sim\nu_k}\Big[\underbrace{\langle Q^{\pi_{\theta_k}}(s,\cdot), \pi_\theta(\cdot\,|\,s)\rangle}_{\text{improvement}} - \beta_k \underbrace{\mathrm{KL}(\pi_\theta(\cdot\,|\,s)\,\|\,\pi_{\theta_k}(\cdot\,|\,s))}_{\text{regularization}}\Big]$$

which is related to REINFORCE, (S)AC, DPG, SBEED, ...

- temporal-difference learning (Sutton; Tsitsiklis & Van Roy)

$$\omega(t+1) \leftarrow \omega(t) - \eta \underbrace{\big(Q_{\omega(t)}(s,a) - r(s,a) - \gamma Q_{\omega(t)}(s',a')\big)}_{\text{Bellman residual}} \nabla_\omega Q_{\omega(t)}(s,a)$$

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# Challenges & Agenda

- two of deadly triad: nonlinearity + bootstrapping + off-policy (Sutton & Barto)
- sources of nonconvexity: "bad" stationary points
  - $J(\pi_\theta)$ nonconvex in $\pi_\theta \in \Delta^{|\mathcal{S}|}$ (infinite dimensions)
  - critic: $Q_\omega$ nonlinear in $\omega$ (finite dimensions)
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- causes of divergence: instability in practice
  - actor + critic: bilevel optimization (Pfau et al.; Heusel et al.)
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# Global Convergence of “Neural” TRPO/PPO

# An Infinite-Dimensional Optimization View

- ideal case: nonconvex infinite-dimensional mirror descent

$$\pi_{k+1} \leftarrow \arg\max_{\pi} \mathbb{E}_{s\sim\nu_k}\big[\langle Q^{\pi_k}(s,\cdot), \pi(\cdot,s)\rangle - \beta_k \mathrm{KL}(\pi(\cdot\,|\,s)\,\|\,\pi_k(\cdot\,|\,s))\big]$$

which factorizes across $\pi(\cdot\,|\,s) \in \Delta$ with $s \in \mathcal{S}$

- geometry via performance difference (Kakade & Langford)

$$0 \geq J(\pi) - J(\pi^*) = (1-\gamma)^{-1}\mathbb{E}_{s\sim\nu^*}\big[\langle \underbrace{Q^{\pi}(s,\cdot)}_{\text{dual}}, \underbrace{\pi(\cdot\,|\,s) - \pi^*(\cdot\,|\,s)}_{\text{primal}}\rangle\big]$$

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# Primal-Dual Geometry: One-Point Monotone

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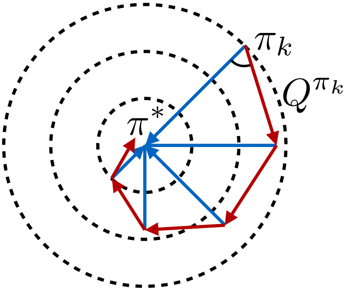


infinite-dimensional mirror descent converges to global optimum!

# Proof Sketch: Sublinear Rate of Convergence

- primal-dual geometry via performance difference (Puterman)

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- one-step descent under one-point monotonicity

$$\begin{aligned} &2(1-\gamma)\beta_k^{-1}(J(\pi^*) - J(\pi_k)) \\ &\quad \leq 2\mathbb{E}_{s\sim\nu^*}\big[\mathrm{KL}(\pi^*(\cdot\,|\,s)\,\|\,\pi_k(\cdot\,|\,s)) - \mathrm{KL}(\pi^*(\cdot\,|\,s)\,\|\,\pi_{k+1}(\cdot\,|\,s))\big] \\ &\qquad + \beta_k^{-2}\mathbb{E}_{s\sim\nu^*}\big[\|Q^{\pi_k}(s,\cdot)\|_\infty^2\big] \end{aligned}$$

- telescoping one-step descent yields global convergence

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# From Infinite Dimensions to Finite Dimensions

- parameterize actor $\pi_k$ (primal) & critic $Q^{\pi_k}$ (dual) as

$$\pi_{\theta_k}(a \,|\, s) \propto \exp(\tau^{-1} u_{\theta_k}(s,a)), \quad Q_{\omega_k} = u_{\omega_k}(s,a)$$

where $u_\alpha(s,a) = \dfrac{1}{\sqrt{m}} \sum_{i=1}^{m} b_i \sigma([\alpha]_i^\top (s,a))$

- incorporating primal & dual errors still gives $1/\sqrt{K}$ rate
  - policy improvement (actor): mean squared error of GD
  - policy evaluation (critic): mean squared Bellman error of TD

- unified analysis for GD & TD: focus on TD in the sequel

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# Global Convergence of "Neural" TD

# TD: Bias & Nonlinearity Leads to Divergence

- policy evaluation by minimizing mean squared Bellman error

$$\min_{\omega} \mathrm{MSBE}(\omega) := \mathbb{E}_{(s,a)\sim\nu_k}\left[\left(Q_\omega(s,a) - \mathcal{T}^{\pi_{\theta_k}} Q_\omega(s,a)\right)^2\right]$$

where $\mathcal{T}^{\pi_{\theta_k}} Q(s,a) := \mathbb{E}_{s'\sim\mathcal{P}(\cdot\,|\,s,a),a'\sim\pi_{\theta_k}(\cdot\,|\,s')}\left[r(s,a) + \gamma Q(s',a')\right]$

- TD: stochastic semigradient descent (Sutton)

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# Overparametrization Tames Divergence

- implicit linearization: linearize $u_\omega(s, a)$ as

$$v_\omega(s, a) := \frac{1}{\sqrt{m}} \sum_{i=1}^{m} b_i \, \mathbb{1}\Big\{[\omega(0)]_i^\top (s, a) > 0\Big\} [\omega]_i^\top (s, a)$$
$$\approx \frac{1}{\sqrt{m}} \sum_{i=1}^{m} b_i \, \mathbb{1}\Big\{[\omega]_i^\top (s, a) > 0\Big\} [\omega]_i^\top (s, a) := u_\omega(s, a)$$

- error of implicit linearization decreases in $m$

$$\mathbb{E}_{(s,a)\sim\nu_k, \omega(0)\sim N(0,I)} \Big[ \big| u_\omega(s, a) - v_\omega(s, a) \big|^2 \Big] \lesssim m^{-1/2}$$

  which implies error of implicit linearization in semigradient

- same role as explicit linearization in nonlinear gradient TD (Maei et al.) $\to$ TD converges to global optimum of MSBE!

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# Overparametrization Tames Divergence

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$$v_\omega(s,a) := \frac{1}{\sqrt{m}} \sum_{i=1}^{m} b_i \, \mathbb{1}\Big\{[\omega(0)]_i^\top (s,a) > 0\Big\} [\omega]_i^\top (s,a)$$

$$\approx \frac{1}{\sqrt{m}} \sum_{i=1}^{m} b_i \, \mathbb{1}\Big\{[\omega]_i^\top (s,a) > 0\Big\} [\omega]_i^\top (s,a) := u_\omega(s,a)$$

- error of implicit linearization decreases in $m$

$$\mathbb{E}_{(s,a)\sim\nu_k, \omega(0)\sim N(0,I)}\Big[\big|u_\omega(s,a) - v_\omega(s,a)\big|^2\Big] \lesssim m^{-1/2}$$

  which implies error of implicit linearization in semigradient

- same role as explicit linearization in nonlinear gradient TD (Maei et al.) $\rightarrow$ TD converges to global optimum of MSBE!

# Proof Sketch: Sublinear Rate of Convergence

- approximate one-point monotonicity via implicit linearization

$$\langle \mathbb{E}[g(t)], \omega(t) - \omega^* \rangle \geq (1-\gamma)\mathbb{E}\big[|u_{\omega(t)}(s,a) - u_{\omega^*}(s,a)|^2\big] + O(m^{-1/4})$$

- telescoping one-step descent yields global convergence of TD

$$\min_{t \in [T]} \mathrm{MSBE}(\omega(t)) \lesssim 1/\sqrt{T} + O(m^{-1/4})$$

with overparametrization: $m \to \infty$

- policy improvement (actor): same rate of convergence for MSE of GD $\to$ global convergence of TRPO/PPO

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- ideal case: TRPO/PPO as infinite-dimensional mirror descent under one-point monotonicity
  - $1/\sqrt{K}$ rate of convergence to global optimal policy
- from infinite to finite dimensions: primal & dual errors of policy improvement (GD) & policy evaluation (TD)
  - TD: overparametrization tames divergence
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# Summary

- policy gradient and natural policy gradient?
  — Neural Policy Gradient Methods: Global Optimality and Rates of Convergence
  (joint work with Lingxiao Wang, Qi Cai, Zhuoran Yang)

- exploration for sample efficiency?
  — Provably Efficient Reinforcement Learning with Linear Function Approximation
  (joint work with Chi Jin, Zhuoran Yang, Micheal Jordan)