Important Things

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1 Method of Nathan

The row space is orthogonal to the null space, so for a 2×2 matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{1}$$

and if the nullity is 1, then the null space is:

$$\text{span}\left\{\begin{bmatrix} -b \\ a \end{bmatrix}\right\} \tag{2}$$

since:

$$\begin{bmatrix} a & b \end{bmatrix}\begin{bmatrix} -b \\ a \end{bmatrix} = \vec{0} \tag{3}$$

2 Eigenvalues

- If A has eigenvalue λ, then A^k has eigenvalue λ^k.
- A is noninvertible if and only if at least one of its eigenvalues are zero.
- The geometric multiplicity m_{λ_1} is equal to:

$$m = \dim \operatorname{null}(B - \lambda_1 I) = n - \operatorname{rank}(B - \lambda_1 I) \tag{4}$$

- For a $n \times n$ matrix, the algebraic multiplicities sum up to n:

$$n_1 + n_2 + \cdots = n \tag{5}$$

  and for geometric multiplicities, it is *bounded* by n

$$m_1 + m_2 + \cdots \leq n \tag{6}$$

- Geometric multiplicities are smaller or equal to the algebraic multiplicites:

$$1 \leq m_i \leq n_i \tag{7}$$

- The trace of a matrix is the sum of its eigenvalues. (Medici)

3 Diagonalization

- If A is diagonalizable, it can be written as $A = PDP^{-1}$ and:

$$A^k = PD^kP^{-1} \tag{8}$$

- If D is a diagonal matrix, then we can calculate D^k by taking each element to the power of k.
- In general, if D_1 and D_2 are diagonal matrices, $D_1D_2 = D_2D_1$ where each element is the element-wise product of the two.

- If D is a diagonal, then AD is equivalent to scaling the columns of A by the elements of D. Similarly, DA scales the rows of A be the elements of D.
- A matrix A is diagonalizable if the eigenvectors of A form a basis for ${}^n\mathbb{R}$.
- The eigenvectors of a diagonal matrix are the standard basis.
- If two matrices have the same eigenvalues with the same linearly independent eigenvectors, then they are equal.

4 Important Matrices

Try these 2×2 matrices when looking for counterexamples:

- Nilpotent Matrix: $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$
- Rotation matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$