AER210: Vector Calc and Fluid Mechanics

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1 Double Integrals

- **Integrals Involving a Parameter**

Example 1: Let $\int_0^1 Cx^3 \, \mathrm{d}x$ where C is a constant. Then it gives

$$\int_0^1 Cx^3 \, \mathrm{d}x = \frac{1}{4}C \tag{1}$$

The result contains C.

- Suppose we have something like

$$\int_a^b f(x, y) \, \mathrm{d}x = g(y) \tag{2}$$

and therefore y is a parameter

Definition: A variable which is kept constant during an integration is called a parameter.

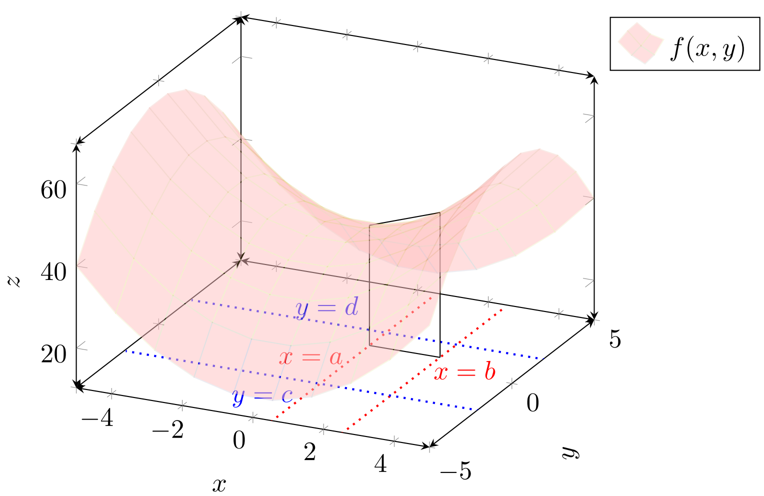
- Partial integration wrt x

Example 2: An example of partial integration wrt x is

$$\int_0^1 x^3 y \, \mathrm{d}x = y \int_0^1 x^3 \, \mathrm{d}x = \frac{1}{4}y \tag{3}$$

- Notice the similarity between partial differentiation wrt x, $f_x(x, y)$ and the partial integration wrt x, $\int_a^b f(x, y) \, \mathrm{d}x$.
- **Iterated Integrals** (Integral of an Integral)
- Consider $x = f(x, y)$ where $x \in [a, b]$, $y \in [c, d]$. This defines a rectangular region.
- Assume that $f(x, y) \geq 0$. This can be represented as a surface, as shown below:

Example Slice



If we take the integral $\int_{y=c}^d f(x, y) \, \mathrm{d}y = A(x)$, we see that the area of the slice depends on x.

If we suppose that the surface has a tiny thickness Δx, then the volume is

$$\Delta V(x) = A(x) \cdot \Delta x = \left(\int_{y=c}^d f(x, y) \, \mathrm{d}y \right) \Delta x \tag{4}$$

If we break up the interval $[a, b]$ into N segments

$$x_0 = a \leq x_1 \leq x_2 \leq \dots x_{i-1} \leq x_i \leq \dots \leq x_{N-1} \leq x_N = b \tag{5}$$

with $\Delta x_i = x_i - x_{i-1}$. We can then approximate the volume as

$$V \approx \sum_{i=1}^{N} \Delta V_i = \sum_{i=1}^{N} A(x_i)\Delta x_i \tag{6}$$

which is known as a **Riemann sum**.

> **Idea**: As we take the limit as $N \to \infty$ which implies $\Delta x_i \to 0$, we get the double integral:
>
> $$V = \int_a^b \int_c^d f(x, y)\, \mathrm{d}y\, \mathrm{d}x \tag{7}$$
>
> which can be determined by calculating two integrals.

- Similarly, we can find the volume by taking slices parallel to the xz plane.

  The area of each slice is a function of y:

$$A(y) = \int_a^b f(x, y)\, \mathrm{d}x \tag{8}$$

so we have $\Delta V(y) = A(y) \cdot \Delta y$. Again, summing up all slices and taking the limit, we get

$$V = \int_c^d A(y)\, \mathrm{d}y = \int_c^d f(x, y)\, \mathrm{d}x\, \mathrm{d}y \tag{9}$$

> **Theorem**: **Fubini's Theorem** tells us that
>
> $$\int_a^b \int_c^d f(x, y)\, \mathrm{d}y\, \mathrm{d}x = \int_c^d \int_a^b f(x, y)\, \mathrm{d}x\, \mathrm{d}y \tag{10}$$

The analog for equality of mixed partial derivatives is known as **Clairut's Theorem.**

Example 3: Find the volume under the surface $z = x^2y$ where $x \in [1, 3]$ and $y \in [0, 1]$. We first form the integral by integrating wrt y. We have

$$V = \int_1^3 \int_0^1 x^2 y\, \mathrm{d}y\, \mathrm{d}x \tag{11}$$

$$= \int_1^3 x^2(1^2/2 - 0^2/2)\, \mathrm{d}x \tag{12}$$

$$= \int_1^3 \frac{x^2}{2}\, \mathrm{d}x \tag{13}$$

$$= \frac{13}{3} \tag{14}$$

We can also form the integral by integrate it wrt x:

$$V = \int_0^1 \int_1^3 x^2 y\, \mathrm{d}x\, \mathrm{d}y \tag{15}$$

$$= \int_0^1 \frac{26}{3} y\, \mathrm{d}y \tag{16}$$

$$= \frac{13}{3} \tag{17}$$

so we can confirm they give the same answer.

Example 4: Evaluate the double integral of $f(x, y) = x - 3y^2$ over region R where

$$R = \{(x, y)|0 \leq x \leq 2, 1 \leq y \leq 2\} \tag{18}$$

To do this, we have

$$\int_0^2 \int_1^2 (x - 3y^2)\,\mathrm{d}y\,\mathrm{d}x = \int_0^2 (xy - y^3)\big|_{y=1}^{y=2}\,\mathrm{d}x \tag{19}$$

$$= \int_0^2 (x - 7)\,\mathrm{d}x \tag{20}$$

$$= -12 \tag{21}$$

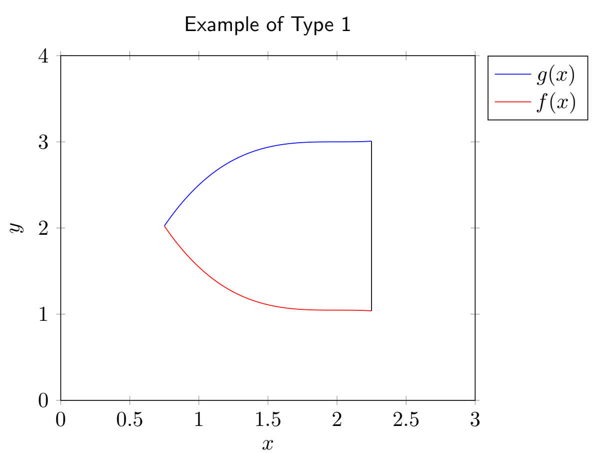
- Note that in the special case where the function $f(x, y)$ is $f(x, y) = g(x) \cdot h(y)$, then

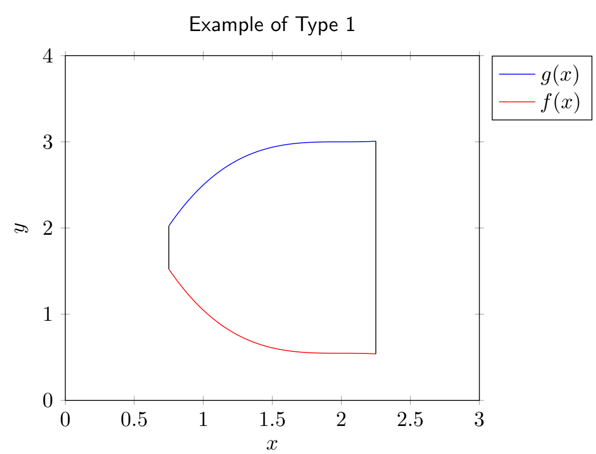
$$\int_c^d \int_a^b f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int_c^d \left[h(y) \int_a^b g(x)\,\mathrm{d}x\right] \mathrm{d}y = \int_a^b g(x)\,\mathrm{d}x \cdot \int_c^d h(y)\,\mathrm{d}y \tag{22}$$

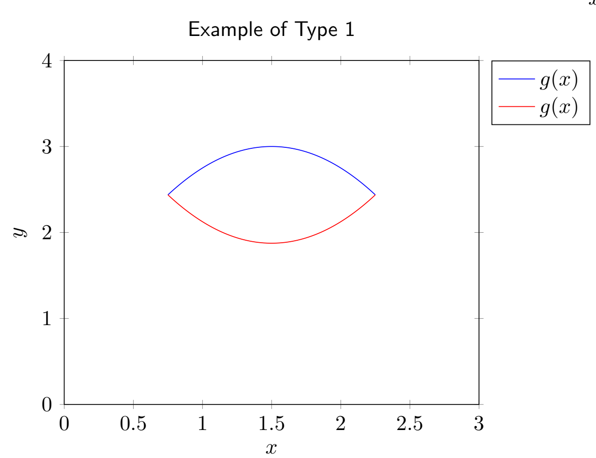
  This gives us a shortcut of evaluating double integrals in this form.
- Double integrals over general regions (What if region is non-rectangular?)
- **Type 1 Region** is in the form of

$$R = \{(x, y)|a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\} \tag{23}$$

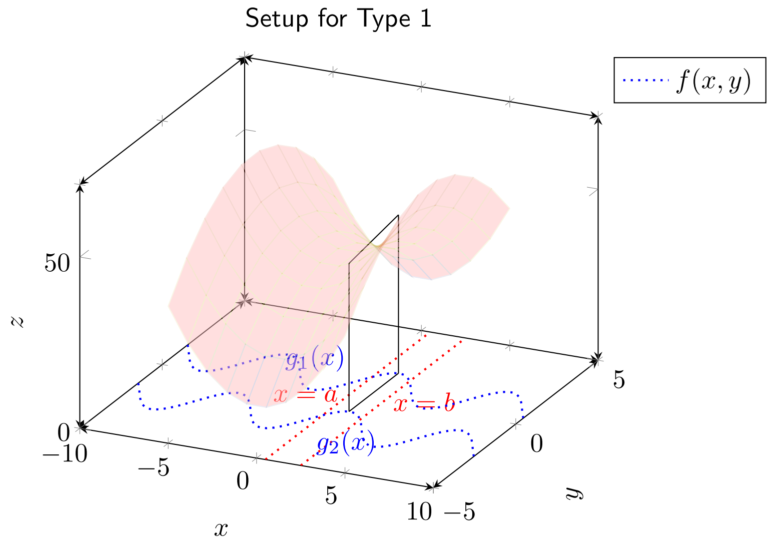
  Here are some examples







- Let's think about the case where $f(x, y) \geq 0$ on a type-1 region. Suppose we have the following illustration



We find the area of slices, so

$$A(x) = \int_{g_1(x)}^{g_2(x)} f(x, y)\, \mathrm{d}y \tag{24}$$

and the volume is thus

$$V = \int_a^b A(x)\, \mathrm{d}X = \int_a^b \int_{g_1(x)}^{g_2(X)} f(x, y)\, \mathrm{d}y\, \mathrm{d}x \tag{25}$$

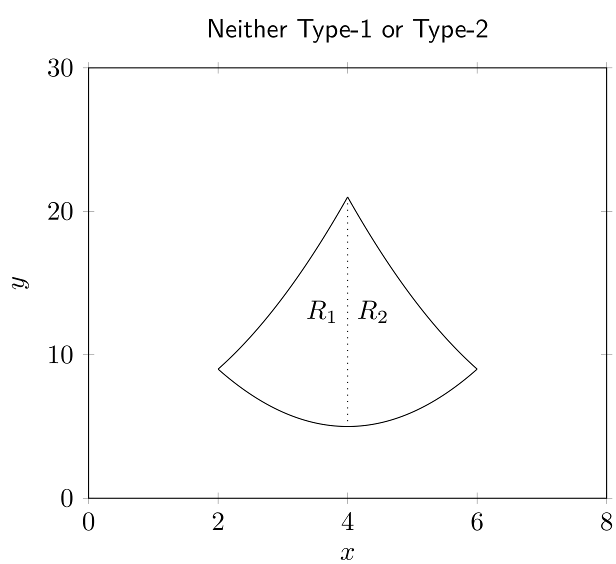
- Type-2 regions have the form

$$R = \{(x, y) | c \le y \le d \text{ and } h_1(y) \le x \le h_2(y)\} \tag{26}$$

In a similar way, the volume bounded by this region is

$$V = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, \mathrm{d}x\, \mathrm{d}y \tag{27}$$

- Type-3 regions are neither type-1 nor type-2. It is possible to break up the region into parts that can be classified as either type-1 or type-2:



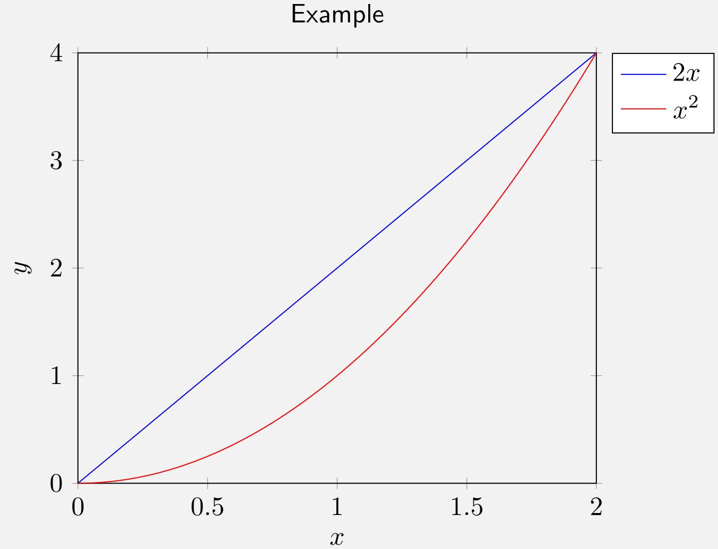
> **Idea**: While these formulas are derived by assuming a positive volume (and thus cannot work if $f < 0$), they still work in general.

Example 5: Find the volume of the solid that lies under the surface

$$z = f(x, y) = x^2 + y^2 \tag{28}$$

and above the region R in the xy-plane. The region R is bounded by the straight line $y = 2x$ and the parabola $y = x^2$.

1. First we draw a diagram of the planar region R over which the surface is defined.



2. We then draw a line parallel to the axis of first integration (i.e. vertical lines for integrating in the y-direction first)
3. This gives us

$$\begin{aligned} V &= \int_{x=0}^{x=2} \int_{y=x^2}^{y=2x} f(x, y)\,\mathrm{d}y\,\mathrm{d}x && (29)\\ &= \int_0^2 \int_{x^2}^{2x} (x^2 + y^2)\,\mathrm{d}y\,\mathrm{d}x && (30)\\ &= \frac{216}{35} && (31) \end{aligned}$$

Alternatively, we can find the volume by integrating in the x direction first. In this case, we need to obtain boundary curves in the $x = x(y)$ form:

$$\begin{aligned} y = x^2 &\implies x = \sqrt{y} && (32)\\ y = 2x &\implies x = y/2 && (33) \end{aligned}$$

This then gives us

$$\begin{aligned} V &= \int_{y=0}^{y=4} \int_{x=y/2}^{x=\sqrt{y}} f(x, y)\,\mathrm{d}x\,\mathrm{d}y && (34)\\ &= \frac{216}{35} && (35) \end{aligned}$$

Warning: Do not just pick the minimum and maximum points. For example, the following is *incorrect*

$$\int_{y=0}^{y=4} \int_{x=0}^{x=2} f(x, y)\,\mathrm{d}x\,\mathrm{d}y \tag{36}$$

as that corresponds with a rectangular region.

Example 6: Integrate the surface given by $z = e^{x^2}$ over the following region:

We can first integrate wrt x

$$V = \in_{y=0}^{y=1} \int_{x=y}^{x=1} e^{x^2} \,\mathrm{d}x \,\mathrm{d}y \tag{37}$$

This is a hard problem since we don't know the anti-derivative of e^{x^2}. To solve this, we can first integrate wrt y, which gives us

$$V = \int_{x=0}^{x=1} \int_{y=0}^{y=x} e^{x^2} \,\mathrm{d}y \,\mathrm{d}x \qquad = \int_{x=0}^{1} e^{x^2} y\Big|_{y=0}^{y=x} \,\mathrm{d}x \tag{38}$$

$$= \int_{0}^{1} e^{x^2} x \,\mathrm{d}x \tag{39}$$

This integral can be more easily solved using the u-sub $u = x^2$, $\mathrm{d}u = 2x \,\mathrm{d}x$ to get

$$V = \frac{1}{2}(e - 1) \tag{40}$$

2 Formal Definition of Double Integrals

- We will see two ways of defining double integrals.
- First, let us review the formal definition of definite integrals for functions of a single variable.

  To determine the area under a curve in the region $x \in [a, b]$, we can break the region up into intervals Δx_i, so the Riemann sum is

$$A \approx \sum_{i=1}^{n} f(x_i^*) \Delta x_i \tag{41}$$

  Let $m_i \le f(x_i^*) \le M_i$ for $x_i^* \in \Delta x_i$. Then:

$$\sum_{i=1}^{n} m_i \Delta x_i \le \underbrace{\sum_{i=1}^{n} f(x_i^*) \Delta x_i} Estimateoftheentireareacalculatedby Riemann Sum \le \sum_{i=1}^{n} M_i \Delta x_i \tag{42}$$

  If the Δx_i are of equal length and we take the limit, we can define:

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x_i = \int_a^b f(x) \,\mathrm{d}x \tag{43}$$

  If they are not of equal length, we need to define the norm of the partition $\|P\| = (\Delta x_i)_{\text{max}}$ for $i = 1, 2, \ldots, n$. This way, the integral can be alternatively defined as

$$A = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i^*) \Delta x_i = \int_a^b f(x) \,\mathrm{d}x \tag{44}$$

- Consider a double integral over rectangular region. Let $z = f(x, y)$ be defined on $R = \{(x, y) | a \le x \le b, c \le y \le d\}$. Assume $f(x, y) \ge 0$ over R.
- **Formal Definition 1:** We can approximate the volume as

$$\Delta v_i \approx f(x_i^*, y_i^*) \Delta A_i \tag{45}$$

  where $\Delta A_i = \Delta x_i \cdot \Delta y_i$. The Riemann sum is then

$$V \approx \sum_{i=1}^{N} f(x_i^*, y_i^*) \Delta A_i \tag{46}$$

We can pick x_i^*, y_i^* such that $f(x_i^*, y_i^*)$ is the smallest and largest value in the region, we can bound the Riemann sum by:

$$\sum_{i=1}^{N} m_i \Delta x_i \Delta y_i \leq \sum_{i=1}^{N} f(x_i^*, y_i^*) \Delta x_i \Delta y_i \leq \sum_{i=1}^{N} M_i \Delta x_i \Delta y_i \tag{47}$$

Warning: Taking the limit where $N \to \infty$ is not sufficient, as it does not necessarily mean the size of all partitions approach zero.

We define the norm of the partition to be

$$\|P\| = \max(\Delta d_i) \tag{48}$$

for $i = 1, 2, \ldots, N$. Therefore:

$$V = \lim_{\|P\| \to 0} \sum_{i=1}^{N} f(x_i^*, y_i^*) \Delta A_i = \iint_R f(x, y) \, \mathrm{d}A = \iint_R f(x, y) \, \mathrm{d}x \, \mathrm{d}y \, . \tag{49}$$

Idea: Functions that are continuous are integrable over that region.

- **Formal Definition 2:** We are free to divide the region R into any tiling, we can use uniform divisions.

  As a result, the area of each tile is

$$\Delta A_{ij} = \Delta x_i \Delta y_j \tag{50}$$

  where the (i, j) represent the coordinate of the tile. The double Riemann sum is then:

$$V \approx \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta x_i \Delta y_j \tag{51}$$

  Again, we can define m_{ij} and M_{ij} such that

$$\sum_{j=1}^{m} \sum_{i=1}^{n} m_{ij} \Delta x_i \Delta y_j \leq \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta x_i \Delta y_j \leq \sum_{j=1}^{m} \sum_{i=1}^{n} M_{ij} \Delta x_i \Delta y_j \tag{52}$$

  Since these intervals are equally partitioned, we can define

$$V = \lim_{m,n \to \infty} \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A_{ij} = \iint_R f(x, y) \, \mathrm{d}A \, . \tag{53}$$

  If they were not, we would have to define the norm again.

  Example 7: Estimate the volume of the solid that lies above the square $R = [0, 2] \times [0, 2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$. Divide R into four equal squares & choose the sample point to be the upper corner of each square.

  We would then have:

$$\begin{aligned} V &\approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(x_{ij}^*, y_{ij}^*) \Delta A && (54) \\ &\approx f(1, 1)\Delta A + f(1, 2)\Delta A + f(2, 1)\Delta A + f(2, 2)\Delta A && (55) \\ &\approx 34 && (56) \end{aligned}$$

  Note that the actual answer is 48. The approximation will improve as the number of regions increase.

- We can also define double integrals over non-rectangular regions.
- **Definition 1:** We can again tile a region using rectangular regions in two ways:
  - Each tile is contained within R and there are some space.

- Some tiles extend past the boundary of R, which is completely covered.

When we take the limit as $\|P\| \to 0$, both of these tiling methods will approach the actual area, so using any of these tilings will cause the double integral to approach the actual volume.

If $f(x, y)$ is a continuous function over R, then

$$V = \lim_{\|P\|} \sum f(x_i^*, y_i^*)\Delta A_i = \lim_{\|P\| \to 0} \sum_{j=1}^{N} f(x_j^*, y_j^*)\Delta A_j = \iint_R f(x, y)\,\mathrm{d}x\,\mathrm{d}y \tag{57}$$

- **Definition 2:** Similarly, we can use uniform partitions that either leave gaps or extend past the region. We can again define m_{ij} and M_{ij} for each tile R_{ij} such that

$$V = \iint_R f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \lim_{\|P\| \to 0} \sum_{j=1}^{M} \sum_{i=1}^{N} f(x_{ij}^*, y_{ij}^*)\Delta x_i \Delta y_j \tag{58}$$

3 Double Integrals in Polar Coordinates

- Using polar coordinates is helpful when integrating over circular regions.
- Recall that we can convert between rectangular and polar coordinates via

$$x = r\cos\theta, \qquad y = r\sin\theta \tag{59}$$

and that the area of a sector is

$$A = \frac{1}{2}r^2\theta \tag{60}$$

- Suppose we have some function $f(x, y)$ defined on $R = \{(r, \theta) | a \le r \le b, \alpha \le \theta \le \beta\}$. We can then define:

$$f(x, y) = f(r\cos\theta, r\sin\theta) = g(r, \theta). \tag{61}$$

Assume $f(x, y) = g(r, \theta) \ge 0$ on R. Then we can approximate the volume as

$$\Delta V_i \approx g(r_i^*, \theta_i^*) \cdot \Delta A_i = f(r_i^* \cos\theta_i^*, r_i^* \sin\theta_i^*) \cdot r_i \Delta r_i \Delta\theta_i \left(1 + \frac{\Delta r_i}{2r_i}\right). \tag{62}$$

Taking the limit, we have

$$V = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(r_i^* \cos\theta_i^*, r_i^* \sin\theta_i^*) r_i \Delta r_i \Delta\theta_i \tag{63}$$

$$* = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r\,\mathrm{d}r\,\mathrm{d}\theta\,. \tag{64}$$

We can generalize this finding regardless of whether the function is positive or negative over R.

Idea: In a region bounded by $\alpha \le \theta \le \beta$, $a \le r \le b$, we have

$$\iint_R f(x, y)\,\mathrm{d}A = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r\,\mathrm{d}r\,\mathrm{d}\theta\,. \tag{65}$$

- We can extend this to more complicated regions. Suppose R is bounded by $\alpha \le \theta \le \beta$ and $g(\theta) \le r \le g_2(\theta)$. Then the volume would be

$$\iint_R f(x, y)\,\mathrm{d}A = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta) r\,\mathrm{d}r\,\mathrm{d}\theta \tag{66}$$

- Similarly, if R is bounded by $a \le r \le b$ and $h_1(r) \le \theta \le h_2(r)$, we have

$$\iint_R f(x, y)\,\mathrm{d}A = \int_a^b \int_{h_1(r)}^{h_2(r)} f(r\cos\theta, r\sin\theta) r\,\mathrm{d}r\,\mathrm{d}\theta\,. \tag{67}$$

Example 8: Evaluate $\displaystyle\iint_R (3x+4y^2)\,\mathrm{d}A$ where R is the region in the upper half-plane bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$.

This leads to the region $R = \{(r,\theta) | 1 \le r \le 2, 0 \le \theta \le \pi\}$. Then:

$$I = \iint_R (3x+4y^2)\,\mathrm{d}A \tag{68}$$

$$= \int_0^\pi \int_1^2 (3r\cos\theta + 4r^2\sin^2\theta) r\,\mathrm{d}r\,\mathrm{d}\theta \tag{69}$$

Solving this integral gives $\dfrac{15}{2}\pi$.

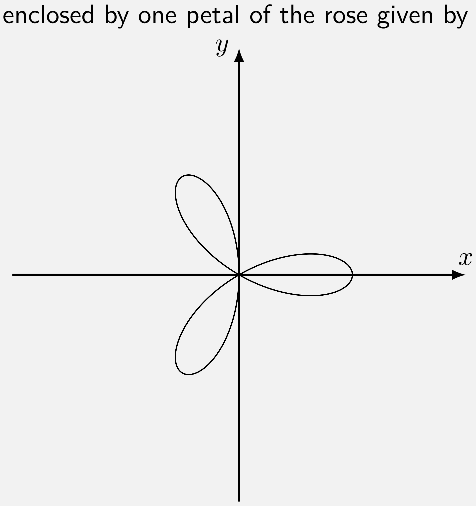
Example 9: Find the volume of the solid bounded by the $z=0$ plane and the paraboloid $z = 1-x^2-y^2$.

Note that at $z=0$, we get $0 = 1-x^2-y^2 \implies x^2+y^2=1$. We can write our region as $R = \{(r,\theta)|0 \le r \le 1, 0 \le \theta \le 2\pi\}$. Our double integral is then

$$V = \iint_R (1-x^2-y^2)\,\mathrm{d}A = \int_0^{2\pi}\int_0^1 (1-r^2) r\,\mathrm{d}r\,\mathrm{d}\theta \tag{70}$$

Solving this gives $V = \pi/2$.

Example 10: Find the area enclosed by one petal of the rose given by $r = \cos 3\theta$.



The area is

$$A = \int_{-\pi/6}^{\pi/6}\int_0^{\cos 3\theta} 1 \cdot r\,\mathrm{d}r\,\mathrm{d}\theta \tag{71}$$

which evaluates to $\dfrac{1}{12}$.

Example 11: Find the volume trapped between the cone $z = \sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2=1$.

First, let us find the intersection using cartesian coordinates. We have

$$\sqrt{x^2+y^2} = \sqrt{1-x^2-y^2} \implies x^2+y^2 = \frac{1}{2}. \tag{72}$$

This can be written as $r = \frac{1}{\sqrt{2}}$ in polar coordinates. The volume is thus

$$\int_0^{2\pi} \int_0^{1/\sqrt{2}} f(x,y) r \,\mathrm{d}r \,\mathrm{d}\theta \tag{73}$$

where $f(x,y) = \sqrt{1 - x^2 - y^2} - \sqrt{x^2 + y^2}$. This gives $\frac{2\pi}{3}\left(1 - \frac{1}{\sqrt{2}}\right)$.

- **Applications of Double Integrals**
- We can determine the mass of a plate with nonuniform density $\rho = \rho(x,y)$. The mass is then

$$\iint\limits_R \rho(x,y) \,\mathrm{d}A \,. \tag{74}$$

- We can find the center of mass of a particle. Imagine we break a plate into small pieces. Each small piece has a moment about the x axis:

$$(M_x)_i = m_i y_i^* \approx \rho(x_i^*, y_i^*) \Delta A_i \cdot y_i^* \tag{75}$$

  The total x and y moments are thus

$$M_x = \iint\limits_R y\rho(x,y) \,\mathrm{d}A \tag{76}$$

$$M_y = \iint\limits_R x\rho(x,y) \,\mathrm{d}A \tag{77}$$

  These are equal to the moment $\bar{y}m$ and $\bar{x}m$, respectively, where m is the mass of the object. Thus:

$$\bar{x} = \frac{\iint\limits_R x\rho(x,y) \,\mathrm{d}A}{\iint_R \rho(x,y) \,\mathrm{d}A} \tag{78}$$

  and similarly for $\bar{y}$.

- Consider a rotating object. A point mass has a kinetic energy $K = \frac{1}{2} m r^2 \omega^2$. However, mr^2 would be different for different points on a solid object.

  We can consider:

$$K = \frac{1}{2}\left(\sum_{i=1}^{n} m_i r_i^2\right)\omega^2. \tag{79}$$

  The quantity inside the parentheses is known as the moment of inertia I. While this may be true for a series of point masses, for a continuous distribution we need to take the limit:

$$I = \iint\limits_R \rho(x,y)[r(x,y)]^2 \,\mathrm{d}x \,\mathrm{d}y \,. \tag{80}$$