Phy450: Relativistic Electrodynamics
Quiz 6

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We can use Leibniz's rule,

$$\begin{aligned}
\partial_\mu(x_\nu A^{\mu\nu}S) &= (\partial_\mu x_\nu)A^{\mu\nu}S + x_\nu(\partial_\mu A^{\mu\nu})\partial_\mu S + x_\nu A^{\mu\nu}(\partial_\mu S) &(1)\\
&= (\partial_\mu x^\lambda g_{\lambda\nu})A^{\mu\nu}S + x_\nu A^{\mu\nu}(\partial_\mu S) &(2)\\
&= \delta^\lambda_\mu g_{\lambda\nu}A^{\mu\nu}S + x_\nu A^{\mu\nu}(\partial_\mu S) &(3)\\
&= g_{\mu\nu}A^{\mu\nu}S + x_\nu A^{\mu\nu}(\partial_\mu S), &(4)
\end{aligned}$$

where $\partial_\mu A^{\mu\nu} = 0$ since A is a constant tensor. We claim that this first term is zero. Let $\xi = g_{\mu\nu}A^{\mu\nu}$. Then:

$$\begin{aligned}
\xi &= \frac{1}{2}(\xi + \xi) &(5)\\
&= \frac{1}{2}\left(g_{\mu\nu}A^{\mu\nu} + g_{\nu\mu}A^{\mu\nu}\right) &(6)\\
&= \frac{1}{2}\left(g_{\mu\nu}A^{\mu\nu} - g_{\nu\mu}A^{\nu\mu}\right) &(7)\\
&= \frac{1}{2}\left(g_{\mu\nu}A^{\mu\nu} - g_{\mu\nu}A^{\mu\nu}\right) &(8)\\
&= \frac{1}{2}(\xi - \xi) &(9)\\
&= 0. &(10)
\end{aligned}$$

The second line is due to the symmetry of g, third line is due to anti-symmetry of A. In general, contracting a symmetric tensor with an antisymmetric one will give us zero. Therefore, only the second term of equation 4 survives, and we are left with

$$\partial_\mu(x_\nu A^{\mu\nu}S) = x_\nu A^{\mu\nu}(\partial_\mu S). \tag{11}$$

In this problem, we only assumed the anti-symmetry of $A^{\mu\nu}$, the symmetry of $g^{\mu\nu}$, and the fact that A is constant. Therefore, this derivation will also hold for general curved spacetimes.