# MAT185 Test 2 Review

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## 1 Eigenvectors and Diagonalization

**Definition**: Let $A$ be an $n \times n$ matrix. A vector $\mathbf{x}$ is an eigenvector of $A$ if $\mathbf{x} \neq \mathbf{0}$ and:

$$A\mathbf{x} = \lambda \mathbf{x} \tag{1}$$

for some scalar $\lambda$, known as the **eigenvalue** of $A$ corresponding to $\mathbf{x}$. For a given eigenvalue $\lambda$, the **eigenspace** of $A$ corresponding to eigenvalue $\lambda$ is:

$$E_\lambda(A) = \{\mathbf{x} \in {}^n\mathbb{R} | A\mathbf{x} = \lambda\mathbf{x}\} \tag{2}$$

Note that we also have:

$$E_\lambda(A) = \operatorname{null}(\lambda I - A) \tag{3}$$

It can be helpful to think about eigenvectors geometrically. If we interpret multiplying vectors by $A$ as a linear transformation, then the eigenvectors are vectors that undergo only a stretching under $A$.

Let $A$ be an $n \times n$ matrix. The following statements are equivalent:

$$\begin{aligned} \lambda \text{ is an eigenvalue of } A &\iff A\mathbf{x} = \lambda\mathbf{x} \text{ for some } \mathbf{x} \in {}^n\mathbb{R} \\ &\iff (\lambda I - A)\mathbf{x} = \mathbf{0} \text{ has infinitely many solutions} \\ &\iff \dim \operatorname{null}(\lambda I - A) \neq 0 \\ &\iff \lambda I - A \text{ is not invertible.} \end{aligned}$$

**Proposition** 1: Let $\lambda$ and $\mu$ be distinct eigenvalues of $\mathbf{A} \in {}^n\mathbb{R}^n$. Then:

$$E_\lambda \cap E_\mu = \{\mathbf{0}\} \tag{4}$$

**Definition**: Let $A$ be an $n \times n$ matrix. The **characteristic polynomial** of $A$ is:

$$p_A(\lambda) = \det(\lambda I - A) \tag{5}$$

The eigenvalues are the solutions to $p_A(\lambda) = 0$.

**Theorem**: Let $A$ be an $n \times n$ matrix. The characteristic polynomial of $A$ has the form:

$$p_A(\lambda) = \lambda^n + c_{n-1}\lambda^{n-1} + c_{n-2}\lambda^{n-2} + \cdots + c_1\lambda + c_0 \tag{6}$$

where $c_{n-1} = -\operatorname{tr} A$ and $c_0 = (-1)^n \det A$. Recall that the trace is the sum of the main diagonal.

**Definition**: The matrix $\mathbf{P} \in {}^n\mathbb{R}^n$ is said to diagonalize $\mathbf{A} \in {}^n\mathbb{R}^n$ if $\mathbf{P}$ is invertible such that:

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{\Lambda} \tag{7}$$

where $\mathbf{\Lambda} = \operatorname{diag}\{\lambda_\alpha\}$ is the diagonal matrix of the eigenvalues of $\mathbf{A}$.

**Theorem**: The matrix $\mathbf{P} \in {}^n\mathbb{R}^n$ diagonalizes $\mathbf{A} \in {}^n\mathbb{R}^n$ where:

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \tag{8}$$

if and only if the columns of $\mathbf{P}$ form a basis for ${}^n\mathbb{R}$ consisting of the eigenvectors $\mathbf{p}_\alpha$ of $\mathbf{A}$ where $\mathbf{A}\mathbf{p}_\alpha = \lambda_\alpha \mathbf{p}_\alpha$.

The corollary is that the matrix $\mathbf{A} \in {}^n\mathbb{R}^n$ is diagonalizable if and only if ${}^n\mathbb{R}$ has a basis consisting of eigenvectors of $\mathbf{A}$.

**Proposition 2**: Let $\mathbf{A} \in {}^n\mathbb{R}^n$ and let $\mathbf{T} \in {}^n\mathbb{R}^n$ be invertible. Then the characteristic polynomials of $\mathbf{A}$ and of $\mathbf{T}^{-1}\mathbf{A}\mathbf{T}$ are identical and so the eigenvalues of the two matrices are identical.

**Theorem**: Let $\mathbf{P} \in {}^n\mathbb{R}^n$ diagonalize $\mathbf{A} \in {}^n\mathbb{R}^n$ and let $\lambda_1 \cdots \lambda_n$ be the eigenvalues of $\mathbf{A}$. Then:
(a) $c_{\mathbf{A}}(\lambda) = c_\Lambda(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots$
(b) $\det \mathbf{A} = \det \mathbf{\Lambda} = \lambda_1\lambda_2\cdots\lambda_n$.
(c) $\operatorname{tr} \mathbf{A} = \operatorname{tr} \Lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_n$.

Not all matrices are diagonalizable.

**Theorem**: Let $\mathbf{A} \in {}^n\mathbb{R}^n$ with distinct eigenvalues $\lambda_1 \cdots \lambda_r$, $r \leq n$. If $\mathbf{x}_\alpha \in E_{\lambda_\alpha} \backslash \{\mathbf{0}\}, \alpha = 1 \cdots r$, then $\{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_r\}$ is linearly independent.

**Theorem**: If $\mathbf{A} \in {}^n\mathbb{R}^n$ has $n$ distinct eigenvalues, then $\mathbf{A}$ is diagonalizable.

**Theorem**: Let $\mathbf{A} \in {}^n\mathbb{R}^n$ with distinct eigenvalues $\lambda_1 \cdots \lambda_r$, $r \leq n$. If $\mathbf{x}_\alpha \in E_{\lambda_\alpha}$ and

$$\mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_r = \mathbf{0} \tag{9}$$

then:

$$\mathbf{x}_\alpha = \mathbf{0}, \quad /\alpha = 1 \cdots r \tag{10}$$

**Theorem**: Let $\mathbf{A} \in {}^n\mathbb{R}^n$ with distinct eigenvalues $\lambda_1 \cdots \lambda_r$, $r \leq n$. If $H_{\lambda_\alpha}$ is a linearly independent set in $E_{\lambda_\alpha}$, then:

$$H = H_{\lambda_1} \cup H_{\lambda_2} \cup \cdots \cup H_{\lambda_r} \equiv \bigcup_{\alpha=1}^{r} H_{\lambda_\alpha} \tag{11}$$

is linearly independent and:

$$m_1 + m_2 + \cdots + m_r \leq n \tag{12}$$

where $m_\alpha = \dim E_{\lambda_\alpha}$.

**Definition**: Let $\mathbf{A} \in {}^n\mathbb{R}^n$ with eigenvalues $\lambda_\alpha$. The highest power $n_\alpha$ of $\lambda - \lambda_\alpha$ that divides the characteristic polynomial $p(\lambda)$ such that $p(\lambda) = (\lambda - \lambda_\alpha)^{n_\alpha} g(\lambda)$ is the algebraic multiplicity of $\lambda_\alpha$. The dimension $m_\alpha$ of $E_{\lambda_\alpha}$ is the geometric multiplicity of $\lambda_\alpha$.

**Proposition 3:** Let:

$$\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{B} \\ \mathbf{O} & \mathbf{C} \end{bmatrix} \in {}^{n}\mathbb{R}^{n} \tag{13}$$

where $\mathbf{B} \in {}^{r}\mathbb{R}^{n-r}$, $\mathbf{C} \in {}^{n-r}\mathbb{R}^{n-r}$, and $\mathbf{1} \in {}^{r}\mathbb{R}^{n}$ is the $r \times r$ identity matrix. Then: $\det \mathbf{A} = \det \mathbf{C}$.

**Theorem: Multiplicity Theorem:** Let $\lambda_\alpha$ be an eigenvalue of $\mathbf{A} \in {}^{n}\mathbb{R}^{n}$. Then $1 \leq m_\alpha \leq n_\alpha$, where $m_\alpha$ and $n_\alpha$ are respectively, the geometric and algebraic multiplicities of $\lambda_\alpha$. In particular, if $n_\alpha = 1$, then $m_\alpha = n_\alpha = 1$.

**Theorem: Diagonalization Test:** Let $\mathbf{A} \in {}^{n}\mathbb{R}^{n}$ with distinct eigenvalues $\lambda_\alpha, \alpha = 1 \cdots r$. Then, $\mathbf{A}$ is diagonalizable if and only if $m_\alpha = n_\alpha$, $\alpha = 1 \cdots R$, i.e. the geometric and algebraic multiplicities of each eigenvalue are equal.