PHY293: Waves and Modern Physics

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Contents

1 Simple Harmonic Oscillators: Energy and Damping

- Simple harmonic motion takes place about an equilibrium position and the motion is periodic.
- The most simple system is a mass attached to a spring. The restoring force given by Hooke's Law.

$$\vec{F} = -k\Delta\vec{x} \tag{1}$$

- The period (time/period) and frequency (amount of cycles/second) is related via

$$f = \frac{1}{T} \tag{2}$$

Note: This also applies for other oscillatory systems, such as pendulums, LC circuits, i.e.

> **Warning**: Since acceleration is not constant, the standard kinematic equations do not apply.

> **Example 1:** Consider a mass on a spring (and only affected by it). We get a second order differential equation:
>
> $$m\frac{d^2x}{dt^2} = -kx \implies \frac{d^2x}{dt^2} + \frac{k}{m}x = 0. \tag{3}$$
>
> This is a second order ODE. After solving, we get
>
> $$\omega^2 = \frac{k}{m}. \tag{4}$$

- For a mass on a *vertical spring*, we have a slightly different situation.

First, note that the equilibrium location shifts. The spring will extend by a distance $\Delta y = (y_1 - y_0) = \frac{mg}{k}$, and will then oscillate around this new equilibrium point.

Let y_0 be the equilibrium height (measured from top) and let y be the location of the mass (measured from top). The spring force will then be

$$\begin{aligned}\sum F_y = ma &= k(y - y_0) - mg &(5)\\ &= k(y - y_0 - y_1 + y_0) &(6)\\ &= k(y - y_1) &(7)\end{aligned}$$

Since y_1 is a constant, we can turn the differential equation to be

$$\frac{d^2(y - y_1)}{dt^2} = \frac{k}{m}(y - y_1) \tag{8}$$

and therefore the angular frequency stays the same.

> **Warning**: There are issues related to energy when using this trick. Be careful!

- The general solution to $\frac{d^2x}{dt^2} + \omega^2 x = 0$ is

$$x(t) = x_0 + A\cos(\omega t + \phi_0) \tag{9}$$

Oftentimes, $x_0 = 0$ is set as convention. However, the other constants A and ϕ_0 are determined by initial conditions.

- The maximum speed is given by

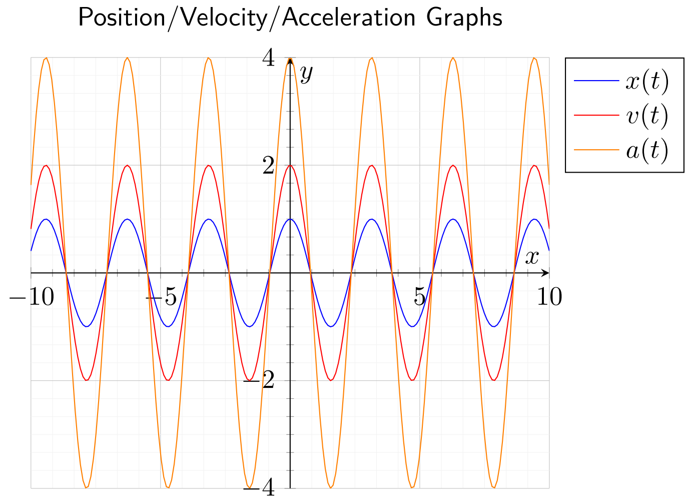
$$v_{\text{max}} = A\omega \tag{10}$$

and the maximum acceleration is

$$a_{\text{max}} = A\omega^2 \tag{11}$$

These can be derived by taking the first and second derivatives of $x(t)$.

- We can visualize the position, velocity and acceleration graphs:



- There are other ways to express the general solution:

$$x(t) = a\cos\omega t + b\sin\omega t \tag{12}$$

Example 2: Determine the amplitude and phase constant of a pendulum moving with a motion described by a sum of two functions, $x_1(t) = 0.25\cos\omega t$ and $x_2(t) = -0.5\sin\omega t$.

To figure out ϕ_0, we take the ratio to get

$$\frac{0.5}{0.25} = \tan\phi_0 \implies \tan\phi_0 = 2 \tag{13}$$

Be careful with the assignment of the angle; $\tan(x)$ takes value of 2 twice during one period, at $x_1 = 1.107$ rad and $x_2 = x_1 + \pi = 4.25$ rad.

To determine which one is correct, you need to look at your original functions: both $\sin\phi_0 > 0$ and $\cos\phi_0 > 0$. Therefore, $\phi_0 = 1.107$ rad.

- Energy of a simple harmonic oscillator. It has a kinetic energy $K = \frac{1}{2}mv(t)^2$ and potential energy.
- The potential energy is related to the restoring force and, by a definition of potential energy, the cahnge in system's potential energy when moving from a position x_i to position x_f is

$$\Delta U = -\int_{x_i}^{x_f} (-kx')\,\mathrm{d}x' = \frac{1}{2}kx_f^2 - \frac{1}{2}kx_i^2. \tag{14}$$

For a reference, let $x_i = 0$ such that we can define $U = \frac{1}{2}kx^2$.

- Energy is conserved, i.e.

$$U + K = \frac{1}{2}kx^2 + \frac{1}{2}mv^2 = \text{constant} \tag{15}$$

- Note that we can show conservation of energy explicitly. The potential energy is

$$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega t) \tag{16}$$

and the kinetic energy is

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 A^2\sin^2(\omega t). \tag{17}$$

Note that $\omega^2 = \frac{k}{m}$ so the total energy is

$$E = K + U = \frac{1}{2}kA^2 = \frac{1}{2}m\omega^2 \tag{18}$$

- When an object is $\frac{1}{\sqrt{2}}A$ from the maximum compression/extension, the kinetic energy equals the potential energy.
- **Physics of small vibrations:** Most systems will oscillate with SHM when the amplitude is small. But what if it's not?

Example 3: The potential energy of a pendulum is given by

$$U = mgy = mgL(1 - \cos\theta) \tag{19}$$

We can use the Taylor series:

$$f(x) = f(a) + \frac{x-a}{1!}\left(\frac{\partial f}{\partial x}\right)_{x=a} + \frac{(x-a)^2}{2!}\left(\frac{\partial^2 f}{\partial x^2}\right)_{x=a} + \cdots \tag{20}$$

The energy about $x = 0$ is then:

$$U(x) = U(0) + x\frac{dU}{dx}\bigg|_{x=0} + \frac{x^2}{2}\frac{d^2U}{dt^2}\bigg|_{x=0} + \cdots . \tag{21}$$

Using this, we can make the lowest order (nonzero) approximation to be

$$U = \frac{1}{2}mgL\theta^2 \tag{22}$$

- In this course, any angle smaller than 10° would be considered a *small angle.*
- For example, the differential equation that describes a pendulum is

$$L\frac{d^2\theta}{dt^2} = -g\sin\theta \tag{23}$$

If $\theta \ll 1$, then we can approximate $\theta \approx \sin\theta$ such that the differential equation becomes

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta \tag{24}$$

which is the familiar second order differential equation. The period is:

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}. \tag{25}$$

- The energy of a pendulum is given by

$$E = K + U = \frac{1}{2}mv^2 + mg\left(\frac{x^2}{2L}\right). \tag{26}$$

2 Damped Harmonic Oscillator

- The drag force is proportional to the speed $F = -kv$.
- The equation of motion for dampened harmonic oscillator is given by

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = 0 \tag{27}$$

where $\omega_0^2 = \frac{k}{m}$ and $\gamma = \frac{k}{m}$.

- **Light Damping** occurs when $\gamma < \omega_0$. The general solution for lightly damped (underdamped) oscillator is

$$x(t) = A_0\exp\left(-\frac{\gamma t}{2}\right)\cos(\omega t + \phi_0) \tag{28}$$