AER210: Vector Calc and Fluid Mechanics

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Fall 2021

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1 Introduction

• Integrals Involving a Parameter

Example 1: Let $\int_0^1 Cx^3 dx$ where C is a constant. Then it gives

$$\int_0^1 Cx^3 \, \mathrm{d}x = \frac{1}{4}C \tag{1}$$

The result contains C.

• Suppose we have something like

$$\int_{a}^{b} f(x,y) \, \mathrm{d}x = g(y) \tag{2}$$

and therefore y is a parameter

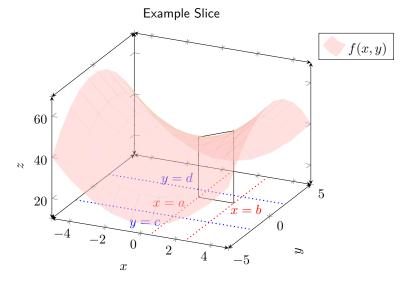
Definition: A variable which is kept constant during an integration is called a parameter.

ullet Partial integration wrt x

Example 2: An example of partial integration wrt x is

$$\int_0^1 x^3 y \, \mathrm{d}x = y \int_0^1 x^3 \, \mathrm{d}x = \frac{1}{4} y \tag{3}$$

- Notice the similarity between partial differentiation wrt x, $f_x(x,y)$ and the partial integration wrt x, $\int_a^b f(x,y) \, \mathrm{d}x$.
- Iterated Integrals (Integral of an Integral)
- Consider x = f(x, y) where $x \in [a, b], y \in [c, d]$. This defines a rectangular region.
- Assume that $f(x,y) \ge 0$. This can be represented as a surface, as shown below:



If we take the integral $\int_{y=c}^d f(x,y)\,\mathrm{d}y = A(x)$, we see that the area of the slice depends on x.

If we suppose that the surface has a tiny thickness Δx , then the volume is

$$\Delta V(x) = A(x) \cdot \Delta x = \left(\int_{y=c}^{d} f(x, y) \, \mathrm{d}y \right) \Delta x \tag{4}$$

If we break up the interval [a, b] into N segments

$$x_0 = a \le x_1 \le x_2 \le \dots x_{i-1} \le x_i \le \dots \le x_{N-1} \le x_N = b$$
 (5)

with $\Delta x_i = x_i - x_{i-1}$. We can then approximate the volume as

$$V \approx \sum_{i=1}^{N} \Delta V_i = \sum_{i=1}^{N} A(x_i) \Delta x_i$$
 (6)

which is known as a Riemann sum.

Idea: As we take the limit as $N \to \infty$ which implies $\Delta x_i \to 0$, we get the double integral:

$$V = \int_a^b \int_c^d f(x, y) \, \mathrm{d}y \, \mathrm{d}x \tag{7}$$

which can be determined by calculating two integrals.

ullet Similarly, we can find the volume by taking slices parallel to the xz plane.

The area of each slice is a function of y:

$$A(y) = \int_{a}^{b} f(x, y) \, \mathrm{d}x \tag{8}$$

so we have $\Delta V(y) = A(y) \cdot \Delta y$. Again, summing up all slices and taking the limit, we get

$$V = \int_{c}^{d} A(y) \, \mathrm{d}y = \int_{c}^{d} f(x, y) \, \mathrm{d}x \, \mathrm{d}y \tag{9}$$

Theorem: Fubini's Theorem tells us that

$$\int_{a}^{b} \int_{a}^{d} f(x, y) \, \mathrm{d}y \, \mathrm{d}x = \int_{a}^{d} \int_{a}^{b} f(x, y) \, \mathrm{d}x \, \mathrm{d}y \tag{10}$$

The analog for equality of mixed partial derivatives is known as Clairut's Theorem.

Example 3: Find the volume under the surface $z = x^2y$ where $x \in [1,3]$ and $y \in [0,1]$. We first form the integral by integrating wrt y. We have

$$V = \int_{1}^{3} \int_{0}^{1} x^{2} y \, \mathrm{d}y \, \mathrm{d}x \tag{11}$$

$$= \int_{1}^{3} x^{2} (1^{2}/2 - 0^{2}/2) \, \mathrm{d}x \tag{12}$$

$$= \int_{1}^{3} \frac{x^{2}}{2} \, \mathrm{d}x \tag{13}$$

$$=\frac{13}{3}\tag{14}$$

We can also form the integral by integrate it wrt x:

$$V = \int_0^1 \int_1^3 x^2 y \, \mathrm{d}x \, \mathrm{d}y \tag{15}$$

$$= \int_0^1 \frac{26}{3} y \, \mathrm{d}y \tag{16}$$

$$=\frac{13}{3}\tag{17}$$

so we can confirm they give the same answer.

Example 4: Evaluate the double integral of $f(x,y) = x - 3y^2$ over region R where

$$R = \{(x,y)|0 \le x \le 2, 1 \le y \le 2\}$$
(18)

To do this, we have

$$\int_0^2 \int_1^2 (x - 3y^2) \, \mathrm{d}y \, \mathrm{d}x = \int_0^2 (xy - y^3) \Big|_{y=1}^{y=2} \, \mathrm{d}x$$
 (19)

$$= \int_0^2 (x - 7) \, \mathrm{d}x \tag{20}$$

$$= -12 \tag{21}$$

• Note that in the special case where the function f(x,y) is $f(x,y)=g(x)\cdot h(y)$, then

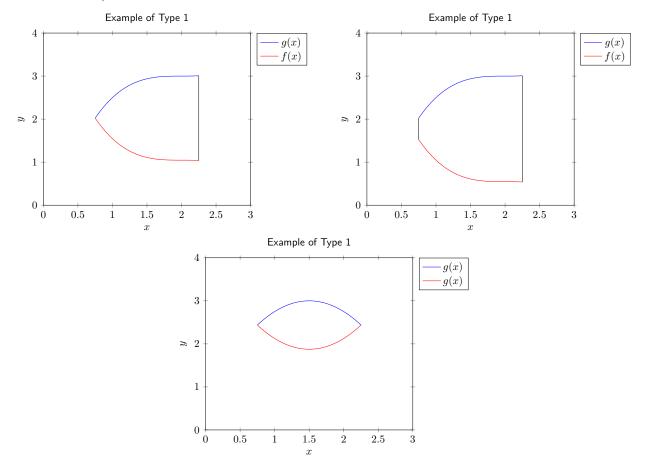
$$\int_{c}^{d} \int_{a}^{b} f(x, y) \, \mathrm{d}x \, \mathrm{d}y = \int_{c}^{d} \left[h(y) \int_{a}^{b} g(x) \, \mathrm{d}x \right] \, \mathrm{d}y = \int_{a}^{b} g(x) \, \mathrm{d}x \cdot \int_{c}^{d} h(y) \, \mathrm{d}y \tag{22}$$

This gives us a shortcut of evaluating double integrals in this form.

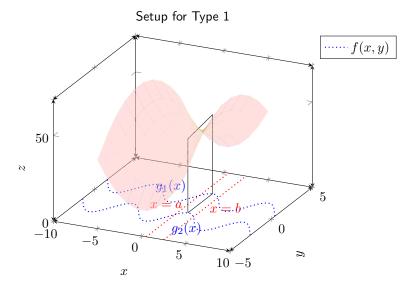
- Double integrals over general regions (What if region is non-rectangular?)
- Type 1 Region is in the form of

$$R = \{(x, y) | a \le x \le b, g_1(x) \le y \le g_2(x) \}$$
(23)

Here are some examples



ullet Let's think about the case where $f(x,y)\geq 0$ on a type-1 region. Suppose we have the following illustration



We find the area of slices, so

$$A(x) = \int_{g_1(x)}^{g_2(x)} f(x, y) \, \mathrm{d}y$$
 (24)

and the volume is thus

$$V = \int_{a}^{b} A(x) \, dX = \int_{a}^{b} \int_{g_{1}(x)}^{g_{2}(X)} f(x, y) \, dy \, dx$$
 (25)

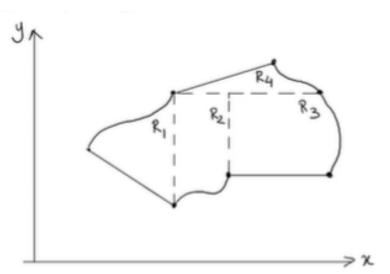
• Type-2 regions have the form

$$R = \{(x,y)|c \le y \le d \text{ and } h_1(y) \le x \le h_2(y)\}$$
 (26)

In a similar way, the volume bounded by this region is

$$V = \int_{c}^{d} \int_{h_{1}(y)}^{h_{2}(y)} f(X, y) \, \mathrm{d}x \, \mathrm{d}y$$
 (27)

• Type-3 regions are neither type-1 nor type-2. It is possible to break up the region into parts that can be classified as either type-1 or type-2:



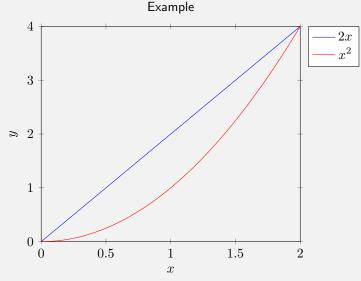
Idea: While these formulas are derived by assuming a positive volume (and thus cannot work if f < 0), they still work in general.

Example 5: Find the volume of the solid that lies under the surface

$$z = f(x, y) = x^2 + y^2 (28)$$

and above the region R in the xy-plane. The region R is bounded by the straight line y=2x and the parabola $y = x^2$.

1. First we draw a diagram of the planar region R over which the surface is defined.



- 2. We then draw a line parallel to the axis of first integration (i.e. vertical lines for integrating in the y-direction first)
- 3. This gives us

$$V = \int_{x=0}^{x=2} \int_{y=x^2}^{y=2x} f(x,y) \, \mathrm{d}y \, \mathrm{d}x$$
 (29)

$$= \int_0^2 \int_{x^2}^{2x} (x^2 + y^2) \, \mathrm{d}y \, \mathrm{d}x \tag{30}$$

$$=\frac{216}{35} \tag{31}$$

Alternatively, we can find the volume by integrating in the x direction first. In this case, we need to obtain boundary curves in the x = x(y) form:

$$y = x^2 \implies x = \sqrt{y} \tag{32}$$

$$y = 2x \implies x = y/2 \tag{33}$$

This then gives us

$$V = \int_{y=0}^{y=4} \int_{x=y/2}^{x=\sqrt{y}} f(x,y) \, dx \, dy$$

$$= \frac{216}{35}$$
(34)

$$=\frac{216}{35} \tag{35}$$

Warning: Do not just pick the minimum and maximum points. For example, the following is incorrect

$$\int_{y=0}^{y=4} \int_{x=0}^{x=2} f(x,y) \, \mathrm{d}x \, \mathrm{d}y \tag{36}$$

as that corresponds with a rectangular region.

Example 6: Integrate the surface given by $z=e^{x^2}$ over the following region:

We can first integrate wrt \boldsymbol{x}

$$V = \in_{y=0}^{y=1} \int_{x=y}^{x=1} e^{x^2} dx dy$$
 (37)

This is a hard problem since we don't know the anti-derivative of e^{x^2} . To solve this, we can first integrate wrt y, which gives us

$$V = \int_{x=0}^{x=1} \int_{y=0}^{y=x} e^{x^2} dy dx \qquad = \int_{x=0}^{1} e^{x^2} y \Big|_{y=0}^{y=x} dx$$
 (38)

$$=\int_0^1 e^{x^2} x \, \mathrm{d}x \tag{39}$$

This integral can be more easily solved using the u-sub $u=x^2$, $du=2x\,dx$ to get

$$V = \frac{1}{2}(e-1) \tag{40}$$