MAT301 Notes

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Contents

1 Lecture One

- Groups are everywhere in mathematics and nature in one of two forms:
  - as groups of symmetries
  - as groups of "numbers" or quantities
- We will call a subset $F \subseteq \mathbb{R}^n$ a **figure** in $\mathbb{R}^n$ when we consider F not just as a set, but as a set together with the structure of its distance functions:

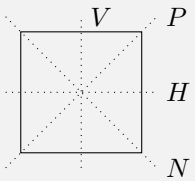
$$d : F \times F \to \mathbb{R}_{\geq 0}, \quad d(x, y) = \|x - y\| \tag{1}$$

A figure is then defined as the pair (F, d).

> **Definition**: A **symmetry** of a figure $F \subseteq \mathbb{R}^n$ is a bijection $\sigma : F \to F$ such that σ and σ^{-1} preserve distances:
>
> $$\forall x, y, \in F, \quad d(\sigma(x), \sigma(y)) = d(x, y) \tag{2}$$
> $$\iff d(\sigma^{-1}(x), \sigma^{-1}(y)) = d(x, y) \tag{3}$$
>
> Therefore:
>
> $$\text{Sym}(F) \equiv \{\sigma : F \to F | \sigma \text{ is a symmetry}\} \tag{4}$$

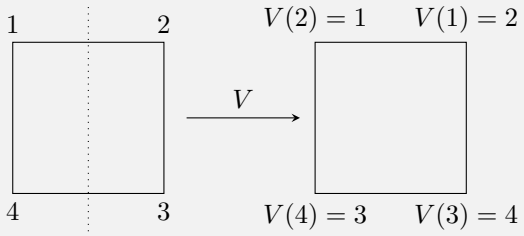
- For example, any point, line, shape, or form is a figure. However, we are only interested in figures that have interesting symmetries.

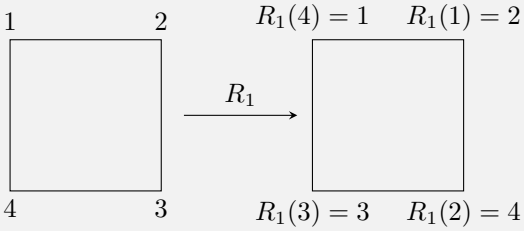
Example 1: Let F be a square in $\mathbb{R}^2$. There are four different lines of reflections:



and there are three rotations: R_1, R_2, and R_3, which represent $90°$, $180°$, and $270°$ clockwise rotations. I represents the identity transformation (do nothing).

We can combine symmetries. For example, what is $R_1 \circ V$? To do so, we can label the vertices:





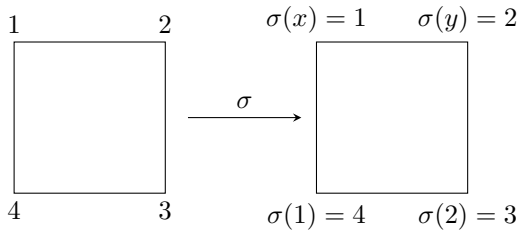
Applying the computations:

$$(R_1 \circ V)(1) = R_1(V(1)) = R_1(2) = 3 \tag{5}$$
$$(R_1 \circ V)(2) = R_1(V(2)) = R_1(1) = 2 \tag{6}$$
$$(R_1 \circ V)(3) = 1 \tag{7}$$
$$(R_1 \circ V)(4) = 4 \tag{8}$$

Check that $V \circ R_1 = N$. Also notice that these operations are not commutative: $R_1 \circ V \neq V \circ R_1$.

- In the above example, how are we sure that these are all of the symmetries of a square? To answer this, we will need the following facts:

  1. A symmetry maps vertices to vertices. The vertices are the points of the square that are furthest from the center.
  2. Symmetries map adjacent vertices tto adjacent vertices. If x, y are adjacent vertices, then $\sigma(x)$, $\sigma(y)$ are vertices, and $d(\sigma(x), \sigma(y)) = d(x, y) =$ side length.
  3. A symmetry σ is completely determined by $(\sigma(1), \sigma(2))$. For example, suppose we have the symmetry σ on a square such that:



     From this, we know that we must have $y = 3$, from fact 1, as well as $x = 4$.

  4. For all $x, y \in \{1, 2, 3, 4\}$ such that x is adjacent to y, $\exists!$ symmetry σ of the square such that:

$$(\sigma(1), \sigma(2)) = (x, y) \tag{9}$$

By the above facts, we must count the ordered pairs (x, y) such that $x, y \in \{1, 2, 3, 4\}$ and x is adjacent to y:

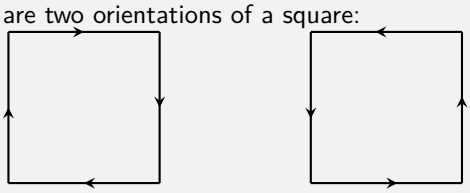
- There are 4 choices for x.
- For each choice of x, there are two choices of y. Therefore, there are $4 \times 2 = 8$ symmetries.

Since we listed 8 different symmetries of a square, we have therefore defined all of them.

2 Lecture Two

- Let X be a set with some **structures**. Then a symmetry of X (w.r.t. the structures) is a bijection $\sigma : X \mapsto X$, such that σ and σ^{-1} preserve the structures.
- The set of symmetries of X is denoted as $\mathrm{Sym}\,(X)$.

Example 2: We can consider a square not only with the structure of its distance function but with additional structure of its orientations. There are two orientations of a square:



A symmetry of the square with respect to its orientation is a bijection from the square to itself that maps each orientation to itself.

– Rotations preserve orientations, but reflections don't.

Therefore, the symmetries preserving orientations are $\{I, R_1, R_2, R_3\}$.

- In general:

0. If σ_1, σ_2:$X \to X$ are symmetries, then:
$$\sigma_1 \circ \sigma_2 : X \to X \tag{10}$$
is also a symmetry. Consequently, composition of symmetries restrict a map:
$$\mathrm{Sym}\,(X) \times \mathrm{Sym}\,(X) \mapsto \mathrm{Sym}\,(X), \quad (\sigma_1, \sigma_2) \mapsto \sigma_1 \circ \sigma_2 \tag{11}$$
Remarks: A map $m : S \times S \to S$ is called a binary operation on S.

1. Associativity: For all $\sigma_1, \sigma_2, \sigma_3 \in \mathrm{Sym}\,(X)$, we have:
$$(\sigma_1 \circ \sigma_2) \circ \sigma_3 = \sigma_1 \circ (\sigma_2 \circ \sigma_3) \tag{12}$$
2. The identity $\mathrm{id} : X \mapsto X$ is a symmetry and $\mathrm{id} \in \mathrm{Sym}\,(X)$.
3. Immediately from the "definition," we have: $\sigma \in \mathrm{Sym}\,(X) \implies \sigma^{-1} \in \mathrm{Sym}\,(X)$

- The notion of a group is an abstraction of $\mathrm{Sym}\,(X)$ and its properties.

Definition: A group is an ordered pair $(G, *)$ consisting of a set G and a binary operation $* : G \times G \to G$ such that:

1. $*$ is associative, $\forall g_1, g_2, g_3 \in G$, we have:
$$(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3) \tag{13}$$
2. There exists an element $e \in G$ such that for all $g \in G$, we have $g * e = g = e * g$.
3. For all $g \in G$, there exists an element $h \in G$ such that $g \star h = e = h \star g$.

These numberings are abstractions of the properties listed above.

- The binary operator $*$ is called the **group law** or **group operation**. It is often denoted by a dot $\cdot$ or by juxtaposition (gh instead of $g * h$).
- The *cardinality* of G, $|G|$, is called the **order** of G.
- It is common to denote e by 1 or I.

Warning: A common *misconceptions* is saying "G is a group" instead of "$(G, *)$ is a group."

- These are equivalent statements:
$$(G, *) \text{ is a group} \tag{14}$$
$$\iff G \text{ is a group under } * \tag{15}$$

Definition: A group $(G, *)$ is **abelian** (or commutative) if for all $g, h \in G$, we have:

$$g * h = h * g \tag{16}$$

- Here are some examples of groups:
  - $(\mathrm{Sym}\,(X), \circ)$
  - $(\mathbb{Z}, +)$
  - $(\mathbb{R}^x, \cdot)$ where:
$$F^x = \{x \in F : \exists y \in F \text{ with } xy = 1 = yx\} \tag{17}$$
  - $(\mathbb{Q}_{>0}, \cdot)$, $(\mathbb{R}_{>0}, \cdot)$.
  - $(\mu_n, \cdot)$ where for $n \in \mathbb{Z}_{>0}$, let
$$\mu_n = \{z \in \mathbb{C} | z^n = 1\} = \{e^{2\pi k i/n} | k = 0, 1, \ldots, n-1\} \tag{18}$$
  - $(\mathbb{R}^n, +)$
  - $(\mathrm{GL}_n(F), \cdot)$ where $\mathrm{GL}_n(F) = \{A \in \mathrm{Mat}_{n\times n}(F) | A \text{ invertible}\}$, $F = \mathbb{Q}, \mathbb{R}, \mathbb{C}$. For all $n \geq 2$, $\mathrm{GL}_n(F)$ is non-abelian. Note that GL stands for *general linear*
  - $(\mathrm{SL}_n(F), \cdot)$ where $\mathrm{SL}_n(F) = \{A \in \mathrm{GL}_n(F) | \det A = 1\}$. Note that SL stands for special linear.
  - $(\mathrm{Mat}_{n\times n}(F), +)$

  and non-groups:
  - $(\mathbb{Z}, \cdot)$
  - $(\mathbb{Z}_{>0}, +)$
  - $(\mathbb{Z}, -)$, $(\mathbb{Q}^x, \div)$.
  - $(\mathrm{Mat}_{n\times n}(F), \cdot)$

Proposition 1: Let $(G, *)$ be a group. If $e, e' \in G$ such that $\forall g \in G$ we have

$$g * e = g = e * g \tag{19}$$

and

$$g * e' = g = e' * g, \tag{20}$$

then $e = e'$.

Proof. Consider $e * e'$. By 19, we have:

$$e * e' = e' \tag{21}$$

Similarly, by 20, we have:

$$e * e' = e \tag{22}$$

Therefore, $e = e * e' = e'$. □

- We call the unique element $e \in G$ satisfying the second property in the definition of a group, the identity element of G.
- The **trivial group:** For any singleton $\{e\}$, there exists a unique binary operation $\cdot$ such that:

$$\{e\} \times \{e\} \mapsto \{e\}, \quad (e, e) \mapsto e \tag{23}$$

and $(\{e\}, \cdot)$ is a group, called a trivial group.

Proposition 2: Let $(G, *)$ be a group and let $g \in G$. If $h, h' \in G$ satisfies:

$$g * h = e = h * g \tag{24}$$

and

$$g * h' = e = h' * g \tag{25}$$

then $h = h'$. By 24, we have:

$$h * g = e. \tag{26}$$

By 25, we have:

$$g * h' = e. \tag{27}$$

Therefore:

$$\begin{aligned} h &= h * e && \text{(property 2)} && (28)\\ &= h * (g * h') && (27) && (29)\\ &= (h * g) * h' && \text{(property 1)} && (30)\\ &= e * h' && (26) && (31)\\ &= h' && \text{(property 2)} && (32) \end{aligned}$$

- For each $g \in G$, the unique element $h \in G$ such that $g * h = e = h * g$ is called the inverse of g and denoted by g^{-1}.

Lemma 1: Let $(G, *)$ be a group and let $x, y, z \in G$. Then, right cancellation tells us:

$$x * z = y * z \implies x = y \tag{33}$$

and left cancellation tells us:

$$z * x = z * y \implies x = y \tag{34}$$

Proof. If $z * x = z * y$, then:

$$\begin{aligned} & z^{-1} * (z * x) = z^{-1} * (z * y) && (35)\\ \implies & (z^{-1} * z) * x = (z^{-1} * z) * y && (36)\\ \implies & e * x = e * y && (37)\\ \implies & x = y && (38) \end{aligned}$$

The other implication is similar. □

Warning: The notation $\frac{a}{b}$ is ambiguous. Does it mean $a * b^{-1}$ or $b^{-1} * a$? These can be different in a non-abelian group.

Lemma 2: Let $(G, *)$ be a group and let $g_1, \ldots, g_n \in G$. Every way of way inserting parentheses into $g_1 * g_2 * \cdots * g_n$ to determine a well defined product in G results in the same element of G.

Proof. Proved in tutorial worksheet. □

- The consequence of the above lemma is that the notation $g_1 * g_2 * \cdots * g_n$ is unambiguous.

Definition: Let $(G, *)$ be a group and let $n \in \mathbb{Z}$. We define:

$$g^n = \begin{cases} \underbrace{g * g * \cdots * g}_{n \text{ copies}}, & n > 0 \\ e, & n = 0 \\ \underbrace{g^{-1} * \cdots * g^{-1}}_{n \text{ copies}} = (g^{-1})^{-n}, & n < 0 \end{cases} \tag{39}$$

Lemma 3: Let $(G, *)$ be a group. For all $g \in G$ and $m, n \in \mathbb{Z}$, we have:

$$g^m * g^n = g^{m+n} \tag{40}$$

and:

$$(g^m)^n = g^{mn} \tag{41}$$

- To prove the above lemma, we can use induction.

Warning: If G is a non-abelian group and $a, b \in G$ and $n \in Z$, then it can happen that:

$$(ab)^n \neq a^n b^n \tag{42}$$

Lemma 4: Let G be a group and let $a, b \in G$. Then:

$$(ab)^{-1} = b^{-1}a^{-1} \tag{43}$$

Proof. We just need to check the two conditions:

$$(ab)(b^{-1}a^{-1}) = aea^{-1} = aa^{-1} = e \tag{44}$$

and:

$$(b^{-1}a^{-1})(ab) = b^{-1}eb = b^{-1}b = e \tag{45}$$

Therefore, it is the inverse. □

- **Dihedral Groups**. Let $n \in \mathbb{Z}$, $n \geq 3$. Let P_n be a regular $n-$gon.

Definition: The group of symmetries of the regular n-gon P_n is called the dihedral group of order $2n$ and is denoted by D_n.

Warning: Some people use D_{2n} instead of D_n.

Lemma 5: The order of D_n is $2n$.

Proof. Label the vertices of P_n by $v_1, v_2, \ldots, v_n$ in some clockwise order. By the same reasoning from the case $n = 4$ when we were considering a square, we have a bijection:

$$D_n = \mathrm{Sym}\,(P_n) \to \{(v_i, v_j) | v_i \text{ adjacent to } v_j\} \tag{46}$$

$$\sigma \mapsto (\sigma(v_1), \sigma(v_2)) \tag{47}$$

Note that $\{(v_i, v_j | v_i \text{ adjacent to } v_j)\} = \{(v_i, v_j) | j \equiv i \pm 1 \pmod n\}$. We have:

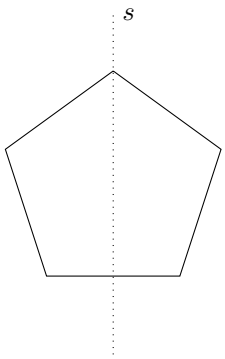
$$|D_n| = |\{(v_i, v_j) | j \equiv i \pm 1 \pmod n\}| = n \cdot 2 \tag{48}$$

□

- For example, consider D_5. There are 5 lines of reflection, 4 rotational symmetries, and the identity. We can further compose transformations, for example:

$$rs = sr^4, \quad r^2 s = sr^3, \quad r^3 s = sr^2, \quad r^4 s = sr, \quad r^5 s = sr \tag{49}$$

where s represents a reflection and r is a 72° clockwise rotation.



Lemma 6: Let P_n be a regular n-gon. Let r be either a clockwise or counterclockwise rotation about the center of P_n by $\frac{2\pi}{n}$, and let s be any reflectional symmetry of P_n. Then:

1. $r^n = 1$, $s^2 = 1$
2. For all $k = 0, 1, \ldots, n-1$, sr^k is a reflection and:

$$sr^k = r^{-k}s = r^{n-k}s \tag{50}$$

3. $1, r, \ldots, r^{n-1}, s, sr, \ldots, sr^{n-1}$ are all distinct.
4. $D_n = \{1, r, \ldots, r^{n-1}, s, sr, \ldots, sr^{n-1}\}$.

Proof. We will prove all four:

1. r is a rotation by $2\pi/n$ CW or CCW so $r^n = 1$. Since s is a reflection, $s^2 = 1$.
2. The composition of a reflection and a rotation in the plane is a reflection. Therefore, $\forall k = 0, 1, \ldots, n-1$, sr^k is a reflection (orientation is not preserved). Therefore:

$$(sr^k)^2 = 1 \tag{51}$$
$$sr^k sr^k = 1 \tag{52}$$
$$sr^k s = r^{-k} \tag{53}$$
$$sr^k = r^{-k}s^{-1} \tag{54}$$

   Since $s^2 = 1$, $s^{-1} = s$, this is proved. Furthermore, since $r^n = 1$, we must also have:

$$sr^k = r^{n-k}s \tag{55}$$

3. Since r^k is a rotation CW or CCW by $2\pi k/n$, then $1, r, \ldots, r^{n-1}$ are all distinct. Since rotations preserve orientation and reflections do not, then $r^i \neq sr^j$ for all i, j. If $sr^i = sr^j$, then $r^i = r^j$ so $i = j$ if $i, j \in \{0, \ldots, n-1\}$.

   Therefore, $1, r, \ldots, r^{n-1}, s, sr, \ldots, sr^{n-1}$ are distinct.
4. This follows directly from the previous property and the order of the dihedral group is $|D_n| = 2n$.

□

3 Lecture Three

- **Notation:** Sometimes the group operation for an **abelian** group is denoted by $+$.

  If $(A, +)$ is an abelian group, then:

  - The identity is denoted by 0
  - a^{-1} is denoted by $-a$
  - a^n is denoted by na
  - $a + (-b)$ is denoted by $a - b$.

- One way to get a better understanding of a group G is to find a group "inside of" G that you understand better.

Definition: Let $(G, *_G)$ be a group. A subset $H \subseteq G$ is a subgroup if:

1. For all $h_1, h_2 \in H$, $h_1 *_G h_2 \in H$, and therefore the operation of G:

$$*_G : G \times G \to G \tag{56}$$

   restricts to a binary operation on H:

$$*_H : H \times H \to H, \quad (h_1, h_2) \mapsto h_1 *_H h_2 := h_1 *_G h_2 \tag{57}$$

2. $(H, *_H)$ is a group.

- We write $H \leq G$ as a shorthand for "H is a subgroup of G." If $(G, *)$ is a group and $H \subseteq G$, we often denote the group operator for H by $*$ as well.

Example 3: Let G be a group. Then $G \leq G$ and $\{e\} \leq G$. We call $\{e\}$ the trivial subgroup of G.

- If $H \leq G$ and $H \neq G$, we write $H < G$ and call H a **proper subgroup** of G.

Example 4: Let D_n be the symmetric group of the regular n-gon with vertices $\{(\cos(2\pi k/n), \sin(2\pi k/n)) | k = 0, \ldots, n-1\}$.

From last lecture, we have $D_n = \{1, r, \ldots, r^{n-1}, s, rs, \ldots, r^{n-1}s$. Then: $H := \{1, r, \ldots, r^{n-1}\} \leq D_n$.

Proposition 3: Let G be a group and $H \leq G$.

1. The identity of H is the identity of G.
2. For all $h \in H$, the inverse of h in H is the inverse of h in G.

Proof. 1. Let e_H be the identity of H and e_g is that of G. Since e_H is the identity of H, we have:

$$e_H e_H = e_H \tag{58}$$

Let x be the inverse of e_H in G, then:

$$e_H e_H x = e_H x \tag{59}$$
$$\implies e_H e_G = e_G \tag{60}$$
$$\implies e_H = e_G \tag{61}$$

The first implication follows since x is the inverse of e_H in G and the second follows since e_G is the identity in G.

2. Let $h \in H$, let x be the inverse of h in H, and let y be the inverse of h in G. Then:

$$hx = e_H = e_G \tag{62}$$

and

$$xh = e_H = e_G \tag{63}$$

so x is the inverse of h in G.

□

Theorem: **Two-step subgroup test:** Let H be a nonempty subset of a group G. If:
1. $a, b \in H \implies ab \in H$ (H is closed under the group operator)
2. $a \in H \implies a^{-1} \in H$ (H is closed under taking inverses)

then H is a subgroup of G.

Proof. Assume that H is as in the theorem. We will prove that $(H, *_H)$ is a group.

- Associative: Let $h_1, h_2, h_3 \in H$

$$h_1 *_H (h_2 *_H h_3) = h_1 *_G (h_2 *_G h_3) \tag{64}$$
$$= (h_1 *_G h_2) *_G h_3 \tag{65}$$
$$= (h_1 *_H h_2) *_H h_3 \tag{66}$$

- H has an identity: Since $H \neq \phi$, there exists $x \in H$. By (2), we have $x^{-1} \in H$. By (1), we have $e_G = xx^{-1} \in H$ since $x, x^{-1} \in H$.

  For all $h \in H$, we have:
$$he_G = h = e_G h \tag{67}$$
  since e_G is the identity of G. Therefore e_G is an identity of H.

- H has inverses: Let $h \in H$. By (2), we have that $h^{-1} \in H$. Since h^{-1} is the inverse of h in G, we have $hh^{-1} = e_G = h^{-1}h$. Therefore h^{-1} is an inverse of h in H.

□

Theorem: **One-step subgroup test:** Let G be a group and let H be a nonempty subset of G. Suppose that:
1. $a, b \in H \implies ab^{-1} \in H$

then $H \leq G$.

Proof. Let H be as in the theorem statement. Since $H \neq \phi$, $\exists h \in H$. Taking $a = b = h$ in (1) gives $e = hh^{-1} \in H$. Taking $a = e$, $b = h$ in (1) gives $h^{-1} = eh^{-1} = ab^{-1} \in H$. Therefore, $h \in H \to h^{-1} \in H$.

Let $h_1, h_2 \in H$. Then $h_2^{-1} \in H$. Taking $a = h$, $b = h_2^{-1}$ in (1) gives $h_1, h_2 = ab^{-1} \in H$. Therefore, $h_1, h_2 \in H \implies h_1 h_2 \in H$. By the two-step subgroup test, $H \leq G$. □

Example 5: Let G be an abelian group. Prove that $H = \{x \in G | x^2 = e\}$ is a subgroup of G.

Proof. Let $a, b \in H$. Then $a^2 = b^2 = e$. Since G is abelian:

$$(ab^{-1})^2 = a^2 b^{-2} = a^2 (b^2)^{-1} = ee^{-1} = e \tag{68}$$

Therefore, $ab^{-1} \in H$ by the one-step subgroup test, $H \leq G$. □

Example 6: Prove that matrices in the form of $\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$ where $x, y, z \in \mathbb{R}$ is a subgroup of $\mathrm{SL}_3(\mathbb{R})$ using either subgroup test.

Proof. Using the one-step subgroup test. Let $g_1 = \begin{pmatrix} 1 & x_1 & y_1 \\ 0 & 1 & z_1 \\ 0 & 0 & 1 \end{pmatrix}$ and $g_2 = \begin{pmatrix} 1 & x_2 & y_2 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix}$. The inverse of g_2 is:

$$g_2^{-1} = \begin{pmatrix} 1 & -x & xz - y \\ 0 & 1 & -z \\ 0 & 0 & 1 \end{pmatrix} \tag{69}$$

and carrying out the computation:

$$g_1 g_2^{-1} = I \tag{70}$$

Since I is in the given group, we are done. □

4 Lecture Four

- We begin with the **Finite Subgroup Test**

> **Theorem**: Let G be a group and let H be a finite nonempty subset of G. If H is closed under the group operation of G, then $H \leq G$.

Proof. By the 2-step subgroup test, it suffices to prove that H is closed under taking inverses. Let $a \in H$:

- If $a = e$, then $a^{-1} = e \in H$.
- If $a \neq e$, consider the set:

$$\{a^n | n \geq 1\} = \{a, a^2, a^3, \dots\} \tag{71}$$

Since H is closed under the group operation and $a \in H$, we have $\{a^n | n \geq 1\} \subseteq H$ by a short induction argument. Since H is finite, so is $\{a^n | m \geq 1\}$. Therefore, $\exists m, n \geq 1, m \neq n$ such that:

$$a^m = a^n \tag{72}$$

WLOG, we may assume that $m > n$, so $m - n > 0$. We have:

$$a^{m-n} = e \tag{73}$$

Since $a \neq e$, $m - n \neq 1$. Therefore, $m - n \geq 2$, so $m - n - 1 \geq 1$. Thus:

$$a^{m-n-1} \in \{a^k | k \geq 1\} \subseteq H \tag{74}$$

and

$$a^{m-n-1}a = a^{m-n} = e \tag{75}$$

so:

$$a^{m-n-1} = a^{-1} \tag{76}$$

□

- We will look at a special class of subgroups: **subgroups generated by one element.**

> **Definition**: Let G be a group and let $a \in G$. Define:
>
> $$\langle a \rangle = \{a^n | n \in \mathbb{Z}\} \tag{77}$$
>
> We call $\langle a \rangle$ the subgroup of G generated by a.

- We propose that $\langle a \rangle \leq G$.

Proof. Since $e = a^0 \in \langle a \rangle$, we have $\langle a \rangle \neq \varnothing$.

If $g, h \in \langle a \rangle$, then $g = a^m$ and $h = a^n$ for some $m, n \in \mathbb{Z}$ and:

$$gh^{-1} = a^m(a^n)^{-1} = a^m a^{-n} = a^{m-n} \in \langle a \rangle \tag{78}$$

□

Example 7: Let $G = (\mathbb{Z}/14\mathbb{Z})^\times = \{1, 3, 5, 9, 11, 13\}$. We have:

$$a = 3,\ a^2 = 9,\ a^3 = 27 = 13 = -1 = -1,\ a^4 = -3 = 11,\ a^5 = -9 = 5,\ a^6 = 15 = 1 \tag{79}$$

Similarly:

$$a^0 = 1,\ a^{-1} = 5,\ a^{-2} = 11,\ a^{-3} = 13,\ a^{-4} = 9,\ a^{-5} = 3,\ a^{-6} = 1 \tag{80}$$

Therefore:

$$\langle a \rangle = \{1, 3, 5, 9, 11, 13\} = (\mathbb{Z}/14\mathbb{Z})^x \tag{81}$$

Therefore, $(\mathbb{Z}/14\mathbb{Z})^\times$ is cyclic. **Remarks:** If $a^n = e$, then for all $k \in \mathbb{Z}$, we have:

$$a^{-k} = a^{n-k} \tag{82}$$

so we can easily figure out negative exponents.

Example 8: Let $G = \mathbb{Z}/12\mathbb{Z}$ and $a = 2$. We have:

$$-a = 10,\ 0a = 0,\ 2a = 4,\ 3a = 6,\ 4a = 8,\ 5a = 10,\ 6a = 12 = 0,\ 7a = 2 \tag{83}$$

so:

$$\langle a \rangle = \{0, 2, 4, 6, 8, 10\}. \tag{84}$$

Example 9: Let $G = \mathbb{R}$ and $a = 2\pi$. Here,

$$\langle a \rangle = \{n2\pi | n \in \mathbb{Z}\} = 2\pi\mathbb{Z} \tag{85}$$

Definition: Let G be a group and $a \in G$. If there exists $n \in \mathbb{Z}_{>0}$ such that $a^n = e$, then we say that a has **finite order** and the **order of a** is defined to be the smallest $n \in \mathbb{Z}_{>0}$ such that $a^n = e$.

If there does not exist $n \in \mathbb{Z}_{>0}$ such that $a^n = e$, then we say that a has infinite order.

The order of a is denoted by $o(a)$ or $|a|$. If a has infinite order, we write $o(a) = \infty$.

- Note that:
  - $o(a) = 1 \iff a = e$
  - If $o(a) = \infty$, then $a^n = e \iff n = 0$.
- Let G be a group and $a \in G$.
  1. If $o(a) = \infty$, then $\forall i, j \in \mathbb{Z}$ we have:

$$a^{i-j} = e \iff i - j = 0 \tag{86}$$
$$\iff i = j \tag{87}$$

  2. If $o(a) = n < \infty$, then $\forall_{i,j} \in \mathbb{Z}$ we have:

$$a^i = a^j \iff n | i - j \tag{88}$$
$$\iff i \equiv j \pmod{n} \tag{89}$$

     In particular, $a^i = e (= a^0) \iff n | i$.

Proof. Let $i, j \in \mathbb{Z}$. Note $a^i = a^j \implies a^{i-j} = e$.

1. Suppose $o(a) = \infty$. Then $a^{i-j} = e$ iff $i - j = 0 \iff i = j$.
2. Suppose $o(a) = n < \infty$. We must show that $a^{i-j} = e \iff n | i - j$.

   (Backwards): If $n | i - j$, then $\exists k \in \mathbb{Z}$ such that $i - j = kn$ so $a^{i-j} = a^{kn} = (a^n)^k = e^k = e$.

   (Forwards) Now suppose $a^{i-j} = e$. By the division algorithm, $\exists!$ q and $0 \leq r < n$ such that:

$$i - j = qn + r \tag{90}$$

   We have:

$$e = a^{i-j} = a^{qn+r} = a^{qn}a^r = (a^n)^q a^r = e^q a^r = a^r \tag{91}$$

   Since n is the smallest positive integer with $a^n = e$ and $0 \leq r < n$ and satisfies $a^r = e$, we must have $r = 0$.

   Therefore, $i - j = qn$ so $n | i - j$.

□

Corollary 1: Let G be a group and $a \in G$.
1. If $o(a) = \infty$, then $\ldots, a^{-2}, a^{-1}, e, a, a^2, \ldots$ are distinct (and $\langle a \rangle = \{a^n | n \in \mathbb{Z}\}$)
2. If $o(a) = n < \infty$, then $e, a, \ldots, a^{n-1}$ are distinct and $\langle a \rangle = \{e, a, \ldots, a^{n-1}\}$.

Corollary 2: Let G be a group and $a \in G$. Then $o(a) = |\langle a \rangle|$ where $|\langle a \rangle| = \infty$ when $\langle a \rangle$ is infinite.

Corollary 3: Let G be a group and $a, b \in G$. If $ab = ba$ and $o(a), o(b) < \infty$, then

$$o(ab) | o(a)o(b) \tag{92}$$

Proof. Suppose $ab = ba$ and $o(a), o(b) < \infty$. Since:

$$\begin{aligned}
(ab)^{o(a)o(b)} &= a^{o(a)o(b)} b^{o(a)o(b)} && (93)\\
&= (a^{o(a)})^{o(b)} (b^{o(b)})^{o(a)} && (94)\\
&= e^{o(b)} e^{o(a)} && (95)\\
&= e && (96)
\end{aligned}$$

Therefore, $o(ab) | o(a)o(b)$. □

- **Remarks about notation:**
  - $\mathbb{Z}/n\mathbb{Z}$ is sometimes denoted by $\mathbb{Z}_n$ or $\mathbb{Z}/(n)$.
  - $(\mathbb{Z}/n\mathbb{Z})^\times = \{[a] \in \mathbb{Z}/n\mathbb{Z} | [b] \in \mathbb{Z}/n\mathbb{Z} \text{ with } [a][b] = 1\} = \{[a] | \gcd(n, a) = 1\}$.

Theorem: Let G be a group and $a \in G$ with $o(a) = n < \infty$. For any $k \in \mathbb{Z}$, we have:

$$o(a^k) = \frac{o(a)}{\gcd(o(a), k)} = \frac{n}{\gcd(n, k)} \tag{97}$$

Proof. By definition, $o(a^k)$ is the smallest $m \in \mathbb{Z}_{>0}$ such that

$$\begin{aligned}
(a^k)^m = e &\iff a^{mk} = e && (98)\\
&\iff n | mk && (99)
\end{aligned}$$

Since mk is a multiple of k, we have $n | mk \iff mk$ is common multiple of n and k.

If there exists $m \in \mathbb{Z}_{>0}$ such that $mk = \text{lcm}(n, k)$, then $m = o(a^k)$. Recall that:

$$\frac{nk}{\gcd(n, k)} = \text{lcm}(n, k) \tag{100}$$

Since $\gcd(n, k) | n$, then $\dfrac{n}{\gcd(n, k)} \in \mathbb{Z}_{>0}$ with

$$\left(\frac{n}{\gcd(n, k)}\right) k = \text{lcm}(n, k) \tag{101}$$

Therefore:

$$o(a^k) = \frac{n}{\gcd(n, k)} \tag{102}$$

□

Corollary 4: In a finite group G, the order of every element divides the order of the group:

$$\forall x \in G, \quad o(x) \Big| |G| \tag{103}$$

Example 10: $\mathbb{Z} = \langle 1 \rangle$ is an infinite cyclic group. Meanwhile, $\mathbb{Z}/n\mathbb{Z} = \langle 1 \rangle$ is a finite cyclic group.

- Next we will study subgroups of cyclic groups. Choose a generator $a \in G$ and $G = \langle a \rangle$.
- For each $k \in \mathbb{Z}$, $a^k \in \langle a \rangle$. Therefore $\langle a^k \rangle \subseteq \langle a \rangle$.

Proposition 4: Let G be a group and let $a \in G$. If $H \leq G$ and $a \in H$, then $\langle a \rangle \subseteq H$.

- One natural question is: *Do we get every subgroup in this way?* If $k, \ell \in \mathbb{Z}$, when is $\langle a^k \rangle = \langle a^\ell \rangle$?

Theorem: Classification of subgroups of cyclic groups: Let $G = \langle a \rangle$ be a cyclic group:
1. If $|G| = \infty$ ($\iff o(a) = \infty$) then every subgroup of G is of the from $\langle a^m \rangle$ for a unique $m \in \mathbb{Z}_{\geq 0}$.
   Remarks: $\langle a^m \rangle = \langle a^{-m} \rangle$.
2. If $|G| = n < \infty$ ($\iff o(a) = n < \infty$) then every subgroup of G is of the form $\langle a^m \rangle$ for a unique $m \in \mathbb{Z}_{>0}$ with $m|n$.
   Said differently, the order of every subgroup of G divides n and for each $d \in \mathbb{Z}_{>0}$ with $d|n$ there is a unique subgroup of G of order d, namely $\langle a^{n/d} \rangle$.

Proof. Let $H \leq G = \langle a \rangle$ with $H \neq \{e\}$. Then $\exists k \in \mathbb{Z} \setminus \{0\}$ such that a^k, $a^{-k} \in H$. Therefore, $a^{|k|} \in H$ so $\exists k' \in \mathbb{Z}_{>0}$ such that $a^{k'} \in H$. Let m be the smallest positive integer such that $a^m \in H$ (which exists by the well-ordering principle).

We will prove that $H = \langle a^m \rangle$. Since $a^m \in H$, we have $\langle a^m \rangle \subseteq H$. To prove $H \subseteq \langle a^m \rangle$, it suffices to prove:

- If $a^k \in H$ where $k \in \mathbb{Z}$, then $m|k$.

  Let $k \in \mathbb{Z}$ and assume $a^k \in H$. By the division algorithm, $\exists! q, r \in \mathbb{Z}$ such that $0 \leq r < m$ and:

  $$k = qm + r \tag{104}$$

  Then:

  $$a^k = a^{qm+r} = (a^m)^q a^r \implies a^r = (a^m)^{-q} a^k \tag{105}$$

  Since $(a^m)^{-q}, a^k \in H$.

  Since $\langle a^m \rangle \subseteq H$, $(a^m)^{-q} \in H$. We assumed $a^k \in H$. Therefore, $a^r \in H$.

  Since m is the smallest positive integer with $a^m \in H$ and $a^r \in H$ and $0 \leq r < m$, we have $r = 0$. Therefore $k = qm$ so $m|k$.

If $|G| = n < \infty$, then $o(a) = n$, so $a^n = e \in H$. Therefore by the above point, $m|n$. Now we look at the two cases:

1. Suppose $|G| = \infty$. We prove that every nontrivial subgroup of G is of the form $\langle a^m \rangle$ for some $m \in \mathbb{Z}_{>0}$. Since $\{e\} = \langle a^0 \rangle$, we have that every subgroup of G is of the form $\langle a^m \rangle$ for some $m \in \mathbb{Z}_{\geq 0}$.

   To prove that m is unique, suppose $H \leq G$ and $H = \langle a^m \rangle = \langle a^{m'} \rangle$ for some $m, m' \in \mathbb{Z}_{\geq 0}$.

   Since $a^m \in \langle a^m \rangle = \langle a^{m'} \rangle$, $a^m \in \langle a^{m'} \rangle$, so $a^m = a^{m'k}$ for some $k \in \mathbb{Z}$. Since $o(a) = \infty$, we must have $m = m'k$ so $m'|m$. Similarly, $m|m'$. Thus, $m = m'$.

2. Suppose $|G| = n < \infty$. Then $o(a) = n$, so $a^n = e$ and therefore $\{e\} = \langle a^n \rangle$. We proved above that every nontrivial subgroup of G is of the form $\langle a^m \rangle$ for some $m \in \mathbb{Z}_{>0}$ with $m|n$.

3. Therefore, every subgroup of G is of the form $\langle a^m \rangle$ for some $m \in \mathbb{Z}_{>0}$ with $m|n$.

   To prove that m is unique, suppose $H \leq G$ with $H = \langle a^m \rangle = \langle a^{m'} \rangle$ where $m, m'' \in \mathbb{Z}_{>0}$ with $m, m'|n$. Then:

   $$o(a^m) = |\langle a^m \rangle| = |\langle a^{m'} \rangle| = o(a \tag{106}$$

   Since $o(a^k) = \dfrac{n}{\gcd(n,k)}$ for all $k \in \mathbb{Z}$, we got:

   $$\frac{n}{\gcd(n,m)} = \frac{n}{\gcd(n,m')} \tag{107}$$

   which implies $\gcd(n,m) = \gcd(n,m')$. Since $m, m'|n$ we have $\gcd(n,m) = m$ and $\gcd(n,m') = m'$ so $m = m'$.

□

Corollary 5: Criterion for $\langle a^i \rangle = \langle a^j \rangle$ and $o(a^i) = o(a^j)$.

Let $G = \langle a \rangle$ be a cyclic group and let $i, j \in \mathbb{Z}$.
1. If $|G| = \infty$, then $\langle a^i \rangle = \langle a^j \rangle$ if and only if $j = \pm k$.
2. If $|G| = n < \infty$, then the following are equivalent:
   - $\langle a^i \rangle = \langle a^k \rangle$
   - $o(a^i) = o(a^j)$
   - $\gcd(n, i) = \gcd(n, j)$

Corollary 6: (The generators of a cyclic group) Let $G = \langle a \rangle$ be a cyclic group. The generators of G are:

$$\begin{cases} \{a, a^{-1}\} & |G| = \infty \\ \{a^k | \gcd(n, k) = 1\} & |G| = n < \infty \end{cases} \tag{108}$$

This corollary follows from the first corollary.

- If $G = \langle a \rangle$ is cyclic of order $n < \infty$, it follows that there are exactly $\phi(n)$ generators where $\phi(n)$ is Euler's Toitent function.

5 Permutation Groups

- Let X be a set. A is a symmetry of X as a set is just a bijection $\sigma : X \to X$ because there is no structure that σ should preserve.
- We call bijections $\sigma : X \to X$ permutations of X.

Definition: The **symmetric group** on X is the group of all permutations of X with group operation given by composition. It is denoted by S_x.

Example 11: Let $X = \{a, b, c\}$, where a, b, c distinct. The map $\sigma : X \to X$ defined by $\sigma(a) = b$, $\sigma(b) = a$, $\sigma(c) = c$ is a permutation of X, so $\sigma \in S_x$.
Similarly, the map $\tau : X \to X$ defined by $\tau(a) = c$, $\tau(b) = a$, $\tau(c) = b$ is a permutation of X, so $\tau \in S_X$ also.

Proposition 5: For every finite set X, $|S_x| = |X|!$.

- To prove this proposition rigorously, we can prove this via induction on $n \in \mathbb{Z}_{\geq 0}$ with $|X| = |Y| = n$, the set $\{\sigma : X \to Y | \sigma \text{ is a bijection}\}$ has cardinality $n!$. Then apply that in the case $X = Y$.

Definition: A subgroup of S_x is called a permutation group on X.

- We are most interest in the case when $0 < |X| < \infty$.
- By choosing a linear ordering $x_1, \ldots, x_n$ of the elements of X, then we can regard X as the set $\{1, \ldots, n\}$.
- We may as well, and we will, assume that $X = \{1, \ldots, n\}$.
- We denote $S_{\{1,\ldots,n\}}$ by S_n and we call it the symmetric group on n letters.
- The identity of S_n is something denoted by id, 1, e, or ϵ.
- If $\sigma \in S_n$, we write:

$$\sigma = \begin{pmatrix} 1 & 2 & \ldots & n \\ \sigma(1) & \sigma(2) & \ldots & \sigma(n) \end{pmatrix} \tag{109}$$

Example 12: Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$. Then:

$$\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} and \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} \tag{110}$$

- For $n \geq 3$, S_n is non-abelian.

  Proof. Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 1 & 3 & \cdots & n \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 3 & 2 & \cdots & n \end{pmatrix}$. Then:

$$\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 3 & 1 \cdots & n & \end{pmatrix} \tag{111}$$

  but

$$\tau\sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 3 & 1 & 2 & \cdots & n \end{pmatrix} \tag{112}$$

  so $\sigma\tau \neq \tau\sigma$. □

- We will now introduce the notion of a cycle

Definition: Let $r \in \mathbb{Z}$, $r \geq 2$. An **r-cycle** in S_n is a permutation $\gamma \in S_n$ with the following property: There exist r distinct elements $c_1, \ldots, c_r \in \{1, \ldots, n\}$ such that:
(a) $\gamma(c_i) = c_i + 1$ for $1 \leq i \leq r-1$, and $\gamma(c_r) = c_1$.
(b) $\gamma(k) = k$ for all $k \in \{1, \ldots, n\} \setminus \{c_1, \ldots, c_r\}$.
In this case, we write the r-cycle γ as:

$$\gamma = \begin{pmatrix} c_1 & c_2 & \ldots & c_r \end{pmatrix} \tag{113}$$

That is, γ is an r-cycle if it moves precisely r elements of $\{1, \ldots, n\}$ in a cyclic pattern (and leaves every other element fixed).

Example 13: Let $\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 5 & 3 & 2 & 6 & 9 & 7 & 4 & 8 \end{pmatrix} \in S_9$. We claim that γ is a 6-cycle.
Note that γ fixes $1, 3, 7$. We then need to show that the remaining elements are mapped by γ in a cyclic pattern:

$$2 \mapsto 5 \mapsto 6 \mapsto 9 \mapsto 8 \mapsto 4 \mapsto 2 \tag{114}$$

Therefore, $\gamma = \begin{pmatrix} 2 & 5 & 6 & 9 & 8 & 4 \end{pmatrix}$. Note that this is also equivalent to:

$$\gamma = \begin{pmatrix} 5 & 6 & 9 & 8 & 4 & 2 \end{pmatrix}. \tag{115}$$

Proposition 6: Let $r \geq 2$ and let $\gamma = \begin{pmatrix} c_1 & c_2 & \cdots & c_r \end{pmatrix}$ be an r-cycle in S_n.
1. For all $2 \leq i \leq r$ we have:

$$\gamma = \begin{pmatrix} c_i & c_{i+1} & \ldots & c_r & c_1 & c_2 & \ldots & c_{i-1} \end{pmatrix} \tag{116}$$

2. The inverse γ^{-1} is given by:

$$\gamma^{-1} = \begin{pmatrix} c_r & c_{r-1} & \ldots c_1 \end{pmatrix} \tag{117}$$

Proof. We prove both parts of the above proposition.

1. Exercise left to reader.
2. Let $\delta = \begin{pmatrix} c_r & c_{r-1} & \ldots c_1 \end{pmatrix}$. To show that $\delta = \gamma^{-1}$, it suffices to show that $\delta\gamma = \text{id}$. (since S_n is a group). To do so, we must prove that $\forall i \in \{1, \ldots, n\}$, we have $\delta\gamma(i) = i$.

By definition of cycles, we have:

$$\gamma(k) = \begin{cases} k & k \notin \{c_1, \ldots, c_r\} \\ c_{i+1} & k = c_i, 1 \leq i \leq r-1 \\ c_1 & k = c_r \end{cases} \tag{118}$$

and:

$$\delta(k) = \begin{cases} k & k \notin \{c_1, \ldots, c_r\} \\ c_{i-1} & k = c_i, 2 \leq i \leq r \\ c_r & k = c_1 \end{cases} \tag{119}$$

We can then check for $k \notin \{c_1, \ldots, c_r\}$, we have:

$$\delta\gamma(k) = \gamma(k) = k \tag{120}$$

For $k = c_i$, $1 \leq i \leq r-1$, we have:

$$\delta\gamma(k) = \delta\gamma(c_i) = \delta(c_{i+1}) = c_i = k \tag{121}$$

For $k = c_r$, we have:

$$\delta\gamma(k) = \delta(c_1) = c_r = k. \tag{122}$$

□

- Let us investigate the product of two cycles.

Example 14: Let $\gamma = \begin{pmatrix} 1 & 3 & 2 & 4 \end{pmatrix}$ and $\delta = \begin{pmatrix} 2 & 6 & 3 \end{pmatrix}$ where $\gamma, \delta \in S_8$. Then:

$$\delta\gamma = \begin{pmatrix} 1 & 3 & 2 & 4 \end{pmatrix} \begin{bmatrix} 2 & 6 & 3 \end{bmatrix} \tag{123}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 4 & 1 & 5 & 2 & 7 & 8 \end{pmatrix} \tag{124}$$

Notice that:

$$1 \mapsto 3 \mapsto 4 \mapsto 1 \tag{125}$$

However, the other elements are not fixed since $6 \mapsto 2$. Therefore, $\gamma\delta$ is not a cycle.

6 Lecture Six

- We continue our investigation of permutations.

Definition: Let $\sigma \in S_n$. Define:

$$\mathrm{Fix}(\sigma) = \{k \in \{1, \ldots, n\} | \sigma(k) = k\} \tag{126}$$

Definition: Let $\sigma, \tau \in S_n$. We say that σ and τ are disjoint if for all $k \in \{1, \ldots, n\}$,

$$\sigma(k) \neq k \implies \tau(k) = k \tag{127}$$

which means that $k \in \mathrm{Fix}(\tau)$. Similarly:

$$\tau(k) \neq k \implies \sigma(k) = k \tag{128}$$

which means that $k \in \mathrm{Fix}(\sigma)$.

- Note that two cycles $\gamma = \begin{pmatrix} c_1 & \cdots & c_r \end{pmatrix}$ and $\delta = \begin{pmatrix} d_1 & \cdots & d_s \end{pmatrix}$ are disjoint if and only if:

$$\{c_1, \ldots, c_r\} \cap \{d_1, \ldots, d_s\} = \emptyset \tag{129}$$

- This is because

$$\mathrm{Fix}\begin{pmatrix} c_1 & \cdots & c_r \end{pmatrix} = \{1, \ldots, n\} \setminus \{c_1, \ldots, c_r\} \tag{130}$$

and:

$$\mathrm{Fix}\begin{pmatrix} d_1 & \cdots & d_s \end{pmatrix} = \{1, \ldots, n\} \setminus \{d_1, \ldots, d_s\}. \tag{131}$$

> **Lemma 7**: Let $\sigma \in S_n$. Then:
> 1. If $k \in \mathrm{Fix}(\sigma)$, then $k \in \mathrm{Fix}(\sigma^m)$ for all $m \in \mathbb{Z}$.
> 2. If $k \notin \mathrm{Fix}(\sigma)$, then $\sigma^m(k) \notin \mathrm{Fix}(\sigma)$ for all $m \in \mathbb{Z}$.

Proof. We will prove both of the above:

1. Let $k \in \mathrm{Fix}(\sigma)$, i.e. $\sigma(k) = k$. Then $k = \sigma^{-1}(\sigma(k)) = \sigma^{-1}(k)$. Therefore, we have $k \in \mathrm{Fix}(\sigma^{-1})$. It follows by a simple induction argument that $k \in \mathrm{Fix}(\sigma^m)$ for all $m \in \mathbb{Z}_{\geq 0}$ and $k \in \mathrm{Fix}(\sigma^m)$ for all $m \in \mathbb{Z}_{\leq 0}$.

   The induction argument involves the fact that $\sigma(\sigma(k)) = \sigma(k) = k$.

2. Let $k \notin \mathrm{Fix}(\sigma)$. It suffices to prove that $\sigma(k) \notin \mathrm{Fix}\,\sigma$. It suffices to prove that:

$$\sigma(k), \sigma^{-1}(k) \notin \mathrm{Fix}(\sigma) \tag{132}$$

   To show why, suppose that $\sigma(k), \sigma^{-1}(k) \notin \mathrm{Fix}(\sigma)$. Then, the idea is that we cannot have $\sigma^2(k) \in \mathrm{Fix}(\sigma)$ since $\sigma(\sigma(k)) = \sigma(k) \notin \mathrm{Fix}(\sigma)$.

   Alternatively, we can have a direct proof. Let $k \notin \mathrm{Fix}(\sigma)$. Let $m \in \mathbb{Z}$. Suppose for the sake of contradiction that $\sigma^m(k) \in \mathrm{Fix}(\sigma)$. Then:

$$\sigma(\sigma^m(k)) = \sigma^m(k) \tag{133}$$

   Therefore, applying σ^{-m} on both sides gives $\sigma(k) = k$. This contradicts $k \notin \mathrm{Fix}(\sigma)$. Therefore, $\sigma^m(k) \notin \mathrm{Fix}(\sigma)$.

□

> **Theorem**: (Disjoint permutations commute) Let $\sigma, \tau \in S_n$ be disjoint. Then $\sigma\tau = \tau\sigma$.

Proof. Let $k \in \{1, \ldots, n\}$, and let $\sigma, \tau \in S_n$ be disjoint.

For the first case, suppose $k \in \mathrm{Fix}(\sigma) \cap \mathrm{Fix}(\tau)$. Then $\sigma(k) = k = \tau(k)$. Therefore:

$$\sigma\tau(k) = \sigma(k) = k \tag{134}$$

and:

$$\tau\sigma(k) = \tau(k) = k \tag{135}$$

so $\sigma\tau(k) = \tau\sigma(k)$.

For the second case, suppose $k \notin \mathrm{Fix}(\sigma)$. Since σ and τ are disjoint, we have $k \in \mathrm{Fix}(\tau)$. Therefore $\tau(k) = k$ and $\sigma\tau(k) = \sigma(k)$. Since $k \notin \mathrm{Fix}(\sigma)$, we have $\sigma(k) \notin \mathrm{Fix}(\sigma)$ by part (2) of the above lemma. Since σ and τ are disjoint and $\sigma(k) \notin \mathrm{Fix}(\sigma)$, we have $\sigma \in \mathrm{Fix}(\tau)$.

Therefore, $\tau\sigma(k) = \sigma(k)$. As a result:

$$\tau\sigma(k) = \sigma\tau(k) \tag{136}$$

For the last case, we consider $k \notin \mathrm{Fix}(\tau)$. It can be handled in the same way as the second case. □

- We now introduce the notion of an orbit.

> **Definition**: Let $\sigma \in S_n$. For each $k \in \{1, \ldots, n\}$, the set:
>
> $$O_\sigma(k) = \{\sigma^m(k) | m \in \mathbb{Z}\} \tag{137}$$
> $$= \{\ldots, \sigma^{-2}, \sigma^{-1}, k, \sigma(k), \ldots \tag{138}$$
>
> is called the **orbit** of k under the set σ.

- Note that $|O_\sigma(k)| = 1$ if and only if $O_\sigma(k) = \{k\}$ if and only if $k \in \mathrm{Fix}(\sigma)$.

Proposition 7: Let $\sigma \in S_n$. For all $k \in \{1, \dots, n\}$, there exists $\ell \in \mathbb{Z}_{>0}$ such that $\sigma^\ell(k) = k$.

If ℓ is the smallest positive integer such that $\sigma^\ell = k$, then $k, \sigma(k), \sigma^2(k), \dots, \sigma^{\ell-1}(k)$ are distinct and:

$$O_\sigma(k) = \{k, \sigma(k), \dots, \sigma^{\ell-1}(k)\}. \tag{139}$$

Warning: The smallest $\ell \in \mathbb{Z}_{>0}$ such that $\sigma^\ell(k) = k$ is not necessarily the order of σ, which is the smallest $m \in \mathbb{Z}_{>0}$ such that:

$$\sigma^m(j) = j \tag{140}$$

for all $j \in \{1, \dots, n\}$.

Proof. The subset $\{\sigma^m(k) | m \in \mathbb{Z}\}$ of $\{1, \dots, n\}$ is finite.

Therefore, there exist $m_1, m_2 \in \mathbb{Z}$ with $m_1 < m_2$ such that $\sigma^{m_1}(k) = \sigma^{m_2}(k)$. Then $\sigma^{m_2 - m_1}(k) = k$ and $m_2 - m_1 \in \mathbb{Z}_{>0}$.

Let $\ell \in \mathbb{Z}_{>0}$ be the smallest positive integer such that $\sigma^\ell(k) = k$. This exists by the well ordering principle.

If $m_1, m_2 \in \{0, 1, \dots, \ell - 1\}$, $m_1 < m_2$, and $\sigma^{m_1}(k) = \sigma^{m_2}(k)$, then $0 < m_2 - m_1 < \ell$ and $\sigma^{m_2 - m_1}(k) = k$, contradicting the definition of ℓ.

Thus, $k, \sigma(k), \dots, \sigma^{\ell-1}(k)$ are distinct. All we have to do now is to prove all the element sin the orbit of k is one of these.

Let $m \in \mathbb{Z}$. While $m = q\ell + r$ for unique $q, \ell \in \mathbb{Z}$ with $0 \leq r < \ell$ by the division algorithm. Now,

$$\begin{aligned} \sigma^m(k) &= \sigma^{q\ell + r}(k) && (141)\\ &= (\sigma^\ell)^q \sigma^r(k) && (142)\\ &= \sigma^r(\sigma^\ell)^q(k) && (143)\\ &= \sigma^r(k) && (144) \end{aligned}$$

We are able to go through these steps by noting $\sigma^\ell(k) = k \implies (\sigma^\ell)^q(k) = k$. Therefore $\sigma^m(k) = \sigma^r(k) = \{k, \sigma(k), \dots, \sigma^{\ell-1}(k)\}$ and:

$$O_\sigma(k) = \{k, \sigma(k), \dots, \sigma^{\ell-1}(k)\}. \tag{145}$$

□

Proposition 8: Let $\sigma \in S_n$.

1. For all $k \in \{1, \dots, n\}$, then $j \in O_\sigma(k)$, if and only if $O_\sigma(j) = O_\sigma(k)$.
2. Distinct orbits of σ are disjoint. If $O_\sigma(j) \neq O_\sigma(k)$, then:

$$O_\sigma(j) \cap O_\sigma(k) = \emptyset. \tag{146}$$

Consequently, the orbits of σ partition $\{1, \dots, n\}$.

Proof. Again, we prove both parts.

1. Let $k \in \{1, \dots, n\}$. Suppose $j \in O_\sigma(k)$. Then there exists $m \in \mathbb{Z}$ such that $\sigma^m(k) = j$. Therefore, for all $r \in \mathbb{Z}$, $\sigma^r(j) = \sigma^{m+r}(k) \in O_\sigma(k)$. Thus, we have proved that:

$$j \in O_\sigma(k) \implies O_{\sigma(k)} \subseteq O_\sigma(j). \tag{147}$$

Now since $j = \sigma^m(k)$, we have $k = \sigma^{-m}(j) \in O_\sigma(j)$. Therefore, $O_\sigma(k) \subseteq O_\sigma(j)$ by the same argument. Thus, $O_\sigma(j) = O_\sigma(k)$.

Note that we also have to prove the reverse direction. We know that $j \in O_\sigma(k)$ since $j \in O_\sigma(j)$.

2. We will prove the contrapositive. Suppose $O_\sigma(j) \cap O_\sigma(k) \neq \emptyset$. Then, there exist $m_1, m_2 \in \mathbb{Z}$ such that:

$$\sigma^{m_1}(j) = \sigma^{m_2}(k). \tag{148}$$

Therefore, $j = \sigma^{m_2 - m_1}(k) \in O_\sigma(k)$. By part (1), we have $O_\sigma(j) = O_\sigma(k)$.

□

- We introduce the cycle *attached* to an orbit of $\sigma \in S_n$.
- Let $\sigma \in S_n$ and let O be an orbit of σ. Let $\ell = |O|$. Assume $\ell \geq 2$.
- Choose a $k \in O$. Then $O = O_\sigma(k)$ (by part (1) in the proposition.) By an earlier proposition:

$$O = O_\sigma(k) = \{k, \sigma(k), \ldots, \sigma^{\ell-1}(k)\}. \tag{149}$$

- We can define a cycle $\gamma_O = \begin{pmatrix} k & \sigma(k) & \cdots & \sigma^{\ell-1}(k) \end{pmatrix}$ which is an ℓ-cycle in S_n.
- The ℓ-cycle γ_O does not depend on the choice of $k \in O$. Proof is left as an exercise.
- **Note:** If O, O' are distinct orbits of σ, then they are disjoint so the cycles γ_O and $\gamma_{O'}$ are disjoint as well. Therefore, these two cycles commute.

> **Theorem**: (Cycle Decomposition Theorem) Every non-identity permutation can be written as a product of mutually disjoint cycles, i.e. there exist cycles $\gamma_1, \ldots, \gamma_r$ such that γ_i and γ_j are disjoint if $i \neq j$ and $\sigma = \gamma_1 \cdots \gamma_r$.
> Moreover, if $\gamma_1, \ldots, \gamma_r$ are as above,
> then $\{\gamma_1, \ldots, \gamma_r\} = \{\gamma_0 : O \text{ is an orbit of } \sigma \text{ and } |O| \geq 2$. In particular, the set $\{\gamma_1, \ldots, \gamma_r\}$ is unique.

- **Remarks:** We can extend the theorem to the case where $\sigma = \text{id}$ if we define an empty product (or a product of 0 elements of S_n) to be id.

Proof. Let $\sigma \in S_n$ and $\sigma \neq \text{id}$. Let $O_1, \ldots, O_s$ be the distinct orbits of σ of size at least 2. The cycles $\gamma_O, \ldots, \gamma_{O_s}$ are mutually disjoint because the orbits $O_1, \ldots, O_s$ are mutually disjoint.

Define $\tau = \sigma_{O_1} \cdots \sigma_{O_S}$. We will prove that $\sigma = \tau$. Let $O_{s+1}, \ldots, O_t$ be the distinct orbits of σ of size 1. Then:

$$\{1, \ldots, n\} = \left(\dot{\bigcup}_{i=1}^{s} O_i\right) \dot{\bigcup} \left(\dot{\bigcup}_{j=s+1}^{t} O_j\right). \tag{150}$$

Let $k \in \{1, \ldots, n\}$. We must show that $\sigma(k) = \tau(k)$. If $k \notin O_1 \dot{\cup} \cdots \dot{\cup} O_s$. Then $k \in O_j$ for some $j \in \{s+1, \ldots, t\}$. Since O_j is an orbit of size 1, we must have $\sigma(k) = k$.

For each $i = \{1, \ldots, s\}$, $k \notin O_i$, so $\gamma_{O_i}(k) = k$. Therefore:

$$\tau(k) = \gamma_{O_1} \ldots \gamma_{O_s}(k) = k = \sigma(k) \tag{151}$$

If $k \in O_i$ for some $i \in \{1, \ldots, s\}$, then by the definition of γ_{O_i}, we have:

$$\gamma_{O_i}(k) = \sigma(k) \tag{152}$$

for all $j \neq i$. Since $\tau = \gamma_{O_i} \ldots \gamma_{O_s} = \gamma_{O_i} \prod_{j \neq i} \gamma o_j$.

We have:

$$\begin{aligned} \tau(k) &= \gamma_{O_i} \prod_{j \neq i} \gamma_{O_j}(k) && (153)\\ &= \gamma_{O_i}(k) && (154)\\ &= \sigma(k). && (155) \end{aligned}$$

Therefore, $\sigma(k) = \tau(k)$ for all $k \in \{1, \ldots, n\}$, i.e. $\sigma = \tau$.

Now suppose that $\sigma = \gamma_1 \cdots \gamma_r$ where $\gamma_1, \ldots, \gamma_r$ are mutually disjoint cycles. We will prove that:

$$\{\gamma_1, \ldots, \gamma_r\} = \{\gamma_O : O \text{ is an orbit of } \sigma \text{ and } |O| \geq 2\}. \tag{156}$$

Proof of $\subseteq$ Let $i \in \{1, \dots, r\}$ and write $\gamma_i = \begin{pmatrix} c_1 & \dots & c_\ell \end{pmatrix}$. Since $\gamma_1, \dots, \gamma_r$ are mutually disjoint, if $j \neq i$, then $\gamma_j(c_k) = c_k$ for all $k \in \{1, \dots, \ell\}$. Therefore,

$$\sigma(c_k) = \gamma_i \prod_{j \neq i} \gamma_j(c_k) \tag{157}$$

$$= \gamma_i c_k \tag{158}$$

$$= \begin{cases} c_{k+1} & k < \ell \\ c_1 & k = \ell \end{cases} \tag{159}$$

for all $k \in \{1, \dots, \ell\}$. Consequently, $\sigma(c_1) = c_2, \sigma^2(c_1) = c_3, \dots, \sigma^{\ell-1}(c_1) = c_\ell, \sigma^\ell(c_1) = c_1$.

Therefore, $O_\sigma(c_1) = \{c_1, c_2, \dots, c_\ell\}$ and $\gamma_i = \gamma_{O_\sigma(c_1)}$.

Proof of $\supseteq$: Let O be an orbit of σ with $|O| \geq 2$. Let $k \in O$. Then as we have seen before, $O = O_\sigma(k)$. Since $|O| \geq 2$, we have $\sigma(k) \neq k$. Since $\sigma = \gamma_1 \cdots \gamma_r$ and $\sigma(k) \neq k$, there exists $i \in \{1, \dots, r\}$ such that:

$$\gamma_i(k) \neq k. \tag{160}$$

Let us write $\gamma_i = \begin{pmatrix} c_1 & \dots & c_\ell \end{pmatrix}$. Since $\gamma_i(k) \neq k$, we have $k = c_j$ for some $j \in \{1, \dots, \ell\}$. By relabelling $c_1, \dots, c_\ell$, we may assume that $k = c_1$. We showed above that $\gamma_i = \gamma_{O_\sigma(c_1)}$.

Since $c_1 = k$, $O_\sigma)c_1) = O_\sigma(k) = O$. Therefore, $\gamma_i = \gamma_O$. □

> **Lemma** 8: If $\sigma, \tau \in S_n$ are disjoint, then so are σ^{m_1}, τ^{m_2} for all $m_1, m_2 \in \mathbb{Z}$.

Proof. Suppose $\sigma, \tau \in S_n$ are disjoint Let $m_1, m_2 \in \mathbb{Z}$.

If $k \in \{1, \dots, n\}$ and $\sigma^{m_1}(k) \neq k$, then $\sigma(k) \neq k$. Therefore, $\tau(k) = k$ (since σ and τ are disjoint), and therefore $\tau^{m_2}(k) = k$.

Similarly, if $k \in 1, \dots, n$ and $\tau^{m_2}(k) \neq k$, then $\sigma^{m_1}(k) = k$. □

> **Theorem**: (Order of a Permutation) Let $\sigma \in S_n$. Let $\sigma = \gamma_1 \cdots \gamma_r$ be the cycle decomposition of σ. (When $\sigma = \mathrm{id}$, $r = 0$ and σ is an empty product of mutually disjoint cycles.)
>
> Then $o(\sigma) = \mathrm{lcm}(o(\gamma_1), \dots, o(\gamma_r))$. (If $\sigma = \mathrm{id}$, then $o(\sigma) = 1 = \mathrm{lcm}(o(\emptyset))$.)

Proof. Since $\gamma_1, \dots, \gamma_r$ commute, for all $m \in \mathbb{Z}$, we have:

$$\sigma^m = \gamma_1^m \cdots \gamma_r^m. \tag{161}$$

Let $m_i = o(\gamma_i)$ for each i and let $M = \mathrm{lcm}(m_1, \dots,)$. Since $m_i | M$ for each i, we have $\sigma^M = \sigma_1^M \cdots \sigma_r^M = \mathrm{id} \cdots \mathrm{id} = \mathrm{id}$.

Let $m \in \mathbb{Z}$ and suppose $\sigma^m = \mathrm{id}$. Then $\gamma_1^m \cdots \gamma_r^m = \mathrm{id}$. Since $\gamma_1, \dots, \gamma_r$ are mutually disjoint, so are $\gamma_1^m, \dots, \gamma_r^m$ by the above lemma.

If $\gamma_i^m(k) \neq k$, then $\gamma_j^m(k) = k$ for all $j \neq i$, so:

$$\gamma_1^m \cdots \gamma_r^m(k) = \gamma_i^m(k) \neq k, \tag{162}$$

contradicting the fact that:

$$\gamma_1^m \cdots \gamma_r^m = \mathrm{id}. \tag{163}$$

Therefore, $\gamma_i^m(k) = k$ for all i, k. So, $\gamma_i^m = \mathrm{id}$ for all i. Therefore $m_i = o(\gamma_i) | m$ for all i, Thus, $M = \mathrm{lcm}(m_1, \dots, m_r) | m$.

We proved that $\sigma^M = 1$ and $\sigma^m = 1 \implies M | m$. Since $M \in \mathbb{Z}_{>0}$, it follows that $M = o(\sigma)$. □

7 Transpositions

- We start with the definition:

> **Definition**: A transposition is just a 2-cycle

Lemma 9: Let $(c_1 \quad \cdots \quad c_r) \in S_n$ be an $r-$cycle. Then:

$$(c_1 \quad \cdots \quad c_r) = (c_1\, c_2)(c_2\, c_3)\cdots(c_{r-1}\, c_r), \tag{164}$$

a product of $r-1$ transpositions.

Proof. We can prove by induction that for all $i \in \{1,\ldots,r\}$, we have:

$$(c_1\, c_2)(c_2\, c_3)\cdots(c_{i-1}\, c_i)c_i = c_1. \tag{165}$$

Then, let $i \in \{1,\ldots,r-1\}$, and it remains to be shown that:

$$(c_1\, c_2)(c_2\, c_3)\cdots(c_{r-1}\, c_r)c_i = c_{i+1}. \tag{166}$$

For $j \in \{i+1,\ldots,r-1\}$, we have:

$$(c_j\, c_{j+1})c_i = c_i \tag{167}$$

Therefore:

$$\begin{aligned}
&(c_1\, c_2)\cdots(c_{r-1}\, c_r)c_i && (168)\\
&= (c_1\, c_2)\cdots(c_{i-1}c_i)(c_i\, c_{i+1})c_i && (169)\\
&= (c_1\, c_2)\cdots(c_{i-1}\, c_i)c_{i+1} && (170)
\end{aligned}$$

For $j \in \{1,\ldots,i-1\}$ we have:

$$(c_j\, c_{j+1})c_{i+1} = c_{i+1} \tag{171}$$

Therefore:

$$(c_1\, c_2)\cdots(c_{r-1}\, c_r)c_i = c_{i+1} \tag{172}$$

□

Corollary 7: If $\sigma \in S_n$, then σ is a (possibly empty) product of transpositions.

Definition: Let $\sigma \in S_n$. An **inversion** of σ is an ordered pair:

$$(i,j) \in \{1,\ldots,n\}^2 \tag{173}$$

s.t. $i<j$ and $\sigma(j)<\sigma(i)$.

Let $\mathrm{inv}(\sigma) = \{(i,j) \in \{1,\ldots,n\}^2 | i<j, \sigma(j)<\sigma(i)\}$.

Example 15: Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix} \in S_4$. Then:

$$\mathrm{inv}(\sigma) = \{(1,3),(2,3),(2,4)\} \tag{174}$$

Lemma 10: Let $\tau \in S_n$ be a transposition with $n \geq 2$. Write $\tau = (k\,\ell)$ with $1 \leq k < \ell \leq n$. Then:

$$\mathrm{inv}(\tau) = \{(k,k+1),(k,k+2),\ldots,(k,\ell-1),(k,\ell),(k+1,\ell),(k+2,\ell),\ldots,(\ell-1,\ell)\} \tag{175}$$

Thus:

$$|\mathrm{inv}(\tau)| = 2(\ell-k-1)+1 \tag{176}$$

Theorem: (Parity Theorem) Let $\sigma \in S_n$. If $\sigma = \tau_1\cdots\tau_r$, where $\tau_1,\ldots,\tau_r$ are transpositions, then:

$$r \equiv |\mathrm{inv}(\sigma)| \pmod 2 \tag{177}$$

Consequently, if $\sigma = \tau_1\cdots\tau_r = \tau_1'\cdots\tau_s'$, where $\tau_1,\ldots,\tau_r,\tau_1',\ldots,\tau_s'$ are transpositions, then $r \equiv s \pmod 2$.

Definition: If $\sigma \in S_n$ can be written as a product of an even (resp. odd) number of transpositions, we say that σ is even (respectively odd).

Corollary 8: A permutation is either even or odd, but not both. And, the parity of $\sigma \in S_n$ is equal to the parity of the number $|\operatorname{inv}(\sigma)|$.

- Note that $\operatorname{inv}(\sigma) = \emptyset \iff \sigma = \mathrm{id}$.
- Therefore, $|\operatorname{inv}(\mathrm{id})| = 0$, so id is an even permutation, i.e. id can only be written as a product of an even number of transpositions.
- Let $\mathbb{C}[x_1, \ldots, x_n]$ denote the set of polynomials in the variables $X_1, \ldots, X_n$ with complex coefficients. That is,

$$\mathbb{C}[x_1, \ldots, x_r] = \left\{ \sum_{i_1, \ldots, i_n \geq 0} a_{i_1, \ldots, i_n} X_1^{i_1} \cdots X_n^{i_n} \right\} \tag{178}$$

  where $a_{i_1, \ldots, i_n} \in \mathbb{C}$ and all but finitely many of $a_{i_1, \ldots, i_n}$ are zero.

- For each $\sigma \in S_n$. Define:

$$A_\sigma : \mathbb{C}[X_1, \ldots, X_n] \to \mathbb{C}[x_1, \ldots, X_n] \tag{179}$$

  by:

$$A_\sigma \left(\sum_{i_1, \ldots, i_n \geq 0} a_{i_1, \ldots, i_n} X_1^{i_1} \cdots X_n^{i_n} \right) \tag{180}$$

$$= \sum_{i_1, \ldots, i_n \geq 0} a_{i_1, \ldots, i_n} X_{\sigma(1)}^{i_1} \cdots X_{\sigma(n)}^{i_n} \tag{181}$$

Example 16: Let $\sigma = (1\,3\,2)$. Then:

$$A_\sigma(3X_1X_2 + 2X_3^5) = 3X_3X_1 + 2X_2^5 \tag{182}$$

- It has the following properties:
  1. For all $\sigma \in S_n$, we have:
     (a) For all $P, Q \in \mathbb{C}[x+1, \ldots, x_n]$, we have

$$A_\sigma(P+Q) = A_\sigma(P) + A_\sigma(Q)$$

     (b) and:

$$A_\sigma(PQ) = A_\sigma(P)A_\sigma(Q) \tag{183}$$

  2. For all $\sigma, \tau \in S_n$, we have:

$$A_{\sigma\tau} = A_\sigma \circ A_\tau \tag{184}$$

  Proof. Let $P = \sum_{i_1, \ldots, i_n} a_{i_1, \ldots, i_n} X_1^{i_1} \cdots X_n^{i_n}$ be an arbitrary element of $\mathbb{C}[x_1, \ldots, x_n]$. Then:

$$A_\sigma(A_\tau(P)) = A_\sigma \left(\sum_{i_1, \ldots, i_n \geq 0} a_{i_1, \ldots, i_n} X_{\tau(1)}^{i_1} \cdots X_{\tau(n)}^{i_n} \right) \tag{185}$$

$$= \sum_{i_1, \ldots, i_n} \geq 0 a_{i_1, \ldots, i_n \geq 0} a_{i_1, \ldots, i_n} A_\sigma(X_\tau(1)^{i_1}) \cdots A_\sigma(X_{\tau(n)}^{i_n}) \tag{186}$$

$$= A_{\sigma\tau}(P) \tag{187}$$

□

Definition: The Vandermonde polynomial in $\mathbb{C}[X_1, \ldots, X_n]$ is the polynomial $V_n = \prod 1 \leq i < j \leq n(X_j - X_i)$.

- A key observation is that for all $\sigma \in S_n$, we have:

$$A_\sigma(V_n) = \prod (X_{\sigma(j)} - X_{\sigma(i)}) \tag{188}$$

$$= (-1)^{|\operatorname{inv}(\sigma)|} \prod_{1 \leq i < j \leq n} (X_j - X_i) \tag{189}$$

$$= (-1)^{|\operatorname{inv}(\sigma)|} V_n \tag{190}$$