PHY484: General Relativity II

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1 Deriving Einstein's Equation from the Action

Recall that for a non-relativistic particle, the action is given by $S = \int \mathrm{d}t\, \mathcal{L}(q, \dot{q})$, where q is a generalized coordinate. Then the Euler-Lagrange equations gives the equation of motion

$$\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} = 0. \tag{1.1}$$

In field theory, the action becomes

$$S[\Phi^a] = \int d^D x \mathcal{L}(\Phi^a, \partial_\mu \Phi^a)$$

and the E-L equation becomes

$$\frac{\partial \mathcal{L}}{\partial \Phi^a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^a)} \right) = 0. \tag{1.2}$$

1.1 Scalar Gravity

We first, incorrectly, attempt to write the gravitational field as a scalar (spin 0 field). The Einbein action is

$$S_{\text{einbein}} = \frac{1}{2}\int \mathrm{d}\lambda\, (e^{-1}(\lambda)\dot{z}(\lambda) + e(\lambda)m^2). \tag{1.3}$$

For a massive particle, $e^{-1} = m$ and for a massless particle $e^{-1} = 1$ in order to enforce the on shell conditions, $E^2 - p^2c^2 = m_0^2c^4$. To try and recreate Newtonian gravity, the simplest realization is given by

$$S = \int \mathrm{d}^D x \left[\frac{1}{8\pi G_N}\partial^\mu\psi\partial_\mu\psi + \int \mathrm{d}\lambda\, \delta(x - z(\lambda)) \left\{ -\psi(x)e^{-1}\dot{z}^\mu(\lambda)\dot{z}_\mu(\lambda) + \frac{1}{2}e^{-1}\dot{z}^2(\lambda) + \frac{e}{2}m^2 \right\} \right] \tag{1.4}$$

The E-L equations give, when varying ψ.

$$\int \mathrm{d}\lambda\, \delta^D(x - z(\lambda))(-e^{-1}\dot{z}^2) = \frac{1}{4\pi G_n}\partial_\mu\partial^\mu\psi. \tag{1.5}$$

When varying e, we have $\frac{\partial \mathcal{L}}{\partial e} = 0$, or

$$m^2 = \frac{1}{e^2}\dot{z}^2(\lambda)(1 - 2\psi z(\lambda)). \tag{1.6}$$

For a massive particle, we have $\lambda = \tau$ and $e^{-1} = m$ so this reduces to $1 = \dot{z}^2(1 - 2\psi)$. For a massless particle, we can choose $e^{-1} = 1$ so $\dot{z}^2 = 0$. The trajectory of a particle is given by

$$\frac{\partial \mathcal{L}}{\partial z^\nu} = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu z^\mu)}. \tag{1.7}$$

We can compute,

$$\partial_\mu z^\nu = \frac{dz^\nu}{d\lambda}\frac{d\lambda}{dx^\mu} \tag{1.8}$$

$$\implies \frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial(\partial_\mu z^\nu)} = \frac{\partial}{\partial x^\mu}\frac{dx^\mu}{d\lambda}\frac{\partial \mathcal{L}}{\partial \dot{z}^\nu} = \frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial \dot{z}^\nu} \tag{1.9}$$

For the LHS, it looks like we have no explicit z-dependence in S, but there is, in ψ! We have,

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial z^\nu} &= \frac{\partial \mathcal{L}}{\partial \psi}\frac{\partial \psi}{\partial x^\mu}\frac{\partial x^\mu}{\partial z^\nu} && (1.10)\\
&= \left(\int \mathrm{d}\lambda\, \delta^D(x - z(\lambda))(-e^{-1}\dot{z}^2) \right) \partial_\mu\psi \frac{\partial x^\mu}{\partial z^\nu} && (1.11)\\
&= -e^{-1}\dot{z}^2(\lambda)\partial_\nu\psi, && (1.12)
\end{aligned}$$

where we used the fact that

$$\int \mathrm{d}\lambda\, \delta^D(x - z)\frac{\partial x^\mu}{\partial z^\nu} = \delta^\mu_\nu. \tag{1.13}$$

Therefore, we have

$$\frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial \dot{z}^\nu} = -e^{-1}\dot{z}^2(\lambda)\partial_\nu\psi. \tag{1.14}$$

The ψ EoM for massive particles gives

$$\partial_\mu\partial^\mu\psi = 4\pi G m\delta(x-z), \tag{1.15}$$

reproducing Newton. But for massless particles, we have $\dot{z}^2 = 0$, and taking the derivative, we get

$$\frac{d}{d\lambda}\left[\dot{z}_\nu(\lambda)(1-2\psi(z(\lambda)))\right] = 0. \tag{1.16}$$

Therefore, the direction of z cannot change since ψ is a scalar quantity. Therefore, light cannot be deflected by gravitational fields.

2 Questions

- Where did equation 1.9 and 1.13 come from?
- What's the purpose of the integral $\int \mathrm{d}\lambda\, \delta^D(x - z(\lambda))$?