

AER210: Vector Calc and Fluid Mechanics

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1 Double Integrals

- Integrals Involving a Parameter

Example 1: Let $\int_0^1 Cx^3 dx$ where C is a constant. Then it gives

$$\int_0^1 Cx^3 dx = \frac{1}{4}C \quad (1)$$

The result contains C .

- Suppose we have something like

$$\int_a^b f(x, y) dx = g(y) \quad (2)$$

and therefore y is a parameter

Definition: A variable which is kept constant during an integration is called a parameter.

- Partial integration wrt x

Example 2: An example of partial integration wrt x is

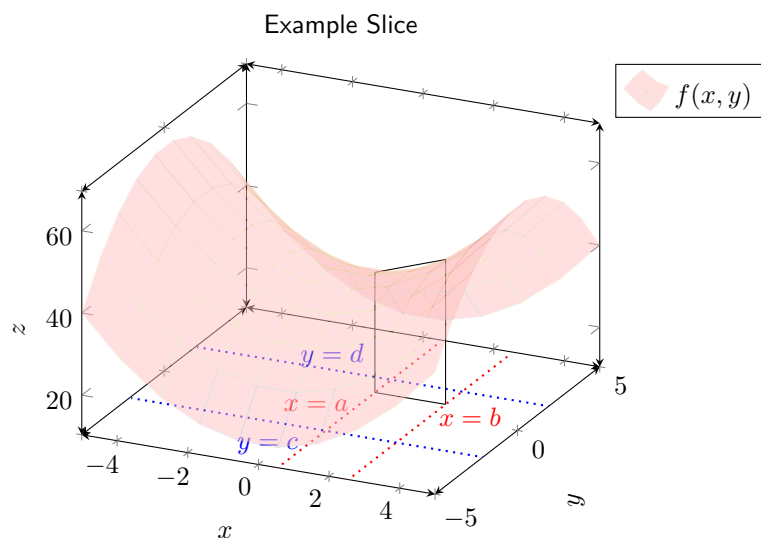
$$\int_0^1 x^3 y dx = y \int_0^1 x^3 dx = \frac{1}{4}y \quad (3)$$

- Notice the similarity between partial differentiation wrt x , $f_x(x, y)$ and the partial integration wrt x , $\int_a^b f(x, y) dx$.

- Iterated Integrals** (Integral of an Integral)

- Consider $x = f(x, y)$ where $x \in [a, b]$, $y \in [c, d]$. This defines a rectangular region.

- Assume that $f(x, y) \geq 0$. This can be represented as a surface, as shown below:



If we take the integral $\int_{y=c}^d f(x, y) dy = A(x)$, we see that the area of the slice depends on x .

If we suppose that the surface has a tiny thickness Δx , then the volume is

$$\Delta V(x) = A(x) \cdot \Delta x = \left(\int_{y=c}^d f(x, y) dy \right) \Delta x \quad (4)$$

If we break up the interval $[a, b]$ into N segments

$$x_0 = a \leq x_1 \leq x_2 \leq \dots x_{i-1} \leq x_i \leq \dots \leq x_{N-1} \leq x_N = b \quad (5)$$

with $\Delta x_i = x_i - x_{i-1}$. We can then approximate the volume as

$$V \approx \sum_{i=1}^N \Delta V_i = \sum_{i=1}^N A(x_i) \Delta x_i \quad (6)$$

which is known as a **Riemann sum**.

Idea: As we take the limit as $N \rightarrow \infty$ which implies $\Delta x_i \rightarrow 0$, we get the double integral:

$$V = \int_a^b \int_c^d f(x, y) dy dx \quad (7)$$

which can be determined by calculating two integrals.

- Similarly, we can find the volume by taking slices parallel to the xz plane.

The area of each slice is a function of y :

$$A(y) = \int_a^b f(x, y) dx \quad (8)$$

so we have $\Delta V(y) = A(y) \cdot \Delta y$. Again, summing up all slices and taking the limit, we get

$$V = \int_c^d A(y) dy = \int_c^d \int_a^b f(x, y) dx dy \quad (9)$$

Theorem: Fubini's Theorem tells us that

$$\int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy \quad (10)$$

The analog for equality of mixed partial derivatives is known as **Clairut's Theorem**.

Example 3: Find the volume under the surface $z = x^2 y$ where $x \in [1, 3]$ and $y \in [0, 1]$. We first form the integral by integrating wrt y . We have

$$V = \int_1^3 \int_0^1 x^2 y dy dx \quad (11)$$

$$= \int_1^3 x^2 (1^2/2 - 0^2/2) dx \quad (12)$$

$$= \int_1^3 \frac{x^2}{2} dx \quad (13)$$

$$= \frac{13}{3} \quad (14)$$

We can also form the integral by integrate it wrt x :

$$V = \int_0^1 \int_1^3 x^2 y dx dy \quad (15)$$

$$= \int_0^1 \frac{26}{3} y dy \quad (16)$$

$$= \frac{13}{3} \quad (17)$$

so we can confirm they give the same answer.

Example 4: Evaluate the double integral of $f(x, y) = x - 3y^2$ over region R where

$$R = \{(x, y) | 0 \leq x \leq 2, 1 \leq y \leq 2\} \quad (18)$$

To do this, we have

$$\int_0^2 \int_1^2 (x - 3y^2) dy dx = \int_0^2 (xy - y^3) \Big|_{y=1}^{y=2} dx \quad (19)$$

$$= \int_0^2 (x - 7) dx \quad (20)$$

$$= -12 \quad (21)$$

- Note that in the special case where the function $f(x, y)$ is $f(x, y) = g(x) \cdot h(y)$, then

$$\int_c^d \int_a^b f(x, y) dx dy = \int_c^d \left[h(y) \int_a^b g(x) dx \right] dy = \int_a^b g(x) dx \cdot \int_c^d h(y) dy \quad (22)$$

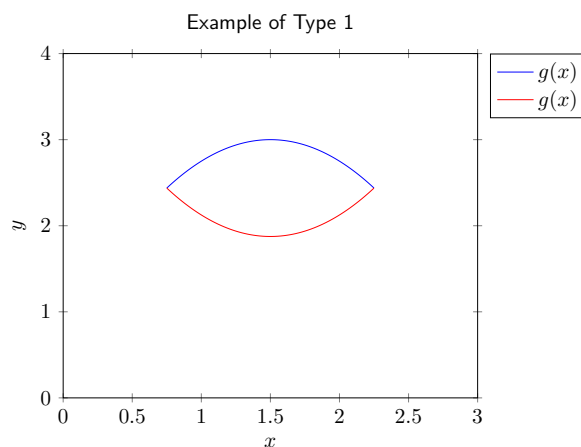
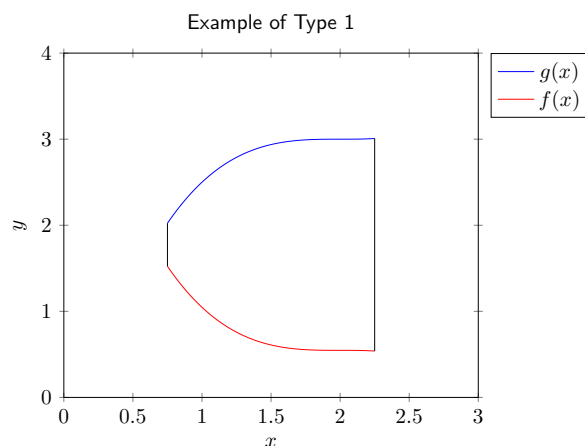
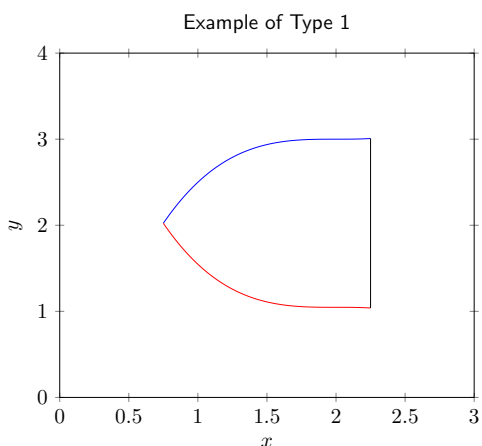
This gives us a shortcut of evaluating double integrals in this form.

- Double integrals over general regions (What if region is non-rectangular?)

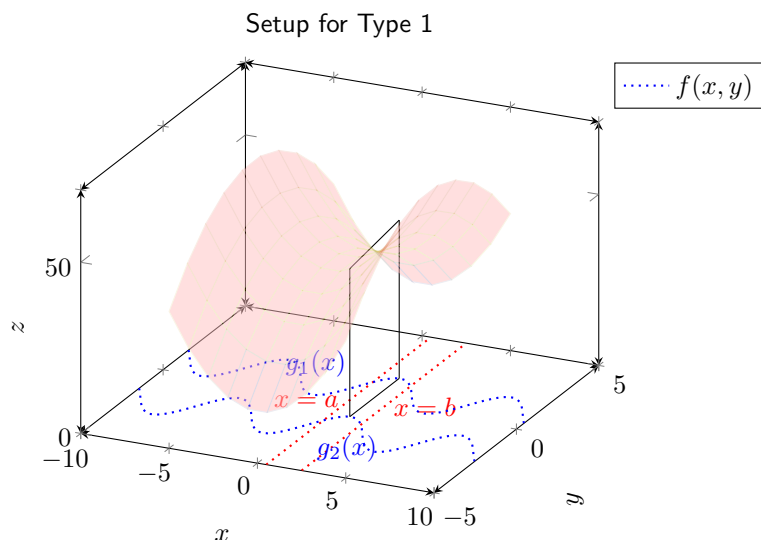
- **Type 1 Region** is in the form of

$$R = \{(x, y) | a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\} \quad (23)$$

Here are some examples



- Let's think about the case where $f(x, y) \geq 0$ on a type-1 region. Suppose we have the following illustration



We find the area of slices, so

$$A(x) = \int_{g_1(x)}^{g_2(x)} f(x, y) dy \quad (24)$$

and the volume is thus

$$V = \int_a^b A(x) dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx \quad (25)$$

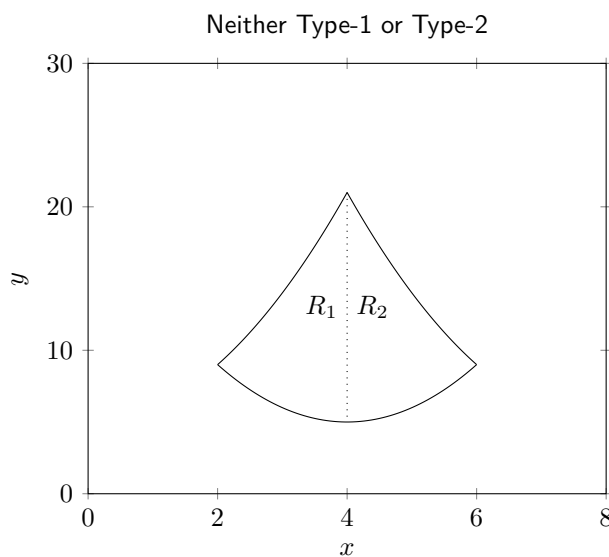
- Type-2 regions have the form

$$R = \{(x, y) | c \leq y \leq d \text{ and } h_1(y) \leq x \leq h_2(y)\} \quad (26)$$

In a similar way, the volume bounded by this region is

$$V = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy \quad (27)$$

- Type-3 regions are neither type-1 nor type-2. It is possible to break up the region into parts that can be classified as either type-1 or type-2:



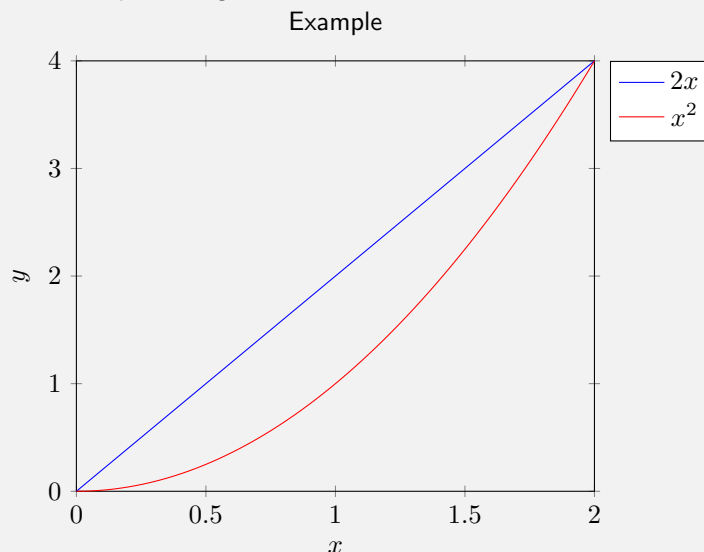
Idea: While these formulas are derived by assuming a positive volume (and thus cannot work if $f < 0$), they still work in general.

Example 5: Find the volume of the solid that lies under the surface

$$z = f(x, y) = x^2 + y^2 \quad (28)$$

and above the region R in the xy -plane. The region R is bounded by the straight line $y = 2x$ and the parabola $y = x^2$.

1. First we draw a diagram of the planar region R over which the surface is defined.



2. We then draw a line parallel to the axis of first integration (i.e. vertical lines for integrating in the y -direction first)
3. This gives us

$$V = \int_{x=0}^{x=2} \int_{y=x^2}^{y=2x} f(x, y) \, dy \, dx \quad (29)$$

$$= \int_0^2 \int_{x^2}^{2x} (x^2 + y^2) \, dy \, dx \quad (30)$$

$$= \frac{216}{35} \quad (31)$$

Alternatively, we can find the volume by integrating in the x direction first. In this case, we need to obtain boundary curves in the $x = x(y)$ form:

$$y = x^2 \implies x = \sqrt{y} \quad (32)$$

$$y = 2x \implies x = y/2 \quad (33)$$

This then gives us

$$V = \int_{y=0}^{y=4} \int_{x=y/2}^{x=\sqrt{y}} f(x, y) \, dx \, dy \quad (34)$$

$$= \frac{216}{35} \quad (35)$$

Warning: Do not just pick the minimum and maximum points. For example, the following is *incorrect*

$$\int_{y=0}^{y=4} \int_{x=0}^{x=2} f(x, y) \, dx \, dy \quad (36)$$

as that corresponds with a rectangular region.

Example 6: Integrate the surface given by $z = e^{x^2}$ over the following region:

We can first integrate wrt x

$$V = \int_{y=0}^{y=1} \int_{x=y}^{x=1} e^{x^2} dx dy \quad (37)$$

This is a hard problem since we don't know the anti-derivative of e^{x^2} . To solve this, we can first integrate wrt y , which gives us

$$V = \int_{x=0}^{x=1} \int_{y=0}^{y=x} e^{x^2} dy dx = \int_{x=0}^1 e^{x^2} y \Big|_{y=0}^{y=x} dx \quad (38)$$

$$= \int_0^1 e^{x^2} x dx \quad (39)$$

This integral can be more easily solved using the u-sub $u = x^2$, $du = 2x dx$ to get

$$V = \frac{1}{2}(e - 1) \quad (40)$$

2 Formal Definition of Double Integrals

- We will see two ways of defining double integrals.
- First, let us review the formal definition of definite integrals for functions of a single variable.

To determine the area under a curve in the region $x \in [a, b]$, we can break the region up into intervals Δx_i , so the Riemann sum is

$$A \approx \sum_{i=1}^n f(x_i^*) \Delta x_i \quad (41)$$

Let $m_i \leq f(x_i^*) \leq M_i$ for $x_i^* \in \Delta x_i$. Then:

$$\sum_{i=1}^n m_i \Delta x_i \leq \underbrace{\sum_{i=1}^n f(x_i^*) \Delta x_i}_{\text{Estimate of the entire area calculated by Riemann Sum}} \leq \sum_{i=1}^n M_i \Delta x_i \quad (42)$$

Estimate of the entire area calculated by Riemann Sum

If the Δx_i are of equal length and we take the limit, we can define:

$$A = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x_i = \int_a^b f(x) dx \quad (43)$$

If they are not of equal length, we need to define the norm of the partition $\|P\| = (\Delta x_i)_{\max}$ for $i = 1, 2, \dots, n$. This way, the integral can be alternatively defined as

$$A = \lim_{\|P\| \rightarrow 0} \sum_{i=1}^n f(x_i^*) \Delta x_i = \int_a^b f(x) dx \quad (44)$$

- Consider a double integral over rectangular region. Let $z = f(x, y)$ be defined on $R = \{(x, y) | a \leq x \leq b, c \leq y \leq d\}$. Assume $f(x, y) \geq 0$ over R .
- **Formal Definition 1:** We can approximate the volume as

$$\Delta v_i \approx f(x_i^*, y_i^*) \Delta A_i \quad (45)$$

where $\Delta A_i = \Delta x_i \cdot \Delta y_i$. The Riemann sum is then

$$V \approx \sum_{i=1}^N f(x_i^*, y_i^*) \Delta A_i \quad (46)$$

We can pick x_i^*, y_i^* such that $f(x_i^*, y_i^*)$ is the smallest and largest value in the region, we can bound the Riemann sum by:

$$\sum_{i=1}^N m_i \Delta x_i \Delta y_i \leq \sum_{i=1}^N f(x_i^*, y_i^*) \Delta x_i \Delta y_i \leq \sum_{i=1}^N M_i \Delta x_i \Delta y_i \quad (47)$$

Warning: Taking the limit where $N \rightarrow \infty$ is not sufficient, as it does not necessarily mean the size of all partitions approach zero.

We define the norm of the partition to be

$$\|P\| = \max(\Delta d_i) \quad (48)$$

for $i = 1, 2, \dots, N$. Therefore:

$$V = \lim_{\|P\| \rightarrow 0} \sum_{i=1}^N f(x_i^*, y_i^*) \Delta A_i = \iint_R f(x, y) \, dA = \iint_R f(x, y) \, dx \, dy. \quad (49)$$

Idea: Functions that are continuous are integrable over that region.

- **Formal Definition 2:** We are free to divide the region R into any tiling, we can use uniform divisions.

As a result, the area of each tile is

$$\Delta A_{ij} = \Delta x_i \Delta y_j \quad (50)$$

where the (i, j) represent the coordinate of the tile. The double Riemann sum is then:

$$V \approx \sum_{j=1}^m \sum_{i=1}^n f(x_{ij}^*, y_{ij}^*) \Delta x_i \Delta y_j \quad (51)$$

Again, we can define m_{ij} and M_{ij} such that

$$\sum_{j=1}^m \sum_{i=1}^n m_{ij} \Delta x_i \Delta y_j \leq \sum_{j=1}^m \sum_{i=1}^n f(x_{ij}^*, y_{ij}^*) \Delta x_i \Delta y_j \leq \sum_{j=1}^m \sum_{i=1}^n M_{ij} \Delta x_i \Delta y_j \quad (52)$$

Since these intervals are equally partitioned, we can define

$$V = \lim_{m, n \rightarrow \infty} \sum_{j=1}^m \sum_{i=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A_{ij} = \iint_R f(x, y) \, dA. \quad (53)$$

If they were not, we would have to define the norm again.

Example 7: Estimate the volume of the solid that lies above the square $R = [0, 2] \times [0, 2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$. Divide R into four equal squares & choose the sample point to be the upper corner of each square.

We would then have:

$$V \approx \sum_{i=1}^2 \sum_{j=1}^2 f(x_{ij}^*, y_{ij}^*) \Delta A \quad (54)$$

$$\approx f(1, 1) \Delta A + f(1, 2) \Delta A + f(2, 1) \Delta A + f(2, 2) \Delta A \quad (55)$$

$$\approx 34 \quad (56)$$

Note that the actual answer is 48. The approximation will improve as the number of regions increase.

- We can also define double integrals over non-rectangular regions.
- **Definition 1:** We can again tile a region using rectangular regions in two ways:
 - Each tile is contained within R and there are some space.

- Some tiles extend past the boundary of R , which is completely covered.

When we take the limit as $\|P\| \rightarrow 0$, both of these tiling methods will approach the actual area, so using any of these tilings will cause the double integral to approach the actual volume.

If $f(x, y)$ is a continuous function over R , then

$$V = \lim_{\|P\|} \sum f(x_i^*, y_i^*) \Delta A_i = \lim_{\|P\| \rightarrow 0} \sum_{j=1}^N f(x_j^*, y_j^*) \Delta A_j = \iint_R f(x, y) \, dx \, dy \quad (57)$$

- **Definition 2:** Similarly, we can use uniform partitions that either leave gaps or extend past the region. We can again define m_{ij} and M_{ij} for each tile R_{ij} such that

$$V = \iint_R f(x, y) \, dx \, dy = \lim_{\|P\| \rightarrow 0} \sum_{j=1}^M \sum_{i=1}^N f(x_{ij}^*, y_{ij}^*) \Delta x_i \Delta y_j \quad (58)$$

3 Double Integrals in Polar Coordinates

- Using polar coordinates is helpful when integrating over circular regions.
- Recall that we can convert between rectangular and polar coordinates via

$$x = r \cos \theta, \quad y = r \sin \theta \quad (59)$$

and that the area of a sector is

$$A = \frac{1}{2} r^2 \theta \quad (60)$$

- Suppose we have some function $f(x, y)$ defined on $R = \{(r, \theta) | a \leq r \leq b, \alpha \leq \theta \leq \beta\}$. We can then define:

$$f(x, y) = f(r \cos \theta, r \sin \theta) = g(r, \theta). \quad (61)$$

Assume $f(x, y) = g(r, \theta) \geq 0$ on R . Then we can approximate the volume as

$$\Delta V_i \approx g(r_i^*, \theta_i^*) \cdot \Delta A_i = f(r_i^* \cos \theta_i^*, r_i^* \sin \theta_i^*) \cdot r_i \Delta r_i \Delta \theta_i \left(1 + \frac{\Delta r_i}{2r_i}\right). \quad (62)$$

Taking the limit, we have

$$V = \lim_{\|P\| \rightarrow 0} \sum_{i=1}^n f(r_i^* \cos \theta_i^*, r_i^* \sin \theta_i^*) r_i \Delta r_i \Delta \theta_i \quad (63)$$

$$* = \int_{\alpha}^{\beta} \int_a^b f(r \cos \theta, r \sin \theta) r \, dr \, d\theta. \quad (64)$$

We can generalize this finding regardless of whether the function is positive or negative over R .

Idea: In a region bounded by $\alpha \leq \theta \leq \beta$, $a \leq r \leq b$, we have

$$\iint_R f(x, y) \, dA = \int_{\alpha}^{\beta} \int_a^b f(r \cos \theta, r \sin \theta) r \, dr \, d\theta. \quad (65)$$

- We can extend this to more complicated regions. Suppose R is bounded by $\alpha \leq \theta \leq \beta$ and $g_1(\theta) \leq r \leq g_2(\theta)$. Then the volume would be

$$\iint_R f(x, y) \, dA = \int_{\alpha}^{\beta} \int_{g_1(\theta)}^{g_2(\theta)} f(r \cos \theta, r \sin \theta) r \, dr \, d\theta \quad (66)$$

- Similarly, if R is bounded by $a \leq r \leq b$ and $h_1(r) \leq \theta \leq h_2(r)$, we have

$$\iint_R f(x, y) \, dA = \int_a^b \int_{h_1(r)}^{h_2(r)} f(r \cos \theta, r \sin \theta) r \, d\theta \, dr. \quad (67)$$

Example 8: Evaluate $\iint_R (3x + 4y^2) dA$ where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

This leads to the region $R = \{(r, \theta) | 1 \leq r \leq 2, 0 \leq \theta \leq \pi\}$. Then:

$$I = \iint_R (3x + 4y^2) dA \quad (68)$$

$$= \int_0^\pi \int_1^2 (3r \cos \theta + 4r^2 \sin^2 \theta) r dr d\theta \quad (69)$$

Solving this integral gives $\frac{15}{2}\pi$.

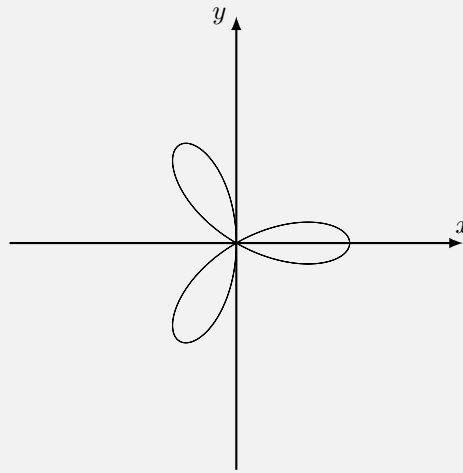
Example 9: Find the volume of the solid bounded by the $z = 0$ plane and the paraboloid $z = 1 - x^2 - y^2$.

Note that at $z = 0$, we get $0 = 1 - x^2 - y^2 \implies x^2 + y^2 = 1$. We can write our region as $R = \{(r, \theta) | 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}$. Our double integral is then

$$V = \iint_R (1 - x^2 - y^2) dA = \int_0^{2\pi} \int_0^1 (1 - r^2) r dr d\theta \quad (70)$$

Solving this gives $V = \pi/2$.

Example 10: Find the area enclosed by one petal of the rose given by $r = \cos 3\theta$.



The area is

$$A = \int_{-\pi/6}^{\pi/6} \int_0^{\cos 3\theta} 1 \cdot r dr d\theta \quad (71)$$

which evaluates to $\frac{1}{12}$.

Example 11: Find the volume trapped between the cone $z = \sqrt{x^2 + y^2}$ and the sphere $x^2 + y^2 + z^2 = 1$.

First, let us find the intersection using cartesian coordinates. We have

$$\sqrt{x^2 + y^2} = \sqrt{1 - x^2 - y^2} \implies x^2 + y^2 = \frac{1}{2}. \quad (72)$$

This can be written as $r = \frac{1}{\sqrt{2}}$ in polar coordinates. The volume is thus

$$\int_0^{2\pi} \int_0^{1/\sqrt{2}} f(x, y) r \, dr \, d\theta \quad (73)$$

where $f(x, y) = \sqrt{1 - x^2 - y^2} - \sqrt{x^2 + y^2}$. This gives $\frac{2\pi}{3} \left(1 - \frac{1}{\sqrt{2}}\right)$.

• Applications of Double Integrals

- We can determine the mass of a plate with nonuniform density $\rho = \rho(x, y)$. The mass is then

$$\iint_R \rho(x, y) \, dA. \quad (74)$$

- We can find the center of mass of a particle. Imagine we break a plate into small pieces. Each small piece has a moment about the x axis:

$$(M_x)_i = m_i y_i^* \approx \rho(x_i^*, y_i^*) \Delta A_i \cdot y_i^* \quad (75)$$

The total x and y moments are thus

$$M_x = \iint_R y \rho(x, y) \, dA \quad (76)$$

$$M_y = \iint_R x \rho(x, y) \, dA \quad (77)$$

These are equal to the moment $\bar{y}m$ and $\bar{x}m$, respectively, where m is the mass of the object. Thus:

$$\bar{x} = \frac{\iint_R x \rho(x, y) \, dA}{\iint_R \rho(x, y) \, dA} \quad (78)$$

and similarly for \bar{y} .

- Consider a rotating object. A point mass has a kinetic energy $K = \frac{1}{2} m r^2 \omega^2$. However, $m r^2$ would be different for different points on a solid object.

We can consider:

$$K = \frac{1}{2} \left(\sum_{i=1}^n m_i r_i^2 \right) \omega^2. \quad (79)$$

The quantity inside the parentheses is known as the moment of inertia I . While this may be true for a series of point masses, for a continuous distribution we need to take the limit:

$$I = \iint_R \rho(x, y) [r(x, y)]^2 \, dx \, dy. \quad (80)$$

4 Surface Area and Triple Integrals

- Suppose we wish to find the surface area.
- Method 1: Given $z = f(x, y)$ we can estimate the area as

$$S \iint_S dT \quad (81)$$

where dT gives the area of the tangent plane and S is the region of the curve. The projection of dT is given by

$$\Delta A = \Delta T |\cos \alpha| \implies \frac{\Delta A}{|\cos \alpha|} \quad (82)$$

where α is the angle between \vec{n} (normal to plane) and \vec{k} (normal to xy plane), such that

$$S = \iint_R \frac{dA}{|\cos \alpha|} \quad (83)$$

where R is the projection of S . To determine $\cos \alpha$, we can write $z = f(x, y)$ in explicit form as

$$F(x, y, z) = z - f(x, y) = 0, \quad (84)$$

which is the 0th level surface. Since $\vec{\nabla}$ is perpendicular to it, we have

$$\vec{n} = \frac{\vec{\nabla} F}{\|\vec{\nabla} F\|}. \quad (85)$$

Recall that

$$\vec{\nabla} F \cdot \vec{k} = \left(\frac{\partial F}{\partial x} \hat{i} + \frac{\partial F}{\partial y} \hat{j} + \frac{\partial F}{\partial z} \hat{k} \right) \cdot \vec{k} \quad (86)$$

so

$$|\cos \alpha| = |\vec{n} \cdot \vec{k}| = \frac{|\vec{\nabla} F \cdot \vec{k}|}{\|\vec{\nabla} F\|} = \frac{\left| \frac{\partial F}{\partial z} \right|}{\|\vec{\nabla} F\|}. \quad (87)$$

Therefore, we have

$$S = \iint_R \frac{\sqrt{\left(\frac{\partial F}{\partial x} \right)^2 + \left(\frac{\partial F}{\partial y} \right)^2 + \left(\frac{\partial F}{\partial z} \right)^2}}{\left| \frac{\partial F}{\partial z} \right|} dA \quad (88)$$

which can be simplified to

$$S = \iint_R \sqrt{\left(\frac{\partial F}{\partial x} \right)^2 + \left(\frac{\partial F}{\partial y} \right)^2 + 1} dA \quad (89)$$

- Method 2: Consider a rectangular subregion R_i with area $\Delta A_i = \Delta y_i \times \Delta x_i$. Projecting this onto $z = f(x, y)$ gives a parallelogram. This parallelogram has sides

$$\vec{a}_i = \Delta x_i \cdot \hat{i} + 0\hat{j} + f_x(x_i, y_i) \Delta x_i \hat{k} \quad (90)$$

$$\vec{b}_i = 0\hat{i} + \Delta y_i \cdot \hat{j} + f_y(x_i, y_i) \Delta x_i \hat{k}. \quad (91)$$

The area of the parallelogram is $\Delta T_i = \|\vec{a}_i \times \vec{b}_i\|$. Taking the cross product, we get

$$S = \iint_R \sqrt{f_x^2(x, y) + f_y^2(x, y) + 1} \quad (92)$$

- All the ideas for double integrals carry over for **triple Integrals**. Formally, we can break it up into sub-volumes, gain an estimate by finding the largest and smallest value in each ΔV_i , which bound the triple integral and approach to it after taking the limit.

Example 12: Suppose $f(x, y, z)$ is a continuous function defined on the box region Q , given by

$$Q = \{(x, y, z) | a \leq x \leq b, c \leq y \leq d, r \leq z \leq s\}. \quad (93)$$

We then have

$$\iiint_Q f(x, y, z) dV = \int_r^s \int_c^d \int_a^b f(x, y, z) dx dy dz. \quad (94)$$

- Suppose we have something more complicated like $Q = \{(x, y, z) | (x, y) \in R \text{ and } g_1(x, y) \leq z \leq g_2(x, y)\}$. We will then have

$$\iint_R \int_{g_1(x, y)}^{g_2(x, y)} f(x, y, z) dz dA \quad (95)$$

Example 13: Evaluate $\iiint_Q 6xy \, dV$ where Q is the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $2x + y + z = 4$. We then have

$$\int_{x=0}^{x=2} \int_{y=0}^{y=4-2x} \int_{z=0}^{z=4-2x-y} 6xy \, dz \, dy \, dx. \quad (96)$$

If we want to first integrate with respect to x , we have

$$\int_{y=0}^{y=4} \int_{z=0}^{z=4-y} \int_{x=0}^{x=1/2(4-y-z)} 6xy \, dx \, dz \, dy \quad (97)$$