

# PHY293: Waves and Modern Physics

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# 1 Simple Harmonic Oscillators: Energy and Damping

- Simple harmonic motion takes place about an equilibrium position and the motion is periodic.
- The most simple system is a mass attached to a spring. The restoring force given by Hooke's Law.

$$\vec{F} = -k\Delta\vec{x} \quad (1)$$

- The period (time/period) and frequency (amount of cycles/second) is related via

$$f = \frac{1}{T} \quad (2)$$

*Note:* This also applies for other oscillatory systems, such as pendulums, LC circuits, i.e.

**Warning:** Since acceleration is not constant, the standard kinematic equations do not apply.

**Example 1:** Consider a mass on a spring (and only affected by it). We get a second order differential equation:

$$m \frac{d^2x}{dt^2} = -kx \implies \frac{d^2x}{dt^2} + \frac{k}{m}x = 0. \quad (3)$$

This is a second order ODE. After solving, we get

$$\omega^2 = \frac{k}{m}. \quad (4)$$

- For a mass on a *vertical spring*, we have a slightly different situation.

First, note that the equilibrium location shifts. The spring will extend by a distance  $\Delta y = (y_1 - y_0) = \frac{mg}{k}$ , and will then oscillate around this new equilibrium point.

Let  $y_0$  be the equilibrium height (measured from top) and let  $y$  be the location of the mass (measured from top). The spring force will then be

$$\sum F_y = ma = k(y - y_0) - mg \quad (5)$$

$$= k(y - y_0 - y_1 + y_0) \quad (6)$$

$$= k(y - y_1) \quad (7)$$

Since  $y_1$  is a constant, we can turn the differential equation to be

$$\frac{d^2(y - y_1)}{dt^2} = \frac{k}{m}(y - y_1) \quad (8)$$

and therefore the angular frequency stays the same.

**Warning:** There are issues related to energy when using this trick. Be careful!

- The general solution to  $\frac{d^2x}{dt^2} + \omega^2x = 0$  is

$$x(t) = x_0 + A \cos(\omega t + \phi_0) \quad (9)$$

Oftentimes,  $x_0 = 0$  is set as convention. However, the other constants  $A$  and  $\phi_0$  are determined by initial conditions.

- The maximum speed is given by

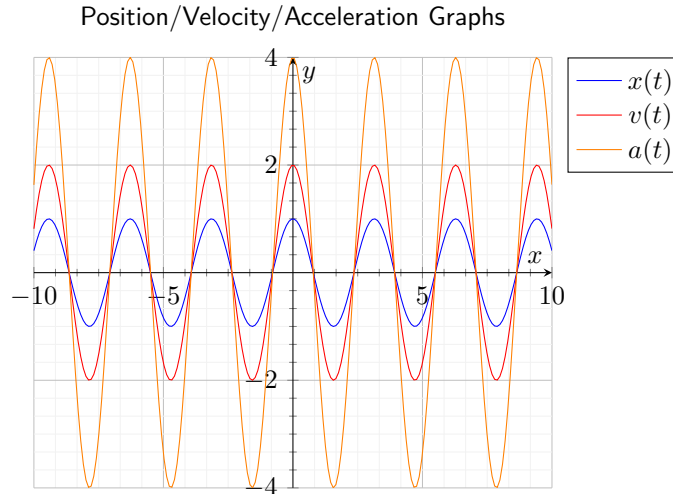
$$v_{\max} = A\omega \quad (10)$$

and the maximum acceleration is

$$a_{\max} = A\omega^2 \quad (11)$$

These can be derived by taking the first and second derivatives of  $x(t)$ .

- We can visualize the position, velocity and acceleration graphs:



- There are other ways to express the general solution:

$$x(t) = a \cos \omega t + b \sin \omega t \quad (12)$$

**Example 2:** Determine the amplitude and phase constant of a pendulum moving with a motion described by a sum of two functions,  $x_1(t) = 0.25 \cos \omega t$  and  $x_2(t) = -0.5 \sin \omega t$ .

To figure out  $\phi_0$ , we take the ratio to get

$$\frac{0.5}{0.25} = \tan \phi_0 \implies \tan \phi_0 = 2 \quad (13)$$

Be careful with the assignment of the angle;  $\tan(x)$  takes value of 2 twice during one period, at  $x_1 = 1.107$  rad and  $x_2 = x_1 + \pi = 4.25$  rad.

To determine which one is correct, you need to look at your original functions: both  $\sin \phi_0 > 0$  and  $\cos \phi_0 > 0$ . Therefore,  $\phi_0 = 1.107$  rad.

- Energy of a simple harmonic oscillator. It has a kinetic energy  $K = \frac{1}{2}mv(t)^2$  and potential energy.
- The potential energy is related to the restoring force and, by a definition of potential energy, the change in system's potential energy when moving from a position  $x_i$  to position  $x_f$  is

$$\Delta U = - \int_{x_i}^{x_f} (-kx') dx' = \frac{1}{2}kx_f^2 - \frac{1}{2}kx_i^2. \quad (14)$$

For a reference, let  $x_i = 0$  such that we can define  $U = \frac{1}{2}kx^2$ .

- Energy is conserved, i.e.

$$U + K = \frac{1}{2}kx^2 + \frac{1}{2}mv^2 = \text{constant} \quad (15)$$

- Note that we can show conservation of energy explicitly. The potential energy is

$$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2 \cos^2(\omega t) \quad (16)$$

and the kinetic energy is

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t). \quad (17)$$

Note that  $\omega^2 = \frac{k}{m}$  so the total energy is

$$E = K + U = \frac{1}{2}kA^2 = \frac{1}{2}m\omega^2 \quad (18)$$

- When an object is  $\frac{1}{\sqrt{2}}A$  from the maximum compression/extension, the kinetic energy equals the potential energy.
- **Physics of small vibrations:** Most systems will oscillate with SHM when the amplitude is small. But what if it's not?

**Example 3:** The potential energy of a pendulum is given by

$$U = mgy = mgL(1 - \cos \theta) \quad (19)$$

We can use the Taylor series:

$$f(x) = f(a) + \frac{x-a}{1!} \left( \frac{\partial f}{\partial x} \right)_{x=a} + \frac{(x-a)^2}{2!} \left( \frac{\partial^2 f}{\partial x^2} \right)_{x=a} + \dots \quad (20)$$

The energy about  $x = 0$  is then:

$$U(x) = U(0) + x \left. \frac{dU}{dx} \right|_{x=0} + \frac{x^2}{2} \left. \frac{d^2U}{dx^2} \right|_{x=0} + \dots \quad (21)$$

Using this, we can make the lowest order (nonzero) approximation to be

$$U = \frac{1}{2}mgL\theta^2 \quad (22)$$

- In this course, any angle smaller than  $10^\circ$  would be considered a *small angle*.
- For example, the differential equation that describes a pendulum is

$$L \frac{d^2\theta}{dt^2} = -g \sin \theta \quad (23)$$

If  $\theta \ll 1$ , then we can approximate  $\theta \approx \sin \theta$  such that the differential equation becomes

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L} \theta \quad (24)$$

which is the familiar second order differential equation. The period is:

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{L}{g}}. \quad (25)$$

- The energy of a pendulum is given by

$$E = K + U = \frac{1}{2}mv^2 + mg \left( \frac{x^2}{2L} \right). \quad (26)$$

## 2 Damped Harmonic Oscillator

- The drag force is proportional to the speed  $F = -kv$ .
- The equation of motion for dampened harmonic oscillator is given by

$$\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = 0 \quad (27)$$

where  $\omega_0^2 = \frac{k}{m}$  and  $\gamma = \frac{k}{m}$ .

- **Light Damping** occurs when  $\gamma < \omega_0$ . The general solution for lightly damped (underdamped) oscillator is

$$x(t) = A_0 \exp\left(-\frac{\gamma t}{2}\right) \cos(\omega t + \phi_0) \quad (28)$$

- **Overdamping (Heavy)** occurs when  $\gamma > 2\omega_0$  and we have

$$C_1 e^{-\mu_1 t} + C_2 e^{-\mu_2 t} \tag{29}$$

where

$$\mu_1 = \frac{\gamma}{2} \tag{30}$$