Introduction to Dynamical Systems

Consider the following system of N variables that evolve in time through coupled ODEs:

$$\begin{cases} \frac{dx_1}{dt} = f_1(x_1, \dots, x_N) \\ \vdots \\ \frac{dx_N}{dt} = f_N(x_1, \dots, x_N) \end{cases} \longrightarrow \text{What is the qualitative behaviour of this system?}$$

Differential Equations: Find closed-form solutions ~~analytically/numerically~~

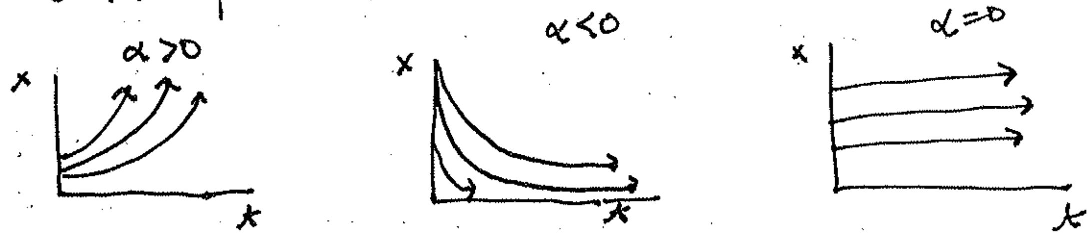
Dynamical Systems: ~~Use~~ Figure out long-term behaviour without solving anything

Example: 1D Linear System

$$\frac{dx}{dt} = \alpha x$$

We already know how to solve this! $x(t) = x(0)e^{\alpha t}$

What is the qualitative behaviour?



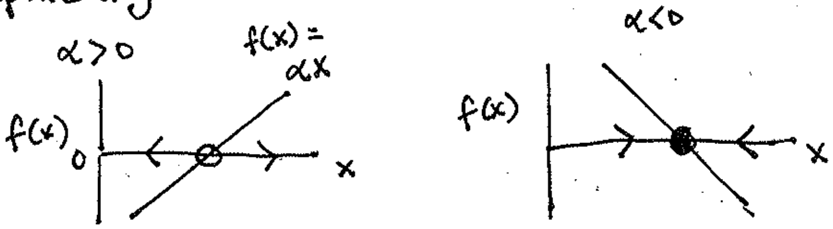
exponential growth	exponential decay	$x(t) = x(0)$
unstable	stable → 0	neutrally stable

How can we analyse this using dynamical systems theory, without explicitly solving the equation?

· 1st note that in each case above, if $x(0) = 0$, $x(t) = 0$

$\boxed{\text{Fixed point: } \frac{dx}{dt} = 0 \quad \text{Zero "velocity"}}$ → In example above, solve $\frac{dx}{dt} = 0$ to get $x^* = 0$

Graphically:



unstable fixed point

stable fixed point

Stable: perturbations get smaller

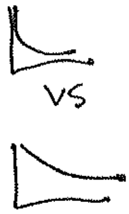
Unstable: perturbations get larger

Neutrally stable: perturbations don't get larger

$\boxed{\text{Attractor: a stable * trajectory *}}$ → not necessarily a fixed point! (we will see later)

General 1D linear equation:

$$\frac{dx}{dt} = \alpha x + \beta \quad \longrightarrow \quad x^* = -\beta/\alpha, \text{ timescale } \tau = \frac{1}{|\alpha|}$$



Can also write as: $\tau \frac{dx}{dt} = -(x - x^*)$ only 1 fixed point

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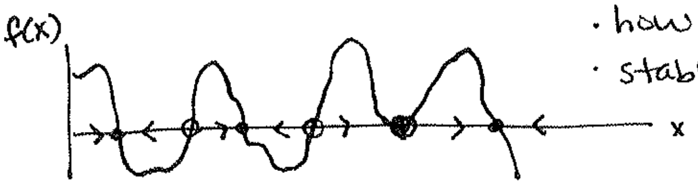
In general, 1D dynamics are defined by fixed points

Very nonlinear example:



· how many fixed points?
· stable or unstable?

This is an example of a multistable system: has many ~~fps~~ attractors

Which initial conditions end up in which attractors? sukbin

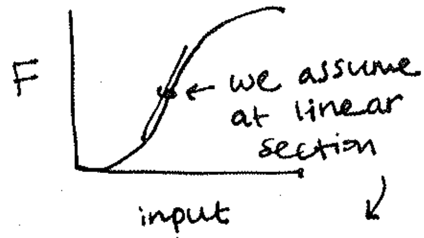
Basin of attraction: range of x for which an initial condition will evolve to a given attractor

* Note that in this nonlinear example, can see easily that the slope $\frac{df}{dx}$ defines the stability of a fixed point *

* Also, note that the basin of attraction is defined by the unstable fixed points. * $\boxed{\text{Bonus example?}}$

Example: Linear rate network with recurrent excitation

$$\underset{\substack{\uparrow \\ \text{intrinsic} \\ \text{time constant}}}{\tau_r} \frac{dr}{dt} = \underset{\substack{\uparrow \\ \text{leak}}}{-r} + F(\underbrace{wr}_{\substack{\text{recurrent} \\ \text{self-} \\ \text{excitation}}} + \underset{\text{input current}}{I(t)})$$



$F(x) = x$

Assuming linearity:

$$\tau_r \frac{dr}{dt} = -r + wr + I(t)$$
$$= -(1-w)r + I(t)$$

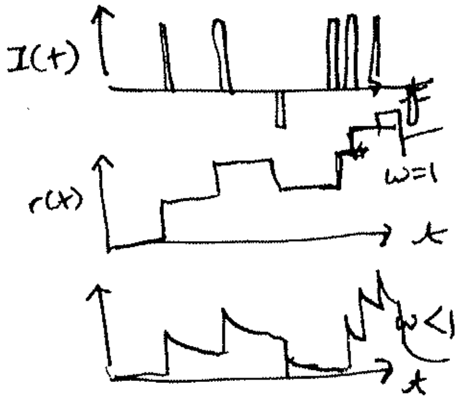
$$\frac{dr}{dt} = -\frac{r}{\tau_{eff}} + \frac{1}{\tau_r} I(t) \quad \text{where } \tau_{eff} = \frac{\tau_r}{|1-w|}$$

→ Recurrent excitation leads to longer effective time constant
(more persistant activity as in Xiao-Jing's lecture)

w=1: Perfect integrator: $\frac{dr}{dt} = \frac{1}{\tau_r} I(t)$

$\rightarrow r(t) = \frac{1}{\tau_r} \int_0^t I(s)\,ds$

(perfectly tuned as in Sukbin's lecture)



w<1: Leaky integrator

Integrates with decay with time constant τ_{eff}

w>1: Unstable integrator

exponential growth with time constant τ_{eff}

Example: Bistability in a single-neuron membrane

Shout out XJ Wang Encyclopedia of Neuroscience

Passive membrane with leak + external input:

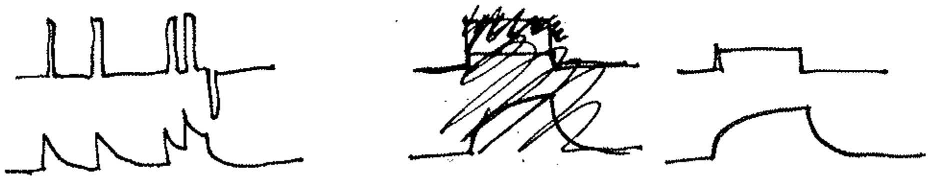
$$C_m \frac{dV}{dt} = -g_L \underbrace{(V - E_L)}_{\text{driving force}} + I_{app}$$

from Ohm's Law $V = IR$ or $I = gV$

C_m: capacitance; $\frac{dV}{dt}$: Δ membrane potential; g_L: conductance; I_{app}: input current (compare to LIF neuron assignment)

Fixed point: $V_* = E_L + \frac{I_{app}}{g_L}$ $\rightarrow$ Stable b/c of sign

Time constant: $\tau_m = C/g_L$



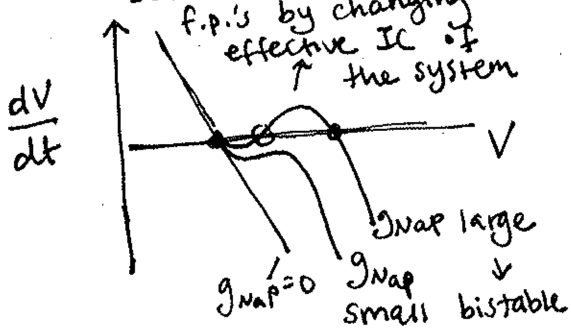
Now, add voltage-dependent persistent sodium (NaP) current to give the system positive feedback

$$C \frac{dV}{dt} = -g_L(V - E_L) - g_{NaP}\, m_{NaP}(V)(V - E_{Na}) + I_{app}$$

$m_{NaP}(V)$: represents opening of channel (voltage-dependent)

sigmoidal

Iapp can switch between stable f.p.'s by changing effective IC of the system



As g_{NaP} increases, system changes from one stable fixed point to two. This is an example of a bifurcation: change in qualitative behaviour as some parameter changes.

$\boxed{\text{BREAK}}$

The Hodgkin-Huxley Model:

- developed in 1952 by H&H to explain action potential generation in the squid giant axon (Nobel prize in 1963)
- most famous/succesful mathematical model in neuroscience
- famously predicted structure of Na + K ion channels

$$C\frac{dV}{dt} = -\underbrace{g_L(V-E_L)}_{I_{leak}} - \underbrace{g_{Na}m^3h(V-E_{Na})}_{I_{Na}} - \underbrace{g_K n^4(V-E_K)}_{I_K} + I_{app}(t)$$

$E_{Na} = 50mV$

$E_K = -77mV$

$V_{rest} = -65mV$

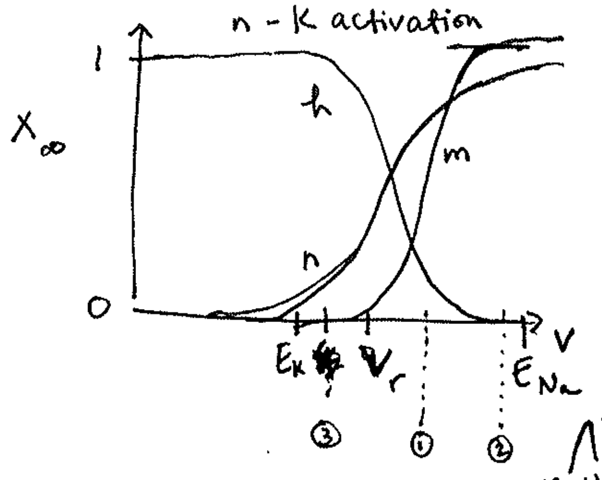
h- Na inactivation

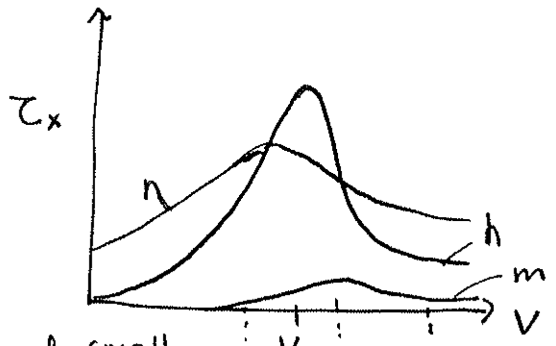
m- Na activation

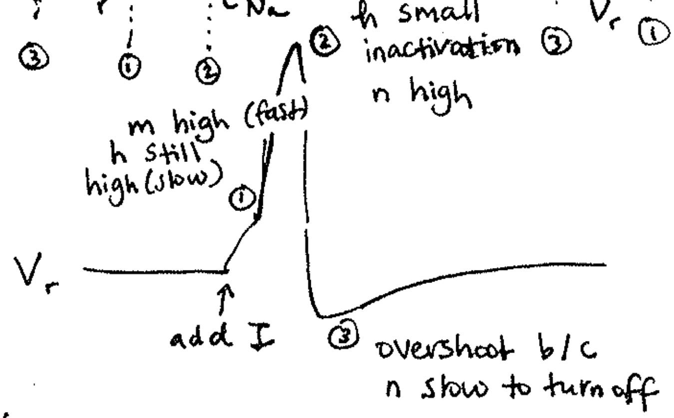
n - K activation

m, n, h all evolve as:

$$\frac{dx}{dt} = \frac{1}{\tau_x(V)}(x - x_\infty(V)) \rightarrow \text{System of 4 ODE's}$$







non linear positive feedback

Let's now use dynamical systems theory to analyze a reduced form of HH dynamics.

Fitzhugh-Nagumo Model: Makes 2 assumptions

1. Assume $m = m_\infty(V)$ since τ_m is small
2. Collapse n, h into same variable since $\tau_n(V) \approx \tau_h(V)$, $h_\infty(V) \sim 1 - n_\infty(V)$

Then convert to unitless form for easier analysis.

$$\underset{4D}{HH} \longrightarrow \underset{2D}{FN}$$

$$\begin{cases} \dfrac{du}{dt} = u - \frac{1}{3}u^3 - w + I & \leftarrow \text{analogous to } V \quad \text{"fast" dynamics} \\ \dfrac{dw}{dt} = \varepsilon(b_0 + b_1 u - w) & \leftarrow \text{analogous to } n \quad \text{"slow" dynamics} \end{cases}$$

$b_1 > 0$

$\varepsilon = \tau/\tau_w$ = ratio of timescales

Assume $\varepsilon < 1$ $\tau_w > \tau$ so u fast, w slow

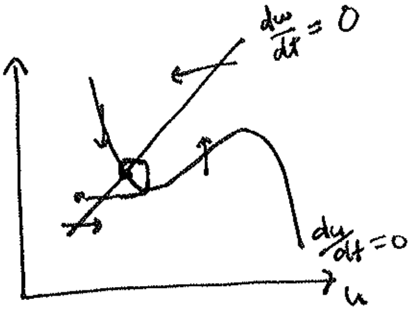
What fixed points are there?

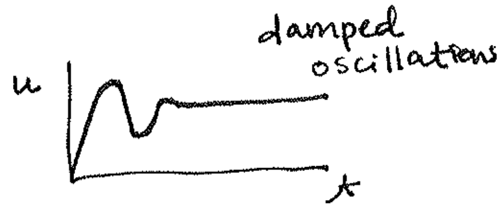
$\frac{du}{dt} = 0 \rightarrow w = u(1 - u^2/3) + I$

$\frac{dw}{dt} = 0 \rightarrow w = b_0 + b_1 u$

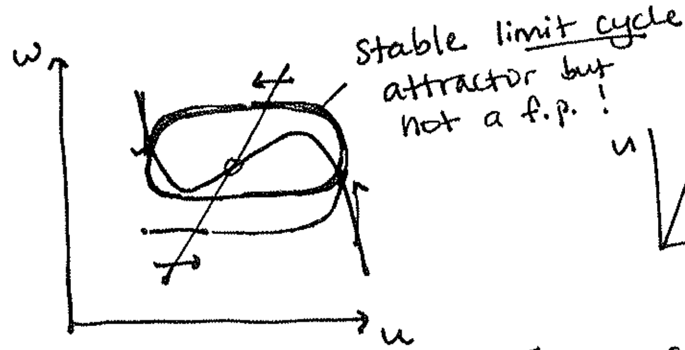
instead of solving for f.p. algebraically, can plot these nullclines and look for intersection

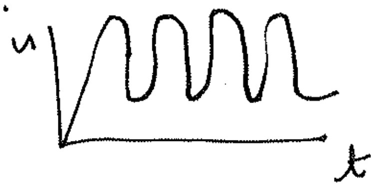
$\boxed{I = 0}$



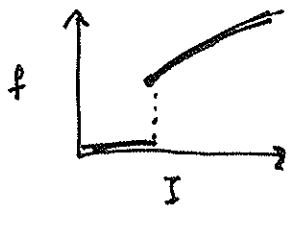


$\boxed{\text{increase } I}$



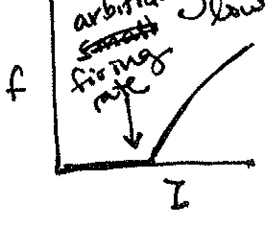


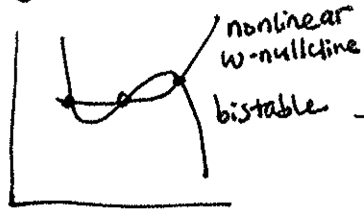
This is an example of Type II neuron

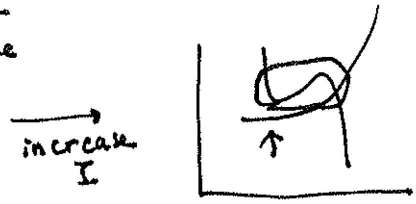


Question: Can you ever have a limit cycle in 1D?

How can you get Type I neuron?







In this bifurcation, 2 f.p.'s collide trajectories "slow" down near this neighbourhood, creating arbitrarily low firing rates