MAT448: Algebraic Geometry

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Contents

1 Introduction

Algebraic geometry is the study of geometric objects defined by polynomials. Consider $x^2 + y^2 = 1$ and $x^2 + y^2 = -1$ plotted in $\mathbb{R}^2$. These two algebraic varieties are different as real varieties but the same (namely, isomorphic) as complex varieties. For example, consider the coordinate change $(x, y) \leftrightarrow (ix, iy)$.

Let k be an algebraically closed field, i.e. $k = \mathbb{C}$. Consider an element of the affine space

$$a \in \mathbb{A}^n = \{a = (a_1, \ldots, a_n) | a \in k\} = \text{affine space}/k \tag{1.1}$$

and polynomial

$$f \in R = k[x_1, \ldots, x_n] \tag{1.2}$$

acting on $\mathbb{A}^n$. Then, $f(a) = f(a_1, \ldots, a_n) \in k$ and we wish to examine

$$V(f) = \{a \in \mathbb{A}^n | f(a) = 0\}, \tag{1.3}$$

or perhaps a collection of polynomials, known as an **affine algebraic set**

$$V(f_1, \ldots, f_r) = \{a \in \mathbb{A}^n | f_i(a) = 0 \forall i\} \subseteq \mathbb{A}^n. \tag{1.4}$$

We could also consider a subset $S \subset R_n$, so we can take the ideal generated by S in R_n as $I(S)$. We claim that

$$V(S) = V(I(S)). \tag{1.5}$$

Note that if $S = \{0\}$, then $V(S) = \mathbb{A}^n$. Also, $V((1)) = V(R_n) = \emptyset$. Here, (1) refers to the ideal generated by the identity 1.

2 Commutative Algebra

All our rings will be commutative with identity $1 \neq 0$.

Proposition 1: R is Noetherian iff equivalently:
(a) every ideal $I \subset R$ is finitely generated,
(b) every ascending chain (AAC) of ideals terminates,
(c) every non-empty set of ideals contains maximal elements.

Recall that an ascending chain of ideals is the following $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$.

Proof. (a) $\Longrightarrow$ (b): Suppose we have an ascending chain. Let $I = \bigcup_n I_n$ be an ideal in R. We know that I is finitely generated, i.e. $I = (r_1, \ldots, r_k)$. There exists an N such that $r_i \in I_N$ for all i, so that we must have $I_N = I$.

(b) $\Longrightarrow$ (c): Let Σ be a non-empty set of ideals. Suppose for the sake of contradiction there are no maximal elements. Take any $I_0 \in \Sigma$. Since it is not maximal, we can find an I_1 such that $I_0 \subsetneq I_1$, and this doesn't terminate. But by assumption, we know that the ACC terminates, so contradiction.

(c) $\Longrightarrow$ (a): Given $f \in I \subset R$ and $\Sigma = \{J = \text{ideal in R s.t. } J \subset I, J = f.g\}$. Take $J \in \Sigma$ to be a maximum element. We know that $J \subset I$. Take $f \in I \setminus J$. Then $J \subset (J, f) \subset I$, which is a contradiction. The only way to resolve this is if $J = I$ (i.e. not a proper subset). $\square$

Also,

Proposition 2:
(i) If R is Noetherian, and $I \subset R$ is an ideal, then R/I is Noetherian.
(ii) If R is a Noetherian integral domain (i.e. we can make a fraction field $R \subset Frac(R)$) with $S \subset R$ and $0 \neq S$. Then $S^{-1}R$ is Noetherian.

Theorem: Hilbert's Theorem: If R is Noetherian, then $R[x]$ is Noetherian.

Corollary 1: Suppose A is a finitely generated noetherian R-algebra, then A is noetherian. Here, we mean that $A = R(a_1, \ldots, a_n)$.

Proof. Consider $J \subset R[x]$ is an ideal. Our goal is to show that this is finitely generated. Consider

$$C_n = \{a \in R | \exists f \in J, f(x) = ax^n + lower\} \tag{2.1}$$

Clearly, C_n is an ideal in R. We also have an ACC

$$C_n \subset C_{n+1} \subset C_{n+2} \subset \cdots,$$

because we can take $f(x)$ and multiply it by x. Since R is Noetherian, this terminates at some $C_N = C_{N+1}$. For $n \leq N$, take $\{a_{n_j}\}$ given by C and $f_{n_j} = a_{n_j} x^n + \text{lower} \in J$.

If $f(x) = cx^n + \cdots + \text{lower}$ in J, then $c \in C_n$ if $n \leq N$. Let $c = \sum_j b_{n_j} a_{n_j}$ with $b_{n_j} \in R$, so

$$f(x) - \sum_j b_{n_j} f_{n_j}(x) \in C_{n-1},$$

and we can use induction to finish. If $n > N$, we have

$$f(x) - \sum_j b_{n_j} f_{N_j}(x) x^{n-N} \tag{2.2}$$

which has degree smaller or equal to $n - 1$. □

What this says is that given any set of polynomial equations, there exists a finite set of polynomial equations such that their simultaneous solutions are the same.

2.1 Relations among algebraic affine sets

(a) $I_1 \subset I_2 \implies V(I_2) \subseteq V(I_1)$

(b) $V(I_1 \cap I_2) = V(I_1) \cup V(I_2)$

(c) $V\left(\sum_\lambda I_\lambda\right) = \bigcap_\lambda V(I_\lambda)$

Definition: The Zariski topology on $\mathbb{A}^n$ has as closed sets the affine algebraic sets $V(I)$.

This is the topology we will be using for algebraic geometry, but it is also very lousy and course. For example, consider $\mathbb{A}^1$. Two open sets will always have a nonzero intersection.

We can go in the reverse direction. LEt $X \subset \mathbb{A}^n$. Let $I(x) = \{f \in R_n | f(P) = a \forall P \in X\}$. Note:

(i) $I(x)$ is an ideal in R_n

(ii) $I(X)$ is a radical ideal.

Proposition 3:
(a) $X \subseteq Y \implies I(Y) \subseteq I(X)$.
(b) $X \subset \mathbb{A}^n$ for any $X \subset V(I(X))$.
(c) If $X = V(I)$ then $X = V(I(X))$.