TJPhO Solutions

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1 Background and Problems

Circular Motion and Orbits

1 Answer the following questions on uniform rotational motion:

a) A proton orbiting a black hole can be parameterized by $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$. Assume $x(t)$, $y(t)$ satisfy $x^2+y^2 = 4D^2$, for some constant length D (in meters). Determine the centripetal acceleration of the proton assuming the velocity is $3D$ m/s.

b) A distrubance alters the path of the proton, now moving it in a circle with twice the radius of the original. Assuming the velocity of the proton remains constant as it is in prat a), determine the new centripetal acceleration of the particle.

Solution. a) The equation

$$x^2 + y^2 = 4D^2$$

gives the equation of a circle with radius $r = 2D$. The centripetal acceleration is given by:

$$a = \frac{v^2}{r}$$

so substituting in the radius and the speed, we get:

$$a = \frac{(3D)^2}{2D} \implies \boxed{a = \frac{9D}{2}}$$

b) Similarly, since everything stays constant except the radius gets doubled, the acceleration will be halved. Therefore:

$$\boxed{a = \frac{9D}{4}}$$

Gravity

2 Answer the following questions checking for your understanding of gravity:

a) A star undergoes many changes as it ages. Near the end of a star's lifetime, it may collapse gravitationally into a black hole. What do you think will happen to the orbits of our solar system's planets if the Sun were to become a black hole?

b) Let us assume that there are two bodies that have a gravitational field of 64 Newtons. If the distance between the two bodies is doubled, and the masses are tripled, what is the resulting gravitational force between the objects?

c) Consider an infinite line of mass with mass per unit length λ. Using Newton's law of Universal Gravitation and assuming the principle of superposition for gravitational fields, calculate the gravitational field a distance r from the line.

d) Appealing to some familiarity with electromagnetism, derive an analogous form of Gauss's Law for gravitational fields.

Solution. a) Ignoring all other effects, and focusing on just the orbits: they would not change. The magnitude of the force of gravity is given by:

$$F_g = \frac{GMm}{r^2}$$

where r is measured from the center of mass of the two objects. Notice this does not depend on the radius of the star. Suddenly becoming a black hole when the mass but keeping the mass the same, changes none of the four parameters so the force of gravity will be held constant and the motion of the orbits will stay the same.

b) The proportionality equation is:

$$F_g \propto \frac{m^2}{r^2}$$

If we triple masses and double the distance, then we have:

$$F_{g,new} = \frac{9m^2}{4r^2} = \frac{9}{4}F_g = \boxed{144 \text{ N}}$$

c) Let us place a test particle a distance r away. Consider a differential piece of the line a distance $d = r/cos\theta$ away from this test particle. The length of this differential piece is $d(d\theta)$ and therefore the mass is $\lambda d(d\theta)$. The gravitational field from this differential piece would then be:

$$dg = \frac{G(\lambda(r/\cos\theta)(d\theta)}{(r/\cos\theta)^2} = \frac{G\lambda\cos\theta}{r}d\theta$$

Integrating from $\pi/2 \to -\pi/2$ to cover the entire length of the rod, we get:

$$\begin{aligned}
g &= \int_{-\pi/2}^{\pi/2} \frac{G\lambda \cos\theta}{r} d\theta \\
&= \frac{G\lambda}{r} \left[\sin \pi/2 - \sin -\pi/2\right] \\
&= \boxed{\frac{2G\lambda}{r}}
\end{aligned}$$

d) Gauss's law states that for some field $\vec{g}$ that drops off with the square of its distance, then the flux through any closed surface containing it will be constant. For example, we might want to think about it as a source shooting off imaginary particles that cannot be destroyed or duplicated. Thus do to conservation laws, the amount going through any closed surface encompassing it will stay constant. For our gravitational field $\vec{g}$ we have:

$$\Phi = \oint \vec{g} \cdot d\vec{A}$$

We can determine Φ by applying it to a single point particle. Let our closed surface be a sphere of radius r centered around and encompassing a point mass M. The gravitational field will be constant at and parallel to this surface so the flux is:

$$\Phi = g(4\pi r^2) \implies g = \frac{\Phi}{4\pi r^2}$$

However, we know that due to Newton's gravitation, we have:

$$g = \frac{GM}{r^2}$$

so we must have:

$$\Phi = 4\pi GM$$

Thus the Gauss's Law variation for gravitation is:

$$\boxed{4\pi GM = \oint \vec{g} \cdot d\vec{A}}$$

Remarks: Using this theorem we can find an elegant solution for part c) of this problem. Let us build a cylindrical Gaussian surface of radius r and length ℓ where the flat ends are centered and perpendicular to the line. The gravitational field will be parallel to the flat parts, so the dot product will be zero. However, they will be constant at and parallel to the rounded part of the cylinder which has an area of $2\pi r\ell$. Thus we have:

$$4\pi GM = g(2\pi r\ell$$

The mass that it encloses is:

$$M = \lambda\ell$$

Substituting this in gives:

$$4\pi G(\lambda\ell) = g(2\pi r\ell) \implies g = \frac{2G\lambda}{r}$$

Mass Density

3 Consider the mass density $\lambda(\rho) = Ae^{-k\rho^3}$, where k, A are constants with appropriate units and ρ is the spherical radius from the center of the origin. Although this may not be completely accurate to the actual mass density of a black hole, this mass density is meant to model the extremely dense center and the steep drop of density of a black hole. With this given mass density, find the radius of the event horizon ρ_{event} in terms of A, k, and other physical constants. Assume that A is sufficiently large to achieve an event horizon.

Solution. Consider breaking up the mass into infinitesimally thin shells of radius $d\rho$. The mass of this shell is:

$$dm = \lambda(4\pi\rho^2)(d\rho) = Ae^{-k\rho^3}(4\pi\rho^2)(d\rho)$$

Integrating from $\rho = 0$ to $\rho = \rho$ gives:

$$\begin{aligned} M &= \int dm \\ &= \int_0^\rho Ae^{-k\rho^3}(4\pi\rho^2)(d\rho) \\ &= 4\pi A \int_0^\rho e^{-k\rho^3}\rho^2(d\rho) \end{aligned}$$

Let $u = -k\rho^3$ so $du = -3k\rho^2(d\rho)$. Making this u-substitution gives:

$$\begin{aligned} M &= \frac{-4\pi A}{3k} \int_0^{-k\rho^3} e^u(du) \\ &= \frac{-4\pi A}{3k}\left[e^{-k\rho^3} - 1\right] \\ &= \frac{4\pi A}{3k}\left(1 - e^{-k\rho^3}\right) \end{aligned}$$

The event horizon is the distance at which one will need an escape velocity of c in order to escape. At $\rho \to \infty$, both the kinetic and potential energy will be zero. Conservation of energy tells us:

$$\frac{1}{2}c^2 - \frac{GM}{\rho_{\text{event}}} = 0 \implies \rho_{\text{event}} = \frac{2GM}{c^2}$$

or

$$\rho_{\text{event}} = \frac{2G}{c^2}\left(\frac{4\pi A}{3k}\left(1 - e^{-k\rho_{\text{event}}^3}\right)\right)$$

There is no closed form solution to solve this. We can either compute numerically or make the assumption that $e^{-k\rho_{\text{event}}^3} \ll 1$. This is a reasonable assumption to make as the mass quickly drops off with distance and after a certain radius, the mass would essentially be constant. Therefore:

$$\boxed{\rho_{\text{event}} = \frac{8\pi GA}{3c^2k}}$$

Angular Momentum

4 Consider a system of two identical spherical objects in orbit around each other. Each object has a mass of m_0, is moving with a tangential velocity of v_0, and the two are separated by a distance of d.

a) Find the magnitude of the total angular momentum of the system about the center of mass of the two objects in terms of the given quantities.

b) Calculate the value of v_0 in terms of the other given quantities. You may use the gravitation constant G in your answer.

c) Suppose one of the spheres is suddenly grabbed and held in place. Use energy conservation to calculate the distance of closest approach by the other sphere in terms of d.

Solution. a) The total angular momentum of the system about the center of mass is simply $L = m_0 v_0 (d/2) + m_0 v_0 (d/2)$ or

$$\boxed{L = m_0 v_0 d}$$

b) To do this part, we can just sum the forces in the centripetal direction. We obtain

$$\sum F_c = \frac{mv^2}{R} \implies \frac{Gm_0^2}{d^2} = \frac{m_0 v_0^2}{d/2} \implies \boxed{v_0 = \sqrt{\frac{Gm_0}{2d}}}$$

c) As hinted, we will utilize conservation of energy to calculate the distance of closest approach by the other sphere. Initially, only one sphere is in motion and they are both gravitationally acting on each other. This implies that

$$\begin{aligned} E_i &= \frac{1}{2} m_0 v_0^2 - \frac{Gm_0^2}{d} \\ &= \frac{1}{2} m_0 \left(\frac{Gm_0}{2d} \right) - \frac{Gm_0^2}{d} \\ &= -\frac{3Gm_0^2}{4d} \end{aligned}$$

This energy term is negative, which means that the object will move in an ellipse. It starts off at aphelion, and we wish to find its closest distance at perihelion. The final energy is again the kinetic energy of the second sphere and the gravitational potential energy between the two spheres at their closest point. Call this distance r.

$$E_f = \frac{1}{2} m_0 v_f^2 - \frac{Gm_0^2}{r}$$

At perihelion, the mass will once again by moving with a velocity component perpendicular to the radial distance between the two masses. By conservation of angular momentum, we find the final velocity of the moving mass to be

$$\begin{aligned} m_0 v_0 d &= m_0 v_f r \\ v_f &= \frac{d}{r} v_0 \\ &= \frac{d}{r}\sqrt{\frac{Gm_0}{2d}} \end{aligned}$$

Using conservation of angular momentum, we can rewrite this final energy as:

$$\begin{aligned} E_f &= \frac{1}{2} m_0 \left(\frac{d^2}{r^2} \cdot \frac{Gm_0}{2d} \right) - \frac{Gm_0^2}{r} \\ &= \frac{Gm_0^2 d}{4r^2} - \frac{4Gm_0^2 r}{4r^2} \\ &= \frac{Gm_0^2}{4r^2} (d - 4r) \end{aligned}$$

Using conservation of energy:

$$\begin{aligned} E_i &= E_f \\ -\frac{3Gm_0^2}{4d} &= \frac{Gm_0^2}{4r^2} (d - 4r) \\ -\frac{3}{d} &= \frac{d - 4r}{r^2} \\ 3r^2 - 4dr + d^2 &= 0 \\ (3r - d)(r - d) &= 0 \end{aligned}$$

The root $r = d$ is trivial and corresponds to the initial condition. The other root $\boxed{r = d/3}$ corresponds to the distance at perihelion and thus when the distance is minimized.

Escape Velocity

5

a) Calculate the escape velocity on the surface of a spherical body in terms of its mass M, radius R, and G.

b) Calculate the escape velocity on the surface of spherical planet X if its radius is 10500 km and it has a uniform density of 3700 kg/m^3

c) Compare this to the speed of light. First, disregarding the fact that the planet has a nonzero radius, how far out would one have to be from the center of X to reach an escape velocity equal to the speed of light? If one was to somehow be at this distance from the center of planet X, would we need to actually have a speed of c in order to escape?

Solution.

a) The escape velocity is the minimum speed that initially needs to be given to an object such that it is able to completely escape the pull of gravity such that at $r \to \infty$ it has a speed of zero and a potential energy of zero. Conservation of energy tells us:

$$\frac{1}{2}mv^2 - \frac{GMm}{R} = 0$$

Solving for v gives:

$$\boxed{v = \sqrt{\frac{2GM}{R}}}$$

b) We have:

$$M = \frac{4}{3}\pi\rho R^3$$

so the escape velocity is:

$$v = R\sqrt{\frac{8\pi G\rho}{3}}$$

Plugging in numbers gives us:

$$\boxed{v = 15098 \text{ m/s}}$$

c) The speed of light is $c = 2.99 \times 10^8$, about four orders of magnitude higher. Let us calculate at what distance our escape velocity will be c given that all the mass is concentrated in an infinitesimally small point. Again, we use conservation of energy to give us:

$$\frac{1}{2}c^2 - \frac{GM}{d} = 0$$

Rearranging for d gives us:

$$d = \frac{2GM}{c^2} = \frac{8\pi\rho R^3 GM}{3c^2}$$

Plugging in values give:

$$\boxed{d = 0.0268 \text{ m}}$$

Note that at this distance, an initial speed of c is needed to escape given that the only external force acting on it is gravity from that point on-wards. If we are able to initially start the object with a lower speed but give it a nonzero acceleration outwards (for example rocket thrusters) then we can (according to Newtonian mechanics) escape without ever reaching a speed of c. This is because as the distance from the center increases, the escape velocity will decrease.

For example, one possible solution would be to give our object an initial speed of $v/2$. Conservation of energy gives:

$$\frac{1}{2}\frac{c^2}{4} - \frac{GM}{d} = -\frac{GM}{r}$$

At a radius of $r > d$, the speed will be zero. At this point, we give the rocket another initial velocity, this time sufficient to escape. However at this point the escape velocity will be less than c so the rocket doesn't need a speed of c to escape!

An even plainer argument is this. Suppose we have a rocket that can create a radially-dependent force of:

$$F = \frac{GMm}{r^2}$$

where r is the distance from the center. Neglect the loss of mass in this case. Then the net force experienced from the particle will be zero and it can escape with a speed of something as slow as 1m/s!

However: while these may work in Newtonian mechanics, effects from relativity forbids this. *Nothing* can escape the event horizon.

Thermodynamics

6 Consider the Sun/Earth system. The Sun has temperature $T_\odot = 5800$ K and radius $R_\odot = 6.96 \times 10^8$ m, and the Earth has radius $R_\oplus = 6.37 \times 10^6$ m. The distance from the Earth to the Sun is $D = 1.50 \times 10^{11}$ m.

a) Find the power per unit area radiated by the Sun and the total power radiated by the Sun, in terms of constants given.

b) What fraction of the power emitted by the Sun does the Earth perceive? Express your answer in terms of constants given.

c) The Earth's *albedo* is the fraction of the incoming energy that is reflected by the surface, and this value is about 0.3, accounting for clouds. How does this affect your answer to part b)?

d) Let the equilibrium temperature of the Earth by $T_\oplus$. Use energy conservation to find an expression for $T_\oplus$ in terms of constants given, and evaluate to find what the equilibrium temperature is. *Account for the Earth's albedo when doing this problem.*

e) In real life, it turns out the equilibrium temperature of the Earth is 298 K. What factors account for this discrepancy?

Solution.

a) According to Stefan-Boltzmann's Law, the power per unit area emitted from the surface of an object at temperature T is σT^4. Thus it can be accurately stated that the power per unit area emitted is σT^4.

The total power emitted from the sun is therefore:

$$P_\odot = \sigma T_\odot^4 (4\pi R_\odot^2)$$

b) Due to the inverse square law, the solar flux stays constant through any closed surface. The portion of energy that reaches the Earth is given by the ratio between the cross-sectional area of Earth and the area of an imaginary sphere centered around the Sun with a radius of $R_\oplus$:

$$\gamma = \frac{\pi R_\oplus^2}{4\pi D^2} = \left(\frac{R_\oplus}{2D}\right)^2$$

c) If the reflectance (or albedo) of the Earth is α then the power that the Earth is receiving is then

$$\gamma_{\text{eff}} = (1-\alpha)\gamma = \boxed{(1-\alpha)\left(\frac{R_\oplus}{2D}\right)^2}$$

d) If we assume that Earth is a perfect blackbody, then in order to maintain thermal equilibrium, it must emit the same energy as it receives. If this was not true, then it would continuously lose or gain energy until it is at equilibrium. Thus:

$$P_{\text{in}} = P_{\text{out}}$$

By Stefan-Boltzmann Law, we have

$$P_{\text{output}} = 4\pi R_{\oplus}^2 \sigma T^4$$

Equating P_{in} to P_{out} gives us

$$\begin{aligned}
P_{\text{output}} &= P_{\odot}\gamma_{\text{eff}} \\
4\pi R_{\oplus}^2 \sigma T^4 &= \sigma T_{\odot}^4 (4\pi R_{\odot}^2)(1-\alpha)\left(\frac{R_{\oplus}}{2D}\right)^2 \\
T^4 &= T_{\odot}^4 (R_{\odot}^2)(1-\alpha)\left(\frac{1}{2D}\right)^2 \\
T^4 &= \frac{T_{\odot}^4 R_{\odot}^2 (1-\alpha)}{4D^2} \\
T &= T_{\odot}\left(\frac{R_{\odot}^2(1-\alpha)}{4D^2}\right)^{1/4}
\end{aligned}$$

After susbstituting in the figures, we get $\boxed{T = 255.5 \text{ K} = -17.7°\text{C}}$.

e) In the problem it was stated that the Stephan Boltzmann Law is given by $P = \sigma T^4$. While this is accurate to a certain extent, it is still not the complete formula. What was forgotten in the formula is the factor of emmisivisity or ϵ. If we include this in then we have:

$$P_{\text{output}} = 4\pi R_{\oplus}^2 \sigma \epsilon T^4$$

Following the same steps gives us:

$$T = T_{\odot}\left(\frac{R_{\odot}^2(1-\alpha)}{4\epsilon D^2}\right)^{1/4}$$

Choosing a typical value $\epsilon = 0.85$ gives $T = -7.03°$ C. However, this is not the only factor.

However, it would be naive to compare this to the ground temperature of 298 K. This is because the temperature we calculated above is the average of everything in the atmosphere from both ground objects to objects at high altitudes (or if we take into account emmisivisity, it would be the average in the atmosphere above a specific region)

The ground temperature would be higher because there are other complex and dynamic heating processes at play. The greenhouse effect can trap in heat and make the surface temperature hotter than what it would've been had it been modelled as a black body.

2 Applied Problems

Hawking Radiation and Thermodynamics

7 This problem is about an approximation for the time it takes for a black hole of a mass M to evaporate. Stephen Hawking hypothesized the existence of this radiation in the 1970s that slowly dissipates mass and energy from the black hole back into its surroundings. Assume for our scenario we have a *Schwarzschild* black hole, meaning that it has no angular momentum or charge, and that it is a perfect blackbody. (In general, a black hole is determined by its mass, angular momentum, and charge.) The radius of a Schwarzschild black hole of mass M is $r_S = \dfrac{2GM}{c^2}$, and the temperature of such a black hole is related to its surface gravity g by $T_S = \dfrac{\hbar}{2\pi c k_B} g$. Recall the Stefan-Boltzmann constant $\sigma = \dfrac{2\pi^2 k_B^4}{15 h^3 c^2}$

a) Using a Newtonian calculation, compute the temperature of a black hole in terms of its mass.

b) Using the Stefan-Boltzmann Law, find the power emitted by the black hole per unit area, and the total power emitted by the black hole in terms of M and the fundamental constants c, G, k_B, h.

c) Assume the energy of the black hole is related to its mass per Einstein's equation $E = Mc^2$. Set up and solve a differential equation to find the total time it takes for the black hole to evaporate in terms of M and the fundamental physical constants c, G, k_B, h.

d) How long would a black hole with the mass of the Earth take to evaporate? The Earth's mass is approximately 5.97×10^{24} kg.

Solution.

a) The surface gravity g is given by:

$$g = \frac{GM}{r_s^2}$$

and

$$r_s = \frac{2GM}{c^2}$$

substituting this into g gives:

$$g = \frac{c^4}{4GM}$$

We can finally substitute this in for our expression for T_S to get:

$$\boxed{T_S = \frac{\hbar c^3}{8\pi G M k_B}}$$

b) Firstly, we note that $\hbar = h/2\pi$, thus we can simplify temperature to be

$$T_S = \frac{\hbar c^3}{8\pi G M k_B} = \frac{hc^3}{16\pi^2 G M k_B}$$

Now we use the Stefan-Boltzmann Law, $I = \sigma T^4$ to get

$$\begin{aligned} I &= \frac{2\pi^5 k_B^4}{15h^3c^2} \cdot \left(\frac{hc^3}{16\pi^2 G M k_B}\right)^4 \\ &= \boxed{\frac{1}{491520\pi^3} \cdot \frac{hc^{10}}{G^4 M^4}} \end{aligned}$$

c) We know that the time derivative of energy is equal to power. Thus, by Stefan-Boltzmann's Law we have

$$\frac{dE}{dt} = -\sigma T^4 A.$$

The energy of the black hole is given by $E = Mc^2$, thus

$$\frac{dE}{dt} = \frac{d}{dt}Mc^2 = c^2\frac{dM}{dt}.$$

Since the Swarzchild radius is $\frac{2GM}{c^2}$, then the area of the black hole is given as

$$A = 4\pi r^2 = 16\pi\frac{G^2M^2}{c^4}$$

Now that we have all our variables, setting our differential equal to power gives us

$$\begin{aligned} c^2\frac{dM}{dt} &= -\frac{1}{491520\pi^3} \cdot \frac{hc^{10}}{G^4M^4} \cdot 16\pi\frac{G^2M^2}{c^4} \\ \frac{dM}{dt} &= -\frac{1}{30720\pi^2} \cdot \frac{hc^4}{G^2M^2} \\ \int_M^0 M^2 dM &= \int_0^t \left(-\frac{1}{30720\pi^2} \cdot \frac{hc^4}{G^2}\right) dt \\ \frac{1}{3}M^3 &= \left(\frac{1}{30720\pi^2} \cdot \frac{hc^4}{G^2}\right) t \end{aligned}$$

Solving for time gives:

$$\boxed{t = 10240\pi^2 \cdot \frac{G^2M^3}{hc^4}}$$

d) Plugging in

$$M = 5.97 \times 10^{24}\ \text{kg}$$

$$G = 6.67 \times 10^{-11}\text{m}^3/\text{kg} \cdot \text{s}^2$$

$$h = 6.626 \times 10^{-34}\text{m}^2\text{kg/s}$$

$$c = 2.99 \times 10^8\ \text{m/s}$$

gives: $t = 1.81 \times 10^{58}$ sec or $\boxed{t = 5.74 \times 10^{50}\ \text{years}}$

8 Assume again we are dealing with a Schwarzschild black hole, so as a result of ignoring any contributions to the angular momentum or charge, we assume that any heat or energy added goes straight to increasing the black hole's mass and area.

a) Assume that for our black hole, δW is always 0. Using the First Law of Thermodynamics, and letting the internal energy be equal to Mc^2, where M is the mass of the black hole, write down dS in terms of dM, including any previously-mentioned physical constants as needed.

b) Relate the area A of the black hole to the mass of a black hole, using the Schwarzchild radius r_S, and attempt to simplify dS. The expression won't simplify immensely, but you'll be able to relate dS to dA. From here, write down the entropy of a Schwarzschild black hole in terms of its area.

c) Explain in your words, why this result makes intuitive sense from a qualitative perspective.

d) What would you expect the thermodynamic volume V to be? (Hint: it's easy.) Why is this an issue for the internal energy? In other words, why is $U = U(S, V)$ a problematic statement?

Solution.

a) The first law of Thermodynamics gives

$$dE = dQ + dW.$$

By our assumption, $dW = 0$, thus by using the definition of entropy, $dS = \frac{dQ}{T}$, gives us,

$$dE = T_S dS$$

Since $E = Mc^2$, then $dE = c^2 dM$. Therefore,

$$\boxed{c^2 dM = T_S dS.}$$

b) Since the Schwarzschild radius is $\frac{2GM}{c^2}$, then the area of the black hole is given as

$$A = 4\pi r^2 = 16\pi \frac{G^2 M^2}{c^4}$$

Differentiating both sides with respect to t gives us

$$\frac{dA}{dt} = 32\pi \frac{G^2 M}{c^4} \frac{dM}{dt}$$

multiplying both sides by dt, we get:

$$32\pi \frac{G^2 M}{c^4} dM \implies dM = \frac{c^4}{32\pi G^2 M} dA$$

Multiplying by c^2 and using the result from part a) gives us the equation:

$$c^2 dM = \frac{c^6}{32\pi G^2 M} dA = T_S dS.$$

Now integrating and simplifying gives us

$$\int_0^A \frac{c^6}{32\pi G^2 M} dA = \int_0^S \frac{hc^3}{16\pi^2 G M k_B}$$
$$\frac{c^3}{2G} A = \frac{h}{\pi k_B} S$$
$$\boxed{S = \frac{\pi c^3 k_B}{2Gh} A}$$

c) When examining the answer of entropy from part (b), one may realize that the entropy given is really big. Overall, this makes sense for a variety of reasons. One reason, is because a black hole is a mass sucking machine. Whatever goes beyond the event horizon will disappear and turn into part of the black hole. To not violate the second law of thermodynamics, the black hole must absorb all the entropy of the things that it absorbs which yields in its massive amount of entropy as calculated from part b.

Furthermore, let us see how entropy depends on the physical constants. Consider an alternate universe where the value of h is large. Heisenberg's uncertainty principle tells us that there will be a high level of uncertainty in determining a particle's position and momentum. As a result, the possible number of "locations" that a particle can be in will decrease, so the micro-states will decrease and so will the overall entropy.

Next up, we look at G. If the gravitational force were to get stronger, then the area A will increase and by our equation, the entropy will also increase. Now instead of adding in more mass to increase the force, we somehow increase the value of G. While the denominator will be doubled, the numerator will be quadrupled, and the overall entropy increases. This is because $A \propto r^2$ and $r_{\text{event horizon}} \propto G$ so $A \propto G^2$ and thus $S \propto G$.

The speed of light c has the opposite effect of h. It sets the upper bound of the maximum speed of any particle. If we take the case as $c \to 0$, then we will know the speed of every particle with absolute certainty ($v = 0!$) and as a result, the number of micro-states will decrease and entropy will decrease.

The boltzmann constant is the easiest to explain mathematically, as entropy is directly given by:

$$S = k_B \ln \Omega$$

where Ω is the number of microstates. Another way to see this without this equation is that the boltzmann constant is just a conversion factor to convert temperature to energy! Thus in almost every single case, it is inversely proportional to the temperature. Entropy is also inversely proportional to the temperature so it would make sense entropy is proportional to k_B.

Last but not least, entropy is proportional to the area. This is a subtlety that will be explored more in depth in the next part.

d) The thermodynamic volume V would be $\frac{4}{3}\pi r^3$ where the Schwarzchild radius is given by $r = \frac{2GM}{c^2}$. Thus we have:

$$V = \frac{32\pi G^3 M^3}{3c^6}$$

Writing $U(S, V)$ is problematic for multiple reasons. First, as we've already seen: entropy is proportional to area, not volume! The main reason this is true is because past the event horizon, a Schwarzchild black hole is defined by only its mass. Thus any object that enters the black hole will somehow lose all of its entropy, making it seemingly appear as if information is destroyed. The resolution is that instead of the volume storing information, it's the actions that happen at the event horizon that are able to conserve information.

We can of course also solve for S as a function of internal energy and volume to be $S(U, V)$, but as we've just stated entropy isn't proportional to the volume! While this may not sound too much of a problem as it is theoretically possible to write area in terms of volume, it gets very messy. More importantly, there are some physical flaws when talking about the implications of the volume of a black hole.

In this problem, we've established that integration $T_S dS$ over the entire black hole will give a change of energy of $-Mc^2$ since we have $dW = 0$. We can rewrite this as:

$$PdV = 0$$

this means that if we somehow change the volume by dV, the change in that volume does not contribute to any of the internal energy!

Model of Black Hole Magnetic Field

9

a) The relativistic equation of motion for a particle in a magnetic field is:

$$\frac{d}{dt}(\gamma m\vec{v}) = q\vec{v} \times \vec{B}$$

Show that the radius of the orbit (called the *Larmor radius*) is $\frac{\gamma mv}{qB}$ assuming that $\vec{v}$ is perpendicular to the magnetic field.

b) The Larmor formula gives the power radiated from an accelerating electron as:

$$P = \frac{e^2a^2}{6\pi\epsilon_0 c^3}$$

in SI units, where a is the proper acceleration. For a non-relativistic orbit, a is simply the centripetal acceleration $\frac{v^2}{r}$. Given that the relativistic momentum is γmv and that $a = \frac{1}{m}\frac{dp}{d\tau}$, where τ is the proper time t/γ, show that the power radiated at relativistic speeds is:

$$P = \frac{e^2\gamma^4 v^4}{6\pi\epsilon_0 c^3 r^2}$$

Assume that γ is constant over time.

c) The LEP electron synchrotron at CERN has a rated energy of 45 GeV and a radius of 4300 m. Calculate the synchotron radiation power per electron.

d) Because of the radiation electrons in a synchrotron emit, they cool down over time. Calculate the time scale for this process by dividing the electron energy γmc^2 by the power radiated per electron (as calculated in question 2).

e) Finally, let's apply the results we've derived to estimate the magnetic field of a black hole. It turns out that

$$\gamma^2 = \frac{2\pi m\nu}{eB}$$

where ν is the critical frequency of the radiation emitted. Given that the cooling timescale of a black hole is 272 s at a frequency of 1.39×10^{14} Hz, estimate the magnetic field strength of the black hole. Assume that $v \to c$.

Solution.

a) The velocity is perpendicular to the magnetic field. We can take the derivative of both sides and determine the magnitude of the acceleration to be

$$\gamma ma = qvB \implies a = \frac{qvB}{\gamma m}$$

We also know that $a = \frac{v^2}{r}$ so:

$$\frac{v^2}{r} = \frac{qvB}{\gamma m} \implies \boxed{r = \frac{\gamma mv}{qB}}$$

b) The magnitude of the proper acceleration is:

$$a = \frac{1}{m}\left(\frac{d(\gamma mv)}{d(t/\gamma)}\right) = \frac{\gamma}{m}(qvB)$$

From part a) we can rearrange our answer to get:

$$\frac{qB}{m} = \frac{\gamma v}{r}$$

Substituting this into our expression for proper acceleration gives:

$$a = \gamma v\left(\frac{\gamma v}{r}\right) = \frac{\gamma^2 v^2}{r}$$

Using the Larmor formula gives:

$$P = \frac{e^2}{6\pi\epsilon_0 c^3}\left(\frac{\gamma^2 v^2}{r}\right)^2 = \boxed{\frac{e^2\gamma^4 v^4}{6\pi\epsilon_0 c^3 r^2}}$$

as desired.

c) The energy of an electron is given by:

$$E = (\gamma - 1)mc^2 \implies \gamma = \left(\frac{E}{mc^2} + 1\right).$$

We can solve for the lorentz factor to be:

$$\gamma = \frac{K}{mc^2} + 1$$

The mass of an electron is 0.511 MeV/c^2. Multiplying it by the square of the speed of light will give us energy. So we have:

$$\gamma = \frac{45000\text{MeV}}{0.511\text{ MeV}} + 1 = 88064$$

This high speed corresponds to a speed very close to the speed of light. We can verify this by expanding out the lorentz factor:

$$\frac{1}{\gamma} \approx 0 = \sqrt{1 - \frac{v^2}{c^2}}$$

which is valid when $v \approx c$. Thus, the radiation power per electron is:

$$P = \frac{e^2\gamma^4 c}{6\pi\epsilon_0 r^2}$$

Substituting in the appropriate values gives $\boxed{P = 1.437 \times 10^{-7} \text{ W/electron}}$

d) The time scale is given by:

$$\tau = \frac{\gamma m c^2}{P} = \frac{6\pi\epsilon_0 r^2 mc}{e^2\gamma^3}$$

Similarly, plugging in the appropriate values gives $\boxed{\tau = 0.0463 \text{ s}}$

e) Using the equation derived above, we can rewrite it by using the relation $r = \frac{\gamma m v}{qB}$:

$$\tau = \frac{6\pi\epsilon_0 mc}{e^2\gamma^3} \cdot \frac{\gamma^2 m^2 c^2}{e^2 B^2} = \frac{6\pi\epsilon_0 m^3 c^3}{e^4 \gamma B^2}$$

Solving for γ gives us:

$$\gamma = \frac{6\pi\epsilon_0 m^3 c^3}{\tau e^4 B^2}$$

Squaring it, and using the relation given leads us to:

$$\frac{1}{B^4}\left(\frac{6\pi\epsilon_0 m^3 c^3}{\tau e^4}\right)^2 = \frac{2\pi m \nu}{eB} \implies B^3 = \frac{e}{2\pi m \nu}\left(\frac{6\pi\epsilon_0 m^3 c^3}{\tau e^4}\right)^2$$

or simplified even more:

$$B = \sqrt[3]{\frac{18\pi\epsilon_0^2 m^5 c^6}{\tau^2 \nu e^7}}$$

this gives $\boxed{B = 41.6 \text{ gauss}}$.

3 Analysis of V404 Cygni Black Hole

1. Using your work from section 3.1, estimate the escape velocity for this black hole. The radius is about 6 times the radius of the Sun, which is 6.957×10^5 km.
2. The mass of the sun is 1.989×10^{30} kg. Based on section 4.1, find the power emitted per unit area and total power emitted by the black hole.
3. Estimate the evaporation time for the V404 Cygni black hole.
4. At a frequency of 6.25×10^{14} Hz, the cooling timescale for V404 Cygni is 131 ± 5 s. Estimate the magnetic field of this black hole in gauss.
5. At a frequency of 1.55×10^{18} Hz, the timescale is 1.8 ± 0.7 s. Make another estimate of the black hole's magnetic field from this data point.
6. Estimate the errors in the magnetic field estimates in questions 4 and 5. Are the magnetic field estimates from the two questions consistent?

1. It is told in the problem statement that V404 Cygni Black hole has a mass 9 times the amount of mass of the sun (1.989×10^{30} kg) and that the radius is 6 times the length of the sun's radius (6.957×10^5 km). From problem 5, it was given that

$$v_{\text{escape}} = \sqrt{\frac{2GM}{R}}$$

Substituting our new values we have

$$v_{\text{escape}} = \sqrt{\frac{18GM}{6R}} = \sqrt{\frac{3GM}{R}}$$

We know that

$$G = 6.67 \times 10^{-11} \text{m}^3/\text{kg} \cdot \text{s}^2$$

Thus,

$$v_{\text{escape}} \approx \boxed{7.177287376 \cdot 10^7 \text{ m/s}}$$

2. According to Stefan-Boltzmann's Law, the power per unit area emitted from the surface of an object at temperature T is σT^4. The temperature T is given by:

$$T_S = \frac{\hbar c^3}{8\pi G M_{\text{black hole}} k_B} = \frac{\hbar c^3}{72\pi G M_\odot k_B}$$

We have $T_S = 6.81 \times 10^{-9}$ K so evaluating σT_S^4 gives: $\boxed{P/A = 1.216 \times 10^{-10} \text{ W/m}^2}$ The total power emitted from the sun is therefore:

$$P_\odot = \sigma T_\odot^4 (4\pi R_\odot^2)$$

or $\boxed{P = 2.66 \times 10^{-20} \text{ W}}$

3. From problem 7, we have that

$$t = 10240\pi^2 \cdot \frac{G^2 M^3}{hc^4}$$

thus, by plugging in

$$M = 1.989 \times 10^{30} \text{ kg}$$
$$G = 6.67 \times 10^{-11} \text{m}^3/\text{kg} \cdot \text{s}^2$$
$$h = 6.626 \times 10^{-34} \text{m}^2 \text{kg/s}$$
$$c = 2.99 \times 10^8 \text{ m/s}$$

gives: $\boxed{t \approx 6.68066 \text{ years}}$

4. From 9 (d) we have that the magnitude of the magnetic field is given by:

$$B = \left(\frac{\tau^2 18\pi\epsilon_0^2 m^5 c^6}{\nu e^7} \right)^{1/3}$$

Evaluating gives $B = 41.0$ gauss. The uncertainty is given by:

$$\begin{aligned} \delta B &= \delta(\tau^{2/3}) \cdot B/(\tau^{2/3}) \\ &= \frac{2}{3}\tau^{-1/3}\delta\tau \cdot B/(\tau^{2/3}) \\ &= \frac{2B}{3}\frac{\delta\tau}{\tau} \end{aligned}$$

Evaluating, we can write our final figure as $\boxed{B = 41.0 \pm 1.04 \text{ gauss}}$.

5. Similarly, applying the two formulas will give us: $\boxed{B = 52 \pm 13 \text{ gauss}}$.

6. Yes, they are consistent. The range of values for the first and second case overlap.

4 Schwarzschild Metric

a) The metric in the form

$$g_{\mu\nu} = \begin{pmatrix} e^{-2\alpha(r)} & 0 & 0 & 0 \\ 0 & e^{2\beta(r)} & 0 & 0 \\ 0 & 0 & e^{2\gamma(r)}r^2 & 0 \\ 0 & 0 & 0 & e^{2\gamma(r)}r^2 \sin^2\theta \end{pmatrix}$$

satisfies our two assumptions: that it is spherically symmetric and that it is static in time. To see why, let us examine a line element of the Minowski metric

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

If we introduce a spherical coordinate system, then via a change of variables we have

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$$

We want to generalize this line element in a way such that it is spherically symmetric and static. Meaning that the metric should not depend on time or depend on the angles θ and φ. Thus it makes sense that the metric has to be in the form

$$ds^2 = -e^{-2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + e^{2\gamma(r)}r^2 d\theta^2 + e^{2\gamma(r)}r^2 \sin^2\theta d\varphi^2$$

b) We can define $\gamma(r)$ to be zero because we are just relabeling the term as r^2.

c)

$$\begin{aligned}
\Gamma^0_{01} &= \Gamma^0_{10} = \partial_1\alpha \\
\Gamma^1_{00} &= e^{2(\alpha-\beta)}\partial_1\alpha \\
\Gamma^1_{11} &= \partial_1\beta \\
\Gamma^2_{12} &= \Gamma^2_{12} = \frac{1}{r} \\
\Gamma^1_{22} &= -re^{-2\beta} \\
\Gamma^3_{13} &= \frac{1}{r} \\
\Gamma^1_{33} &= -re^{-2\beta}\sin^2\theta \\
\Gamma^2_{33} &= -\cos\theta\sin\theta \\
\Gamma^3_{23} &= \cot\theta
\end{aligned}$$

d)

$$\begin{aligned}
R^0_{101} &= e^{2(\beta-\alpha)}((\partial_0\beta)^2 - \partial_0\alpha\partial_0\beta + \partial_0^2\beta) + \partial_1\alpha\partial_1\beta - \partial_1^2\alpha - (\partial_1\alpha)^2) \\
R^0_{203} &= -re^{-2\beta}\partial_0\alpha \\
R^0_{303} &= -re^{-2\alpha}\sin^2\theta\partial_0\beta \\
R^1_{213} &= -re^{-2\beta}\partial_1\beta \\
R^1_{313} &= -re^{-2\beta}\sin^2\theta\partial_1\beta \\
R^2_{323} &= \left(1 - e^{-2\beta}\right)\sin^2\theta\partial_1\beta
\end{aligned}$$

e) The non-zero components of the Ricci tensor are

$$
\begin{aligned}
R_{00} &= (\partial_0\beta)^2 - \partial_0\alpha\partial_0\beta + \partial_0^2\beta + e^{2(\alpha-\beta)}\left(\partial_1^2\alpha - \partial_1\alpha\partial_1\beta + (\partial_1\alpha)^2 + \frac{2}{r}\partial_1\alpha\right) \\
R_{01} &= R_{10} = \frac{2}{r}\partial_0\beta \\
R_{11} &= \partial_1\alpha\partial_1\beta + \frac{2}{r}\partial_1\beta - \partial_1^2\alpha - (\partial_1\alpha)^2 + e^{2(\beta-\alpha)}\left(\partial_0^2\beta - \partial_0\alpha\partial_0\beta + (\partial_0\beta)^2\right) \\
R_{22} &= e^{2\beta}(r\partial_1\beta - r\partial_1\alpha - 1) + 1 \\
R_{33} &= \sin^2\theta(e^{2\beta}(r\partial_1\beta - r\partial_1\alpha - 1) + 1)
\end{aligned}
$$

f) Since both R_{00} and R_{11} are zero, then this implies that

$$0 = e^{2(\beta-\alpha)}R_{00} + R_{11} = \frac{2}{r}(\partial_1\alpha + \partial_1\beta).$$

This equation implies that $\alpha = -\beta + c$, setting $\alpha = -\beta$ gives us the answer as given in the exam packet.

g) Replacing $\alpha = -\beta$ in for our expression gives us

$$
\begin{aligned}
R_{22} = 0 = e^{2\alpha}(2r\partial_1\alpha + 1) = 1 \\
\partial_1(re^{2\alpha}) = 1 \implies e^{2\alpha} = 1 + \frac{R_S}{r}
\end{aligned}
$$

h) R_S is the Schwarzchild Radius and it is equal to $\frac{2GM}{c^2}$.

i) The Schwarzchild metric is attained by combining all of our results from different sections throughout this question. First, note that the metric must be both spherically symmetric and asymptotically flat. The full form of the metric is as follows:

$$\boxed{ds^2 = \left(1 - \frac{R_S}{r}\right)c^2dt^2 - \frac{dr^2}{1-\frac{R_S}{r}} - r^2d\theta^2 - r^2\sin^2\theta d\phi^2}$$

This comes directly by looking at our evaluation of all of the tensors in the previous parts. And yay we are done.