MAT301 Notes

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Contents

1 Lecture One

- Groups are everywhere in mathematics and nature in one of two forms:
  - as groups of symmetries
  - as groups of "numbers" or quantities
- We will call a subset $F \subseteq \mathbb{R}^n$ a **figure** in $\mathbb{R}^n$ when we consider F not just as a set, but as a set together with the structure of its distance functions:

$$d : F \times F \mapsto \mathbb{R}_{\geq 0}, \quad d(x, y) = \|x - y\| \tag{1}$$

  A figure is then defined as the pair (F, d).

  Definition: A **symmetry** of a figure $F \subseteq \mathbb{R}^n$ is a bijection $\sigma : F \mapsto F$ such that σ and σ^{-1} preserve distances:

$$\forall x, y, \in F, \quad d(\sigma(x), \sigma(y)) = d(x, y) \tag{2}$$

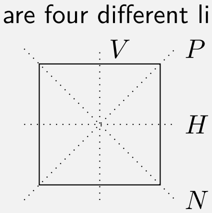
$$\iff d(\sigma^{-1}(x), \sigma^{-1}(y)) = d(x, y) \tag{3}$$

  Therefore:

$$\text{Sym}(F) \equiv \{\sigma : F \to F | \sigma \text{ is a symmetry}\} \tag{4}$$

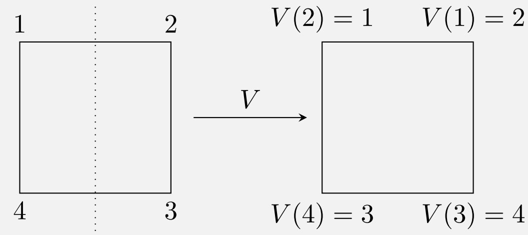
- For example, any point, line, shape, or form is a figure. However, we are only interested in figures that have interesting symmetries.

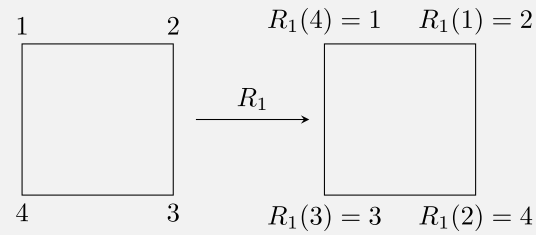
  Example 1: Let F be a square in $\mathbb{R}^2$. There are four different lines of reflections:



  and there are three rotations: R_1, R_2, and R_3, which represent 90°, 180°, and 270° clockwise rotations. I represents the identity transformation (do nothing).

  We can combine symmetries. For example, what is $R_1 \circ V$? To do so, we can label the vertices:





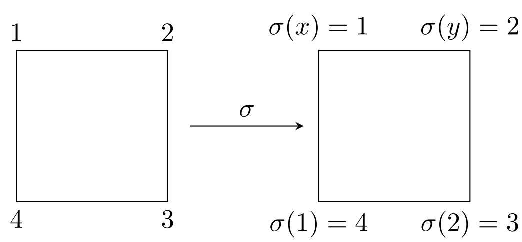
Applying the computations:

$$(R_1 \circ V)(1) = R_1(V(1)) = R_1(2) = 3 \tag{5}$$
$$(R_1 \circ V)(2) = R_1(V(2)) = R_1(1) = 2 \tag{6}$$
$$(R_1 \circ V)(3) = 1 \tag{7}$$
$$(R_1 \circ V)(4) = 4 \tag{8}$$

Check that $V \circ R_1 = N$. Also notice that these operations are not commutative: $R_1 \circ V \neq V \circ R_1$.

- In the above example, how are we sure that these are all of the symmetries of a square? To answer this, we will need the following facts:

  1. A symmetry maps vertices to vertices. The vertices are the points of the square that are furthest from the center.
  2. Symmetries map adjacent vertices tto adjacent vertices. If x, y are adjacent vertices, then $\sigma(x)$, $\sigma(y)$ are vertices, and $d(\sigma(x), \sigma(y)) = d(x, y) =$ side length.
  3. A symmetry σ is completely determined by $(\sigma(1), \sigma(2))$. For example, suppose we have the symmetry σ on a square such that:

     From this, we know that we must have $y = 3$, from fact 1, as well as $x = 4$.
  4. For all $x, y \in \{1, 2, 3, 4\}$ such that x is adjacent to y, $\exists!$ symmetry σ of the square such that:

  $$(\sigma(1), \sigma(2)) = (x, y) \tag{9}$$

  By the above facts, we must count the ordered pairs (x, y) such that $x, y \in \{1, 2, 3, 4\}$ and x is adjacent to y:

  – There are 4 choices for x.

  – For each choice of x, there are two choices of y. Therefore, there are $4 \times 2 = 8$ symmetries.

  Since we listed 8 different symmetries of a square, we have therefore defined all of them.