MAT367: Differential Geometry

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Contents

1 Manifold

Definition: Let $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^n$ be open. A map

$$F : U \to V \tag{1.1}$$

is called **smooth** if it is infinitely differentiable. The collection of all smooth maps from U to V is denoted as $C^\infty(U, V)$.

A map $F \in C^\infty(U, V)$ is a **diffeomorphism** if it has a smooth inverse. For example, $e^x : \mathbb{R} \to \mathbb{R}_{>0}$ is a diffeomorphism. On the other hand, $x^3 : \mathbb{R} \to \mathbb{R}$ is smooth and invertible, but its inverse is not smooth.

Definition: Given a smooth map $F : U \to V$ and $x \in U$, its **Jacobian** matrix is the $n \times m$ matrix of partial derivatives,

$$Df(x) = \left[\frac{\partial F^i}{\partial x^j}(x)\right]_i^j. \tag{1.2}$$

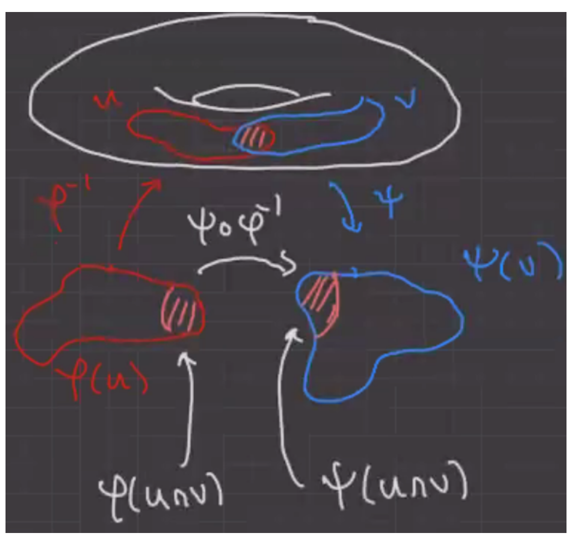
If $n = m$, the determinant of $DF(x)$ is called the **Jacobian determinant.**

Theorem: **Inverse Function Theorem:** A function $F : U \to V$ is a diffeomorphism if and only if,
- F is invertible and smooth.
- $\forall x \in U$, $DF(x)$ is invertible.

The proof is provided in MAT257.

Definition: Let M be a set.
(a) An m-dimensional (coordinate) chart (U, φ) on M is a subset $U \subset M$ together with a map $\varphi : U \to \mathbb{R}^m$ such that $\varphi(U)$ is open and φ is bijective onto its image. Here, U is the **chart domain** and φ is the **coordinate map.**
(b) Two charts (U, φ) and (V, ψ) are **compatible** if
  (a) $\varphi(U \cap V)$ and $\psi(U \cap V)$ are open
  (b) $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is a diffeomorphism.

Note, in MAT257, we defined coordinate charts from $\mathbb{R}^k$ to the manifold M. Here, it is the other way around. The definitions are equivalent since φ is a diffeomorphism. This new perspective is interesting because we are no longer embedding M into $\mathbb{R}^k$, but instead giving it a manifold structure. As a *result* of this structure, we can then get a topology.



Also note that if $U \cap V = \varphi$, then (U, φ) and (V, ψ) are automatically compatible.

Definition: Given a chart (U, φ), the composition

$$U \xrightarrow{\varphi} \varphi(U) \subset \mathbb{R}^n \xrightarrow{pr^i} \mathbb{R} \tag{1.3}$$

is equivalent to $u^i : U \to \mathbb{R}$ and are called the **coordinate functions** of φ. Given $p \in U$, the n-tuple $(u^1(p), \ldots, u^n(p))$ is called the **coordinates** of p in this chart. The transition maps $\psi \circ \varphi^{-1}$ are called **change of coordinates.**

Definition: An m-dimensional **atlas** on a set M is a collection of coordinate charts $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$, such that

1. $\bigcup\limits_\alpha U_\alpha = M$
2. $\forall \alpha, \beta$, $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are compatible.

Let us look at some examples.

Example 1: Let M be the set of **affine** lines in $\mathbb{R}^n$. By affine, we mean it is just a line, and doesn't necessarily need to go through the origin. To find an atlas, define $U = \{\ell | \ell \text{ is not vertical}\}$ and $V = \{\ell | \ell \text{ is not horizontal}\}$. Any $\ell \in U$ can be written as $y = mx + b$, so the map

$$\begin{aligned} \varphi : U &\to \mathbb{R}^2 \\ y = mx + b &\mapsto (m, b) \end{aligned}$$

is a bijection. Similarly, any $\ell \in V$ can be written as $x = my + b$, so the map

$$\begin{aligned} \psi : V &\to \mathbb{R}^2 \\ x = my + b &\mapsto (m, b). \end{aligned}$$

We propose that the above is an atlas. Clearly, this covers M so we just need to check compatibility. Note that $U \cap V$ is the collection of lines which are not horizontal and not vertical. We have,

$$\begin{aligned} \varphi(U \cap V) &= \mathbb{R}^2 - \{\text{y-axis}\} \\ &= \{(m, b) : m \neq 0\}, \end{aligned}$$

so it is open. Similarly, $\psi(U \cap V)$ is also open. Finally, we need to check that the transition function is a diffeomorphism, which is true since it maps

$$(m, b) \mapsto \left(\frac{1}{m}, -\frac{b}{m}\right),$$

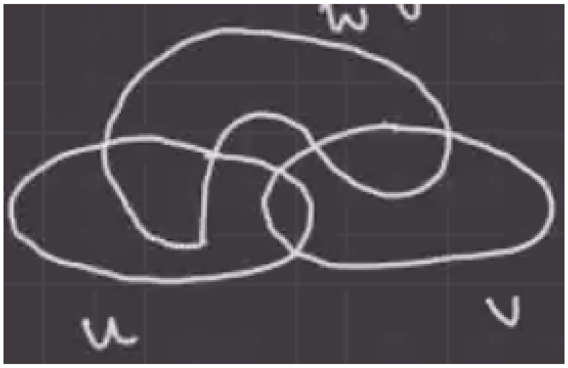
which is smooth. Interestingly, M is the *infinite Mobius band.*

Note that we cannot simply define a manifold to be a set with an atlas. If this was true, then we could potentially have two atlases that describe a single manifold that don't necessarily agree with each other.

Definition: Suppose that $\mathcal{A} = \{(U_\alpha, \psi_\alpha)\}$ is an m-dimensional atlas on M. Let (U, φ) be a chart on M. We say that (U, φ) is compatible with $\mathcal{A}$ if it is compatible with $(U_\alpha, \varphi_\alpha)$, for all α.

Note that (U, φ) is compatible with $\mathcal{A}$ if and only if $\{(U, \varphi)\} \cup \mathcal{A}$ is an atlas on M. This implies that given any atlas, there is a *maximal atlas* that contains it. We want to define $\tilde{\mathcal{A}}$ as the union of all charts which are compatible with $\mathcal{A}$. To do so, we can simply take the union of all charts that are compatible with $\mathcal{A}$.

However, this isn't immediately obvious since the compatibility of charts is not obviously an equivalence relation. In other words, if (U, φ) and (V, ψ) are compatible with $\mathcal{A}$, are they compatible with each other? This is not true since we could get an empty triple intersection, as shown below:



However, since *atlases* covers M, this is not an issue.

Lemma 1: Let $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ be an atlas on M. If (U, φ) and (V, Ψ) are compatible with $\mathcal{A}$, then they are compatible with each other.

Proof. For every chart, the sets $\varphi_\alpha(U \cap U_\alpha)$ and $\varphi_\alpha(V \cap U_\alpha)$ are open, hence their intersection is open. This intersection is

$$\varphi_\alpha(U \cap U_\alpha) \cap \varphi_\alpha(V \cap U_\alpha) = \varphi_\alpha(U \cap V \cap U_\alpha).$$

Since $\varphi \circ \varphi_\alpha^{-1}$ is a diffeomorphism, we have that

$$\varphi(U \cap V \cap U_\alpha) = (\varphi \circ \varphi_\alpha^{-1})(\varphi_\alpha(U \cap V \cap U_\alpha)).$$

is also open. Finally, we have that $\varphi(U \cap V) = \bigcup_\alpha \varphi(U \cap V \cap U_\alpha)$, which implies $\varphi(U \cap V)$ is open.

Why is $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ smooth? This is because

$$\varphi(U \cap V \cap U_\alpha) \xrightarrow{\varphi_\alpha \circ \varphi^{-1}} \varphi_\alpha(U \cap V \cap U_\alpha) \xrightarrow{\psi \circ \varphi_\alpha^{-1}} \psi(U \cap V \cap U_\alpha).$$

Therefore, $\psi \circ \varphi^{-1}\Big|_{U \cap V \cap U_\alpha}$ is smooth, being the composition of smooth maps. Hence, since U_α covers M, $\psi \circ \varphi^{-1}$ is smooth. □

Theorem: Given an atlas $\mathcal{A}$ on M, let $\tilde{A}$ be the collection of all charts that are compatible with $\mathcal{A}$. Then $\mathcal{A}$ is an atlas on M containing $\mathcal{A}$, and be the largest such.

Definition: An atlas on a set M is called **maximal** if it is not properly contained in any larger atlas. Any atlas for M determines a maximal atlas, namely $\tilde{A}$.

We can finally define a manifold,

Definition: A manifold is a set M together with a maximal atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ such that

1. M is covered by countably many charts
2. (Hausdorff condition) For any distinct points $p, q \in M$, there are coordinate charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) such that $p \in U_\alpha$, $q \in U_\beta$, $U_\alpha \cap U_\beta = \emptyset$. The charts $(U_\alpha, \varphi_\alpha) \in \mathcal{A}$ are called the coordinate charts on M.

Let us give some examples.

Example 2: Let $M = \mathbb{R}^n$ with an atlas given by $\{(U_x, \varphi_x)\}$ where for $x \in \mathbb{R}^n$, $U_x = \{x\}$ and $\varphi_x : U_x \to \{0\} = \mathbb{R}^0$ is the unique map. This is an atlas, but fails the countability criteria.

Example 3: Let $X = \mathbb{R} \times \{-1, 1\}$ (i.e. two copies of $\mathbb{R}$). Consider the equivalence relation on X generated by $(x_0, 1) \sim (x_1, -1) \iff x_0 = x_1 < 0$. This is represented by the picture below,

Let $M = X/\sim$. Let $\pi : X \to M$ be the quotient map, and let

$$U = \pi(\mathbb{R} \times \{1\}), \qquad V = \pi(\mathbb{R} \times \{-1\}). \tag{1.4}$$

If $f : X \to \mathbb{R}$ defined by $(x, i) \mapsto x$, then f defines a functions

$$\tilde{f} : M \to \mathbb{R} \tag{1.5}$$

such that $f|_U$ and $f|_V$ are bijectors onto $\mathbb{R}$. So, $(U, f|_U)$ and $(V, f|_V)$ is an atlas on M for which the Hausdoff condition fails.

Lemma 2: Let M be a set with a maximal atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ and suppose that $p, q \in M$ are distinct points contained in a single chart (U, φ). Then there exists α, β such that

1. $p \in U_\alpha$, $q \in U_\beta$
2. $U_\alpha \cap U_\beta = \emptyset$