MAT301 Exercises

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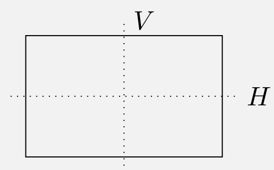
1 Exercise One

Question 01: There are only two possible reflections, one rotation, and one identity. Denote the two reflections as H and V and denote the $180°$ rotation about the center as R and the identity as I. WLOG we can draw the diagram as:



Then the Cayley table is:

	I	H	V	R
I	I	H	V	R
H	H	I	R	H
V	V	R	I	V
R	R	V	H	I

The proof that there are only four symmetries is similar to that of the dihedral group of a square. Label the vertices as $1, 2, 3, 4$ where $1, 2$ are the top two adjacent vertices. Then for all $x, y \in \{1, 2, 3, 4\}$, there exists only one symmetry σ of the proper rectangle such that:

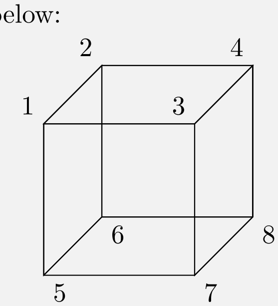
$$(\sigma(1), \sigma(2)) = (x, y) \tag{1}$$

There are four choices for x but for each x, there is only one choice for y. The reason is because we also wish to preserve the position of the side lengths (i.e. shorter sides are always vertical). Since each vertex is only connected to a single "long horizontal side," then there is only one choice for y. Therefore: $4 \times 1 = 4$.

Question 02: Similar to the dihedral group of a square, adjacent vertices must remain adjacent after a symmetric transformation. Unlike the case of a square, we propose that the symmetry σ is now uniquely defined by:

$$(\sigma(1), \sigma(2), \sigma(3)) = (x, y, z) \tag{2}$$

where 2 and 3 are adjacent to 1, as shown below:



Proof. Every vertex is adjacent to three vertices. We know that y and z are adjacent to x, so we can determine the third vertex adjacent to x, which we will call $a = \sigma(5)$. Each of the three pairs of vertices in the set $\{y, z, a\}$ are on a face diagonal, where the face contains the vertex x.

For each vertex v_i, there exists only one other vertex v_j that is the furthest apart from it, and it always occurs at the opposite end. We will denote such a (v_i, v_j) as an *enemy pair*. Since symmetries preserve distances and we know the locations of vertices a, y, z who are all adjacent to x (no pair of the known vertices form an enemy pair) we can uniquely determine each vertex's enemy pair and therefore the locations of all eight vertices. □

We have eight choices for x, three choices for y, and two choices for z for a total of $8 \times 3 \times 2 = 48$ symmetries.

Question 03:

1. a^{10}, a^9, a^6, a^4, a
2. We can write:

$$\begin{aligned} a^{-2}b^{-4} &= (a^2)^{-1}(b^2)^{-1} && (3)\\ &= a^4b^3 && (4) \end{aligned}$$

   and

$$\begin{aligned} (a^2b^4)^{-2} &= (a^2b^4a^2b^4)^{-1} && (5)\\ &= b^{-4}a^{-2}b^{-4}a^{-2} && (6)\\ &= (b^3a^4)^2 && (7) \end{aligned}$$

Question 04:

1. *Proof.* We have:

$$\begin{aligned} 1 &= aa^{-1} && (8)\\ \implies (a^{-1})^{-1}(a^{-1}) &= aa^{-1} && (9)\\ \implies (a^{-1})^{-1} &= a && (10) \end{aligned}$$

   where the first implication is possible since $a^{-1} \in G$ and from the existence of the inverse. The second implication is possible due to right cancellation. □
2. *Proof.* We propose that:

$$(a_1 \cdots a_n)^{-1} = a_n^{-1} \cdots a_1^{-1} \tag{11}$$

   The base case $n = 2$ was already proved in lecture, so we can skip to the inductive step. Suppose this is true for some $n = k$. Let us denote $a_1 \cdots a_k = \alpha_k$. Then for $n = k + 1$, we have:

$$\begin{aligned} (a_1 \cdots a_k a_{k+1})^{-1} &= (\alpha_k a_{k+1})^{-1} && (12)\\ &= a_{k+1}^{-1}\alpha_k^{-1} && \text{(base case)} \quad (13)\\ &= a_{k+1}^{-1}a_k^{-1} \cdots a_1^{-1} && \text{(inductive assumption)} \quad (14) \end{aligned}$$

   and we are done. □

Question 05: Since $G = \{e, a, b\}$ is a set, e, a, b must be distinct. We first propose that $a^{-1} = b$.

Proof. Suppose for the sake of contradiction that $ab \neq e$. Then if $ab = a$, left cancellation shows that $b = e$ and if $ab = b$, right cancellation shows that $a = e$, both of which are not allowed. As a result, we must have $ab = e \implies a^{-1} = b$. □

Next we will show that $a^2 = b$.

Proof. We have already shown that $a^2 \neq a$. Suppose for the sake of contradiction that $a^2 = e$. Then since $a = b^{-1}$, we have:

$$ab^{-1} = e \implies a = b \tag{15}$$

which is not allowed. □

A corollary of this is $a^3 = e$.

Proof. This follows directly:

$$a^2b^{-1} = bb^{-1} \implies a^3 = e \tag{16}$$

□

Finally, since $ab = e = ba$, the group is abelian and any group operation can be represented as:

$$\begin{aligned} a^x b^y &= (ab)^x b^{y-x} && (17)\\ &= b^{y-x} && (18)\\ &= (b^3)^k b^\ell && (19)\\ &= b^\ell && (20) \end{aligned}$$

where k and ℓ are chosen such that $k + \ell = y - x$ and $0 \leq \ell \leq 2$ (division algorithm). We've already shown that this is only equal to the identity e iff $\ell = 0$. This occurs iff:

$$y - x \equiv 0 \pmod 3 \tag{21}$$

Question 06:

- For $x^2 = S$, we know that x cannot be a rotation, since x^2 would also be a rotation. Therefore, let x be the reflection:

  $$x = sr^a \tag{22}$$

  Using the fact that $sr^a = r^{-a}s$, we have:

  $$\begin{aligned} x^2 &= (sr^a)(sr^a) && (23)\\ &= sr^a r^{-a} s && (24)\\ &= s^2 && (25) \end{aligned}$$

  Since s is a reflection, then we must have $s^2 = e$ and this is not possible.
- For $x^3 = S$. we again cannot have a rotation. Let $x = sr^a$ and following the same steps as before:

  $$\begin{aligned} x^3 &= (sr^a)(sr^a)(sr^a) && (26)\\ &= s^3 r^a && (27)\\ &= sr^a && (28)\\ &= x && (29) \end{aligned}$$

  Therefore, this is satisfied iff $x = S$.

Question 07: We have:

$$(\mathbb{Z}/8\mathbb{Z})^\times = \{0, 1, 2, 3, 4, 5, 6, 7\} \tag{30}$$

and the Cayley table is:

	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	0
2	2	3	4	5	6	7	0	1
3	3	4	5	6	7	0	1	2
4	4	5	6	7	0	1	2	3
5	5	6	7	0	1	2	3	4
6	6	7	0	1	2	3	4	5
7	7	0	1	2	3	4	5	6

Question 08:

1. There are three criteria:
   - Reflexive: We have $x \sim x = 0 \in \mathbb{Z}$.
   - Symmetric: If $x \sim y \implies x - y = a$ where $a \in \mathbb{Z}$, then $y - x = -a \in \mathbb{Z} \implies y \sim x$.
   - Transitive: If $x \sim y \implies x - y = a \in \mathbb{Z}$ and $y \sim z \implies y - z = b \in \mathbb{Z}$, then: $x - z = (x - y) + (y - z) =$

$a + b \in \mathbb{Z} \implies x \sim z$.

2. For each element $x \in \mathbb{R}$, the equivalence class would be $\{x + k | k \in \mathbb{Z}\}$.
3. The unique representation of each equivalence class of x is $x - \lfloor x \rfloor$. Since $\lfloor x \rfloor \in \mathbb{Z}$, then this is part of the equivalence class.
4. Let $a, b \in \mathbb{Z}$ such that $x_1 - x_2 = a$ and $y_1 - y_2 = b$. Adding the two equations, we get:
$$x_1 - x_2 + y_1 - y_2 = a + b \implies (x_1 + y_1) - (x_2 + y_2) = a + b \tag{31}$$
Since $a + b \in \mathbb{Z}$, $x_1 + y_1 \sim x_2 + y_2$.
5. All elements in $\mathbb{R}/\mathbb{Z}$ can be represented as $\lfloor x \rfloor$ where $x \in \mathbb{R}$. Since addition is commutative, the group must be abelian.

Question 09: First, note that $a^2 = e \implies a = a^{-1}$. Then:

$$\begin{aligned} & ab = ba && (32)\\ \iff & (ab)(ba)^{-1} = e && (33)\\ \iff & aba^{-1}b^{-1} = e && (34)\\ \iff & abab = e && (35)\\ \iff & (ab)^2 = e && (36) \end{aligned}$$

Since $ab \in G$, we must have $(ab)^2 = e \implies ab = ba$.

Question 10: We propose that if $x \neq e$, then $x \neq x^2$ and $x^2 \neq e$.

Proof. Suppose for the sake of contradiction that $x = x^2$. Then it immediately follows from cancellation that $x = e$.

For the second claim, if $x^2 = e$, then this would imply that $x^3 = x$. But since $x^3 = e$, this means that $x = e$ as well. □

We wish to partition G into disjoint sets:

$$G = \left(\bigcup_{x \in G, x \neq e} \{x, x^2\} \right) \cup \{e\} \tag{37}$$

For any two sets $\{x, x^2\}$ and $\{y, y^2\}$ where $x, y \in G$, we have:

$$\{x, x^2\} \cap \{y, y^2\} = \begin{cases} \{x, x^2\} & x = y \text{ or } x = y^2 \\ \varnothing & \text{otherwise} \end{cases} \tag{38}$$

This shows that $|G| = 2k + 1$ for some integer k, which is odd.

Question 11: We can perform a similar analysis to the previous question, except we can only conclude that $x^2 \neq x$ if $x \neq e$. We can again partition the group into:

$$G = \left(\bigcup_{x \in G, x \neq e} \{x, x^2\} \right) \cup \{e\} \tag{39}$$

where $\left| \bigcup_{x \in G, x \neq e} \{x, x^2\} \right| = 2k$. However, since we are given that $|G|$ is even, we must have $|G| = 2k$ which implies $e \in \bigcup_{x \in G, x \neq e} \{x, x^2\}$. Since $x \neq e$, we must have $x^2 = e$ for some $x \in G$.