An Analytic Model of Stable and Unstable Orbital Resonance

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The purpose of this paper is to develop a model for orbital resonance, both when it leads to unstable chaotic orbits and when it leads to stable configurations. The study of resonance is a great interest in astronomy as it can give clues to the formation of the early solar system by analyzing where certain objects are, and where there is a characteristic lack of objects. This paper applies elementary techniques from Newtonian mechanics to accurately and reliably predict the relative strengths of Kirkwood gaps in the main asteroid belt. It also builds off of prior work to create a more updated and complete approximation of the liberation period in stable resonant systems such as Saturn's moons Titan and Hyperion.

I. INTRODUCTION

The solar system is a complex dynamic system. In short time scales, it appears to be a reliable and predictable clockwork system. However, on the timescale of millions of years, the pulls from even the tiniest of planets can lead to chaotic patterns. But through this planetary anarchy, stable patterns arise.

Orbital resonance occurs when the orbital periods of two orbiting bodies are in the ratio of two integers. As a result, they will exhibit periodic motion relative to each other. Typically, gentle tugs from other planets can be mostly ignored because the direction of the force is random and after a very long time, the time average net force would be zero. However in resonant orbits, even the tiniest tugs can be amplified to large degrees as they will always occur at the same location in each resonant period.

One of the most well known examples are the Kirkwood Gaps in the asteroid belt (a list of key terms is located in the glossary at the back). At certain semi-major axes, their orbital frequencies will form an integer ratio with that of Jupiter. For example, a resonant frequency of 2:1 means that, for every two orbits the inner asteroid makes, Jupiter will make one orbit. In figure 1, selected resonant frequencies are plotted against a histogram showing the abundance of asteroids in the 2 au to 2.5 au range. At these locations, there is a scarcity of asteroids, caused by the unstable nature of these resonances and will migrate away. These gaps, while clearly noticeable, are not all the same strength. This paper will attempt to build a model to explain these apparent relative strengths.

This unstable resonance is caused when objects cannot maintain a continuous resonance with other planets. This is seen in the Kirkwood gaps as asteroids in resonance with Jupiter will receive so much energy by passing by it regularly that it will fly out of orbit. This causes the the gaps inside the asteroid belt. Only the Trojan Asteroids, (which are in a 1:1 ratio with Jupiter) found at the Lagrange points of Jupiter, are able to maintain a continuous resonance.

There are also cases in which orbital resonance leads to stable configurations. For example, Neptune and Pluto are locked in a 3:2 resonance. Even more striking, the Galilean moons: Ganymede, Europa, and Io are locked in a 1:2:4 resonance. There are many other instances of moons and even planets in stable resonance with each other. See table I for a list of major stable resonant systems in our solar system.

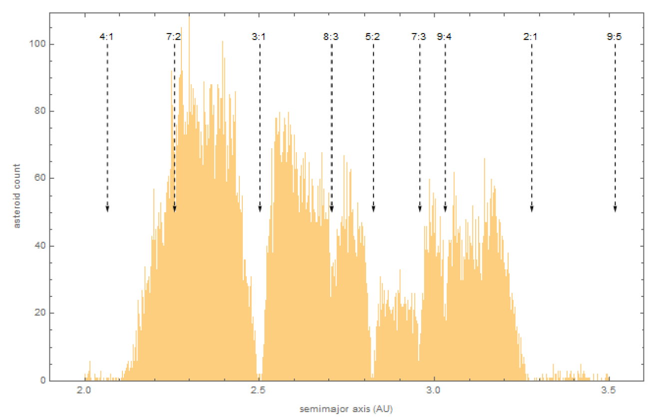

FIG. 1. A histogram of Kirkwood Gaps and the locations of orbital resonances, plotted with Mathematica[4]

Planet	System	Resonance
Jupiter	Io - Europa - Ganymede	4:2:1
Saturn	Cassini Division - Mimas	2:1
	Mimas - Tethys	2:1
	Enceladus - Dione	2:1
	Titan - Hyperion	4:3
	Neptune - Pluto	3:2

TABLE I. Selected examples of resonant systems in the solar system.[5]

II. AN ELEMENTARY MODEL

First, we shall present a simplistic model that can explain the relative strengths of the Kirkwood Gaps. Consider an asteroid of mass m in a circular orbit with a radius of r_1. Moving into a frame co-rotating with the asteroid, the net force is zero. Then, introduce Jupiter, with mass $M \gg m$, in another circular orbit with radius r_2. By Kepler's third law, the ratio of their orbital frequencies is given by

$$\frac{f_1}{f_2} = \left(\frac{r_2}{r_1}\right)^{3/2} = \frac{m}{n} \tag{1}$$

The asteroid will experience resonance if m and n can be expressed as relatively prime integers. For every orbit the asteroid makes, Jupiter will complete n/m of an orbit. Let us assume that Jupiter and the asteroid start off at conjunction; in other words, the three bodies are co-linear. Every time the asteroid makes one full revolution, Jupiter will cycle between m locations, equally spaced around its orbit. We base our model on the assumption:

> If we superimpose the larger object at all the locations it can be in when the smaller object makes one full revolution starting from conjunction, then the strength of orbital resonance directly depends on the average of all the superimposed forces.

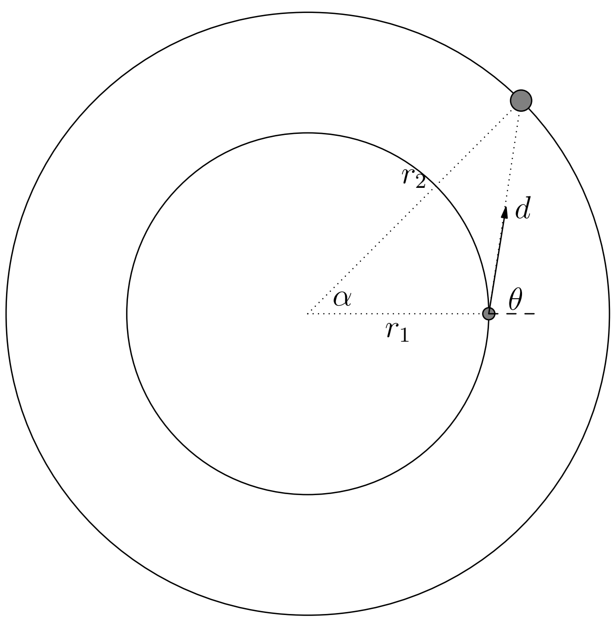

FIG. 2. The inner mass, on average, experiences an outward radial force due to the outer mass.

By symmetry, only the radial components of the force will show up in the average. The radial force at an arbitrary location is:

$$F_r = \frac{GMm}{d^2} \cdot \cos\theta \tag{2}$$

We have:

$$d = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos\alpha} \tag{3}$$

$$\cos\theta = \frac{r_2\cos\alpha - r_1}{d} \tag{4}$$

Using equation 1, we can eliminate r_1:

$$F_r = \frac{GMm}{r_2^2}\left(\frac{\cos\alpha - (n/m)^{2/3}}{\left((n/m)^{4/3} + 1 - 2(n/m)^{2/3}\cos\alpha\right)^{3/2}}\right) \tag{5}$$

The angle α can only take certain discrete values when resonance is reached. We have:

$$\alpha = \frac{2\pi k}{m} \tag{6}$$

for $0 < k < m$. Substituting this in and getting rid of the constant, we have:

$$F_r \propto \frac{\cos\frac{2\pi k}{m} - f^{2/3}}{\left(f^{4/3} + 1 - 2f^{2/3}\cos\frac{2\pi k}{m}\right)^{3/2}} \tag{7}$$

where $f \equiv \frac{n}{m}$. Taking the average of the radial forces experienced at all m locations give us:

$$F_{\mathrm{r,avg}} \propto \frac{1}{m}\sum_{k=1}^{m} \frac{\cos\frac{2\pi k}{m} - f^{2/3}}{\left(f^{4/3} + 1 - 2f^{2/3}\cos\frac{2\pi k}{m}\right)^{3/2}} \tag{8}$$

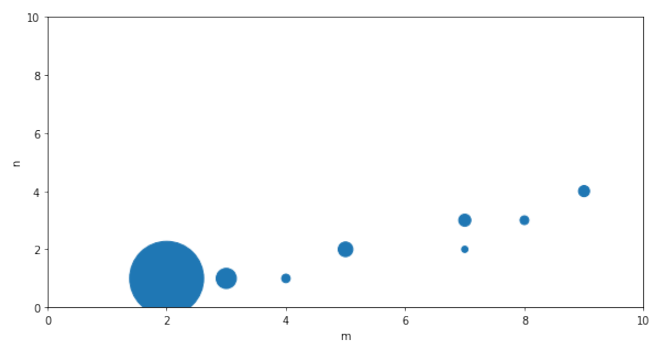

FIG. 3. Average force at all possible resonant frequencies with $m, n < 10$. The size of the bubble represents the relative strengths. Frequencies that are not relatively prime, or exist outside of the main belt are not included.

The results are plotted in figure 3. The resonant frequencies shown, listed from strongest to weakest are 2:1, 3:1, 5:2, 7:3, 9:4, 8:3, 4:1, 7:2. Qualitatively, they line up with observations, as shown in figure 1; however, there are a few discrepancies when compared to the quantitative data.

The number of asteroids within a 0.05 AU range of each Kirkwood gap was recorded and listed and sorted in table II. While there is no direct parameter that gives the strength of each resonant frequency, we can assume the number of asteroids within a certain resonant frequency is inversely proportional to the strength. We see that the elementary model built is extremely accurate in predicting the relative strengths! Everything is in the correct order, with the exception of the 4:1 resonant frequency.

Resonant Frequency	Semi-major axis (AU)	Number of Asteroids	Superimposed Strength (force)
4:1	2.06	29	0.39
2:1	3.27	67	3.46
3:1	2.49	459	0.94
5:2	2.82	614	0.68
7:3	2.95	752	0.56
9:4	3.02	1495	0.51
8:3	2.70	1705	0.40
7:2	2.25	2777	0.29

TABLE II. The resonant frequencies and semi-major axes of major kirkwood gaps and the number of asteroids (± 0.05 AU) in the range, pulled from Mathematica. The average superimposed forced is also shown for comparison and has units where $G = M = r_{\text{earth}} = 1$

This discrepancy can be explained by considering the effects of other planets. One peculiar interest is Earth. It is relatively large and is quite close to this Kirkwood gap, which roughly only 1 AU away. However, the most important factor is that Earth is in a $12 : 1$ resonant orbital with Jupiter. Therefore, not only are asteroids in this gap resonant with Jupiter, it is also in a $3 : 1$ resonance with Earth, which we have already established has a strong effect.

Another explanation of why observations show so little asteroids in this region is simply because this is the lower limit of the asteroid belt. Any asteroids that go past this point can be strongly perturbed by the pull of Mars. While other gaps have asteroids migrating across it from *both* sides, this gap only has asteroids migrating from one side.

III. STABLE RESONANCES

This periodic amplification can wreck havoc to objects in a perfectly circular orbit, but it can lead to stable resonance as well in certain cases where the orbits have a nonzero eccentricity. This is common between different planets and moons.

A famous example of stable resonances is found in the moons of Saturn, or to be more specific, Hyperion and Titan. Hyperion gains an uncertainty of 0.123ϵ in its orbit due to the alluding orbital resonance caused from Titan. In turn, we can see a near perfect circular orbit produced from Titan, and an elliptical orbit created by Hyperion. To get deeper into our analysis, let us figure out what happens at the conjunction points of our model. If the conjunction point is located when moving from pericenter to apocenter, Hyperion receives an outward velocity component while the radial component of Titans gravitational attraction pulls Hyperion inwards. Because of this, Hyperion's energy and angular momentum decrease which, in turn, also decreases the semi-major axis and period of Hyperion. This effect is also enhanced because the point where the pull is the strongest

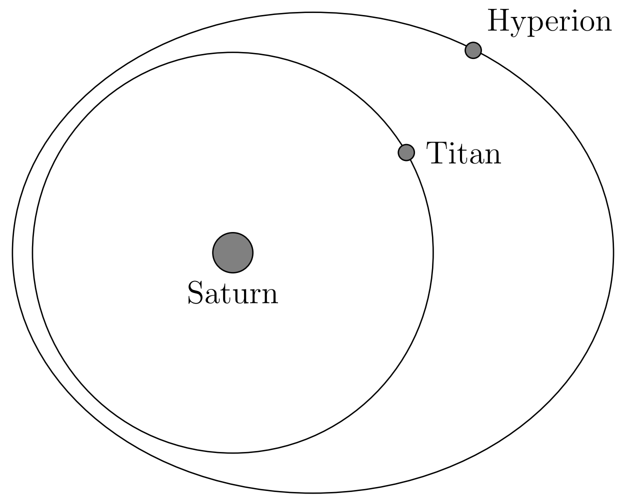

FIG. 4. Hyperion gains or loses angular momentum depending on the setup, which can rotate its orbit clockwise or counterclockwise

is right before the conjunction point, while Titan, which is faster, trails right behind Hyperion. As a consequence, Hyperion speeds up, which makes the conjunction move towards the apocenter of the orbit. Another case exists if the conjunction exists when Hyperion moves from apocenter to pericenter. In this scenario, Titan's pull increases the gravitational energy and angular momentum, slowing the moon down. However, this effect still moves the conjunction to apocenter.

A. Liberation Period

The underlying dynamics of the restricted three body problem is complex, and it is difficult to find a closed form expression for the period of liberations discussed above. We shall follow the techniques outlined by Agol et al. 2005. However, they provided only the general proportionality between liberation period, eccentricity, and the resonant frequency. We wish to take a step forward and investigate the general proportionality factor, and how it depends on the semi-major axis and if and how it depends on the amplitude of oscillations.[1]

Unlike the main asteroid belt example, Titan's circular orbit is *inside* Hyperion's orbit, which has a high eccentricity. We shall choose units such that $G = M_{\text{Saturn}} = r_{\text{Titan}} = 1$. Titan has a mass μ and it is in a $4 : 3$ resonant orbital with Hyperion. For the sake of generalization, let their orbital resonances be $j+1 : j$. This is an example of a first-order resonance and as shown in figure 3, it is one of the strongest types. One further modification is that instead of having a constant angular frequency, we shall denote ω_T and ω_H for the *average* angular frequencies of

Titan and Hyperion, respectively. We have:

$$\frac{\omega_T}{\omega_H} = \frac{j+1}{j} = \left(\frac{a_H}{a_T}\right)^{3/2} \tag{9}$$

We can rewrite the ratio as

$$a_H/a_T = 1 + r/a_T \equiv 1 + x \tag{10}$$

where r is the difference between the two semi-major axes. Using a first-order expansion, we get:

$$\frac{\omega_T}{\omega_H} = \frac{j+1}{j} = (1+x)^{3/2} \approx 1 + \frac{3x}{2} \tag{11}$$

This first order expansion works because we are only interested in an *order of magnitude* calculation. In most cases, r_H and r_T are both significantly far away from the central body, such as Saturn, so their percent difference is negligible. In our case, it leads to a 1% error ($x = 0.22$)! Rearranging and solving for x, we get:

$$x = \frac{2}{3j} \tag{12}$$

Next, we calculate the time between conjunctions. We can accomplish this by switching into a frame co-rotating with Hyperion. In this new frame, Titan's mean angular velocity would be $\omega_T - \omega_H$, and the time it takes for Titan to make one full revolution (equivalent to the time it takes to meet up with Hyperion again) is:

$$T_{\text{conjunction}} = \frac{2\pi}{\omega_T - \omega_H} \tag{13}$$

Let us assume that on average, the angle of Hyperion's orbit changes by $\Delta\theta$ between successive conjunctions. Thus, $\Delta\theta$ is given by:

$$\Delta\theta = \omega_H T_{\text{conjunction}} = \frac{2\pi\omega_H}{\omega_T - \omega_H} \tag{14}$$

However, note that equation 9 no longer holds after Hyperion's orbit changes by $\Delta\theta$. As discussed earlier, this is because its period will slightly increase or decrease as angular momentum is removed or added from the system. After dividing, we write

$$\frac{\omega_H}{\omega_T} = \frac{j+1}{j+\epsilon} \tag{15}$$

where $\epsilon \ll 1$. Plugging this in:

$$\begin{aligned}\Delta\theta &= \frac{2\pi}{\omega_T/\omega_H - 1} \\ &= \frac{2\pi}{\frac{j+1}{j+\epsilon} - 1} \\ &= \frac{2\pi(j+\epsilon)}{1-\epsilon}\end{aligned}$$

Using a first-order power expansion:

$$\Delta\theta = 2\pi j + 2\pi j\epsilon \tag{16}$$

Note that the first term is irrelevant, since m is an integer, the first-term tells us that Hyperion will return to its original position a total of m times. However, we are interested in the net change in its angle between successive conjunctions, and that's given by the second term. From now on, we can say:

$$\Delta\theta = 2\pi j\epsilon \tag{17}$$

Let the total angle subtended in one liberation be ϕ. Note that this will be four times the angular amplitude. Therefore, we need

$$N = \phi/\Delta\theta \tag{18}$$

successive conjunctions in order to complete one full revolution. Therefore, the total time to complete one cycle is:

$$T_{\text{liberation}} = T_{\text{conjunction}} N = \frac{\phi}{j\epsilon(\omega_T - \omega_H)} \tag{19}$$

We can eliminate ω_T by substituting in equation 11 and equation 12. This gives us:

$$T_{\text{conjunction}} = \frac{4\pi}{3x\omega_H} = \frac{2\pi j}{\omega_H} \tag{20}$$

and

$$T_{\text{liberation}} = T_{\text{conjunction}} N = \frac{2\pi j}{\omega_H} \cdot \frac{\phi}{\Delta\theta} = \frac{\phi}{\omega_H \epsilon} \tag{21}$$

Following Agol et al, in the high eccentricity limit of Tisserand's relation (this is satisfied for $e > \mu^{1/3}$, which is valid in our case), we have:

$$\begin{aligned}\Delta(x^2) &= \Delta(e^2) \\ (2x)\Delta x &= (2e)\Delta e \\ \Delta x &= \frac{e}{x}\Delta e\end{aligned}$$

Determining Δe is a difficult task. One method proposed by Agol et al is to use an average impulse-based approach.[1] They claim the change in eccentricity between successive conjunctions is given by:

$$\Delta e/\text{conjunction} = \frac{\mu}{x^2} t_{\text{encounter}} \tag{22}$$

We have $t_{\text{encounter}} = 1$ which is a reasonable assumption due to the units we are working in, so the total change in eccentricity is:

$$\Delta e = \frac{\mu N}{x^2} \tag{23}$$

From equation 12 we see that $x \sim \frac{1}{j}$. Plugging this in gives:

$$\Delta x = e\mu j^3 N \tag{24}$$

Using equation 18 we can substitute for N to get:

$$\Delta x = \frac{e\mu j^2 \phi}{2\pi\epsilon} \tag{25}$$

From Kepler's third law and equation 9, we have:

$$j \sim r_H^{-3/2} \tag{26}$$

The differential change would be:

$$dj = \epsilon \sim \frac{-3}{2} r_H^{-5/2} dr_H \implies \epsilon \sim \frac{-3}{2} a^{-5/2} \Delta x \tag{27}$$

Dividing the two equations, taking the absolute value gives:

$$\frac{\epsilon}{j} \sim \frac{3\Delta x}{2r_H} \tag{28}$$

Isolating for Δx and setting it equal to 25 gives:

$$\frac{2\epsilon r_H}{3j} = \frac{e\mu j^2 \phi}{2\pi\epsilon} \implies \epsilon^2 = \frac{3e\mu j^3 \phi}{4\pi r_H} \tag{29}$$

Plugging it into equation 21 gives:

$$T_{\text{liberation}} = \frac{2\phi\sqrt{\pi r_H}}{\omega_H \sqrt{3e\mu j^3 \phi}} \tag{30}$$

We can simplify this even further if we substitute, according to Kepler's third law:

$$\omega_H = r^{-3/2} \tag{31}$$

to get:

$$T_{\text{liberation}} = \frac{2\sqrt{\pi\phi} r_H^2}{\sqrt{3e\mu j^3}} \tag{32}$$

Hyperion has an eccentricity of $e = 0.123$ [6], Titan has a mass $\mu = 0.000237 m_{\text{Saturn}}$, semi-major axis of $r_H = 1.21 r_{\text{Titan}}$.[3] We also have $\phi = 2.51$ [2] and the 4:3 resonance means $mj3$. Plugging these values in, we get:

$$T_{\text{liberation}} = 169.33$$

Using the natural units we've chosen, the orbital period of Titan is:

$$T_{\text{Titan}} = 6.283 \rightarrow 15.95 \text{ days}$$

Using this conversion factor, we get:

$$T_{\text{liberation}} = 1.18 \text{ years}$$

Although the actual liberation period is $t = 1.75$ years, this is surprisingly close![2] It is correct to an order of magnitude, though it is an underestimation.

Multiple crude estimates and approximations and they could have factored in varying amounts of error. However, the main source of error is probably from equation 22. It is unlikely that the change in eccentricity is *directly* proportional to the impulse that is provided and the use of the force as:

$$F = \mu/x^2$$

is overly generous. For one, this gives the force only closest approach of the two orbits. It is unlikely that conjunctions occur at these close approaches so the force is an overestimation. Next, as the two moons move around Saturn, the angles they make relative to their respective paths constantly change. Thus, the force calculated would give us only the magnitude, not the strength in any particular direction. Because of this, it is once again another overestimation.

By overestimating the effects a single conjunction can have, the final answer will show less conjunctions are needed for the angle ϕ to be subtended and thus complete one whole liberation cycle.

IV. GLOSSARY

Conjunction Point - The point at which all three celestial objects are collinear.

Kirkwood Gaps - The gaps in the distribution of asteroids in the Asteroid Belt with respect to their semi-major axis.

Liberation Period - The period of time for the orbits in a resonant system to return to its initial state.

Orbital Frequency - The amount of orbits completed per unit of time.

Apocenter - The point in an elliptical orbit that is farthest from the inner body.

Pericenter - The point in an elliptical orbit that is closest to the inner body.

V. CONCLUSION

In this paper, we have analyzed and created a mathematical model for two cases of orbital resonance in the Solar System: Kirkwood Gaps in the Asteroid Belt, and Saturn's two moons Hyperion and Titan. The former was modeled as two concentric and circular orbits with the outer mass, Jupiter, much larger than the inner mass, an asteroid. By taking the average radial component of force that Jupiter exerts on the asteroid at regular time intervals, we found that, at certain resonant frequencies, the average force is much stronger than at others. To a degree of remarkable accuracy, these points corresponded with the points at which the Kirkwood Gaps occur.

Secondly, the Saturn-Titan-Hyperion system was modeled as a large mass, Titan, in a circular orbit, and a smaller mass, Hyperion, in a larger and elliptical orbit. During the subsequent 4:3 resonant motion, Hyperion periodically gains and loses energy, causing its orbit to change shape. The time for the system to completely return to its initial state, also known as the liberation period, was successfully approximated to an order of magnitude.

During the process of finding an appropriate model, one option was using the Lagrange Planetary Equations, which give a much more accurate description of the motion of the system. However, the equations were too long and complicated, even when approximated to the first order. However, a complete analysis would require them and the next step in our research is to implement the Lagrange Equations in order to create a much more accurate model of both stable and unstable orbital resonance.

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