Important Things

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1 Analogy of Nathan

Let $E = \{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ be the standard basis for ${}^n\mathbb{R}$ and let $F = \{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_n\}$ be another basis for ${}^n\mathbb{R}$. Then $\boldsymbol{Q}$ is the transformation matrix from F to E if and only if $\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{f}_1 & \boldsymbol{f}_2 & \cdots & \boldsymbol{f}_n \end{bmatrix}$.

This might sounds confusing since Q converts the standard basis e_j to the other basis f_j as

$$\boldsymbol{f}_j = \boldsymbol{Q}\boldsymbol{e}_j, \tag{1}$$

but it takes a vector written in F and converts it to a vector written in the standard basis E. Why is this the case?

We invoke the **Analogy of Nathan**,[1] where we look at using a ruler to measure a distance. Suppose the standard unit of measurement is the foot (i.e. the standard basis), and we want to convert it to another unit of measurement, the inch (i.e. the other basis), then the conversion factor (i.e. transformation matrix) would be $\frac{1}{12}$. The basis is getting smaller, so the final number we report (i.e. the coordinate in the other basis) would be larger, specifically by a factor of 12.

In short, decreasing the size of the basis *increases* the actual number we report at the end. As a result, if we want to convert a vector written in the E basis to the F basis, we need to multiply the coordinate of that vector by the inverse of the transformation matrix.

2 Method of Nathan

The row space is orthogonal to the null space, so for a 2×2 matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{2}$$

and if the nullity is 1, then the null space is:

$$\text{span}\left\{\begin{bmatrix} -b \\ a \end{bmatrix}\right\} \tag{3}$$

since:

$$\begin{bmatrix} a & b \end{bmatrix}\begin{bmatrix} -b \\ a \end{bmatrix} = \vec{0} \tag{4}$$

3 Eigenvalues

- If A has eigenvalue λ, then A^k has eigenvalue λ^k.
- A is noninvertible if and only if at least one of its eigenvalues are zero.
- The geometric multiplicity m_{λ_1} is equal to:

$$m = \dim \operatorname{null}(B - \lambda_1 I) = n - \operatorname{rank}(B - \lambda_1 I) \tag{5}$$

- For a $n \times n$ matrix, the algebraic multiplicities sum up to n:

$$n_1 + n_2 + \cdots = n \tag{6}$$

[1] Yes, that Nathan.

and for geometric multiplicities, it is *bounded* by n

$$m_1 + m_2 + \cdots \leq n \tag{7}$$

- Geometric multiplicities are smaller or equal to the algebraic multiplicites:

$$1 \leq m_i \leq n_i \tag{8}$$

- The trace of a matrix is the sum of its eigenvalues. (Medici)

4 Diagonalization

- If A is diagonalizable, it can be written as $A = PDP^{-1}$ and:

$$A^k = PD^kP^{-1} \tag{9}$$

- If D is a diagonal matrix, then we can calculate D^k by taking each element to the power of k.
- In general, if D_1 and D_2 are diagonal matrices, $D_1D_2 = D_2D_1$ where each element is the element-wise product of the two.
- If D is a diagonal, then AD is equivalent to scaling the columns of A by the elements of D. Similarly, DA scales the rows of A be the elements of D.
- A matrix A is diagonalizable if the eigenvectors of A form a basis for ${}^n\mathbb{R}$.
- The eigenvectors of a diagonal matrix are the standard basis.
- If two matrices have the same eigenvalues with n linearly independent eigenvectors, then they are equal.

5 Important Matrices

Try these 2×2 matrices when looking for counterexamples:

- Nilpotent Matrix: $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$
- Rotation matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$