MAT301 Practice Exam 1

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1 Exercise One

Question 01:

(a) **T**
(b) **T**
(c) **F**
(d) **T**
(e) **F**
(f) **???**
(g) **T**
(h) **F**

Question 02:

(a) We have
  (i) $rs^k = s^{-k}r$
  (ii) $s^2 = e$
  (iii) $r^n = e$

(b) Note that
$$\mathbb{Z}_{18} \times \mathbb{Z}_{25} \times \mathbb{Z}_{14} \cong \mathbb{Z}_{30} \times \mathbb{Z}_{210}.$$
We know that $(0,1)$ and $(1,0)$ generates this second product so we should find the isomorphism mapping. Let ϕ be the isomorphism from $\mathbb{Z}_{18} \times \mathbb{Z}_{25} \times \mathbb{Z}_{14} \to \mathbb{Z}_{30} \times \mathbb{Z}_{210}$. Then the generating elements are $\phi^{-1}(0,1)$ and $\phi^{-1}(1,0)$.

(c) $(1\,2)$ and $(1\,\cdots\,n)$.

Question 03:

1. Note that both α and β is made of disjoint permutations so $|\alpha| = \text{lcm}(3,3) = 3$ and $\beta = \text{lcm}(4,3) = 12$.
2. We have
$$\alpha\beta = (1\,9)(2\,8\,3\,6\,5)$$
3. We have $\alpha^{-1} = (1\,7\,4)(3\,8\,6)$. Then using the result from the previous question, we have
$$(\alpha\beta)\alpha^{-1} = (1\,7\,4\,9)(2\,8\,5).$$

Question 04: Since $a \in G$, then $a^{-1} \in G$. Multiplying the left side of $ab = ac$ by a^{-1} gives
$$a^{-1}ab = a^{-1}ac$$
Because the group operation is distributive, we know that this is equivalent to
$$(a^{-1}a)b = (a^{-1}a)c \implies eb = ec,$$
where e is the identity. We use the property that for any $x \in G$, $ex = x$ to finally conclude that $b = c$.

Question 05: If G is a finite abelian group with order $|G| = 2^6 = 64$, then the classification theorem tells us that

$$G \cong \mathbb{Z}_{2^{k_1}} \times \cdots \times \mathbb{Z}_{2^{k_n}}$$

where

$$|G| = 2^6 = 2^{k_1} \cdots 2^{k_n}$$

which is true if and only if

$$k_1 + \cdots + k_n = 6$$

The tuples $(k_1, \ldots, k_n)$ that satisfy this (and therefore encode an isomorphism class) are the partitions of 6:

- (6)
- (5,1)
- (4,2)
- (4,1,1)
- (3,3)
- (3,2,1)
- (2,2,2)
- (2,2,1,1)
- (2,1,1,1,1)
- (1,1,1,1,1,1)

Question 06:

1. Note that by definition, we must have $\phi(0) = \phi(30) = 0$. Therefore, we must have $\phi(5 \cdot 6) = 5\phi(6) = 0$ where we can take out the 5 as exponentiation respects the group operation. Therefore, $5\phi(6)$ is a multiple of 30, i.e. there exists an integer k such that $5\phi(6) = 30k \implies \phi(6) = 6k$. In other words $\phi(6)$ is a multiple of 6.

   We also know that isomorphisms preserve order. The generating element of $\mathbb{Z}_{30}$ is 1 and has order 30. We use the theorem that an element k has order 30 if and only if k and 30 are relatively prime.

   Since $\phi(1)$ must also have order 30, we can conclude that $\phi(1)$ is relatively prime to 30, so the candidates are $1, 7, 11, 13, 17, 19, 23, 29$. Using properties of isomorphisms, we can write: $\phi(7-6) = \phi(7) - \phi(6) = 23 - \phi(6)$, and therefore $\phi(6) = 23 - \phi(1)$. Using the candidates of $\phi(1)$, the candidates of $\phi(6)$ are then

   $$\phi(6) \in \{22, 16, 12, 10, 6, 4, 0, 24\}.$$

   The only possible candidates are thus $\phi(6) = 0, 6, 24$. We also know that the order of 6 is 5, so $|\phi(6) = 5|$ also. This eliminates 0 as a candidate. Note that if $\phi(6) = 6$, then $\phi(1) = 23$. But we are already told that $\phi(7) = 23$. Since ϕ is injective, we must have $\phi(1) \neq 23$ and therefore $\phi(6) = 24$.
2. It corresponds to $\phi(k) = 30 - k$.

Question 07: Let $G = \langle a \rangle$ such that $|G| = n$. Then we claim we can construct an isomorphism ϕ:

$$\begin{aligned} \phi : G &\to \mathbb{Z}_n \\ a^k &\mapsto k \end{aligned}$$

We show this is well defined by showing that if $a^k = a^j$ (which is true if and only if $k \equiv j \pmod{n}$), then $k = j$. Since the codomain is in $\mathbb{Z}_n$ where each element is taken to be modulo n, then we have $k \equiv j \pmod{n} \iff k = j$. These steps are all reversible, so reversing these show that ϕ is injective as well. To show that ϕ is surjective, note that $k \in \mathbb{Z}_n$ is mapped to by a^k.

We then show this is a homomorphism. Note that

$$\begin{aligned} \phi(a^k a^j) &= \phi(a^{k+j}) \\ &= k + j \\ &= \phi(a^k) + \phi(a^j) \end{aligned}$$

and we're done: ϕ is an isomorphism.

Question 08: Since $\mathbb{Z}_{77}$ is abelian, the image of ϕ must be abelian. The group D_{32} (symmetries of a 32-gon) only has three subgroup classes that are cyclic:

- Case 1: $\{0, s\}$, where s is an arbitrary reflection. Note that s has order 2, but there are no elements of order 2 in $\mathbb{Z}_{77}$, so it can't be homomorphic to this.
- Case 2: $\langle r^i \rangle$ where r is the smallest nonzero rotation and $0 < i < 32$. where $i|2^5$. Therefore, the order of r^i is equal to $32/i$, which will always be even. However, there are no elements of even order in $\mathbb{Z}_{77}$, so it can't be homomorphic to this.
- Case 3: $\{e\}$. The trivial homomorphism will always exist, and since this is the only other case (there are only reflections and rotations so cyclic groups can only be generated by a reflection or rotation), ϕ must be the trivial homomorphism.

Question 09:

(a) Pick $H = \{e, s\}$, where s is an arbitrary reflection and pick $a = r$ where r is the smallest nonzero rotation. Then:

$$aH = rH = \{r, rs\}$$

and

$$Ha = Hr = \{r, sr\}$$

We know that $rs = sr^5 \neq sr$, and we are done.

(b) All the left cosets are

- $rH = \{r, rs\}$
- $r^2H = \{r^2, r^2s\}$
- $r^3H = \{r^3, r^3s\}$
- $r^4H = \{r^4, r^4s\}$
- $r^5H = \{r^5, r^5s\}$
- $eH = \{e, s\}$

Note that the coset r^ksH, where r^ks is a reflection, is already contained above since $sH = H$ so $r^ksH = r^kH$. The index is 6 as we should expect, since

$$\frac{|D_6|}{|H|} = \frac{6 \cdot 2}{2} = 6.$$

Question 10:

(a) The action is $G \times X \to X$. Then $\text{orb}(1) = \{\rho(g)(1) : g \in G\}$. Note that

- $\rho(a) = (1\,4\,6)(2\,7)$
- $\rho(a^2) = (1\,6\,4)$
- $\rho(a^3) = (2\,7)$
- $\rho(a^4) = (1\,4\,6)$
- $\rho(a^5) = (1\,6\,4)(2\,7)$
- $\rho(a^6) = e$.

Therefore,

$$\text{orb}(1) = \{1, 4, 6\}.$$

(b) We have

$$\text{stab}(1) = \{e, a^6\}.$$

(c) For all $g \in G$, we know that $3, 5$ are fixed. We can count the total number of other fixed points to be $0 + 2 + 3 + 2 + 0 + 5 = 12$ (corresponding to $a, a^2, \ldots, a^6$ respectively) giving a total of $12 + 6(2) = 24$ fixed points. Therefore, on average there are $24/6 = 4$ fixed points so there are 4 distinct orbits.

Question 11:

(a) Let the distance between the black dots be ℓ. One tile is consisted of one leaf and its corresponding black dot. The entire figure can be generated by applying a translation of ℓ horizontally (represented by H), a translation of ℓ vertically (represented by V), and a 90° rotation (represented by R), with respect to the black dot.

For example, we can create a coordinate system such that $(0, 0)$ corresponds to the center of a black dot. To generate the flower pattern centered at $(i\ell, j\ell)$, we can perform the four operations:

$$(jV)(iH), R(jV)(iH), R^2(jV)(iH), R^3(jV)(iH).$$

Therefore, the symmetry group is

$$\langle R, H, V \rangle.$$

(b) Let ρ be a bijective mapping that maps one leaf+dot to another leaf+dot. Then we can write

$$\langle R \rangle \times \langle H, V \rangle = \langle R \rangle \rtimes \langle H, V \rangle$$

with the binary operation:

$$(R^a, H^b V^c)(R^\alpha, H^\beta V^\gamma) = (R^a \rho(H^b)(R^\alpha), H^{b+\beta} V^{c+\gamma}) \tag{1}$$