MAT257: Real Analysis II (Manifolds)

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1 Tensors

Definition: A **k-tensor** is a multilinear function $T : V^k \to \mathbb{R}$ and the set of all k-tensors, denoted $\mathcal{T}^k(V)$ becomes a vector space if we define scalar multiplication and vector addition in natural ways, i.e. if for $S, T \in \mathcal{T}^k(V)$ and $a \in \mathbb{R}$, we have:

$$(S+T)(v_1, \dots, v_k) = S(v_1, \dots, v_k) + T(v_1, \dots, v_k) \tag{1.1}$$
$$(aS)(v_1, \dots, v_k) = a \cdot S(v_1, \dots, v_k) \tag{1.2}$$

Definition: We define the **tensor product** $S \otimes T \in \mathcal{T}^{k+\ell}(V)$ by

$$S \otimes T(v_1, \dots, v_k, v_{k+1}, \dots, v_{k+\ell}) = S(v_1, \dots, v_k) \cdot T(v_{k+1}, \dots, v_{k+\ell}). \tag{1.3}$$

Note that $S \otimes T \neq T \otimes S$. Here are the properties of the tensor product:

$$\begin{aligned}(S_1 + S_2) \otimes T &= S_1 \otimes T + S_2 \otimes T \\ S \otimes (T_1 + T_2) &= S \otimes T_1 + S \otimes T_2 \\ (aS) \otimes T &= S \otimes (aT) = a(S \otimes T) \\ (S \otimes T) \otimes U &= S \otimes (T \otimes U)\end{aligned}$$

Note that $\mathcal{T}^1(V)$ is just the dual space V^*. Therefore, it makes sense that we can use the dual elements to create a basis for $\mathcal{T}^k(V)$.

Theorem: Let $v_1, \dots, v_n$ be a basis for V and let $\varphi_1, \dots, \varphi_n$ be the dual basis, i.e. $\varphi_i(v_j) = \delta_{ij}$. Then the set of all k-fold tensor products:

$$\varphi_I = \varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k} \tag{1.4}$$

is a basis for $\mathcal{T}^k(V)$, where $1 \leq i_1, \dots, i_k \leq n$ and therefore has dimension n^k.

For notation purposes, let us define $\underline{n} = \{1, \dots, n\}$ and $\underline{n}^k = \{(i_1, \dots, i_k) : i_\alpha \in \underline{n}\}$. An element $I \in \underline{n}^k$ is thus known as a **multi-index.**

Proof. We need to show three things:

- If $T_1, T_2 \in \mathcal{T}^k$ then $T_1 = T_2 \iff \forall I, T_1(V_I) = T_2(V_I)$.
- The set $\{\varphi_I\}$ spans $\mathcal{T}^k(V)$.
- The elements of the set $\{\varphi_I\}$ are linearly independent.

$\square$

Definition: If $f : V \to W$ is a linear transformation, we can define the **pullback** to be the linear transformation $f^* : \mathcal{T}^k(W) \to \mathcal{T}^k(V)$, defined by:

$$f^*T(v_1, \dots, v_k) = T(f(v_1), \dots, f(v_k)). \tag{1.5}$$

It is easy to verify that $f^*(S \otimes T) = f^*S \otimes f^*T$.

Definition: A k-tensor $T \in \mathcal{T}^k$ is **alternating** if

$$T(\dots, u, \dots, w, \dots) = -T(\dots, w, \dots, u, \dots) \tag{1.6}$$

for any u, w. We then define $\Lambda^k(V) := \{T \in \mathcal{T}^k V : T \text{ is alternating}\}$.