Brief overview of issue: During lecture, we showed that

$$\frac{dE}{dt} = -bv^2 \tag{1}$$

and

$$\frac{dE}{dt} = -\gamma E. \tag{2}$$

However, by setting these two equal to each other and using the substitution $\gamma = \dfrac{b}{m}$, we get $E = mv^2$ which is clearly a contradiction.

Resolution: If we talk about the actual energy, it turns out that equation (2) is incorrect, even after making the approximation that $\gamma \ll \omega_0$. To be more precise, substituting $x(t)$ and $v(t)$ into $E = \dfrac{1}{2}mv^2 + \dfrac{1}{2}kx^2$, we get (with no approximations):

$$E(t) = \frac{1}{2}mA_0^2 \exp\left(-\gamma t\right)\left(\frac{\gamma^2}{4}\sin(2\omega_0 t) + \frac{\gamma\omega}{2}\cos(2\omega_0 t) + \omega_0^2\right). \tag{3}$$

While $E(t) = E_0 \exp(-\gamma t)$ is valid for $\gamma \ll \omega$, it's not necessarily true that we can determine $E'(t)$ by taking the derivative of this approximation. If we start with equation (3) and take the derivative, it turns out that

$$\frac{dE}{dt} = -\gamma E(t) - \gamma E(t)\cos(\omega t) \tag{4}$$

where I have ignored higher order γ terms. The maximum value occurs at $t = 0$, and setting this equal to $-bv^2$ gives

$$E = \frac{1}{2}mv^2 \tag{5}$$

as expected. To fix this equation, we can write

$$\boxed{\frac{d\langle E\rangle}{dt} = -\gamma\langle E\rangle} \tag{6}$$

where $\langle E\rangle$ is the time average energy (across one period). This is true since the time average values of the sinusoidal terms will go to zero.

Reference: Morin