CHE260: Heat Transfer

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1 Mechanisms of Heat Transfer

- Mechanisms involve:
  - Conduction: Transfer of heat through a medium that is stationary.
  - Convection: Transfer of heat from a solid surface and an adjacent fluid that is moving.
  - Radiation: Energy emitted by matter in the form of electromagnetic waves.
- Conduction follows **Fourier's Law:**
$$\dot{Q} = -kA\nabla T \tag{1}$$
- The rate of heat transfer from the surface of a blackbody is given by the Stefan-Boltzmann Law
$$\dot{Q}_{\text{emit,max}} = \sigma A T_s^4 \tag{2}$$
where σ is the Boltzmann constant $\sigma = 5.67 \times 10^{-8} \frac{W}{m^2K^4}$
- For an ideal body, we have
$$\dot{Q}_{\text{emit}} = \epsilon\sigma A T_s^4 \tag{3}$$
where $0 \leq \epsilon \leq 1$ is the emissivity.
- When radiation is incident on a surface, some will be absorbed and some reflected. The **absorptivity** α is defined such that
$$\dot{Q}_{\text{absorbed}} = \alpha\dot{Q}_{\text{incident}} \tag{4}$$
$$\dot{Q}_{\text{reflected}} = (1-\alpha)\dot{Q}_{\text{incident}} \tag{5}$$
- Kirchoff's Law says that
$$\alpha = \epsilon \tag{6}$$
- For a small surface completed surrounded by a much larger surface net radiation is
$$\dot{Q}_{\text{net}} = \epsilon\sigma A(T_s^4 - T_{\text{surrounding}}^4) \tag{7}$$
- Natural convection tells us the heat transfer coefficient is
$$h = c(T_S - T_\infty)^{1/4} \tag{8}$$
where $c = 4.2\text{W}m^2K^{5/4}$ and
$$q_{\text{conv}} = hA(T_S - T_\infty) \tag{9}$$
- Forced convection gives a constant $h = 250W/m^2K$
- Let's look at the **one-dimensional case:** If we look at a segment of length Δx, the rate of the increase of enthalpy is
$$\dot{H} = mc_p\frac{\partial T}{\partial t} \tag{10}$$
$$= \rho c_p A\Delta x\frac{\partial T}{\partial t} \tag{11}$$

The energy balance in this small segment gives

$$\rho c_p A \Delta x \frac{\partial T}{\partial t} = \dot{Q}_x - \dot{Q}_{x+\Delta x} \tag{12}$$

$$= (\dot{q}A)_x - (\dot{q}A)_{x+\Delta x} \tag{13}$$

Dividing by Δx and taking the limit gives

$$\rho c_p \frac{\partial T}{\partial t} = -\frac{1}{A}\frac{\partial(\dot{q}A)}{\partial x} \tag{14}$$

Note that A depends on the coordinate system, $\dot{q}$ depends on Fourier's Law $\dot{q} = -k\frac{dT}{dx}$.

- This gives us the heat conduction equation

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{15}$$

  where $\alpha = \frac{k}{\rho c_p}$ is the thermal diffusivity.

- At constant state, $\frac{\partial T}{\partial t} = 0$, so

$$\frac{d^2 T}{dx^2} = 0 \tag{16}$$

- In cylindrical coordinates, we have

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \tag{17}$$

- In spherical coordinates

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) \tag{18}$$

- In general, we have

$$\frac{1}{r^n}\frac{\partial}{\partial r}\left(r^n\frac{\partial T}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{19}$$

  where

  - $n = 0$ for cartesian
  - $n = 1$ for cylindrical
  - $n = 2$ for spherical

> **Idea**: These equations can alternatively derived by applying the divergence formula in different coordinate systems. In particular, it is done by considering the heat equation
>
> $$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \nabla \cdot T \tag{20}$$
>
> and applying spherical / cylindrical symmetry.

- To solve problems, we also need boundary conditions, and we often make the steady state assumption.

2 Thermal resistance Networks and Contact resistance

read Chapter 17.2, 17.3

We can think of a wall with a temperature difference across the two sides as a **thermal resistor** with a thermal resistance of $r_1 = \frac{T_1 - T_2}{\dot{Q}}$.

In this way, we can put thermal resistors in series and get the equivalent thermal resistance as:

$r_{equivalent} = \sum_i r_i$

What about the situation where there are three materials stacked up on top of each other bridging two different temperatures, each with their own cross-sectional area?

If we call the amount of heat transfer in each of these $\dot{Q}_1$, $\dot{Q}_2$, $\dot{Q}_3$, then the total heat transfer is

$\dot{Q} = \dot{Q}_1 + \dot{Q}_2 + \dot{Q}_3$

$\dot{Q} = \frac{T_1 - T_2}{\dot{r}_1} + \frac{T_1 - T_2}{\dot{r}_2} + \frac{T_1 - T_2}{\dot{r}_3}$

$= (T_1 - T_2)[\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}]$

$= \frac{T_1 - T_2}{r_{total}}$

So our electrical analogy fits thermal resistance completely. Now we can apply this analogy on networks of thermal resistance.

However, a surface in real life is never completely smooth so never has a perfect contact. Between two rough surfaces, air is trapped and affects the heat conduction.

Heat flux $\dot{q}$ $[\frac{w}{m^2}]$

Define a thermal contact resistance $r_c = \frac{\delta T}{\dot{q}}$ $[\frac{m^2 * C}{W}]$

Which is the resistance per unit area.

reciprocal of r_c is the h_c, the "thermal contact conductance".

$h = \frac{1}{r_c} = \frac{\dot{q}}{\delta T}$

$\dot{q} = h_c \delta T$

Typical values of h_c:

Metal Pairs	h_c $(\frac{w}{m^2 * C})$
Steel/Steel	10^3
Al/Al	10^4
Cu/Cu	10^5

3 Contact resistance in Cylinders and Spheres

For a long pipe, $\frac{dT}{dZ} << \frac{dT}{dr}$. So we approximate by assuming 1-dimensional conduction in the Z direction.

$\frac{\partial}{\partial r}(r\frac{\partial T}{\partial r}) = 0$

Boundary conditions:
$r = r_1, T = T_1$
$r = r_2, T = T_2$

Integrate:

$r\frac{dT}{dr} = c_1$

$\rightarrow \frac{dT}{dr} = \frac{c_1}{r}$
$T(r) = c_1 ln(r) + c_2$

Applying boundary conditions:

$T_1 = c_1 ln(r_1) + c_2$
$T_2 = c_1 ln(r_2) + c_2$

$\rightarrow T_1 - T_2 = c_1[ln(r_1) - ln(r_2)]$
$\rightarrow c_1 = \frac{T_1 - T_2}{ln(\frac{r_1}{r_2})}$
$T_2 = \frac{T_1 - T_2}{ln(\frac{r_1}{r_2})} ln(r_2) + c_2$
$c_2 = T_2 - \frac{(T_1 - T_2}{ln(\frac{r_1}{r_2})} ln(r_2)$

$T(r) = \frac{T_1 - T_2}{ln(\frac{r_1}{r_2})} ln(r) - \frac{T_1 - T_2}{ln(\frac{r_1}{r_2})} ln(r_2) + T_2$

$T(r) = \frac{T_1 - T_2}{ln(\frac{r_1}{r_2})} ln(\frac{r}{r_2}) + T_2$

$\dot{Q}_{conduction} = -kA_1 \frac{dT}{dr}|_{(r=r_2)}$
$A_1 = 2\pi r_1 L$
$\frac{dT}{dr}|_{(r=r_2)} = \frac{T_1 - T_2}{ln(\frac{r_1}{r_2})} ln(\frac{r_2}{r} - \frac{1}{r_2})|_{r=r_1}$

Finally: $\dot{Q}_{conduction} = 2\pi Lk \frac{T_1 - T_2}{ln(\frac{r_1}{r_2})})$

Define thermal resistance of a cylinder: $r_{cylinder} = \frac{T_1 - T_2}{\dot{Q}_{conduction}}$ $r_{cylinder} = \frac{ln(r_2/r_1)}{2\pi Lk}$

For a sphere: $r_{sphere} = \frac{r_2 - r_1}{4\pi r_1 r_2 k}$

4 Conduction in Cylinders and Spheres

Let h_1 be the thermal contact conductance between the outer surface of the cylinder and the outer surface of the sphere.

$$\frac{dT}{dr}|_{(r=r_2)} = \frac{T_1 - T_2}{ln(\frac{r_1}{r_2})} ln(\frac{r_2}{r} - \frac{1}{r_2})|_{r=r_1}$$