Art of Problem Solving

Unofficial Solutions

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October 16, 2018

Contents

Chapter 1

Logarithms

1.1 Equations

There are 6 main equations:

$$\log_a b^n = n \log_a b \tag{1.1}$$

$$\log_a b + \log_a c = \log_a bc \tag{1.2}$$

$$\log_a b - \log_a a = \log_a b/c \tag{1.3}$$

$$(\log_a b)(\log_c d) = (\log_a d)(\log_c b) \tag{1.4}$$

$$\frac{\log_a b}{\log_a c} = \log_c b \tag{1.5}$$

$$\log_{a^n} b^n = \log_a b \tag{1.6}$$

1.2 Problems

Exercise 1.1 (View Solution)
Evaluate the product $(\log_2 3)(\log_3 4)(\log_4 5)(\log_5 6)(\log_6 7)(\log_7 8)$

Exercise 1.2 (View Solution)
If $\log(36) = a$ and $\log(125) = b$, express $\log(\frac{1}{12})$ in terms of a and b.

Exercise 1.3 (View Solution)
At which point(s) do $y = 2\log(x)$ and $y = \log(2x)$ intersect?

Exercise 1.4 (View Solution)
Find all solutions of:

$$x^{\log(x)} = \frac{x^3}{100}$$

1.3 Solutions

Solution 1.1 (View Question)

Use equation (1.4) we can note that if we have a product of logs and keep the bases the same, we can shift the arguments around however much we want. If we move all the arguments one to the right, and move the final argument to the first one, we get

$$(\log_2 8)(\log_3 3)(\log_4 4)(\log_5 5)(\log_6 6)(\log_7 7)$$

Since $\log_n n = 1$ we can cancel everything and get our answer of $\boxed{3}$

Solution 1.2 (View Question)

First we note that 36 and 125 are both a perfect square and cube, respectively. Therefore, we can simplify these equations by writing them with an exponent: $\log(36) = \log(6^2) = 2\log(6)$ and $\log(125) = \log(5^3) = 3\log(5)$

Start with $\log(\frac{1}{12})$ and note it is equal to $\log(1) - \log(12) = -\log(12)$. We know from the first step we need to somehow include both $\log(5)$ and $\log(6)$, therefore we need to include these expressions in our solution somewhere...

After some experimentation, we find that $-\log(12) = -\log(\frac{6\times 10}{5})$ which is equal to:

$$-(\log(6\times 10) - \log(5) = -(\log(6) + \log(10) - \log(5)$$

$$-(\frac{a}{2} + 1 - \frac{b}{3})$$

The answer is $\boxed{\frac{b}{3} - \frac{a}{2} - 1}$

Solution 1.3 (View Question)

To begin, let's set both equations equal to each other: $2\log(x) = \log(2x)$. Notice that $2\log(x) = \log(x^2)$ We can simplify both sides:

$$10^{\log(x^2)} = 2^{\log(2x)} \to x^2 = 2x$$

Solving this quadratic equation gives $x = 2$ and $x = 0$ Since $\log(0)$ is undefined, $\boxed{x = 2}$ is the only interesection point

Solution 1.4 (View Question)

We can take the base-x logarithm of both sides and using theorem (1.1) and (1.5):

$$\log_x(x^{\log(x)}) = \log_x(\frac{x^3}{100}) \to \log(x)\log_x(x) = \log_x(\frac{x^3}{10^2})$$

$$\log_{10}(x) = \log_x(x^3) - \log_x(10^2) \to \log_{10}(x) = 3 - \frac{2\log_{10}(10)}{\log_{10}(x)} \to \log_{10}(x) = 3 - \frac{2}{\log_{10}(x)}$$

Substituting $u = \log_{10}(x)$ gives:

$$u = 3 - \frac{2}{u}$$

Solving this quadratic equation gives us $u \in \{1, 2\}$ and thus $\boxed{x \in \{10, 100\}}$