MAT257: Term Test III Theorems

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Theorem: **Change of Variables:** Let $A \subset \mathbb{R}^n$ be open, $g : A \to \mathbb{R}^n$ be continuously differentiable, 1-1, and such that $\forall x \in A$, $g'(x)$ is invertible. If $f : g(A) \to \mathbb{R}$ is integrable, then

$$\int_{g(A)} f = \int_A (f \circ g)|\det g'|. \tag{0.1}$$

Lemmas:

1. If $cov(g)$ and $cov(f)$ holds, then $cov(g \circ h)$ holds.
2. Assume $cov(n-1)$. Let $g : U \to \mathbb{R}^n$ where U is open and bounded be a layer-preserving map such that $g(U)$ is bounded. This means that:
$$g(x_1, \dots, x_n) = (\dots, x_n). \tag{0.2}$$
Then a restricted $cov(g)$ holds: If $f : g(U) \to \mathbb{R}$ is continuous and supp $f \subset g(U)$, then
$$\int f = \int (f \circ g)|\det g'|. \tag{0.3}$$
3. For every $a \in A$, there is some open neighbourhood $U \ni a$ such that on U, g is a composition of linear maps and coordinate swaps:
$$T_{ij} : \mathbb{R}^n \to \mathbb{R}^n \qquad T_{ij}(x_1, , \dots, x_i, \dots, x_j, \dots, x_n) = (x_1, \dots, x_j, \dots, x_i, \dots, x_n). \tag{0.4}$$
4. Local RCOV implies Global RCOV (where R denotes that it is a restricted COV formula, where we only work with continuous functions)
5. COV(1D) holds
6. RCOV implies COV (if it holds for continuous functions, it holds for integrable functions)
7. COV holds for coordinate swaps T_{ij}

Theorem: **Baby Sard's Theorem:** If $A \in \mathbb{R}^n$ is open and $g : A \to \mathbb{R}^n$ is continuously differentiable, given

$$C := \{x \in A : \det g'(x) = 0\} \tag{0.5}$$

then $g(C)$ is of measure 0.

But more importantly, the corollary:

1. In the COV theorem, we can drop the condition that g' is 1-1.

Definition: Let V be a vector space over $\mathbb{R}$, then

$$T : V^k \to \mathbb{R} \tag{0.6}$$

is called multilinear or k-linear if

$$T(u_1, \dots, \alpha u_i' + \beta u_i'', \dots, u_k) = \alpha T(u_1, \dots, u_i', \dots, u_k) + \beta T(u_1, \dots, u_i'', \dots, u_k). \tag{0.7}$$

Definition: A k-tensor $\mathcal{T}^k(V)$ is the set of k-linear maps on v, i.e.

A few claims about the k-vector:

1. $\mathcal{T}^k(V)$ is a vector space.
2. $T_1 \otimes T_2 = T_1 T_2 \in \mathcal{T}^{k+\ell}$

Theorem: Let V have a basis $v_1, \ldots, v_n$ and a dual basis $\varphi_1, \ldots, \varphi_n$. Then:

$$\varphi_I = \{\varphi_I : I \in \underline{n}^k\} \tag{0.8}$$

is a basis of $\mathcal{T}^k V$ and hence $\dim \mathcal{T}^k(V) = n^k$.

Which is done through lemmas:

1. If $T_1, T_2 \in \mathcal{T}^k(V)$, then
$$T_1 = T_2 \iff \forall I, T_1(v_I) = T_2(v_I) \tag{0.9}$$
2. $\{\varphi_I\}$ spans $\mathcal{T}^k(V)$
3. φ_I are linearly independent.

Definition: Suppose $L : V \to W$ is a linear map. Then there exists a function $L^* : \mathcal{T}^k(W) \to \mathcal{T}^k(V)$, defined by

$$T \mapsto (L^*T)(u_1, \ldots, u_k) = T(Lu_1, \ldots, Lu_k) \tag{0.10}$$

where $T \in \mathcal{T}^k W$ and $u_i \in V$.

We make a few claims:

1. If $T \in \mathcal{T}^k W$, then $L^*T \in \mathcal{T}^* V$.
2. The map $L^* : \mathcal{T}^k W \to \mathcal{T}^k V$ is linear.
3. L^* is compatible with the tensor product $\otimes$. If $T_1 \in \mathcal{T}^k W, T_2 \in \mathcal{T}^\ell W$, then
$$L^*(T_1 T_2) = (L^* T_1)(L^* T_2) \tag{0.11}$$

Definition: $T \in \mathcal{T}^k$ is alternating if $T(\ldots, u, \ldots, w, \ldots) = -T(\ldots, w, \ldots, u, \ldots)$. Then:

$$\Lambda^k(V) = \{T \in \mathcal{T}^k V : T \text{ is alternating}\}. \tag{0.12}$$

We can do a lot of the same things as before, but introducing some group theory permutation notation:

Theorem: There exists a unique function sign : $S_k \to \{\pm\}$ such that

$$\text{sign}(\sigma\tau) = \text{sign}(\sigma)\text{sign}(\tau) \tag{0.13}$$

and

$$\text{sign}(\tau_{ij}) = -1. \tag{0.14}$$

This is alos compatible with pullbacks. Namely:

1. If $T \in \Lambda^k V$ and $\sigma \in S_k$, then $\tau \circ \sigma^* = (-1)^\sigma T$ where
$$\sigma^*(v_1, \ldots, v_k) = (v_{\sigma 1}, \ldots, v_{\sigma k}). \tag{0.15}$$

Definition: If $I \in \underline{n}^k$, then:

$$\omega_I = \sum_{\sigma \in S_k} (-1)^\sigma \cdot \varphi_I \circ \sigma^* \tag{0.16}$$

Definition: $\{\omega_I : I \in \underline{n}_a^k\}$ is a basis for $\Lambda^k V$ and so $\dim \Lambda^k V = \binom{n}{k}$.

We prove through a series of steps:

1. $\omega_I(v_J) = \delta_{IJ}$
2. $\lambda_1, \lambda_2 \in \Lambda^k V$, then $\lambda_1 = \lambda_2 \iff \forall I \in \underline{n}_a^k, \lambda_1(V_I) = \lambda_2(V_I)$
3. Given λ, we can find a_I such that
$$\lambda = \sum a_I \omega_I \tag{0.17}$$
4. the ω_I are linearly independent.

Theorem: There exists a unique family of bilinear operations

$$\wedge : \Lambda^k(V) \times \Lambda^\ell(V) \to \Lambda^{k+\ell}(V) \tag{0.18}$$

such that it is
1. Associative
2. Super-commutative
$$\omega^\lambda = (-1)^{k\ell} \lambda \wedge \omega \tag{0.19}$$
3. $\omega_I = \varphi_{i_1} \wedge \varphi_{i_2} \wedge \cdots \wedge \varphi_{i_k}$

Pullbacks are compatible with the wedge product, namely:

1. $L^*(\lambda \wedge \eta) = (L * \lambda) \wedge (L^* \eta)$