MAT257: Real Analysis II

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Contents

1 Introduction: Inner Product, Cauchy–Schwarz, Triangle Inequality

Lectures 1 and 2 (Sept 10/13)

- Everything in the course will lead to Stokes' Theorem:

$$\int_C \mathrm{d}W = \int_{\partial C} W \tag{1}$$

This generalizes a well-known theorem in one-dimensional calculus, known as the Fundamental Theorem of Calculus:

$$\int_{[a,b]} F'(t) = F(b) - F(a) = \int_{\partial[a,b]} F \tag{2}$$

where $\partial[a,b] = \{\underbrace{b}_{+}, \underbrace{a}_{-}\}$.

- **Continuity in** $\mathbb{R}^n$: Recall that continuity in $\mathbb{R}$ is formally defined via $\delta - \epsilon$. However intuitively it means that if you wiggle the input by a tiny bit, you wiggle the output by a tiny bit.

A similar way can be used to view continuity in $\mathbb{R}^n$.

Definition: For $x, y \in \mathbb{R}^n$, **the standard (Euclidean) inner product** of x and y denoted

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i \tag{3}$$

The **norm squared** is defined as

$$|x|^2 = \langle x, x\rangle \tag{4}$$

and the **norm** of x is defined:

$$|x| = \sqrt{|x^2|} = \sqrt{\sum_{i=1}^{n} x_i^2} \tag{5}$$

Idea: There are multiple ways of defining $\mathbb{R}^n$. Some people will define it as the set of all column vectors while others define it as the set of all row vectors. In linear algebra, the distinction is important but in real analysis, this distinction is not too important.

- A **bilinear** function $f(x, y)$ means that the function is linear in each of the two variables. This means that

$$f(ax + by, z) = af(x, z) + bf(y, z) \tag{6}$$

and similarly the same thing for the other parameter.

- A **semi-linear** function $f(x)$ is one such that

$$f(ax) = |a|f(x) \tag{7}$$

Proposition 1: If $x, y, z \in \mathbb{R}^n$ and $a, b \in \mathbb{R}$, then:

0. $\langle \cdot, \cdot\rangle$ is bilinear and $|\cdot|$ is semi-linear. Also note that $\langle x, y\rangle = \langle y, x\rangle$.
1. $|x| \geq 0$ and $|x| = 0 \iff x = 0$.
2. Cauchy–Schwarz Inequality: $|\langle x, y\rangle| \leq |x||y|$ and equality holds if and only if x, y are dependent.
3. Triangle Inequality: $|x + y| \leq |x| + |y|$
4. Polarization Identity: $\langle x, y\rangle = \dfrac{|x + y|^2 - |x - y|^2}{4}$

Proof. We prove each part separately, and skip 0:

1. We have $|x| = \sqrt{\sum x_i^2} \geq 0$ since every x_i^2 is non-negative. Then $|x| = 0$ if and only if $x_i = 0$, i.e. $x = 0$.

2. Consider and note that $\left||y|^2x - \langle x, y\rangle y\right|^2 \geq 0$. This is equal to 0 if and only if the first term (a multiple of x) equals the second term (a multiple of y), which is equivalent to x, y being dependent.

   Next, note that

$$\begin{aligned} |s+t|^2 &= \langle s+t, s+t\rangle && (8)\\ &= \langle s, s\rangle + \langle s, t\rangle + \langle t, s\rangle + \langle t, t\rangle && (9)\\ &= |s|^2 + 2\langle s, t\rangle + |t|^2 && (10) \end{aligned}$$

   Using this result, we can simplify the earlier expression to get that

$$\begin{aligned} \left||y|^2x - \langle x, y\rangle y\right|^2 &= |y|^4|x|^2 + \langle x, y\rangle^2|y|^2 - 2|y|^2\langle x, y\rangle^2 && (11)\\ &= |y|^2(|y|^2|x|^2 - \langle x, y\rangle^2) && (12) \end{aligned}$$

   Since this quantity is non-negative, it follows that $|y|^2|x|^2 \geq \langle x, y\rangle^2$, which is what we wanted to show.

   Note that there is another part regarding equality, which will not be proven in these notes.

3. As both $|x+y|$ and $|x|+|y|$ are nonnegative, we can square both sides. It suffices to prove that

$$\begin{aligned} |x+y|^2 &\overset{?}{\leq} (|x|+|y|)^2 && (13)\\ \langle x+y, x+y\rangle &\overset{?}{\leq} |x|^2 + |y|^2 + 2|x||y| && (14)\\ \langle x, x\rangle + 2\langle x, y\rangle + \langle y, y\rangle &\overset{?}{\leq} |x|^2 + |y|^2 + 2|x||y| && (15)\\ |x|^2 + 2\langle x, y\rangle + |y|^2 &\overset{?}{\leq} |x|^2 + |y|^2 + 2|x||y| && (16)\\ \langle x, y\rangle &\overset{?}{\leq} |x| \cdot |y| && (17) \end{aligned}$$

   which is true via Cauchy–Schwarz.

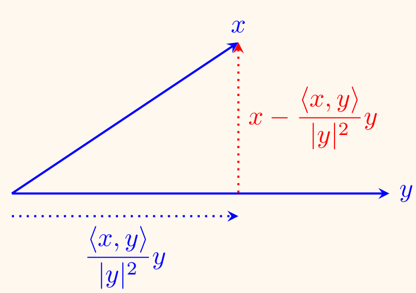
4. The proof is trivial. Expanding everything on the right hand side leads to the left hand side.

   Note: This property is important as it tells us that if we know how to compute norms, we can compute inner products.

□

Idea: In the proof for 2, a weird quantity $\left||y|^2x - \langle x, y\rangle y\right|^2$ was introduced. There is actually an intuition behind it. Recall that a geometric interpretation of Cauchy-Shwarz in $\mathbb{R}^2$ can be given as follows: $\langle x, y\rangle = |x||y|\cos\theta$ where θ is the "angle" between the two vectors x, y. This is smaller than $|x||y|$ since $\cos\theta \leq 1$.

Similarly, we want to find a generalized way to express something equivalent to $\cos\theta \leq 1$ in $\mathbb{R}^n$. One idea to show whether two vectors are dependent or not is to look at the separation between them, i.e. the length of the red line below.



Therefore, we have

$$\left|x - \frac{\langle x, y\rangle}{|y|^2}y\right| \geq 0 \tag{18}$$

Removing the fraction gives

$$\left||y|^2x - \langle x, y\rangle y\right| \geq 0. \tag{19}$$

2 Continuity: Distance

Lecture 2 (Sept 13)

- The properties in the previous section will be important when giving a formal definition of continuity.

Definition: If $x, y \in \mathbb{R}^n$, then:

$$d(x,y) = \text{"Distance between } x \text{ and } y\text{"} = |x - y| \tag{20}$$

Theorem:
1. d is **symmetric**, i.e. $d(x,y) = d(y,x)$.
2. d is **positive definite**, i.e. $d(x,y) \geq 0$ and $d(x,y) = 0 \iff x = y$.
3. Triangle Inequality: $d(x,z) \leq d(x,y) + d(y,z)$.

Proof. We prove each separately:

1. $d(x,y) = |x-y| = |-(y-x)| = |-1| \cdot |y-x| = |y-x| = d(y,x)$
2. $d(x,y) = 0 \iff |x-y| = 0 \iff x - y = 0 \iff x = y$
3. We need to check that

$$d(x,z) \overset{?}{\leq} d(x,y) + d(y,z) \tag{21}$$
$$|x-z| \overset{?}{\leq} |x-y| + |y-z| \tag{22}$$
$$|x-z| \overset{?}{\leq} |x-z| \tag{23}$$

where the third line comes from the previous triangle inequality. The last statement is true, and the steps are reversible, so we are done.

$\square$

- This theorem is significant as these are the only properties that we need to know about distances to formally define continuity.

  Note: In a future section, we will use these properties to *define* a distance function (formally a metric), which is anything that satisfies these properties. This will allow us to generalize continuity to more abstract spaces.

- A note on notation. Our definition of a norm is known as the L^2 or *Euclidean norm,* i.e.

$$|x|_{L^2} = \sqrt{\sum x_i^2} \tag{24}$$

The L^1 norm can be defined as

$$|x|_{L^1} = \sum |x_i| \tag{25}$$

and the infinity norm:

$$|x|_\infty = \max |x_i|. \tag{26}$$

- Similarly, distances for L^1 and the infinity norms can be defined as $d_1(x,y) = |x-y|_1$ and $d_\infty(x,y) = |x-y|_\infty$.

  Exercise: Show that d_1 and d_∞ also satisfy the properties of a distance.