

MAT301 Notes

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May 13, 2021

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1 Lecture One

- Groups are everywhere in mathematics and nature in one of two forms:
 - as groups of symmetries
 - as groups of “numbers” or quantities
- We will call a subset $F \subseteq \mathbb{R}^n$ a **figure** in \mathbb{R}^n when we consider F not just as a set, but as a set together with the structure of its distance functions:

$$d : F \times F \rightarrow \mathbb{R}_{\geq 0}, \quad d(x, y) = \|x - y\| \quad (1)$$

A figure is then defined as the pair (F, d) .

Definition: A **symmetry** of a figure $F \subseteq \mathbb{R}^n$ is a bijection $\sigma : F \rightarrow F$ such that σ and σ^{-1} preserve distances:

$$\forall x, y \in F, \quad d(\sigma(x), \sigma(y)) = d(x, y) \quad (2)$$

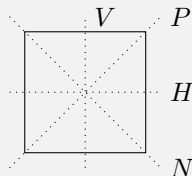
$$\iff d(\sigma^{-1}(x), \sigma^{-1}(y)) = d(x, y) \quad (3)$$

Therefore:

$$\text{Sym}(F) \equiv \{\sigma : F \rightarrow F \mid \sigma \text{ is a symmetry}\} \quad (4)$$

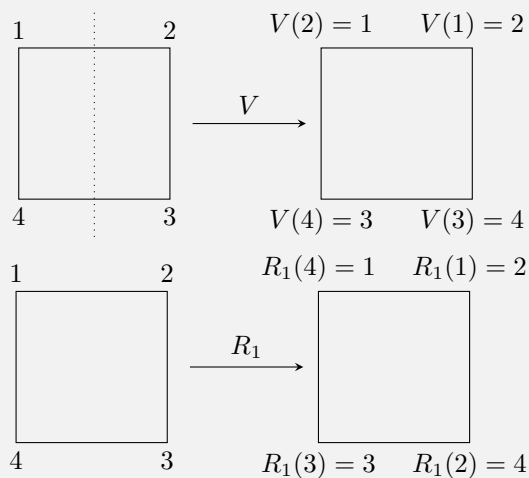
- For example, any point, line, shape, or form is a figure. However, we are only interested in figures that have interesting symmetries.

Example 1: Let F be a square in \mathbb{R}^2 . There are four different lines of reflections:



and there are three rotations: R_1 , R_2 , and R_3 , which represent 90° , 180° , and 270° clockwise rotations. I represents the identity transformation (do nothing).

We can combine symmetries. For example, what is $R_1 \circ V$? To do so, we can label the vertices:



Applying the computations:

$$(R_1 \circ V)(1) = R_1(V(1)) = R_1(2) = 3 \quad (5)$$

$$(R_1 \circ V)(2) = R_1(V(2)) = R_1(1) = 2 \quad (6)$$

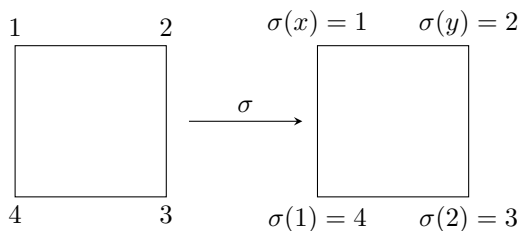
$$(R_1 \circ V)(3) = 1 \quad (7)$$

$$(R_1 \circ V)(4) = 4 \quad (8)$$

Check that $V \circ R_1 = N$. Also notice that these operations are not commutative: $R_1 \circ V \neq V \circ R_1$.

- In the above example, how are we sure that these are all of the symmetries of a square? To answer this, we will need the following facts:

1. A symmetry maps vertices to vertices. The vertices are the points of the square that are furthest from the center.
2. Symmetries map adjacent vertices to adjacent vertices. If x, y are adjacent vertices, then $\sigma(x), \sigma(y)$ are vertices, and $d(\sigma(x), \sigma(y)) = d(x, y) = \text{side length}$.
3. A symmetry σ is completely determined by $(\sigma(1), \sigma(2))$. For example, suppose we have the symmetry σ on a square such that:



From this, we know that we must have $y = 3$, from fact 1, as well as $x = 4$.

4. For all $x, y \in \{1, 2, 3, 4\}$ such that x is adjacent to y , $\exists!$ symmetry σ of the square such that:

$$(\sigma(1), \sigma(2)) = (x, y) \quad (9)$$

By the above facts, we must count the ordered pairs (x, y) such that $x, y \in \{1, 2, 3, 4\}$ and x is adjacent to y :

- There are 4 choices for x .
- For each choice of x , there are two choices of y . Therefore, there are $4 \times 2 = 8$ symmetries.

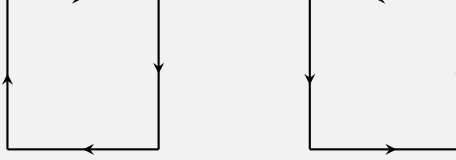
Since we listed 8 different symmetries of a square, we have therefore defined all of them.

2 Lecture Two

- Let X be a set with some **structures**. Then a symmetry of X (w.r.t. the structures) is a bijection $\sigma : X \mapsto X$, such that σ and σ^{-1} preserve the structures.

- The set of symmetries of X is denoted as $\text{Sym}(X)$.

Example 2: We can consider a square not only with the structure of its distance function but with additional structure of its orientations. There are two orientations of a square:



A symmetry of the square with respect to its orientation is a bijection from the square to itself that maps each orientation to itself.

– Rotations preserve orientations, but reflections don't.

Therefore, the symmetries preserving orientations are $\{I, R_1, R_2, R_3\}$.

- In general:

0. If $\sigma_1, \sigma_2: X \rightarrow X$ are symmetries, then:

$$\sigma_1 \circ \sigma_2 : X \rightarrow X \quad (10)$$

is also a symmetry. Consequently, composition of symmetries restrict a map:

$$\text{Sym}(X) \times \text{Sym}(X) \mapsto \text{Sym}(X), \quad (\sigma_1, \sigma_2) \mapsto \sigma_1 \circ \sigma_2 \quad (11)$$

Remarks: A map $m : S \times S \rightarrow S$ is called a binary operation on S .

1. Associativity: For all $\sigma_1, \sigma_2, \sigma_3 \in \text{Sym}(X)$, we have:

$$(\sigma_1 \circ \sigma_2) \circ \sigma_3 = \sigma_1 \circ (\sigma_2 \circ \sigma_3) \quad (12)$$

2. The identity $\text{id} : X \mapsto X$ is a symmetry and $\text{id} \in \text{Sym}(X)$.

3. Immediately from the “definition,” we have: $\sigma \in \text{Sym}(X) \implies \sigma^{-1} \in \text{Sym}(X)$

- The notion of a group is an abstraction of $\text{Sym}(X)$ and its properties.

Definition: A group is an ordered pair $(G, *)$ consisting of a set G and a binary operation $* : G \times G \rightarrow G$ such that:

1. $*$ is associative, $\forall g_1, g_2, g_3 \in G$, we have:

$$(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3) \quad (13)$$

2. There exists an element $e \in G$ such that for all $g \in G$, we have $g * e = g = e * g$.

3. For all $g \in G$, there exists an element $h \in G$ such that $g * h = e = h * g$.

These numberings are abstractions of the properties listed above.

- The binary operator $*$ is called the **group law** or **group operation**. It is often denoted by a dot \cdot or by juxtaposition (gh instead of $g * h$).
- The *cardinality* of G , $|G|$, is called the **order** of G .
- It is common to denote e by 1 or I .

Warning: A common *misconceptions* is saying “ G is a group” instead of “ $(G, *)$ is a group.”

- These are equivalent statements:

$$(G, *) \text{ is a group} \quad (14)$$

$$\iff G \text{ is a group under } * \quad (15)$$

Definition: A group $(G, *)$ is **abelian** (or commutative) if for all $g, h \in G$, we have:

$$g * h = h * g \quad (16)$$

- Here are some examples of groups:

- $(\text{Sym}(X), \circ)$
- $(\mathbb{Z}, +)$
- (\mathbb{R}^x, \cdot) where:

$$F^x = \{x \in F : \exists y \in F \text{ with } xy = 1 = yx\} \quad (17)$$

- $(\mathbb{Q}_{>0}, \cdot), (\mathbb{R}_{>0}, \cdot)$.
- (μ_n, \cdot) where for $n \in \mathbb{Z}_{>0}$, let

$$\mu_n = \{z \in \mathbb{C} | z^n = 1\} = \{e^{2\pi ki/n} | k = 0, 1, \dots, n-1\} \quad (18)$$

- $(\mathbb{R}^n, +)$
- $(\text{GL}_n(F), \cdot)$ where $\text{GL}_n(F) = \{A \in \text{Mat}_{n \times n}(F) | A \text{ invertible}\}$, $F = \mathbb{Q}, \mathbb{R}, \mathbb{C}$. For all $n \geq 2$, $\text{GL}_n(F)$ is non-abelian. Note that GL stands for *general linear*
- $(\text{SL}_n(F), \cdot)$ where $\text{SL}_n(F) = \{A \in \text{GL}_n(F) | \det A = 1\}$. Note that SL stands for special linear.
- $(\text{Mat}_{n \times n}(F), +)$

and non-groups:

- (\mathbb{Z}, \cdot)
- $(\mathbb{Z}_{>0}, +)$
- $(\mathbb{Z}, -), (\mathbb{Q}^x, \div)$.
- $(\text{Mat}_{n \times n}(F), \cdot)$

Proposition 1: Let $(G, *)$ be a group. If $e, e' \in G$ such that $\forall g \in G$ we have

$$g * e = g = e * g \quad (19)$$

and

$$g * e' = g = e' * g, \quad (20)$$

then $e = e'$.

Proof. Consider $e * e'$. By 19, we have:

$$e * e' = e' \quad (21)$$

Similarly, by 20, we have:

$$e * e' = e \quad (22)$$

Therefore, $e = e * e' = e'$. \square

- We call the unique element $e \in G$ satisfying the second property in the definition of a group, the identity element of G .
- The **trivial group**: For any singleton $\{e\}$, there exists a unique binary operation \cdot such that:

$$\{e\} \times \{e\} \mapsto \{e\}, \quad (e, e) \mapsto e \quad (23)$$

and $(\{e\}, \cdot)$ is a group, called a trivial group.

Proposition 2: Let $(G, *)$ be a group and let $g \in G$. If $h, h' \in G$ satisfies:

$$g * h = e = h * g \quad (24)$$

and

$$g * h' = e = h' * g \quad (25)$$

then $h = h'$. By 24, we have:

$$h * g = e. \quad (26)$$

By 25, we have:

$$g * h' = e. \quad (27)$$

Therefore:

$$h = h * e \quad (\text{property 2}) \quad (28)$$

$$= h * (g * h') \quad (27) \quad (29)$$

$$= (h * g) * h' \quad (\text{property 1}) \quad (30)$$

$$= e * h' \quad (26) \quad (31)$$

$$= h' \quad (\text{property 2}) \quad (32)$$

- For each $g \in G$, the unique element $h \in G$ such that $g * h = e = h * g$ is called the inverse of g and denoted by g^{-1} .

Lemma 1: Let $(G, *)$ be a group and let $x, y, z \in G$. Then, right cancellation tells us:

$$x * z = y * z \implies x = y \quad (33)$$

and left cancellation tells us:

$$z * x = z * y \implies x = y \quad (34)$$

Proof. If $z * x = z * y$, then:

$$z^{-1} * (z * x) = z^{-1} * (z * y) \quad (35)$$

$$\implies (z^{-1} * z) * x = (z^{-1} * z) * y \quad (36)$$

$$\implies e * x = e * y \quad (37)$$

$$\implies x = y \quad (38)$$

The other implication is similar. \square

Warning: The notation $\frac{a}{b}$ is ambiguous. Does it mean $a * b^{-1}$ or $b^{-1} * a$? These can be different in a non-abelian group.

Lemma 2: Let $(G, *)$ be a group and let $g_1, \dots, g_n \in G$. Every way of way inserting parentheses into $g_1 * g_2 * \dots * g_n$ to determine a well defined product in G results in the same element of G .

Proof. Proved in tutorial worksheet. \square

- The consequence of the above lemma is that the notation $g_1 * g_2 * \dots * g_n$ is unambiguous.

Definition: Let $(G, *)$ be a group and let $n \in \mathbb{Z}$. We define:

$$g^n = \begin{cases} \underbrace{g * g * \dots * g}_{n \text{ copies}}, & n > 0 \\ e, & n = 0 \\ \underbrace{g^{-1} * \dots * g^{-1}}_{n \text{ copies}} = (g^{-1})^{-n}, & n < 0 \end{cases} \quad (39)$$

Lemma 3: Let $(G, *)$ be a group. For all $g \in G$ and $m, n \in \mathbb{Z}$, we have:

$$g^m * g^n = g^{m+n} \quad (40)$$

and:

$$(g^m)^n = g^{mn} \quad (41)$$

- To prove the above lemma, we can use induction.

Warning: If G is a non-abelian group and $a, b \in G$ and $n \in \mathbb{Z}$, then it can happen that:

$$(ab)^n \neq a^n b^n \quad (42)$$

Lemma 4: Let G be a group and let $a, b \in G$. Then:

$$(ab)^{-1} = b^{-1}a^{-1} \quad (43)$$

Proof. We just need to check the two conditions:

$$(ab)(b^{-1}a^{-1}) = aea^{-1} = aa^{-1} = e \quad (44)$$

and:

$$(b^{-1}a^{-1})(ab) = b^{-1}eb = b^{-1}b = e \quad (45)$$

Therefore, it is the inverse. \square

- **Dihedral Groups.** Let $n \in \mathbb{Z}$, $n \geq 3$. Let P_n be a regular n -gon.

Definition: The group of symmetries of the regular n -gon P_n is called the dihedral group of order $2n$ and is denoted by D_n .

Warning: Some people use D_{2n} instead of D_n .

Lemma 5: The order of D_n is $2n$.

Proof. Label the vertices of P_n by v_1, v_2, \dots, v_n in some clockwise order. By the same reasoning from the case $n = 4$ when we were considering a square, we have a bijection:

$$D_n = \text{Sym}(P_n) \rightarrow \{(v_i, v_j) | v_i \text{ adjacent to } v_j\} \quad (46)$$

$$\sigma \mapsto (\sigma(v_1), \sigma(v_2)) \quad (47)$$

Note that $\{(v_i, v_j) | v_i \text{ adjacent to } v_j\} = \{(v_i, v_j) | j \equiv i \pm 1 \pmod{n}\}$. We have:

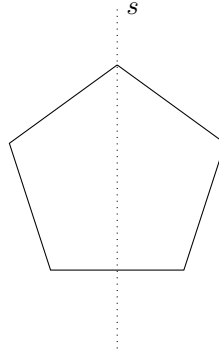
$$|D_n| = |\{(v_i, v_j) | j \equiv i \pm 1 \pmod{n}\}| = n \cdot 2 \quad (48)$$

\square

- For example, consider D_5 . There are 5 lines of reflection, 4 rotational symmetries, and the identity. We can further compose transformations, for example:

$$rs = sr^4, \quad r^2s = sr^3, \quad r^3s = sr^2, \quad r^4s = sr, \quad r^5s = sr \quad (49)$$

where s represents a reflection and r is a 72° clockwise rotation.



Lemma 6: Let P_n be a regular n -gon. Let r be either a clockwise or counterclockwise rotation about the center of P_n by $\frac{2\pi}{n}$, and let s be any reflectional symmetry of P_n . Then:

1. $r^n = 1, s^2 = 1$
2. For all $k = 0, 1, \dots, n-1$, sr^k is a reflection and:

$$sr^k = r^{-k}s = r^{n-k}s \quad (50)$$

3. $1, r, \dots, r^{n-1}, s, sr, \dots, sr^{n-1}$ are all distinct.
4. $D_n = \{1, r, \dots, r^{n-1}, s, sr, \dots, sr^{n-1}\}$.

Proof. We will prove all four:

1. r is a rotation by $2\pi/n$ CW or CCW so $r^n = 1$. Since s is a reflection, $s^2 = 1$.
2. The composition of a reflection and a rotation in the plane is a reflection. Therefore, $\forall k = 0, 1, \dots, n-1$, sr^k is a reflection (orientation is not preserved). Therefore:

$$(sr^k)^2 = 1 \quad (51)$$

$$sr^k sr^k = 1 \quad (52)$$

$$sr^k s = r^{-k} \quad (53)$$

$$sr^k = r^{-k}s^{-1} \quad (54)$$

Since $s^2 = 1, s^{-1} = s$, this is proved. Furthermore, since $r^n = 1$, we must also have:

$$sr^k = r^{n-k}s \quad (55)$$

3. Since r^k is a rotation CW or CCW by $2\pi k/n$, then $1, r, \dots, r^{n-1}$ are all distinct. Since rotations preserve orientation and reflections do not, then $r^i \neq sr^j$ for all i, j . If $sr^i = sr^j$, then $r^i = r^j$ so $i = j$ if $i, j \in \{0, \dots, n-1\}$.

Therefore, $1, r, \dots, r^{n-1}, s, sr, \dots, sr^{n-1}$ are distinct.

4. This follows directly from the previous property and the order of the dihedral group is $|D_n| = 2n$.

□

3 Lecture Three

- **Notation:** Sometimes the group operation for an **abelian** group is denoted by $+$.

If $(A, +)$ is an abelian group, then:

- The identity is denoted by 0
- a^{-1} is denoted by $-a$
- a^n is denoted by na
- $a + (-b)$ is denoted by $a - b$.

- One way to get a better understanding of a group G is to find a group “inside of” G that you understand better.

Definition: Let $(G, *_G)$ be a group. A subset $H \subseteq G$ is a subgroup if:

1. For all $h_1, h_2 \in H$, $h_1 *_G h_2 \in H$, and therefore the operation of G :

$$*_G : G \times G \rightarrow G \quad (56)$$

restricts to a binary operation on H :

$$*_H : H \times H \rightarrow H, \quad (h_1, h_2) \mapsto h_1 *_H h_2 := h_1 *_G h_2 \quad (57)$$

2. $(H, *_H)$ is a group.

- We write $H \leq G$ as a shorthand for “ H is a subgroup of G .” If $(G, *)$ is a group and $H \subseteq G$, we often denote the group operator for H by $*$ as well.

Example 3: Let G be a group. Then $G \leq G$ and $\{e\} \leq G$. We call $\{e\}$ the trivial subgroup of G .

- If $H \leq G$ and $H \neq G$, we write $H < G$ and call H a **proper subgroup** of G .

Example 4: Let D_n be the symmetric group of the regular n -gon with vertices $\{(\cos(2\pi k/n), \sin(2\pi k/n)) \mid k = 0, \dots, n-1\}$.

From last lecture, we have $D_n = \{1, r, \dots, r^{n-1}, s, rs, \dots, r^{n-1}s\}$. Then: $H := \{1, r, \dots, r^{n-1}\} \leq D_n$.

Proposition 3: Let G be a group and $H \leq G$.

1. The identity of H is the identity of G .
2. For all $h \in H$, the inverse of h in H is the inverse of h in G .

Proof. 1. Let e_H be the identity of H and e_G is that of G . Since e_H is the identity of H , we have:

$$e_H e_H = e_H \quad (58)$$

Let x be the inverse of e_H in G , then:

$$e_H e_H x = e_H x \quad (59)$$

$$\implies e_H e_G = e_G \quad (60)$$

$$\implies e_H = e_G \quad (61)$$

The first implication follows since x is the inverse of e_H in G and the second follows since e_G is the identity in G .

2. Let $h \in H$, let x be the inverse of h in H , and let y be the inverse of h in G . Then:

$$hx = e_H = e_G \quad (62)$$

and

$$xh = e_H = e_G \quad (63)$$

so x is the inverse of h in G .

□

Theorem: Two-step subgroup test: Let H be a nonempty subset of a group G . If:

1. $a, b \in H \implies ab \in H$ (H is closed under the group operator)
 2. $a \in H \implies a^{-1} \in H$ (H is closed under taking inverses)
- then H is a subgroup of G .

Proof. Assume that H is as in the theorem. We will prove that $(H, *_H)$ is a group.

– Associative: Let $h_1, h_2, h_3 \in H$

$$h_1 *_H (h_2 *_H h_3) = h_1 *_G (h_2 *_G h_3) \quad (64)$$

$$= (h_1 *_G h_2) *_G h_3 \quad (65)$$

$$= (h_1 *_H h_2) *_H h_3 \quad (66)$$

– H has an identity: Since $H \neq \phi$, there exists $x \in H$. By (2), we have $x^{-1} \in H$. By (1), we have $e_G = xx^{-1} \in H$ since $x, x^{-1} \in H$.

For all $h \in H$, we have:

$$he_G = h = e_G h \quad (67)$$

since e_G is the identity of G . Therefore e_G is an identity of H .

– H has inverses: Let $h \in H$. By (2), we have that $h^{-1} \in H$. Since h^{-1} is the inverse of h in G , we have $hh^{-1} = e_G = h^{-1}h$. Therefore h^{-1} is an inverse of h in H .

□

Theorem: One-step subgroup test: Let G be a group and let H be a nonempty subset of G . Suppose that:

1. $a, b \in H \implies ab^{-1} \in H$
- then $H \leq G$.

Proof. Let H be as in the theorem statement. Since $H \neq \phi$, $\exists h \in H$. Taking $a = b = h$ in (1) gives $e = hh^{-1} \in H$. Taking $a = e, b = h$ in (1) gives $h^{-1} = eh^{-1} = ab^{-1} \in H$. Therefore, $h \in H \rightarrow h^{-1} \in H$.

Let $h_1, h_2 \in H$. Then $h_2^{-1} \in H$. Taking $a = h, b = h_2^{-1}$ in (1) gives $h_1 h_2 = ab^{-1} \in H$. Therefore, $h_1, h_2 \in H \implies h_1 h_2 \in H$. By the two-step subgroup test, $H \leq G$.

□

Example 5: Let G be an abelian group. Prove that $H = \{x \in G \mid x^2 = e\}$ is a subgroup of G .

Proof. Let $a, b \in H$. Then $a^2 = b^2 = e$. Since G is abelian:

$$(ab^{-1})^2 = a^2 b^{-2} = a^2 (b^2)^{-1} = ee^{-1} = e \quad (68)$$

Therefore, $ab^{-1} \in H$ by the one-step subgroup test, $H \leq G$.

□

Example 6: Prove that matrices in the form of $\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$ where $x, y, z \in \mathbb{R}$ is a subgroup of $SL_3(\mathbb{R})$ using either subgroup test.

Proof. Using the one-step subgroup test. Let $g_1 = \begin{pmatrix} 1 & x_1 & y_1 \\ 0 & 1 & z_1 \\ 0 & 0 & 1 \end{pmatrix}$ and $g_2 = \begin{pmatrix} 1 & x_2 & y_2 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix}$. The inverse of g_2 is:

$$g_2^{-1} = \begin{pmatrix} 1 & -x & xz - y \\ 0 & 1 & -z \\ 0 & 0 & 1 \end{pmatrix} \quad (69)$$

and carrying out the computation:

$$g_1 g_2^{-1} = I \quad (70)$$

Since I is in the given group, we are done.

□

4 Lecture Four

- We begin with the **Finite Subgroup Test**

Theorem: Let G be a group and let H be a finite nonempty subset of G . If H is closed under the group operation of G , then $H \leq G$.

Proof. By the 2-step subgroup test, it suffices to prove that H is closed under taking inverses. Let $a \in H$:

- If $a = e$, then $a^{-1} = e \in H$.
- If $a \neq e$, consider the set:

$$\{a^n | n \geq 1\} = \{a, a^2, a^3, \dots\} \quad (71)$$

Since H is closed under the group operation and $a \in H$, we have $\{a^n | n \geq 1\} \subseteq H$ by a short induction argument. Since H is finite, so is $\{a^n | n \geq 1\}$. Therefore, $\exists m, n \geq 1, m \neq n$ such that:

$$a^m = a^n \quad (72)$$

WLOG, we may assume that $m > n$, so $m - n > 0$. We have:

$$a^{m-n} = e \quad (73)$$

Since $a \neq e$, $m - n \neq 1$. Therefore, $m - n \geq 2$, so $m - n - 1 \geq 1$. Thus:

$$a^{m-n-1} \in \{a^k | k \geq 1\} \subseteq H \quad (74)$$

and

$$a^{m-n-1}a = a^{m-n} = e \quad (75)$$

so:

$$a^{m-n-1} = a^{-1} \quad (76)$$

□

- We will look at a special class of subgroups: **subgroups generated by one element**.

Definition: Let G be a group and let $a \in G$. Define:

$$\langle a \rangle = \{a^n | n \in \mathbb{Z}\} \quad (77)$$

We call $\langle a \rangle$ the subgroup of G generated by a .

- We propose that $\langle a \rangle \leq G$.

Proof. Since $e = a^0 \in \langle a \rangle$, we have $\langle a \rangle \neq \emptyset$.

If $g, h \in \langle a \rangle$, then $g = a^m$ and $h = a^n$ for some $m, n \in \mathbb{Z}$ and:

$$gh^{-1} = a^m(a^n)^{-1} = a^m a^{-n} = a^{m-n} \in \langle a \rangle \quad (78)$$

□

Example 7: Let $G = (\mathbb{Z}/14\mathbb{Z})^\times = \{1, 3, 5, 9, 11, 13\}$. We have:

$$a = 3, a^2 = 9, a^3 = 27 = 13 = -1 = -1, a^4 = -3 = 11, a^5 = -9 = 5, a^6 = 15 = 1 \quad (79)$$

Similarly:

$$a^0 = 1, a^{-1} = 5, a^{-2} = 11, a^{-3} = 13, a^{-4} = 9, a^{-5} = 3, a^{-6} = 1 \quad (80)$$

Therefore:

$$\langle a \rangle = \{1, 3, 5, 9, 11, 13\} = (\mathbb{Z}/14\mathbb{Z})^\times \quad (81)$$

Therefore, $(\mathbb{Z}/14\mathbb{Z})^\times$ is cyclic. **Remarks:** If $a^n = e$, then for all $k \in \mathbb{Z}$, we have:

$$a^{-k} = a^{n-k} \quad (82)$$

so we can easily figure out negative exponents.

Example 8: Let $G = \mathbb{Z}/12\mathbb{Z}$ and $a = 2$. We have:

$$-a = 10, 0a = 0, 2a = 4, 3a = 6, 4a = 8, 5a = 10, 6a = 12 = 0, 7a = 2 \quad (83)$$

so:

$$\langle a \rangle = \{0, 2, 4, 6, 8, 10\}. \quad (84)$$

Example 9: Let $G = \mathbb{R}$ and $a = 2\pi$. Here,

$$\langle a \rangle = \{n2\pi | n \in \mathbb{Z}\} = 2\pi\mathbb{Z} \quad (85)$$

Definition: Let G be a group and $a \in G$. If there exists $n \in \mathbb{Z}_{>0}$ such that $a^n = e$, then we say that a has **finite order** and the **order of a** is defined to be the smallest $n \in \mathbb{Z}_{>0}$ such that $a^n = e$.

If there does not exist $n \in \mathbb{Z}_{>0}$ such that $a^n = e$, then we say that a has infinite order.

The order of a is denoted by $o(a)$ or $|a|$. If a has infinite order, we write $o(a) = \infty$.

• Note that:

- $o(a) = 1 \iff a = e$
- If $o(a) = \infty$, then $a^n = e \iff n = 0$.

• Let G be a group and $a \in G$.

1. If $o(a) = \infty$, then $\forall i, j \in \mathbb{Z}$ we have:

$$a^{i-j} = e \iff i - j = 0 \quad (86)$$

$$\iff i = j \quad (87)$$

2. If $o(a) = n < \infty$, then $\forall i, j \in \mathbb{Z}$ we have:

$$a^i = a^j \iff n | i - j \quad (88)$$

$$\iff i \equiv j \pmod{n} \quad (89)$$

In particular, $a^i = e (= a^0) \iff n | i$.

Proof. Let $i, j \in \mathbb{Z}$. Note $a^i = a^j \implies a^{i-j} = e$.

1. Suppose $o(a) = \infty$. Then $a^{i-j} = e$ iff $i - j = 0 \iff i = j$.

2. Suppose $o(a) = n < \infty$. We must show that $a^{i-j} = e \iff n | i - j$.

(Backwards): If $n | i - j$, then $\exists k \in \mathbb{Z}$ such that $i - j = kn$ so $a^{i-j} = a^{kn} = (a^n)^k = e^k = e$.

(Forwards) Now suppose $a^{i-j} = e$. By the division algorithm, $\exists!$ q and $0 \leq r < n$ such that:

$$i - j = qn + r \quad (90)$$

We have:

$$e = a^{i-j} = a^{qn+r} = a^{qn}a^r = (a^n)^qa^r = e^qa^r = a^r \quad (91)$$

Since n is the smallest positive integer with $a^n = e$ and $0 \leq r < n$ and satisfies $a^r = e$, we must have $r = 0$.

Therefore, $i - j = qn$ so $n | i - j$.

□

Corollary 1: Let G be a group and $a \in G$.

1. If $o(a) = \infty$, then $\dots, a^{-2}, a^{-1}, e, a, a^2, \dots$ are distinct (and $\langle a \rangle = \{a^n | n \in \mathbb{Z}\}$)
2. If $o(a) = n < \infty$, then e, a, \dots, a^{n-1} are distinct and $\langle a \rangle = \{e, a, \dots, a^{n-1}\}$.

Corollary 2: Let G be a group and $a \in G$. Then $o(a) = |\langle a \rangle|$ where $|\langle a \rangle| = \infty$ when $\langle a \rangle$ is infinite.

Corollary 3: Let G be a group and $a, b \in G$. If $ab = ba$ and $o(a), o(b) < \infty$, then

$$o(ab) | o(a)o(b) \quad (92)$$

Proof. Suppose $ab = ba$ and $o(a), o(b) < \infty$. Since:

$$(ab)^{o(a)o(b)} = a^{o(a)o(b)} b^{o(a)o(b)} \quad (93)$$

$$= (a^{o(a)})^{o(b)} (b^{o(b)})^{o(a)} \quad (94)$$

$$= e^{o(b)} e^{o(a)} \quad (95)$$

$$= e \quad (96)$$

Therefore, $o(ab) | o(a)o(b)$. □

• **Remarks about notation:**

- $\mathbb{Z}/n\mathbb{Z}$ is sometimes denoted by \mathbb{Z}_n or $\mathbb{Z}/(n)$.
- $(\mathbb{Z}/n\mathbb{Z})^\times = \{[a] \in \mathbb{Z}/n\mathbb{Z} | [b] \in \mathbb{Z}/n\mathbb{Z} \text{ with } [a][b] = 1\} = \{[a] | \gcd(n, a) = 1\}$.

Theorem: Let G be a group and $a \in G$ with $o(a) = n < \infty$. For any $k \in \mathbb{Z}$, we have:

$$o(a^k) = \frac{o(a)}{\gcd(o(a), k)} = \frac{n}{\gcd(n, k)} \quad (97)$$

Proof. By definition, $o(a^k)$ is the smallest $m \in \mathbb{Z}_{>0}$ such that

$$(a^k)^m = e \iff a^{mk} = e \quad (98)$$

$$\iff n | mk \quad (99)$$

Since mk is a multiple of k , we have $n | mk \iff mk$ is common multiple of n and k .

If there exists $m \in \mathbb{Z}_{>0}$ such that $mk = \text{lcm}(n, k)$, then $m = o(a^k)$. Recall that:

$$\frac{nk}{\gcd(n, k)} = \text{lcm}(n, k) \quad (100)$$

Since $\gcd(n, k) | n$, then $\frac{n}{\gcd(n, k)} \in \mathbb{Z}_{>0}$ with

$$\left(\frac{n}{\gcd(n, k)} \right) k = \text{lcm}(n, k) \quad (101)$$

Therefore:

$$o(a^k) = \frac{n}{\gcd(n, k)} \quad (102)$$

□

Corollary 4: In a finite group G , the order of every element divides the order of the group:

$$\forall x \in G, \quad o(x) | |G| \quad (103)$$

Example 10: $\mathbb{Z} = \langle 1 \rangle$ is an infinite cyclic group. Meanwhile, $\mathbb{Z}/n\mathbb{Z} = \langle 1 \rangle$ is a finite cyclic group.

- Next we will study subgroups of cyclic groups. Choose a generator $a \in G$ and $G = \langle a \rangle$.
- For each $k \in \mathbb{Z}$, $a^k \in \langle a \rangle$. Therefore $\langle a^k \rangle \subseteq \langle a \rangle$.

Proposition 4: Let G be a group and let $a \in G$. If $H \leq G$ and $a \in H$, then $\langle a \rangle \subseteq H$.

- One natural question is: *Do we get every subgroup in this way?* If $k, \ell \in \mathbb{Z}$, when is $\langle a^k \rangle = \langle a^\ell \rangle$?

Theorem: Classification of subgroups of cyclic groups: Let $G = \langle a \rangle$ be a cyclic group:

1. If $|G| = \infty$ ($\iff o(a) = \infty$) then every subgroup of G is of the form $\langle a^m \rangle$ for a unique $m \in \mathbb{Z}_{\geq 0}$.

Remarks: $\langle a^m \rangle = \langle a^{-m} \rangle$.

2. If $|G| = n < \infty$ ($\iff o(a) = n < \infty$) then every subgroup of G is of the form $\langle a^m \rangle$ for a unique $m \in \mathbb{Z}_{>0}$ with $m|n$.

Said differently, the order of every subgroup of G divides n and for each $d \in \mathbb{Z}_{>0}$ with $d|n$ there is a unique subgroup of G of order d , namely $\langle a^{n/d} \rangle$.

Proof. Let $H \leq G = \langle a \rangle$ with $H \neq \{e\}$. Then $\exists k \in \mathbb{Z} \setminus \{0\}$ such that $a^k, a^{-k} \in H$. Therefore, $a^{|k|} \in H$ so $\exists k' \in \mathbb{Z}_{>0}$ such that $a^{k'} \in H$. Let m be the smallest positive integer such that $a^m \in H$ (which exists by the well-ordering principle).

We will prove that $H = \langle a^m \rangle$. Since $a^m \in H$, we have $\langle a^m \rangle \subseteq H$. To prove $H \subseteq \langle a^m \rangle$, it suffices to prove:

- If $a^k \in H$ where $k \in \mathbb{Z}$, then $m|k$.

Let $k \in \mathbb{Z}$ and assume $a^k \in H$. By the division algorithm, $\exists! q, r \in \mathbb{Z}$ such that $0 \leq r < m$ and:

$$k = qm + r \tag{104}$$

Then:

$$a^k = a^{qm+r} = (a^m)^q a^r \implies a^r = (a^m)^{-q} a^k \tag{105}$$

Since $(a^m)^{-q}, a^k \in H$.

Since $\langle a^m \rangle \subseteq H$, $(a^m)^{-q} \in H$. We assumed $a^k \in H$. Therefore, $a^r \in H$.

Since m is the smallest positive integer with $a^m \in H$ and $a^r \in H$ and $0 \leq r < m$, we have $r = 0$. Therefore $k = qm$ so $m|k$.

If $|G| = n < \infty$, then $o(a) = n$, so $a^n = e \in H$. Therefore by the above point, $m|n$. Now we look at the two cases:

1. Suppose $|G| = \infty$. We prove that every nontrivial subgroup of G is of the form $\langle a^m \rangle$ for some $m \in \mathbb{Z}_{>0}$. Since $\{e\} = \langle a^0 \rangle$, we have that every subgroup of G is of the form $\langle a^m \rangle$ for some $m \in \mathbb{Z}_{\geq 0}$.

To prove that m is unique, suppose $H \leq G$ and $H = \langle a^m \rangle = \langle a^{m'} \rangle$ for some $m, m' \in \mathbb{Z}_{\geq 0}$.

Since $a^m \in \langle a^m \rangle = \langle a^{m'} \rangle$, $a^m \in \langle a^{m'} \rangle$, so $a^m = a^{m'k}$ for some $k \in \mathbb{Z}$. Since $o(a) = \infty$, we must have $m = m'k$ so $m'|m$. Similarly, $m|m'$. Thus, $m = m'$.

2. Suppose $|G| = n < \infty$. Then $o(a) = n$, so $a^n = e$ and therefore $\{e\} = \langle a^n \rangle$. We proved above that every nontrivial subgroup of G is of the form $\langle a^m \rangle$ for some $m \in \mathbb{Z}_{>0}$ with $m|n$.

3. Therefore, every subgroup of G is of the form $\langle a^m \rangle$ for some $m \in \mathbb{Z}_{>0}$ with $m|n$.

To prove that m is unique, suppose $H \leq G$ with $H = \langle a^m \rangle = \langle a^{m'} \rangle$ where $m, m' \in \mathbb{Z}_{>0}$ with $m, m'|n$. Then:

$$o(a^m) = |\langle a^m \rangle| = |\langle a^{m'} \rangle| = o(a) \tag{106}$$

Since $o(a^k) = \frac{n}{\gcd(n, k)}$ for all $k \in \mathbb{Z}$, we got:

$$\frac{n}{\gcd(n, m)} = \frac{n}{\gcd(n, m')} \tag{107}$$

which implies $\gcd(n, m) = \gcd(n, m')$. Since $m, m'|n$ we have $\gcd(n, m) = m$ and $\gcd(n, m') = m'$ so $m = m'$.



Corollary 5: Criterion for $\langle a^i \rangle = \langle a^j \rangle$ and $o(a^i) = o(a^j)$.

Let $G = \langle a \rangle$ be a cyclic group and let $i, j \in \mathbb{Z}$.

1. If $|G| = \infty$, then $\langle a^i \rangle = \langle a^j \rangle$ if and only if $j = \pm k$.
2. If $|G| = n < \infty$, then the following are equivalent:
 - $\langle a^i \rangle = \langle a^k \rangle$
 - $o(a^i) = o(a^j)$
 - $\gcd(n, i) = \gcd(n, j)$

Corollary 6: (The generators of a cyclic group) Let $G = \langle a \rangle$ be a cyclic group. The generators of G are:

$$\begin{cases} \{a, a^{-1}\} & |G| = \infty \\ \{a^k \mid \gcd(n, k) = 1\} & |G| = n < \infty \end{cases} \quad (108)$$

This corollary follows from the first corollary.

- If $G = \langle a \rangle$ is cyclic of order $n < \infty$, it follows that there are exactly $\phi(n)$ generators where $\phi(n)$ is Euler's Totient function.