Reading Project: Superconductivity

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1 Problem Set Two

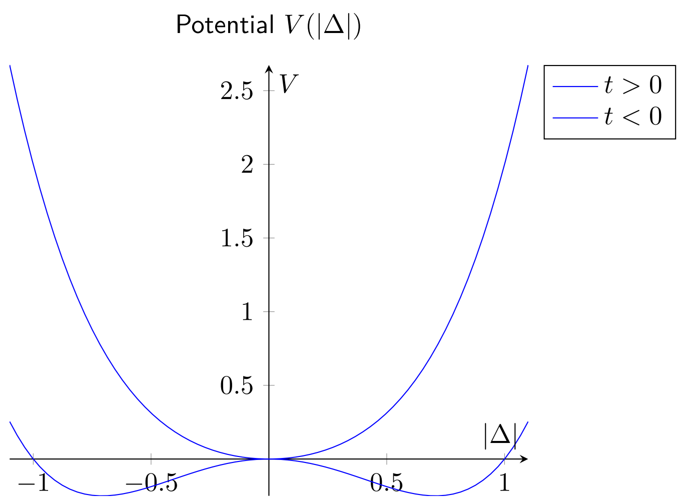
1.1 Ginzburg-Landau Theory of Superconductivity

Consider a potential

$$V(\Delta) = t|\Delta|^2 + u|\Delta|^4, \tag{1.1}$$

which will be derived in the future. Here, Δ is the energy required to break a cooper pair. When t goes from positive to negative, it signifies a phase transition, where

$$t = \frac{T - T_c}{T_c}.$$



For $t > 0$, then $\Delta_* = 0$. For $t < 0$, we have $|\Delta_*|^2 = \frac{-t}{2u}$. Therefore,

$$V(\Delta_*) = \begin{cases} 0 & t > 0 \\ -\dfrac{t^2}{4u} & t < 0. \end{cases}$$

We can use this to determine what the specific heat capacity is (which is something that is measurable). The specific heat capacity is

$$C = -\frac{\partial^2 F}{\partial t^2}.$$

If we assume that we only have potential energy, we have $F = V$, which gives

$$C = \begin{cases} 0 & t > 0 \\ \dfrac{!}{2u} & t < 0, \end{cases}$$

so experimentally there will be a discontinuity. Furthermore, the power law relationship that

$$|\Delta| \sim |t|^{1/2}$$

can be experimentally verified.

1.2 Fermions vs Bosons and 2nd Quantization

Subatomic particles are either fermions (which obey the Pauli exclusion principle), and bosons.

- Electrons are fermions
- Photons are bosons
- Phonons are bosons

Note that no two fermions can be in the same quantum state. For example, consider the two states $|\uparrow\rangle$ and $|\downarrow\rangle$. If we think from a chemistry perspective, no two fermions can be in the s shell with the same spin. This is a *consequence* of the antisymmetry of the wavefunction. Namely, for fermions

$$\psi(x_1, x_2) = -\psi(x_2, x_1),$$

and for bosons,

$$\psi(x_1, x_2) = \psi(x_2, x_1).$$

For **second quantization,** we want to build a Hilbert space for many identical quantum particles. We can define **sectors.**

> **Idea**: In many particle quantum mechanics, the number of particles is not a conserved quantity. For example, we can have annhilationa dn creation of particles (converting it from energy to mass and back). In superconductors, we typically have a bath of electrons where we can freely take and put back electrons.

Instead, we can define **Sectors,** each with a well-defined number of particles. For fermions,

- 0 particles: $|0\rangle$ is the **vacuum vector**
- 1 particle: $\{|i,\sigma\rangle\}_{i=1,\ldots,N_s,\sigma=\uparrow,\downarrow}$ where N_s is the number of sites and σ is the site. We are assuming that there is one occupiable orbital site. This gives $2N_s$ single particle states.
- 2 particles: $|i\sigma\rangle\,|j\tau\rangle$: The naive guess is $(2N_s)^2$ states, but we have to account for the fact that they can't occupy the same state. Instead, there are
$$\binom{2N_s}{2} = \frac{(2N_s)(2N_s-1)}{2}$$
states.
- n particles: We have
$$\binom{2N_s}{n}$$
states, which gets very big, very fast. This is another reason why single-particle quantum mechanics can only get us so far.

We wish to define operators which bring us between different sectors. We can define the **creation operator**

$$c_{i\sigma}^\dagger\,|\sigma\rangle = |i\sigma\rangle$$

and the inverse is the **annihilation operator**, where

$$c_{i\sigma}c_{i\sigma} = |0\rangle\,.$$

Note these identities,

$$\begin{aligned} c_{i\sigma}\,|0\rangle &= 0 \\ (c_{i\sigma})^2 &= 0 \\ (c_{i\sigma}^\dagger) &= 0. \end{aligned}$$

Furthermore, we can deal with the antisymmetric nature of fermions by noting that

$$c_{i\sigma}^\dagger c_{j\sigma'}^\dagger = -c_{j\sigma'}^\dagger c_{i\sigma}^\dagger,$$

so

$$|(i\sigma)(j\sigma')\rangle = -\,|(j\sigma')(i\sigma)\rangle\,.$$

In general, if we define the anti-commutator $\{A, B\} = AB + BA$, then

$$\{c_{i\sigma}, c_{j\sigma'}^\dagger\} = \delta_{ij}\delta_{\sigma\sigma'}. \tag{1.2}$$

Furthermore, annhilation and creation operators anti-commute with themselves, i.e.

$$\begin{aligned} \{c_{i\sigma}, c_{j\sigma'}\} &= 0 \\ \{c_{i\sigma}^\dagger, c_{j\sigma'}^\dagger\} &= 0. \end{aligned}$$

For *bosons,* we have

$$\begin{aligned} [b_{i\sigma}, b_{j\sigma}^\dagger] &= \delta_{ij}\delta_{\sigma\sigma'} \\ [b_{i\sigma}, b_{j\sigma}] &= 0 \\ [b_{i\sigma}^\dagger, b_{j\sigma}^\dagger] &= 0, \end{aligned}$$

where $[A, B] = AB - BA$.

Usually, we can think of an analogy with the quantum harmonic oscillator, where

$$\hat{H} = \hbar\omega \left(\underbrace{a^\dagger a}_{n} + \frac{1}{2} \right), \tag{1.3}$$

where we can treat $a^\dagger a$ as the energy level. As an analogy, we can treat the system as only one energy level, and each new particle we add on a constant energy. This is what motivates the creation and annihilation operators.