PHY293: Waves and Modern Physics
Summary

QiLin Xue

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Disclaimer: A large portion of this document is and will be stolen from David Morin's Waves book, which is currently still a draft.

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1 Damped Harmonic Motion

1.1 Introduction

The Setup: An object undergoing Damped harmonic motion experiences a restoring force $-kx$ and a resistive force $-b\dfrac{dx}{dt}$. The differential equation is:

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x \tag{1}$$

where $\gamma = \dfrac{b}{m}$ and $\omega_0^2 = \dfrac{k}{m}$.

Warning: Most authors prefer to write the differential equation as

$$\frac{d^2x}{dt^2} + \mathbf{2}\gamma\frac{dx}{dt} + \omega_0^2 x$$

as it makes the solution less complicated (i.e. less fractions). Therefore, be very careful when trying to find equations online as we may not all be defining variables the same way.

Motivation for Solution: In general, a solution to a second order linear differential equation is a sum of exponentials, i.e. it is in the form of

$$x(t) = Ae^{\alpha_1 t} + Be^{\alpha_2 t}$$

where α_1, α_2 are solutions to a particular quadratic equation (see Appendix for details), where there are three options:

- Quadratic equation has 2 solutions $\implies$ Then $x(t)$ is a sum of 2 exponential decays.
- Quadratic equation has 1 solution $\implies$ Then $x(t)$ is a single exponential decay.
- Quadratic equation has 0 solutions $\implies$ Then the roots are complex. Recall from ESC194 that complex exponents lead to sinusoidal functions, so $x(t)$ will have a sinusoidal component.

1.2 Underdamping ($\gamma < 2\omega_0$)

We can define

$$\omega^2 = \omega_0^2 - \frac{\gamma^2}{4}, \tag{2}$$

which will be the new angular frequency. *Damping reduces the frequency.*[1] The equation of motion is given by

$$x_{\text{underdamped}}(t) = Ae^{-\gamma t/2}\cos(\omega t + \phi) \tag{3}$$

where A and ϕ are determined by initial conditions.

1.3 Overdamping ($\gamma > 2\omega_0$)

If $\gamma > 2\omega_0$, then the equation of motion is given by

$$x_{\text{overdamped}}(t) = C_1 e^{-\mu_1 t} + C_2 e^{-\mu_2 t} \tag{4}$$

where we have[2]

$$\mu_1 = \frac{\gamma}{2} + \sqrt{\frac{\gamma^2}{4} - \omega_0^2} \tag{5}$$

$$\mu_2 = \frac{\gamma}{2} - \sqrt{\frac{\gamma^2}{4} - \omega_0^2} \tag{6}$$

and C_1, C_2 are determined by initial conditions.

[1] However, this is mostly irrelevant, because if γ is large enough to make ω differ appreciably from ω_0, then the motion becomes negligible after a few cycles anyways. For example, if ω differs from ω_0 be even 20%, then after just 2 cycles, the amplitude would have decrease to 0.01% of the initial.

[2] This shows why most physicists choose to use the 2γ factor, as it reduces a lot of fractions.

1.4 Critical Damping ($\gamma = 2\omega_0$)

Critical damping occurs at $\gamma = 2\omega_0$, then the equation of motion is given by

$$x_{\text{critical}}(t) = (A + Bt)e^{-\omega_0 t} \tag{7}$$

where A and B are determined by initial conditions.

Importance of Critical Damping: Critically damped motion has the property that it *converges* to the origin in the quickest manner, that is, quicker than both the overdamped and underdamped motions.

1.5 Energy of Underdamped Oscillations

Note: We will only focus on very underdamped oscillations.

Underdamped: For simplicity, let us assume $\phi = 0$. The energy of a damped harmonic oscillator is:

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2. \tag{8}$$

Substituting in $x(t)$ gives

$$E = \frac{1}{2}mA_0^2 \exp(-\gamma t)\left(m\omega_0^2 \sin^2(\omega_0 t) + k\cos^2(\omega_0 t) + \omega_0^2\right) \tag{9}$$

This is very messy, so we want to make approximations.

Very Underdamped: If we look at the case where $\gamma \ll \omega_0$, we can reduce this to

$$\boxed{E = \frac{1}{2}mA_0^2\omega_0^2 \exp(-\gamma t) = E_0 \exp(-\gamma t)} \tag{10}$$

as the γ, γ^2 terms approach zero. We can double check that when $\gamma = 0$, this reduces to $E = \frac{1}{2}kA_0^2$.

We can define the lifetime to be $\tau = \dfrac{1}{\gamma}$.

1.5.1 Rate of Energy Loss

The rate of energy loss in a **very underdamped** system is given by

$$\frac{dE}{dt} = -\gamma E \tag{11}$$

Note that E here represents the average energy over a period T.

1.5.2 Q-Factor

We can define the Q-factor to be

$$\boxed{Q = \frac{\omega_0}{\gamma}}. \tag{12}$$

If we consider a very underdamped oscillator (where $\gamma \ll \omega_0$), then

$$\frac{E(t_1) - E(t_1 + T)}{E(t_1)} \approx \gamma T \approx \frac{2\pi\gamma}{\omega} = \frac{2\pi}{Q}. \tag{13}$$

Therefore, we have $Q = \dfrac{E(t_0)}{(E(t_1) - E(t_1+T))/2\pi}$, which is equivalent to the ratio of the initial energy divided by the energy loss per radian:

$$\boxed{Q = \frac{\text{initial energy stored}}{\text{energy loss per radian}}} \tag{14}$$

We can also use the Q-factor to write the differential equation as

$$\frac{d^2x}{dt^2} + \frac{\omega_0}{Q}\frac{dx}{dt} + \omega_0^2 x = 0 \tag{15}$$

and

$$\omega = \omega_0\left(1 - \frac{1}{4}Q^2\right)^{1/2} \tag{16}$$

2 Driven Harmonic Motion

x‘ Suppose there is a driving force of the form $F = F_0 \cos(\omega t)$. Our differential equation becomes

$$F_0 \cos(\omega t) = \frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 \tag{17}$$

2.1 Undamped Forced Oscillations

The solution is in the form of

$$x(t) = A(\omega)\cos(\omega t - \delta) \tag{18}$$

where

$$\tan\delta = 0 \tag{19}$$

so $\delta = 0$ (if $\omega < \omega_0$) or $\delta = \pi$ (if $\omega > \omega_0$). We have

$$A(\omega) = \left|\frac{a}{1 - \omega^2/\omega_0^2}\right| \tag{20}$$

where $a \equiv \dfrac{F_0}{k}$.

2.2 Damped Forced Oscillations

Similarly, the equation is in the same form, except

$$\tan\delta = \frac{\omega y}{\omega_0^2 - \omega^2} \tag{21}$$

and

$$A(\omega) = \frac{\omega_0^2 a}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2}} \tag{22}$$

There are three important regimes:

- $\omega \to 0$ gives $A(\omega) \to a = \dfrac{F_0}{k}$
- $\omega \to \omega_0$ gives $A(\omega) \to \dfrac{a\omega_0}{\gamma}$
- $\omega \to \infty$ gives $A(\omega) \to 0$

The phase shift is separated into three regimes as well:

- $\omega \to 0$ gives $\delta \to 0$
- $\omega \to \omega_0$ gives $\delta \to \dfrac{\pi}{2}$
- $\omega \to \infty$ gives $\delta \to \pi$.

2.3 Power

The power of the damping force is

$$\bar{P}_{\text{damping}} = -\frac{1}{2} b(\omega A)^2 \tag{23}$$

and

$$\bar{P}_{\text{driving}} = \frac{1}{2} F_0 \omega A \sin\delta = \frac{1}{2} b(\omega A)^2 \tag{24}$$

after making the substitution $\sin\delta = \dfrac{\gamma\omega m A}{F_0}$. We can also write $\bar{P}_{\text{driving}}$ in terms of the frequency:

$$\bar{P}_{\text{driving}} = \frac{F_0^2}{2\gamma m} \cdot \frac{\gamma^2\omega^2}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2} \tag{25}$$

2.3.1 The $\bar{P}(\omega)$ Curve

If the driving frequency is close to the natural frequency, we can write

$$\omega^2 - \omega_0^2 \approx -2\omega_0 \Delta\omega, \tag{26}$$

such that

$$\bar{P}(\omega) = \frac{F_0^2}{2m\gamma} \frac{\omega_0^2 \gamma^2}{4\omega_0^2 (\Delta\omega)^2 + \gamma^2 \omega_0^2} \tag{27}$$

3 Appendix

3.1 Derivations

Guessing a solution of the form $x(t) = Ce^{\alpha t}$, and substituting it into the ODE gives the characteristic equation

$$\alpha^2 + \gamma\alpha + \omega_0^2 = 0 \tag{28}$$

which has the solution

$$\alpha = \frac{-\gamma + \sqrt{\gamma^2 - 4\omega_0^2}}{2} \tag{29}$$

The three cases for what the discriminant $\gamma^2 - 4\omega_0^2$ can be gives us the three cases of motion.

Underdamping: In this case, $\sqrt{\gamma^2 - 4\omega_0^2}$ is an imaginary number, so let us write: $\alpha = -\gamma/2 + i\sqrt{4\omega_0^2 - \gamma^2}$. We can define ω such that

$$\alpha = -\gamma/2 + i\omega. \tag{30}$$

Substituting this back into our original guess (and using a linear combination), we get:

$$x_{\text{underdamped}}(t) = C_1 e^{(-\gamma/2 + i\omega)t} + C_2 e^{(-\gamma/2 - i\omega)t} \tag{31}$$

$$= e^{-\gamma t/2}\left(C_1 e^{i\omega t} + C_2 e^{-i\omega t}\right) \tag{32}$$

Since $x(t)$ has to be real, the part inside the parentheses has to be real, which means the two terms are complex conjugates of each other. This means that if $C_1 = Ce^{i\phi}$, then we must have $C_2 = Ce^{-i\phi}$. Making this substitution leads to

$$x_{\text{underdamped}}(t) = 2Ce^{-\gamma t/2}\cos(\omega t + \phi) \tag{33}$$

Overdamping: Note that μ_1 and μ_2 are simply the two solutions to the characteristic equation, so we are left with a simple sum of exponentials:

$$C_1 e^{-\mu_1 t} + C_2 e^{-\mu_2 t}, \tag{34}$$

obtained by straightforward substitution.

Critical Damping: There is only one root, so a naive guess may be that $x(t) = Ce^{-\gamma/2t}$, but this cannot be the case as there is only one parameter, C, which cannot satisfy two freely chosen initial conditions (i.e. initial position and velocity). It turns out (covered in ESC194) that another solution is $te^{-\gamma/2t}$, so the full solution is the linear combination

$$x_{\text{critically damped}}(t) = C_1 e^{-\gamma/2t} + C_2 te^{-\gamma/2t} \tag{35}$$

$$= (C_1 + C_2 t)e^{-\gamma/2t} \tag{36}$$