AER210: Vector Calc and Fluid Mechanics

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Contents

1 Double Integrals

- **Integrals Involving a Parameter**

> **Example 1:** Let $\int_0^1 Cx^3 \, \mathrm{d}x$ where C is a constant. Then it gives
>
> $$\int_0^1 Cx^3 \, \mathrm{d}x = \frac{1}{4}C \tag{1}$$
>
> The result contains C.

- Suppose we have something like

$$\int_a^b f(x, y) \, \mathrm{d}x = g(y) \tag{2}$$

and therefore y is a parameter

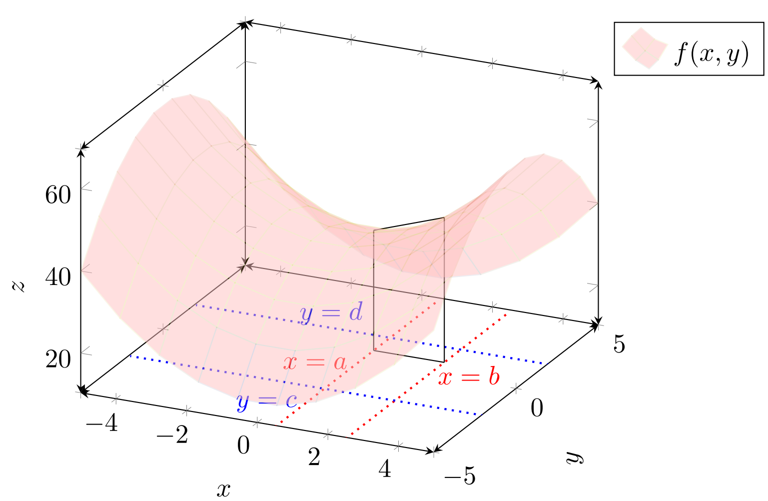
> **Definition**: A variable which is kept constant during an integration is called a parameter.

- Partial integration wrt x

> **Example 2:** An example of partial integration wrt x is
>
> $$\int_0^1 x^3 y \, \mathrm{d}x = y \int_0^1 x^3 \, \mathrm{d}x = \frac{1}{4}y \tag{3}$$

- Notice the similarity between partial differentiation wrt x, $f_x(x, y)$ and the partial integration wrt x, $\int_a^b f(x, y) \, \mathrm{d}x$.
- **Iterated Integrals** (Integral of an Integral)
- Consider $x = f(x, y)$ where $x \in [a, b]$, $y \in [c, d]$. This defines a rectangular region.
- Assume that $f(x, y) \geq 0$. This can be represented as a surface, as shown below:

Example Slice



If we take the integral $\int_{y=c}^d f(x, y) \, \mathrm{d}y = A(x)$, we see that the area of the slice depends on x.

If we suppose that the surface has a tiny thickness Δx, then the volume is

$$\Delta V(x) = A(x) \cdot \Delta x = \left(\int_{y=c}^d f(x, y) \, \mathrm{d}y \right) \Delta x \tag{4}$$

If we break up the interval $[a, b]$ into N segments

$$x_0 = a \leq x_1 \leq x_2 \leq \dots x_{i-1} \leq x_i \leq \cdots \leq x_{N-1} \leq x_N = b \tag{5}$$

with $\Delta x_i = x_i - x_{i-1}$. We can then approximate the volume as

$$V \approx \sum_{i=1}^{N} \Delta V_i = \sum_{i=1}^{N} A(x_i) \Delta x_i \tag{6}$$

which is known as a **Riemann sum**.

Idea: As we take the limit as $N \to \infty$ which implies $\Delta x_i \to 0$, we get the double integral:

$$V = \int_a^b \int_c^d f(x, y) \, \mathrm{d}y \, \mathrm{d}x \tag{7}$$

which can be determined by calculating two integrals.

- Similarly, we can find the volume by taking slices parallel to the xz plane.

  The area of each slice is a function of y:

$$A(y) = \int_a^b f(x, y) \, \mathrm{d}x \tag{8}$$

so we have $\Delta V(y) = A(y) \cdot \Delta y$. Again, summing up all slices and taking the limit, we get

$$V = \int_c^d A(y) \, \mathrm{d}y = \int_c^d f(x, y) \, \mathrm{d}x \, \mathrm{d}y \tag{9}$$

Theorem: **Fubini's Theorem** tells us that

$$\int_a^b \int_c^d f(x, y) \, \mathrm{d}y \, \mathrm{d}x = \int_c^d \int_a^b f(x, y) \, \mathrm{d}x \, \mathrm{d}y \tag{10}$$

The analog for equality of mixed partial derivatives is known as **Clairut's Theorem.**

Example 3: Find the volume under the surface $z = x^2 y$ where $x \in [1, 3]$ and $y \in [0, 1]$. We first form the integral by integrating wrt y. We have

$$\begin{aligned} V &= \int_1^3 \int_0^1 x^2 y \, \mathrm{d}y \, \mathrm{d}x && (11) \\ &= \int_1^3 x^2 (1^2/2 - 0^2/2) \, \mathrm{d}x && (12) \\ &= \int_1^3 \frac{x^2}{2} \, \mathrm{d}x && (13) \\ &= \frac{13}{3} && (14) \end{aligned}$$

We can also form the integral by integrate it wrt x:

$$\begin{aligned} V &= \int_0^1 \int_1^3 x^2 y \, \mathrm{d}x \, \mathrm{d}y && (15) \\ &= \int_0^1 \frac{26}{3} y \, \mathrm{d}y && (16) \\ &= \frac{13}{3} && (17) \end{aligned}$$

so we can confirm they give the same answer.

Example 4: Evaluate the double integral of $f(x,y) = x - 3y^2$ over region R where

$$R = \{(x,y)|0 \leq x \leq 2, 1 \leq y \leq 2\} \tag{18}$$

To do this, we have

$$\int_0^2 \int_1^2 (x - 3y^2)\,\mathrm{d}y\,\mathrm{d}x = \int_0^2 (xy - y^3)\Big|_{y=1}^{y=2}\,\mathrm{d}x \tag{19}$$

$$= \int_0^2 (x - 7)\,\mathrm{d}x \tag{20}$$

$$= -12 \tag{21}$$

- Note that in the special case where the function $f(x,y)$ is $f(x,y) = g(x) \cdot h(y)$, then

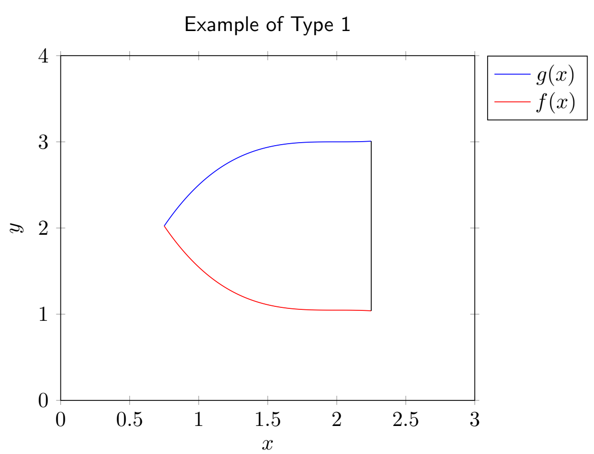
$$\int_c^d \int_a^b f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_c^d \left[h(y) \int_a^b g(x)\,\mathrm{d}x \right] \mathrm{d}y = \int_a^b g(x)\,\mathrm{d}x \cdot \int_c^d h(y)\,\mathrm{d}y \tag{22}$$

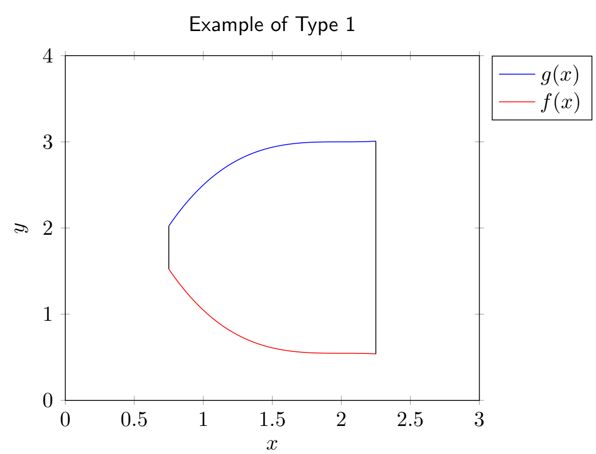
This gives us a shortcut of evaluating double integrals in this form.

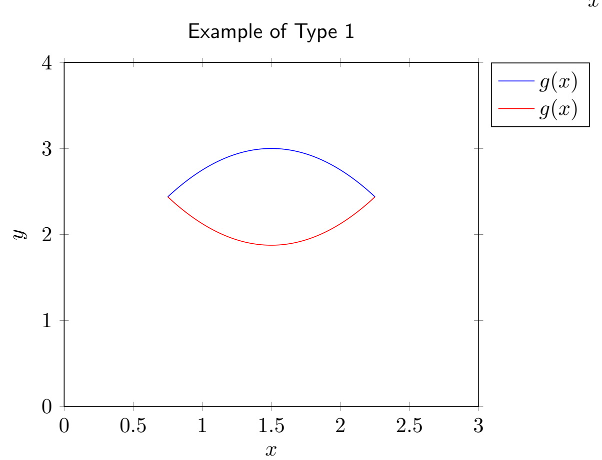
- Double integrals over general regions (What if region is non-rectangular?)
- **Type 1 Region** is in the form of

$$R = \{(x,y)|a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\} \tag{23}$$

Here are some examples

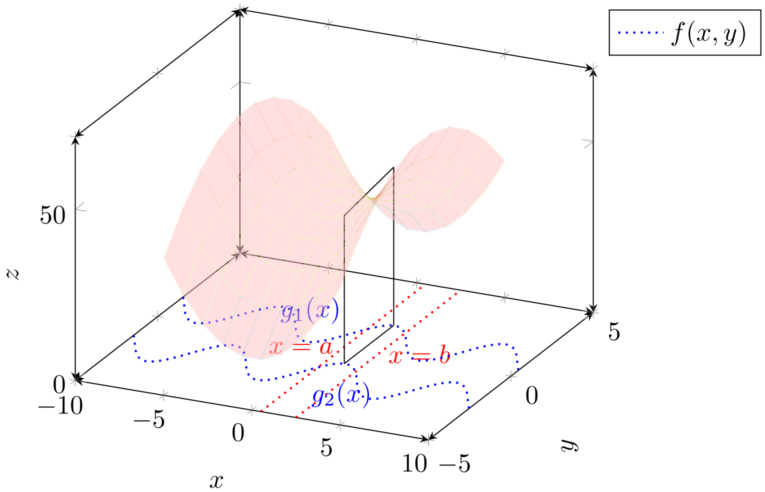






- Let's think about the case where $f(x,y) \geq 0$ on a type-1 region. Suppose we have the following illustration

Setup for Type 1



We find the area of slices, so

$$A(x)=\int_{g_1(x)}^{g_2(x)} f(x,y)\,\mathrm{d}y \tag{24}$$

and the volume is thus

$$V=\int_a^b A(x)\,\mathrm{d}X=\int_a^b\int_{g_1(x)}^{g_2(X)} f(x,y)\,\mathrm{d}y\,\mathrm{d}x \tag{25}$$

- Type-2 regions have the form

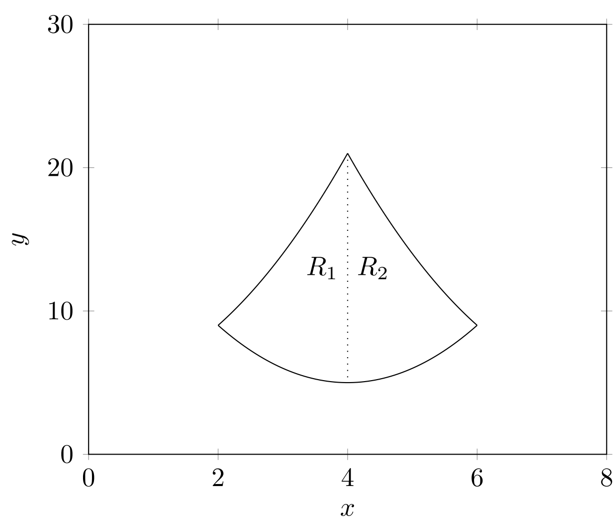
$$R=\{(x,y)|c\le y\le d \text{ and } h_1(y)\le x\le h_2(y)\} \tag{26}$$

In a similar way, the volume bounded by this region is

$$V=\int_c^d\int_{h_1(y)}^{h_2(y)} f(x,y)\,\mathrm{d}x\,\mathrm{d}y \tag{27}$$

- Type-3 regions are neither type-1 nor type-2. It is possible to break up the region into parts that can be classified as either type-1 or type-2:

Neither Type-1 or Type-2



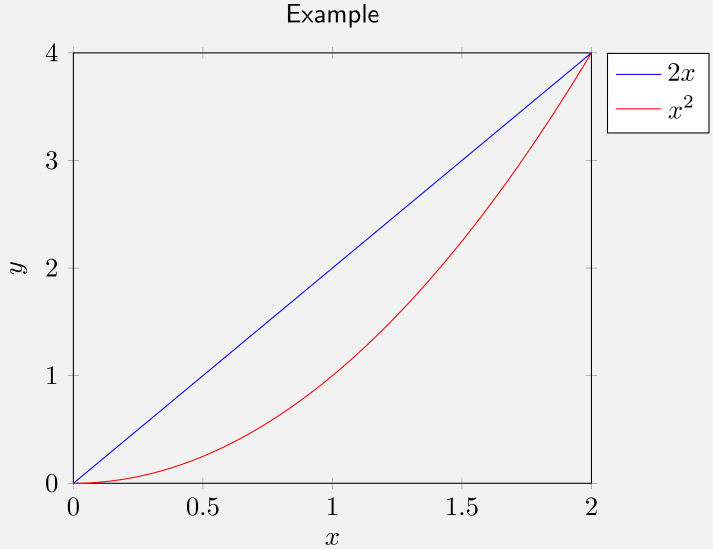
Idea: While these formulas are derived by assuming a positive volume (and thus cannot work if $f<0$), they still work in general.

Example 5: Find the volume of the solid that lies under the surface

$$z = f(x,y) = x^2 + y^2 \tag{28}$$

and above the region R in the xy-plane. The region R is bounded by the straight line $y = 2x$ and the parabola $y = x^2$.

1. First we draw a diagram of the planar region R over which the surface is defined.



2. We then draw a line parallel to the axis of first integration (i.e. vertical lines for integrating in the y-direction first)
3. This gives us

$$V = \int_{x=0}^{x=2} \int_{y=x^2}^{y=2x} f(x,y)\,\mathrm{d}y\,\mathrm{d}x \tag{29}$$

$$= \int_0^2 \int_{x^2}^{2x} (x^2 + y^2)\,\mathrm{d}y\,\mathrm{d}x \tag{30}$$

$$= \frac{216}{35} \tag{31}$$

Alternatively, we can find the volume by integrating in the x direction first. In this case, we need to obtain boundary curves in the $x = x(y)$ form:

$$y = x^2 \implies x = \sqrt{y} \tag{32}$$

$$y = 2x \implies x = y/2 \tag{33}$$

This then gives us

$$V = \int_{y=0}^{y=4} \int_{x=y/2}^{x=\sqrt{y}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y \tag{34}$$

$$= \frac{216}{35} \tag{35}$$

Warning: Do not just pick the minimum and maximum points. For example, the following is *incorrect*

$$\int_{y=0}^{y=4} \int_{x=0}^{x=2} f(x,y)\,\mathrm{d}x\,\mathrm{d}y \tag{36}$$

as that corresponds with a rectangular region.

Example 6: Integrate the surface given by $z = e^{x^2}$ over the following region:

We can first integrate wrt x

$$V = \in_{y=0}^{y=1} \int_{x=y}^{x=1} e^{x^2} \,\mathrm{d}x \,\mathrm{d}y \tag{37}$$

This is a hard problem since we don't know the anti-derivative of e^{x^2}. To solve this, we can first integrate wrt y, which gives us

$$V = \int_{x=0}^{x=1} \int_{y=0}^{y=x} e^{x^2} \,\mathrm{d}y \,\mathrm{d}x = \int_{x=0}^{1} e^{x^2} y\big|_{y=0}^{y=x} \,\mathrm{d}x \tag{38}$$

$$= \int_0^1 e^{x^2} x \,\mathrm{d}x \tag{39}$$

This integral can be more easily solved using the u-sub $u = x^2$, $\mathrm{d}u = 2x\,\mathrm{d}x$ to get

$$V = \frac{1}{2}(e-1) \tag{40}$$

2 Formal Definition of Double Integrals

- We will see two ways of defining double integrals.
- First, let us review the formal definition of definite integrals for functions of a single variable.

  To determine the area under a curve in the region $x \in [a, b]$, we can break the region up into intervals Δx_i, so the Riemann sum is

$$A \approx \sum_{i=1}^{n} f(x_i^*)\Delta x_i \tag{41}$$

  Let $m_i \leq f(x_i^*) \leq M_i$ for $x_i^* \in \Delta x_i$. Then:

$$\sum_{i=1}^{n} m_i \Delta x_i \leq \underbrace{\sum_{i=1}^{n} f(x_i^*)\Delta x_i}_{\text{Estimate of the entire area calculated by Riemann Sum}} \leq \sum_{i=1}^{n} M_i \Delta x_i \tag{42}$$

  If the Δx_i are of equal length and we take the limit, we can define:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i = \int_a^b f(x)\,\mathrm{d}x \tag{43}$$

  If they are not of equal length, we need to define the norm of the partition $\|P\| = (\Delta x_i)_{\mathsf{max}}$ for $i = 1, 2, \ldots, n$. This way, the integral can be alternatively defined as

$$A = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i = \int_a^b f(x)\,\mathrm{d}x \tag{44}$$

- Consider a double integral over rectangular region. Let $z = f(x, y)$ be defined on $R = \{(x, y)|a \leq x \leq b, c \leq y \leq d\}$. Assume $f(x, y) \geq 0$ over R.
- **Formal Definition 1:** We can approximate the volume as

$$\Delta v_i \approx f(x_i^*, y_i^*)\Delta A_i \tag{45}$$

  where $\Delta A_i = \Delta x_i \cdot \Delta y_i$. The Riemann sum is then

$$V \approx \sum_{i=1}^{N} f(x_i^*, y_i^*)\Delta A_i \tag{46}$$

We can pick x_i^*, y_i^* such that $f(x_i^*, y_i^*)$ is the smallest and largest value in the region, we can bound the Riemann sum by:

$$\sum_{i=1}^{N} m_i \Delta x_i \Delta y_i \leq \sum_{i=1}^{N} f(x_i^*, y_i^*) \Delta x_i \Delta y_i \leq \sum_{i=1}^{N} M_i \Delta x_i \Delta y_i \tag{47}$$

Warning: Taking the limit where $N \to \infty$ is not sufficient, as it does not necessarily mean the size of all partitions approach zero.

We define the norm of the partition to be

$$\|P\| = \max(\Delta d_i) \tag{48}$$

for $i = 1, 2, \ldots, N$. Therefore:

$$V = \lim_{\|P\| \to 0} \sum_{i=1}^{N} f(x_i^*, y_i^*) \Delta A_i = \iint_R f(x, y) \, \mathrm{d}A = \iint_R f(x, y) \, \mathrm{d}x \, \mathrm{d}y \, . \tag{49}$$

Idea: Functions that are continuous are integrable over that region.

- **Formal Definition 2:** We are free to divide the region R into any tiling, we can use uniform divisions.

  As a result, the area of each tile is

$$\Delta A_{ij} = \Delta x_i \Delta y_j \tag{50}$$

  where the (i, j) represent the coordinate of the tile. The double Riemann sum is then:

$$V \approx \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta x_i \Delta y_j \tag{51}$$

  Again, we can define m_{ij} and M_{ij} such that

$$\sum_{j=1}^{m} \sum_{i=1}^{n} m_{ij} \Delta x_i \Delta y_j \leq \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta x_i \Delta y_j \leq \sum_{j=1}^{m} \sum_{i=1}^{n} M_{ij} \Delta x_i \Delta y_j \tag{52}$$

  Since these intervals are equally partitioned, we can define

$$V = \lim_{m,n \to \infty} \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A_{ij} = \iint_R f(x, y) \, \mathrm{d}A \, . \tag{53}$$

  If they were not, we would have to define the norm again.

  Example 7: Estimate the volume of the solid that lies above the square $R = [0, 2] \times [0, 2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$. Divide R into four equal squares & choose the sample point to be the upper corner of each square.

  We would then have:

$$\begin{aligned} V &\approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(x_{ij}^*, y_{ij}^*) \Delta A && (54) \\ &\approx f(1, 1)\Delta A + f(1, 2)\Delta A + f(2, 1)\Delta A + f(2, 2)\Delta A && (55) \\ &\approx 34 && (56) \end{aligned}$$

  Note that the actual answer is 48. The approximation will improve as the number of regions increase.

- We can also define double integrals over non-rectangular regions.
- **Definition 1:** We can again tile a region using rectangular regions in two ways:
  - Each tile is contained within R and there are some space.

– Some tiles extend past the boundary of R, which is completely covered.

When we take the limit as $\|P\| \to 0$, both of these tiling methods will approach the actual area, so using any of these tilings will cause the double integral to approach the actual volume.

If $f(x, y)$ is a continuous function over R, then

$$V = \lim_{\|P\|} \sum f(x_i^*, y_i^*)\Delta A_i = \lim_{\|P\|\to 0} \sum_{j=1}^{N} f(x_j^*, y_j^*)\Delta A_j = \iint_R f(x, y)\,\mathrm{d}x\,\mathrm{d}y \tag{57}$$

- **Definition 2:** Similarly, we can use uniform partitions that either leave gaps or extend past the region. We can again define m_{ij} and M_{ij} for each tile R_{ij} such that

$$V = \iint_R f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \lim_{\|P\|\to 0} \sum_{j=1}^{M} \sum_{i=1}^{N} f(x_{ij}^*, y_{ij}^*)\Delta x_i \Delta y_j \tag{58}$$

3 Double Integrals in Polar Coordinates

- Using polar coordinates is helpful when integrating over circular regions.
- Recall that we can convert between rectangular and polar coordinates via

$$x = r\cos\theta, \qquad y = r\sin\theta \tag{59}$$

and that the area of a sector is

$$A = \frac{1}{2}r^2\theta \tag{60}$$

- Suppose we have some function $f(x, y)$ defined on $R = \{(r, \theta) | a \le r \le b, \alpha \le \theta \le \beta\}$. We can then define:

$$f(x, y) = f(r\cos\theta, r\sin\theta) = g(r, \theta). \tag{61}$$

Assume $f(x, y) = g(r, \theta) \ge 0$ on R. Then we can approximate the volume as

$$\Delta V_i \approx g(r_i^*, \theta_i^*) \cdot \Delta A_i = f(r_i^*\cos\theta_i^*, r_i^*\sin\theta_i^*) \cdot r_i \Delta r_i \Delta\theta_i \left(1 + \frac{\Delta r_i}{2r_i}\right). \tag{62}$$

Taking the limit, we have

$$V = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(r_i^*\cos\theta_i^*, r_i^*\sin\theta_i^*) r_i \Delta r_i \Delta\theta_i \tag{63}$$

$$* = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r\,\mathrm{d}r\,\mathrm{d}\theta\,. \tag{64}$$

We can generalize this finding regardless of whether the function is positive or negative over R.

> **Idea**: In a region bounded by $\alpha \le \theta \le \beta$, $a \le r \le b$, we have
>
> $$\iint_R f(x, y)\,\mathrm{d}A = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r\,\mathrm{d}r\,\mathrm{d}\theta\,. \tag{65}$$

- We can extend this to more complicated regions. Suppose R is bounded by $\alpha \le \theta \le \beta$ and $g(\theta) \le r \le g_2(\theta)$. Then the volume would be

$$\iint_R f(x, y)\,\mathrm{d}A = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta) r\,\mathrm{d}r\,\mathrm{d}\theta \tag{66}$$

- Similarly, if R is bounded by $a \le r \le b$ and $h_1(r) \le \theta \le h_2(r)$, we have

$$\iint_R f(x, y)\,\mathrm{d}A = \int_a^b \int_{h_1(r)}^{h_2(r)} f(r\cos\theta, r\sin\theta) r\,\mathrm{d}r\,\mathrm{d}\theta\,. \tag{67}$$

Example 8: Evaluate $\iint_R (3x + 4y^2)\,\mathrm{d}A$ where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

This leads to the region $R = \{(r, \theta) | 1 \leq r \leq 2, 0 \leq \theta \leq \pi\}$. Then:

$$I = \iint_R (3x + 4y^2)\,\mathrm{d}A \tag{68}$$

$$= \int_0^\pi \int_1^2 (3r\cos\theta + 4r^2\sin^2\theta) r\,\mathrm{d}r\,\mathrm{d}\theta \tag{69}$$

Solving this integral gives $\frac{15}{2}\pi$.

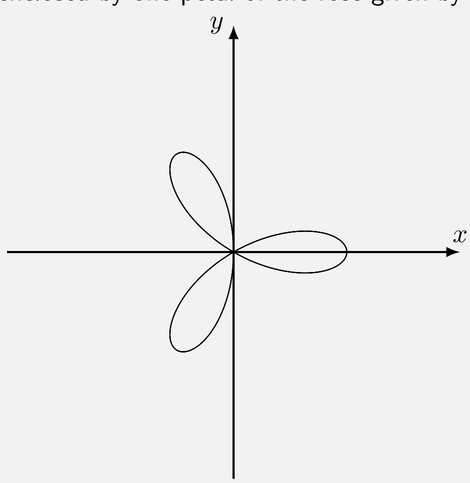
Example 9: Find the volume of the solid bounded by the $z = 0$ plane and the paraboloid $z = 1 - x^2 - y^2$.

Note that at $z = 0$, we get $0 = 1 - x^2 - y^2 \implies x^2 + y^2 = 1$. We can write our region as $R = \{(r, \theta) | 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}$. Our double integral is then

$$V = \iint_R (1 - x^2 - y^2)\,\mathrm{d}A = \int_0^{2\pi} \int_0^1 (1 - r^2) r\,\mathrm{d}r\,\mathrm{d}\theta \tag{70}$$

Solving this gives $V = \pi/2$.

Example 10: Find the area enclosed by one petal of the rose given by $r = \cos 3\theta$.



The area is

$$A = \int_{-\pi/6}^{\pi/6} \int_0^{\cos 3\theta} 1 \cdot r\,\mathrm{d}r\,\mathrm{d}\theta \tag{71}$$

which evaluates to $\frac{1}{12}$.

Example 11: Find the volume trapped between the cone $z = \sqrt{x^2 + y^2}$ and the sphere $x^2 + y^2 + z^2 = 1$.

First, let us find the intersection using cartesian coordinates. We have

$$\sqrt{x^2 + y^2} = \sqrt{1 - x^2 - y^2} \implies x^2 + y^2 = \frac{1}{2}. \tag{72}$$

This can be written as $r = \frac{1}{\sqrt{2}}$ in polar coordinates. The volume is thus

$$\int_0^{2\pi} \int_0^{1/\sqrt{2}} f(x,y) r \, \mathrm{d}r \, \mathrm{d}\theta \tag{73}$$

where $f(x,y) = \sqrt{1-x^2-y^2} - \sqrt{x^2+y^2}$. This gives $\frac{2\pi}{3}\left(1 - \frac{1}{\sqrt{2}}\right)$.

- **Applications of Double Integrals**
- We can determine the mass of a plate with nonuniform density $\rho = \rho(x,y)$. The mass is then

$$\iint\limits_R \rho(x,y) \, \mathrm{d}A \, . \tag{74}$$

- We can find the center of mass of a particle. Imagine we break a plate into small pieces. Each small piece has a moment about the x axis:

$$(M_x)_i = m_i y_i^* \approx \rho(x_i^*, y_i^*) \Delta A_i \cdot y_i^* \tag{75}$$

The total x and y moments are thus

$$M_x = \iint\limits_R y\rho(x,y) \, \mathrm{d}A \tag{76}$$

$$M_y = \iint\limits_R x\rho(x,y) \, \mathrm{d}A \tag{77}$$

These are equal to the moment $\bar{y}m$ and $\bar{x}m$, respectively, where m is the mass of the object. Thus:

$$\bar{x} = \frac{\iint\limits_R x\rho(x,y) \, \mathrm{d}A}{\iint_R \rho(x,y) \, \mathrm{d}A} \tag{78}$$

and similarly for $\bar{y}$.

- Consider a rotating object. A point mass has a kinetic energy $K = \frac{1}{2} m r^2 \omega^2$. However, mr^2 would be different for different points on a solid object.

We can consider:

$$K = \frac{1}{2}\left(\sum_{i=1}^{n} m_i r_i^2\right)\omega^2 . \tag{79}$$

The quantity inside the parentheses is known as the moment of inertia I. While this may be true for a series of point masses, for a continuous distribution we need to take the limit:

$$I = \iint\limits_R \rho(x,y)[r(x,y)]^2 \, \mathrm{d}x \, \mathrm{d}y \, . \tag{80}$$

4 Surface Area and Triple Integrals

- Suppose we wish to find the surface area.
- Method 1: Given $z = f(x,y)$ we can estimate the area as

$$S \iint\limits_S \mathrm{d}T \tag{81}$$

where $\mathrm{d}T$ gives the area of the tangent plane and S is the region of the curve. The projection of $\mathrm{d}T$ is given by

$$\Delta A = \Delta T |\cos\alpha| \implies \frac{\Delta A}{|\cos\alpha|} \tag{82}$$

where α is the angle between $\vec{n}$ (normal to plane) and $\vec{k}$ (normal to xy plane), such that

$$S = \iint_R \frac{\mathrm{d}A}{|\cos\alpha|} \tag{83}$$

where R is the projection of S. To determine $\cos\alpha$, we can write $z = f(x,y)$ in explicit form as

$$F(x,y,z) = z - f(x,y) = 0, \tag{84}$$

which is the 0^{th} level surface. Since $\vec{\nabla}$ is perpendicular to it, we have

$$\vec{n} = \frac{\vec{\nabla}F}{\|\vec{\nabla}F\|}. \tag{85}$$

Recall that

$$\vec{\nabla}F \cdot \vec{k} = \left(\frac{\partial F}{\partial x}\hat{i} + \frac{\partial F}{\partial y}\hat{j} + \frac{\partial F}{\partial z}\hat{k}\right) \cdot \vec{k} \tag{86}$$

so

$$\boxed{|\cos\alpha| = |\vec{n} \cdot \vec{k}| = \frac{|\vec{\nabla}F \cdot \vec{k}|}{\|\vec{\nabla}F\|} = \frac{\left|\frac{\partial F}{\partial z}\right|}{\|\vec{\nabla}F\|}}. \tag{87}$$

Therefore, we have

$$S = \iint_R \frac{\sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}}{\left|\frac{\partial F}{\partial z}\right|} \, \mathrm{d}A \tag{88}$$

which can be simplified to

$$\boxed{S = \iint_R \sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + 1} \, \mathrm{d}A} \tag{89}$$

- Method 2: Consider a rectangular subregion R_i with area $\Delta A_i = \Delta y_i \times \Delta x_i$. Projecting this onto $z = f(x,y)$ gives a parallelogram. This parallelogram has sides

$$\vec{a}_i = \Delta x_i \cdot \hat{i} + 0\hat{j} + f_x(x_i, y_i)\Delta x_i \hat{k} \tag{90}$$

$$\vec{b}_i = 0\hat{i} + \Delta y_i \cdot \hat{j} + f_y(x_i, y_i)\Delta x_i \hat{k}. \tag{91}$$

The area of the parallelogram is $\Delta T_i = \|\vec{a}_i \times \vec{b}_i\|$. Taking the cross product, we get

$$\boxed{S = \iint_R \sqrt{f_x^2(x,y) + f_y^2(x,y) + 1}} \tag{92}$$

- All the ideas for double integrals carry over for **triple Integrals.** Formally, we can break it up into sub-volumes, gain an estimate by finding the largest and smallest value in each ΔV_i, which bound the triple integral and approach to it after taking the limit.

Example 12: Suppose $f(x,y,z)$ is a continuous function defined on the box region Q, given by

$$Q = \{(x,y,z) | a \le x \le b, c \le y \le d, r \le z \le s\}. \tag{93}$$

We then have

$$\iiint_Q f(x,y,z) \, \mathrm{d}V = \int_r^s \int_c^d \int_a^b f(x,y,z) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \,. \tag{94}$$

- Suppose we have something more complicated like $Q = \{(x,y,z) | (x,y) \in R$ and $g_1(x,y) \le z \le (x,y)$. We will then have

$$\iint_R \int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z) \, \mathrm{d}z \, \mathrm{d}A \tag{95}$$

Example 13: Evaluate $\iiint\limits_Q 6xy\,\mathrm{d}V$ where Q is the tetrahedron bounded by the planes $x=0$, $y=0$, $z=0$ and $2x+y+z=4$. We then have

$$\int_{x=0}^{x=2}\int_{y=0}^{y=4-2x}\int_{z=0}^{z=4-2x-y} 6xy\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x\,. \tag{96}$$

If we want to first integrate with respect to x, we have

$$\int_{y=0}^{y=4}\int_{z=0}^{z=4-y}\int_{x=0}^{x=1/2(4-y-z)} \tag{97}$$

5 Cylindrical, Spherical Coordinates, Taylor Series, Jacobian

- In **cylindrical coordinates**, we can represent a point in $\mathbb{R}^3$ as

$$P(x,y,z) = P(r,\theta,z). \tag{98}$$

We can describe a region as

$$Q = \{(x,y,z)|(x,y)\in R, u_1(x,y)\le z\le u_2(x,y)\} \tag{99}$$

where

$$R = \{(r,\theta)|\alpha\le\theta\le\beta, h_1(\theta)\le r\le h_2(\theta)\} \tag{100}$$

and the integral can be written as

$$\iiint\limits_Q f(x,y,z)\,\mathrm{d}V = \iint\limits_R \left[\int_{u_1(x,y)}^{u_2(x,y)}\right]\mathrm{d}A \tag{101}$$

$$= \int_\alpha^\beta\int_{h_1(\theta)}^{h_2(\theta)}\int_{u_1(r\cos\theta,r\sin\theta)}^{u_2(r\cos\theta,r\sin\theta)} f(r\cos\theta,r\sin\theta,z)r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta. \tag{102}$$

- In spherical coordinates, a point can be represented by

$$P(x,y,z) = P(\rho,\theta,\phi) \tag{103}$$

where θ is the same as the one in cylindrical coordinates[1]. We have

$$x = \rho\sin\phi\cos\theta \tag{104}$$
$$y = \rho\sin\phi\sin\theta \tag{105}$$
$$z = \rho\cos\phi \tag{106}$$

and

$$\rho^2 = x^2+y^2+z^2. \tag{107}$$

- The volume in spherical coordinates is given by

$$\mathrm{d}V = \rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta \tag{108}$$

- **Taylor Series for Two-Variable Functions:** Suppose we are given $f(x_0,y_0)$ and want to approximate $f(x_0+\Delta x, y_0+\Delta y)$. Suppose there projections on the xy plane is P and Q. We can parametrize the line segment PQ as

$$x(t) = x_0 + t\Delta x \tag{109}$$
$$y(t) = y_0 + t\Delta y \tag{110}$$

where $0\le t\le 1$. We can then define

$$F(t) = f(x_0+t\Delta x, y_0+t\Delta y) \tag{111}$$

[1] This is the common convention in physics. However, many mathematics texts mix up θ and ϕ

which is a one-variable function, which we can approximate using the one dimensional Taylor Series:

$$F'(t) = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y \tag{112}$$

The second derivative is

$$F''(t) = \frac{\partial^2 f}{\partial x^2}\Delta x^2 + 2\frac{\partial^2 f}{\partial x \partial y}\Delta x \Delta y + \frac{\partial^2 f}{\partial y^2}\Delta y^2 \tag{113}$$

The third derivative is

$$F'''(t) = \frac{\partial^3 f}{\partial x^3}\Delta x^3 + 3\frac{\partial^3 f}{\partial x^2 \partial y}\Delta x^2 \Delta y + 3\frac{\partial^3 f}{\partial x \partial y^2}\Delta x \Delta y^2 + \frac{\partial^3 f}{\partial y^3}\Delta y^3. \tag{114}$$

Therefore:

$$F(t_0 + \Delta t) \approx F(t_0) + F'(t_0)\Delta t + \frac{1}{2!}F''(t_0)\Delta t^2 + \cdots + \frac{F^{(n)}(t_0)\Delta t}{n!} \tag{115}$$

- **Change of Variables in Multiple Integrals:** Consider a bijective mapping between a region S in the uv plane to a region R in the xy plane. We can partition both regions into N regions.

  Specifically, let us partition S into square regions. Consider an arbitrary region with vertices $\bar{A}(u_0, v_0)$, $\bar{B}(u_0 + \Delta u, v_0)$, $\bar{C}(u_0, v_0 + \Delta v)$, and $\bar{D}$. Let the subregion be denoted as S_i with area ΔA_S.

  Suppose we have the mapping

$$x = g(u, v) \tag{116}$$
$$y = h(u, v) \tag{117}$$

  such that $\bar{X} \mapsto X$. If Δu and Δv are sufficiently small, then $R_i = ABCD$ is a parallelogram. Therefore:

$$\Delta A_R \approx \text{area of the parallelogram} = \|\vec{AB} \times \vec{AC}\|. \tag{118}$$

  Note that $\vec{AB} = \Delta x_1 \hat{i} + \Delta y_1 \hat{j}$ and $\vec{AC} = \Delta x_2 \hat{i} + \Delta y_2 \hat{j}$, so their cross product is

$$\|\vec{AB} \times \vec{AC}\| = |\Delta x_1 \Delta y_2 - \Delta x_2 \Delta y_1| \tag{119}$$

  From our linear approximation, we can write

$$\Delta x_1 \approx g_u(u_0, v_0)\Delta u \tag{120}$$
$$\Delta x_2 \approx g_v(u_0, v_0)\Delta v \tag{121}$$
$$\Delta y_1 \approx h_u(u_0, v_0)\Delta u \tag{122}$$
$$\Delta y_2 \approx h_v(u_0, v_0)\Delta v \tag{123}$$

  To sum it up, we have

$$\Delta A_R = \left|\det \begin{bmatrix} g_u(u_0, v_0) & g_v(u_0, v_0) \\ h_u(u_0, v_0) & h_v(u_0, v_0) \end{bmatrix}\right| \Delta u \Delta v \tag{124}$$

Definition: The determinant of the derivative matrix is called the Jacobian (J) of the transformation.

$$J = \det \begin{bmatrix} g_u & g_v \\ h_u & h_v \end{bmatrix} = \det \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} \equiv \frac{\partial(x, y)}{\partial(u, v)} \tag{125}$$

given

$$x = g(u, v) \tag{126}$$
$$y = h(u, v) \tag{127}$$

Therefore,

$$\Delta A_R \approx |J|\Delta A_S \tag{128}$$

Theorem: Assuming that
- f is continuous
- g and h are functions that have continuous first derivatives
- The transformation is $1-1$.
- The Jacobian J is nonzero

we can write

$$\iint_R f(x,y)\,\mathrm{d}A = \iint_S f(g(u,v),h(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\,\mathrm{d}v\,. \tag{129}$$

Note the similarity between this and the single variable case

$$\int_a^b f(x)\,\mathrm{d}x = \int_c^d f(x(u))\frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u \tag{130}$$

Example 14: Suppose we wish to evaluate the integral $\iint_R (x^2+2xy)\,\mathrm{d}A$ where R is the region bounded by the lines

$$y = 2x+3 \tag{131}$$
$$y = 2x+1 \tag{132}$$
$$y = 5-x \tag{133}$$
$$y = 2-x \tag{134}$$

Notice that this is a rotated rectangle, so let's try to switch this into a non-rotated rectangle with the bounds:

$$u = 3 \tag{135}$$
$$u = 1 \tag{136}$$
$$v = 5 \tag{137}$$
$$v = 2 \tag{138}$$

by the transformation

$$x = \frac{1}{3}(v-u) \tag{139}$$
$$y = \frac{1}{3}(2v+u). \tag{140}$$

The Jacobian is

$$J = \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \det\begin{bmatrix} -1/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix} = -\frac{1}{3} \tag{141}$$

which gives

$$\iint_R (x^2+2xy)\,\mathrm{d}A = \iint_S \left[\frac{1}{3}(v-u)^2 + \frac{2}{3}(v-u)(2v+u)\right] |J|\,\mathrm{d}u\,\mathrm{d}v \tag{142}$$

where $S = \{(u,v) | 1 \le u \le 3, 2 \le v \le 5\}$

- For triple integrals, the Jacobian is

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix} \tag{143}$$

- **Successive Transformations:** Suppose we have $x = x(u,v), y = y(u,v)$ and $u = u(s,t)$ and $v = v(s,t)$. Then

$$\frac{\partial(x,y)}{\partial(s,t)} = \frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(s,t)} \tag{144}$$

- **Back Transformations:** Recall that when we transform a region R to a region S with some transformation T, then

$$\mathrm{d}A_R = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}A_S \tag{145}$$

and

$$\mathrm{d}A_S = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| \mathrm{d}A_R \tag{146}$$

Theorem: Jacobians satisfy the property:

$$\frac{\partial(x,y)}{\partial(u,v)} = \left(\frac{\partial(u,v)}{\partial(x,y)}\right)^{-1} \tag{147}$$

or

$$J_{S\to R} = \frac{1}{J_{R\to S}} \tag{148}$$

Idea: This is important since if we know $u = f(x,y)$ and $v = g(x,y)$, then we can calculate the Jacobian without finding the inverse.