PHY354: Advanced Classical Mechanics

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Contents

1 Motivation 2

2 Lagrangian Mechanics 2

1 Motivation

In previous courses, we focused on $\vec{F} = m\vec{a}$ and its consequences. In this course, we will do it in a more elegant way, i.e. via Euler, Lagrange, Hamilton, Jacobi, Noether. Some disadvantages of Newton's approach are:

- Can be difficult to apply to complex situations and extended objects
- Difficult to see how chaos theory arises
- Obscures relationship between quantum and classical mechanics

2 Lagrangian Mechanics

We first start with a few examples showing Lagrangian Mechanics in action, and we will later discuss generalizations and intricacies:

Example 1: Let us consider a simple system with a mass m on a spring with spring constant k. Instead of solving this by writing $F = -kx$, let us write the following strange combination of kinetic and potential energy, known as the **Lagrangian**.:

$$L = T - V, \tag{2.1}$$

where T is the kinetic energy and V is the potential energy. Using $T = \frac{1}{2}m\dot{x}^2$ and $V = \frac{1}{2}kx^2$, we get:

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2. \tag{2.2}$$

Consider the following equation, known as the Euler-Lagrange Equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \tag{2.3}$$

which gives

$$\frac{d}{dt}(m\dot{x}) = -kx \implies m\ddot{x} = -kx. \tag{2.4}$$

Example 2: Suppose we have a general potential $V = V(x)$. Then the E-L equation gives

$$m\ddot{x} = -\frac{\partial V(x)}{\partial x}. \tag{2.5}$$

Notice that $-\frac{\partial V(x)}{\partial x}$ is just the force!

If we have more than one dimension, i.e. x, y, z, then we'll have 3 E-L equations, and we'll have to solve them separately. The Lagrangian would be

$$L = \frac{1}{2}m\sum_i \dot{x}_i^2 - V(x_i) \tag{2.6}$$

and the 3 different E-L equations would be

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i} \implies m\ddot{x}_i = -\frac{\partial V}{\partial x_i} \tag{2.7}$$

for $i = 1, 2, 3$. These equations are just the 3 components of $m\ddot{\vec{r}} = -\vec{\nabla}V(\vec{r})$. The important idea here is that we can determine the equations of motion without knowing the forces! We just need to figure out the potential energy. We can try a more complex example:

Example 3: Suppose we have a mass on a spring that acts as a pendulum. The spring has equilibrium length ℓ and at an angle θ, the spring has length $r = \ell + x$. We will work in polar coordinates x, θ. The kinetic energy is

$$T = \frac{1}{2}m\vec{v}^2 = \frac{1}{2}m\left(\dot{x}^2 + (\ell + x)^2\dot{\theta}^2\right), \tag{2.8}$$

and the potential energy is

$$V = -mg(\ell + x)\cos\theta + \frac{1}{2}kx^2. \tag{2.9}$$

The $E - L$ equation for x gives

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \tag{2.10}$$

$$m\ddot{x} = m(\ell + x)\dot{\theta}^2 + mg\cos\theta - kx. \tag{2.11}$$

The term $m(\ell + x)\dot{\theta}^2$ is the centripetal force, $mg\cos\theta$ is the radial gravitational force, and $-kx$ is the spring force. The θ component is

$$m(\ell + x)^2\dot{\theta} = mg(\ell + x)\sin\theta \tag{2.12}$$

which corresponds to

$$\vec{\tau} = \frac{d\vec{L}}{dt} \tag{2.13}$$

where $\vec{\tau}$ is torque and $\vec{L}$ is the angular momentum.