

# PHY293: Waves and Modern Physics

## Summary

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Fall 2021

*Disclaimer: A large portion of this document is and will be stolen from David Morin's [Waves](#) book, which is currently still a draft.*

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# 1 Damped Harmonic Motion

## 1.1 Introduction

**The Setup:** An object undergoing Damped harmonic motion experiences a restoring force  $-kx$  and a resistive force  $-b\frac{dx}{dt}$ . The differential equation is:

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2x \quad (1)$$

where  $\gamma = \frac{b}{m}$  and  $\omega_0^2 = \frac{k}{m}$ .

**Warning:** Most authors prefer to write the differential equation as

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2x$$

as it makes the solution less complicated (i.e. less fractions). Therefore, be very careful when trying to find equations online as we may not all be defining variables the same way.

**Motivation for Solution:** In general, a solution to a second order linear differential equation is a sum of exponentials, i.e. it is in the form of

$$x(t) = Ae^{\alpha_1 t} + Be^{\alpha_2 t}$$

where  $\alpha_1, \alpha_2$  are solutions to a particular quadratic equation (see Appendix for details), where there are three options:

- Quadratic equation has 2 solutions  $\implies$  Then  $x(t)$  is a sum of 2 exponential decays.
- Quadratic equation has 1 solution  $\implies$  Then  $x(t)$  is a single exponential decay.
- Quadratic equation has 0 solutions  $\implies$  Then the roots are complex. Recall from ESC194 that complex exponents lead to sinusoidal functions, so  $x(t)$  will have a sinusoidal component.

## 1.2 Underdamping ( $\gamma < 2\omega_0$ )

We can define

$$\omega^2 = \omega_0^2 - \frac{\gamma^2}{4}, \quad (2)$$

which will be the new angular frequency. *Damping reduces the frequency.*<sup>1</sup> The equation of motion is given by

$$x_{\text{underdamped}}(t) = Ae^{-\gamma t/2} \cos(\omega t + \phi) \quad (3)$$

where  $A$  and  $\phi$  are determined by initial conditions.

## 1.3 Overdamping ( $\gamma > 2\omega_0$ )

If  $\gamma > 2\omega_0$ , then the equation of motion is given by

$$x_{\text{overdamped}}(t) = C_1 e^{-\mu_1 t} + C_2 e^{-\mu_2 t} \quad (4)$$

where we have<sup>2</sup>

$$\mu_1 = \frac{\gamma}{2} + \sqrt{\frac{\gamma^2}{4} - \omega_0^2} \quad (5)$$

$$\mu_2 = \frac{\gamma}{2} - \sqrt{\frac{\gamma^2}{4} - \omega_0^2} \quad (6)$$

and  $C_1, C_2$  are determined by initial conditions.

<sup>1</sup>However, this is mostly irrelevant, because if  $\gamma$  is large enough to make  $\omega$  differ appreciably from  $\omega_0$ , then the motion becomes negligible after a few cycles anyways. For example, if  $\omega$  differs from  $\omega_0$  by even 20%, then after just 2 cycles, the amplitude would have decrease to 0.01% of the initial.

<sup>2</sup>This shows why most physicists choose to use the  $2\gamma$  factor, as it reduces a lot of fractions.

## 1.4 Critical Damping ( $\gamma = 2\omega_0$ )

Critical damping occurs at  $\gamma = 2\omega_0$ , then the equation of motion is given by

$$x_{\text{critical}}(t) = (A + Bt)e^{-\omega_0 t} \quad (7)$$

where  $A$  and  $B$  are determined by initial conditions.

**Importance of Critical Damping:** Critically damped motion has the property that it *converges* to the origin in the quickest manner, that is, quicker than both the overdamped and underdamped motions.

## 1.5 Energy of Underdamped Oscillations

*Note: We will only focus on very underdamped oscillations.*

**Underdamped:** For simplicity, let us assume  $\phi = 0$ . The energy of a damped harmonic oscillator is:

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2. \quad (8)$$

Substituting in  $x(t)$  gives

$$E = \frac{1}{2}mA_0^2 \exp(-\gamma t) (m\omega_0^2 \sin^2(\omega_0 t) + k \cos^2(\omega_0 t) + \omega_0^2) \quad (9)$$

This is very messy, so we want to make approximations.

**Very Underdamped:** If we look at the case where  $\gamma \ll \omega_0$ , we can reduce this to

$$E = \frac{1}{2}mA_0^2\omega_0^2 \exp(-\gamma t) = E_0 \exp(-\gamma t) \quad (10)$$

as the  $\gamma, \gamma^2$  terms approach zero. We can double check that when  $\gamma = 0$ , this reduces to  $E = \frac{1}{2}kA_0^2$ .

We can define the lifetime to be  $\tau = \frac{1}{\gamma}$ .

### 1.5.1 Rate of Energy Loss

The rate of energy loss in a **very underdamped** system is given by

$$\frac{dE}{dt} = -\gamma E \quad (11)$$

Note that  $E$  here represents the average energy over a period  $T$ .

### 1.5.2 Q-Factor

We can define the  $Q$ -factor to be

$$Q = \frac{\omega_0}{\gamma}. \quad (12)$$

If we consider a very underdamped oscillator (where  $\gamma \ll \omega_0$ ), then

$$\frac{E(t_1) - E(t_1 + T)}{E(t_1)} \approx \gamma T \approx \frac{2\pi\gamma}{\omega} = \frac{2\pi}{Q}. \quad (13)$$

Therefore, we have  $Q = \frac{E(t_0)}{(E(t_1) - E(t_1 + T))/2\pi}$ , which is equivalent to the ratio of the initial energy divided by the energy loss per radian:

$$Q = \frac{\text{initial energy stored}}{\text{energy loss per radian}} \quad (14)$$

We can also use the  $Q$ -factor to write the differential equation as

$$\frac{d^2x}{dt^2} + \frac{\omega_0}{Q} \frac{dx}{dt} + \omega_0^2 x = 0 \quad (15)$$

and

$$\omega = \omega_0 \left(1 - \frac{1}{4Q^2}\right)^{1/2} \quad (16)$$

## 2 Driven Harmonic Motion

x' Suppose there is a driving force of the form  $F = F_0 \cos(\omega t)$ . Our differential equation becomes

$$F_0 \cos(\omega t) = \frac{d^2 x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x \quad (17)$$

### 2.1 Undamped Forced Oscillations

The solution is in the form of

$$x(t) = A(\omega) \cos(\omega t - \delta) \quad (18)$$

where

$$\tan \delta = 0 \quad (19)$$

so  $\delta = 0$  (if  $\omega < \omega_0$ ) or  $\delta = \pi$  (if  $\omega > \omega_0$ ). We have

$$A(\omega) = \left| \frac{a}{1 - \omega^2/\omega_0^2} \right| \quad (20)$$

where  $a \equiv \frac{F_0}{k}$ .

### 2.2 Damped Forced Oscillations

Similarly, the equation is in the same form, except

$$\tan \delta = \frac{\omega \gamma}{\omega_0^2 - \omega^2} \quad (21)$$

and

$$A(\omega) = \frac{\omega_0^2 a}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\omega \gamma)^2}} \quad (22)$$

There are three important regimes:

- $\omega \rightarrow 0$  gives  $A(\omega) \rightarrow a = \frac{F_0}{k}$
- $\omega \rightarrow \omega_0$  gives  $A(\omega) \rightarrow \frac{a\omega_0}{\gamma}$
- $\omega \rightarrow \infty$  gives  $A(\omega) \rightarrow 0$

The phase shift is separated into three regimes as well:

- $\omega \rightarrow 0$  gives  $\delta \rightarrow 0$
- $\omega \rightarrow \omega_0$  gives  $\delta \rightarrow \frac{\pi}{2}$
- $\omega \rightarrow \infty$  gives  $\delta \rightarrow \pi$ .

### 2.3 Power

The power of the damping force is

$$\bar{P}_{\text{damping}} = -\frac{1}{2} b (\omega A)^2 \quad (23)$$

and

$$\bar{P}_{\text{driving}} = \frac{1}{2} F_0 \omega A \sin \delta = \frac{1}{2} b (\omega A)^2 \quad (24)$$

after making the substitution  $\sin \delta = \frac{\gamma \omega m A}{F_0}$ . We can also write  $\bar{P}_{\text{driving}}$  in terms of the frequency:

$$\bar{P}_{\text{driving}} = \frac{F_0^2}{2\gamma m} \cdot \frac{\gamma^2 \omega^2}{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2} \quad (25)$$

### 2.3.1 The $\bar{P}(\omega)$ Curve

If the driving frequency is close to the natural frequency, we can write

$$\omega^2 - \omega_0^2 \approx -2\omega_0\Delta\omega, \quad (26)$$

such that

$$\bar{P}(\omega) = \frac{F_0^2}{2m\gamma} \frac{\gamma^2}{4(\Delta\omega)^2 + \gamma^2} \quad (27)$$

## 2.4 Transient Phenomena

Recall that there are actually two solutions to the differential equation. One is the steady state forced oscillations, however there is another solution known as the transient response:

$$x_1(t) + x_2(t) = A(\omega) \cos(\omega_d t - \delta) + B \exp\left(-\frac{\gamma t}{2}\right) \cos(\omega_{\text{damped}} t + \phi) \quad (28)$$

### 3 Appendix

#### 3.1 Derivations

Guessing a solution of the form  $x(t) = Ce^{\alpha t}$ , and substituting it into the ODE gives the characteristic equation

$$\alpha^2 + \gamma\alpha + \omega_0^2 = 0 \quad (29)$$

which has the solution

$$\alpha = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}}{2} \quad (30)$$

The three cases for what the discriminant  $\gamma^2 - 4\omega_0^2$  can be gives us the three cases of motion.

**Underdamping:** In this case,  $\sqrt{\gamma^2 - 4\omega_0^2}$  is an imaginary number, so let us write:  $\alpha = -\gamma/2 + i\sqrt{4\omega_0^2 - \gamma^2}/2$ . We can define  $\omega$  such that

$$\alpha = -\gamma/2 + i\omega. \quad (31)$$

Substituting this back into our original guess (and using a linear combination), we get:

$$x_{\text{underdamped}}(t) = C_1 e^{(-\gamma/2 + i\omega)t} + C_2 e^{(-\gamma/2 - i\omega)t} \quad (32)$$

$$= e^{-\gamma t/2} (C_1 e^{i\omega t} + C_2 e^{-i\omega t}) \quad (33)$$

Since  $x(t)$  has to be real, the part inside the parentheses has to be real, which means the two terms are complex conjugates of each other. This means that if  $C_1 = C e^{i\phi}$ , then we must have  $C_2 = C e^{-i\phi}$ . Making this substitution leads to

$$x_{\text{underdamped}}(t) = 2C e^{-\gamma t/2} \cos(\omega t + \phi) \quad (34)$$

**Overdamping:** Note that  $\mu_1$  and  $\mu_2$  are simply the two solutions to the characteristic equation, so we are left with a simple sum of exponentials:

$$C_1 e^{-\mu_1 t} + C_2 e^{-\mu_2 t}, \quad (35)$$

obtained by straightforward substitution.

**Critical Damping:** There is only one root, so a naive guess may be that  $x(t) = C e^{-\gamma/2 t}$ , but this cannot be the case as there is only one parameter,  $C$ , which cannot satisfy two freely chosen initial conditions (i.e. initial position and velocity). It turns out (covered in ESC194) that another solution is  $t e^{-\gamma/2 t}$ , so the full solution is the linear combination

$$x_{\text{critically damped}}(t) = C_1 e^{-\gamma/2 t} + C_2 t e^{-\gamma/2 t} \quad (36)$$

$$= (C_1 + C_2 t) e^{-\gamma/2 t} \quad (37)$$