APM346: PDEs

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Contents

1 (2.1) Wave Equation

Wave equation,

$$u_{tt} = c^2 u_{xx}. \tag{1.1}$$

General formula is

$$u(x,t) = f(x+ct) + g(x-ct). \tag{1.2}$$

D'Alambert's formula gives the solution to the IVP $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$.

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,\mathrm{d}s\,. \tag{1.3}$$

In general, note that

$$au_t + bu_x = 0 \tag{1.4}$$

has the general solution

$$\phi\left(ax - bt\right), \tag{1.5}$$

which we can use to solve arbitrary differential equations that can be factored. We have,

$$(\partial_x + a\partial_t)(\partial_x - b\partial_t)u = 0 \implies u(x,t) = f(ax-t) + g(bx+t) \tag{1.6}$$

2 (2.2) Causality and Energy

Consider

$$\rho u_{tt} = T u_{xx}. \tag{2.1}$$

Energy of a wave is given by

$$E = \frac{1}{2}\int_{-\infty}^{+\infty} (\rho u_t^2 + T u_x^2)\,\mathrm{d}x \tag{2.2}$$

and is conserved.

3 (2.3-2.4) Diffusion on Real Line

Diffusion equation is given by

$$u_t = k u_{xx} \tag{3.1}$$

for $x \in \mathbb{R}, t > 0$ and initial condition $u(x,0) = \phi(x)$. General solution is

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-(y-x)^2/4kt}\phi(y)\,\mathrm{d}y\,. \tag{3.2}$$

Sometimes we need to write it in terms of error function,

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,\mathrm{d}t\,. \tag{3.3}$$

4 (3.1-3.3) Reflections and Sources

Consider

$$v_t = k v_{xx}, \tag{4.1}$$

where $x, t > 0$ and $v(x,0) = \phi(x)$ and $v(0,t) = 0$. Can be solved by extending ϕ to be odd and defined over $\mathbb{R}$, to get general solution:

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} \left(e^{-(y-x)^2/4kt} - e^{-(y+x)^2/4kt}\right)\phi(y)\,\mathrm{d}y\,. \tag{4.2}$$

Some boundary conditions:

- Dirichlet: $u(0,t) = c$
- Neumann: $u_x(0,t) = c$

Diffusion with a source: Given $u_t - ku_{xx} = f(x,t)$, we have

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,\mathrm{d}y + \int_0^t \int_{-\infty}^{+\infty} S(x-y,t-s)f(y,s)\,\mathrm{d}y\,\mathrm{d}s \tag{4.3}$$

Suppose we are given the boundary condition $u_x(0,t) = h(t)$, then we solve it for $U(x,t) = u(x,t) - xh(t)$.

5 (3.4) Waves with a Source

Consider

$$u_{tt} - c^2 u_{xx} = f(x,t) \tag{5.1}$$

with the standard initial conditions $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$. The unique solutions is

$$u(x,t) = u_{\mathsf{standard}}(x,t) + \frac{1}{2c}\int_0^t \int_{x-c(t-t_0)}^{x+c(t-t_0)} f, \tag{5.2}$$

where Δ is the area of the characteristic triangle. Recall Green's Theorem:

$$\iint_\Delta (P_x - Q_t)\,\mathrm{d}x\,\mathrm{d}t = \int_{\partial\Delta} P\,\mathrm{d}t + Q\,\mathrm{d}x \tag{5.3}$$

6 (4.1-4.2) Boundary Value Problem and Separation of Variables

The solution to the wave equation given some boundary condition is

$$u(x,t) = \sum_n \left(A_n \cos\frac{n\pi ct}{\ell} + B_n \sin\frac{n\pi ct}{\ell}\right)\sin\frac{n\pi x}{\ell}. \tag{6.1}$$

For diffusion, we have

$$u(x,t) = \sum_n A_n e^{-(n\pi/\ell)^2 kt} \sin\frac{n\pi x}{\ell} \tag{6.2}$$

7 (5.1) Fourier Series Coefficients

8 (5.3-5.4) Orthogonality and Completeness of Fourier Series

9 (5.6) Inhomogeneous Boundary Conditions

10 (6.1) Laplace's Equation

11 (6.2) Rectangles and Cubes

12 (9.1) Energy and Causality in Waves in Space

13 (9.2) Wave Equation in Space-Time

14 (10.1) Fourier's Method, Revisited

15 (11.1,11.3,11.5) General Eigenvalue Problems

16 (12.3) Fourier Transforms + Instructor Notes