MAT292: Ordinary Differential Equations

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Contents

1   Introduction

Covers 1.1: Mathematical Models and Solutions

- Big Idea: Differential equations model physical situations:
    - Take a physical situation and ODE-ify it (How do we model a cooling coffee cup?)
    - Understand an ODE without solving it (What can we deduce directly from $y' = y^2$?)
    - Study, categories, typecast ODEs and solve them

        > **Example 1:** Suppose we have $y' = y/t + \ln t$ and $y' = y^2 + t$. Which of these are harder to solve (without actually solving them)?
        >
        > It turns out that the second one is harder as it is *non-linear*.

    - Handle ODEs numerically (What do we do when we cannot solve an ODE that models a real life phenomenon?)
    - The art of problem solving (How do I work with no strings attached?)
- What is a differential equation?

    > **Definition**: A differential equation relates a function and its derivatives.

- We can understand ODEs without solving it:

    > **Example 2:** Let's consider a cup of coffee in a room. We want to model its change in temperature over time. How do we do this?
    >
    > There are a lot of variables, so we have to simplify our model. The things we care about
    > - The temperature of the coffee cup $y(t)$.
    > - t is in minutes.
    > - $y(t)$ is in Celsius.
    > - The temperature in the room T (in Celsius).
    >
    > The things we ignore / simplify:
    > - Temperature variation within the cup
    > - Temperature variation in the room
    >
    > **Exercise:** Let's consider some suggestions for an ODE describing the temperature of a coffee cup in a room. Each of the following suggested ODEs contradicts our intuition in some way. How?
    > - $y' = y^2$
    >   - T isn't in there
    >   - Temperature would always increase except if $y = 0$.
    >   - The hotter the coffee, the faster it heats up.
    > - $y' = \dfrac{T}{y}$
    >   - If $T > 0, y > 0$, then $y' > 0$
    >   - The model doesn't work for coffee at $0°C$.
    > - $y' = y[e^{y-T} + y^3]$
    > - $y' = y - T$
    > - $y' = T - y$
    >   - There should be a parameter that describes the physical properties (rate of heating/cooling will be different for different materials)

    > **Idea**: Without solving an ODE, you can already make many predictions about its solution
    >
    > (and then, for example, judge your model)

- We introduce a few definitions

> **Definition**: An **ordinary differential equation** (ODE) only considers a function of 1 variable and its derivatives

> **Definition**: A **partial differential equation** considers a function of several variables and its derivatives.

- The most general ODE for a function $y(t)$ is:

  (a) $F[t, y, y'', \ldots, y^{(n)}]$ for $n \in \mathbb{N}$.

  (b) Any function that satisfies this equation is called *a solution*

> **Definition**: The order of an ODE is the highest derivative of an ODE.

- An autonomous ODE is if the independent variable doesn't appear in the ODE.
- Systems of ODEs arise if we study several quantities depending on the same variable and how their changes interact.

  Example 3: Assume that $p(t)$ and $o(t)$ describe the number of twitter followers of two accounts. If there is no interaction, what are reasonable ODEs for these two quantities?

  $$p'(t) = kp(t) \tag{1}$$
  $$o'(t) = \ell o(t) \tag{2}$$

  Suppose that if in addition to "word of mouth," we consider the effects that these two tweets have, what are reasonable ODEs for the number of followers?

  $$p'(t) = k \cdot p(t) - m \cdot o(t) \tag{3}$$
 $$o'(t) = \ell \cdot o(t) - n \cdot p(t) \tag{4}$$

  where all constant are positive. However, this is oversimplified as it assumes the people who follow O also follow P.

- Suppose a differential equation is given by
  $$y'(t) = -1.5(y(t) - 1) \tag{5}$$

> **Definition**: Consider the ODE $y' = f(t, y)$. We can draw a **direction field** as follows:
> 1. Draw a $t - y$ coordinate system.
> 2. Evaluate $f(t, y)$ over a rectangular grid of points.
> 3. Draw a line at each point (t, y) of the grid with slope $f(t, y)$.

Slope field of $y' = -1.5(y - 1)$.
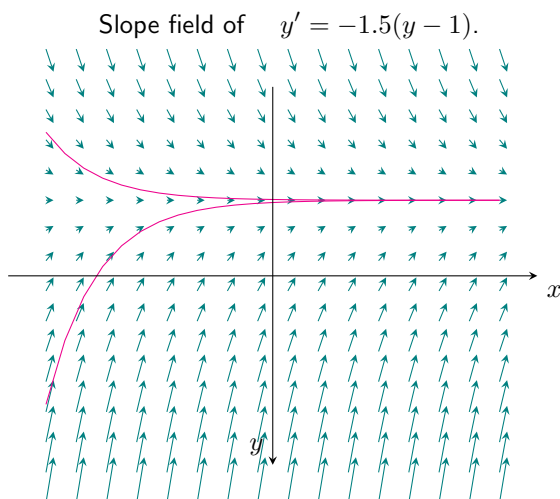
- Consider an autonomous first order ODE $y' = f(y)$. If $f(c) = 0$ for some specific value c, we call c an equilibrium of the ODE. We say it is

1. A **stable equilibrium,** if a solution starting at a value close to c approaches $y = c$ as $t \to \infty$

2. An **equilibrium,** if a solution starting at a value close to c moves away from $y = c$ as $t \to \infty$.

3. A **semistable equilibrium,** if we observe either behaviour, depending on if the solution starts just above or just below c.

Example 4: Consider the differential equation $y' = 3\cos y$. We can construct the phase plot:

Phase Plot
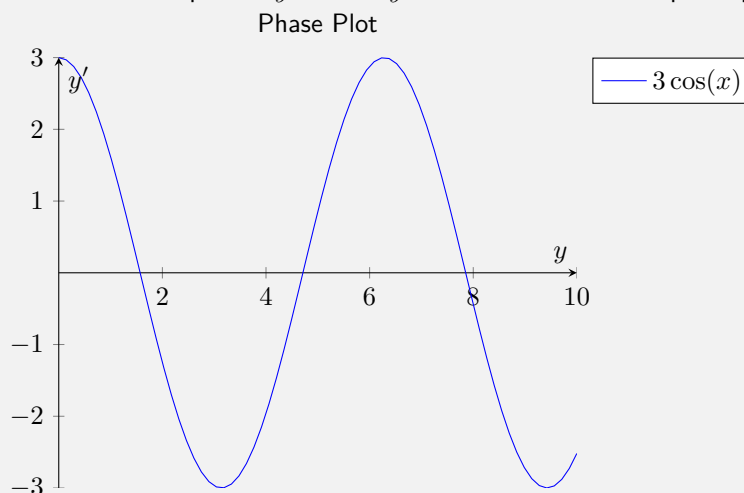
Equilibrium occurs when $y' = 0$ The first equilibrium occurs at $y = \dfrac{\pi}{2}$. This is stable as if we move a bit to the left, y' is positive so that we move back to the right. If we move to the right instead, y' is negative and we move back to the left.

We can also determine this by looking at the second derivative $y'' = -3\sin(y)$. A negative second derivative means that it is stable. A positive second derivative means that it is unstable.

2   to be added