AER210: Vector Calc and Fluid Mechanics

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Contents

1 Double Integrals

- **Integrals Involving a Parameter**

Example 1: Let $\int_0^1 Cx^3\,\mathrm{d}x$ where C is a constant. Then it gives

$$\int_0^1 Cx^3\,\mathrm{d}x = \frac{1}{4}C \tag{1}$$

The result contains C.

- Suppose we have something like

$$\int_a^b f(x,y)\,\mathrm{d}x = g(y) \tag{2}$$

and therefore y is a parameter

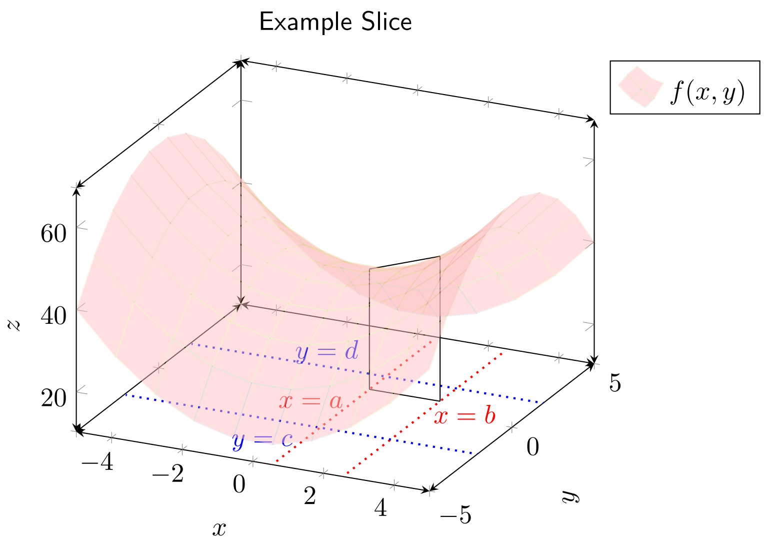
Definition: A variable which is kept constant during an integration is called a parameter.

- Partial integration wrt x

Example 2: An example of partial integration wrt x is

$$\int_0^1 x^3y\,\mathrm{d}x = y\int_0^1 x^3\,\mathrm{d}x = \frac{1}{4}y \tag{3}$$

- Notice the similarity between partial differentiation wrt x, $f_x(x,y)$ and the partial integration wrt x, $\int_a^b f(x,y)\,\mathrm{d}x$.
- **Iterated Integrals** (Integral of an Integral)
- Consider $x = f(x,y)$ where $x \in [a,b]$, $y \in [c,d]$. This defines a rectangular region.
- Assume that $f(x,y) \geq 0$. This can be represented as a surface, as shown below:



If we take the integral $\int_{y=c}^d f(x,y)\,\mathrm{d}y = A(x)$, we see that the area of the slice depends on x.

If we suppose that the surface has a tiny thickness Δx, then the volume is

$$\Delta V(x) = A(x)\cdot\Delta x = \left(\int_{y=c}^d f(x,y)\,\mathrm{d}y\right)\Delta x \tag{4}$$

If we break up the interval $[a, b]$ into N segments

$$x_0 = a \leq x_1 \leq x_2 \leq \dots x_{i-1} \leq x_i \leq \dots \leq x_{N-1} \leq x_N = b \tag{5}$$

with $\Delta x_i = x_i - x_{i-1}$. We can then approximate the volume as

$$V \approx \sum_{i=1}^{N} \Delta V_i = \sum_{i=1}^{N} A(x_i) \Delta x_i \tag{6}$$

which is known as a **Riemann sum**.

Idea: As we take the limit as $N \to \infty$ which implies $\Delta x_i \to 0$, we get the double integral:

$$V = \int_a^b \int_c^d f(x, y) \, \mathrm{d}y \, \mathrm{d}x \tag{7}$$

which can be determined by calculating two integrals.

- Similarly, we can find the volume by taking slices parallel to the xz plane.

  The area of each slice is a function of y:

$$A(y) = \int_a^b f(x, y) \, \mathrm{d}x \tag{8}$$

so we have $\Delta V(y) = A(y) \cdot \Delta y$. Again, summing up all slices and taking the limit, we get

$$V = \int_c^d A(y) \, \mathrm{d}y = \int_c^d f(x, y) \, \mathrm{d}x \, \mathrm{d}y \tag{9}$$

Theorem: **Fubini's Theorem** tells us that

$$\int_a^b \int_c^d f(x, y) \, \mathrm{d}y \, \mathrm{d}x = \int_c^d \int_a^b f(x, y) \, \mathrm{d}x \, \mathrm{d}y \tag{10}$$

The analog for equality of mixed partial derivatives is known as **Clairut's Theorem.**

Example 3: Find the volume under the surface $z = x^2 y$ where $x \in [1, 3]$ and $y \in [0, 1]$. We first form the integral by integrating wrt y. We have

$$\begin{aligned}
V &= \int_1^3 \int_0^1 x^2 y \, \mathrm{d}y \, \mathrm{d}x && (11)\\
&= \int_1^3 x^2 (1^2/2 - 0^2/2) \, \mathrm{d}x && (12)\\
&= \int_1^3 \frac{x^2}{2} \, \mathrm{d}x && (13)\\
&= \frac{13}{3} && (14)
\end{aligned}$$

We can also form the integral by integrate it wrt x:

$$\begin{aligned}
V &= \int_0^1 \int_1^3 x^2 y \, \mathrm{d}x \, \mathrm{d}y && (15)\\
&= \int_0^1 \frac{26}{3} y \, \mathrm{d}y && (16)\\
&= \frac{13}{3} && (17)
\end{aligned}$$

so we can confirm they give the same answer.

Example 4: Evaluate the double integral of $f(x, y) = x - 3y^2$ over region R where

$$R = \{(x, y)|0 \le x \le 2, 1 \le y \le 2\} \tag{18}$$

To do this, we have

$$\int_0^2 \int_1^2 (x - 3y^2)\,\mathrm{d}y\,\mathrm{d}x = \int_0^2 (xy - y^3)\Big|_{y=1}^{y=2}\,\mathrm{d}x \tag{19}$$

$$= \int_0^2 (x - 7)\,\mathrm{d}x \tag{20}$$

$$= -12 \tag{21}$$

- Note that in the special case where the function $f(x, y)$ is $f(x, y) = g(x) \cdot h(y)$, then

$$\int_c^d \int_a^b f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int_c^d \left[h(y) \int_a^b g(x)\,\mathrm{d}x \right] \mathrm{d}y = \int_a^b g(x)\,\mathrm{d}x \cdot \int_c^d h(y)\,\mathrm{d}y \tag{22}$$

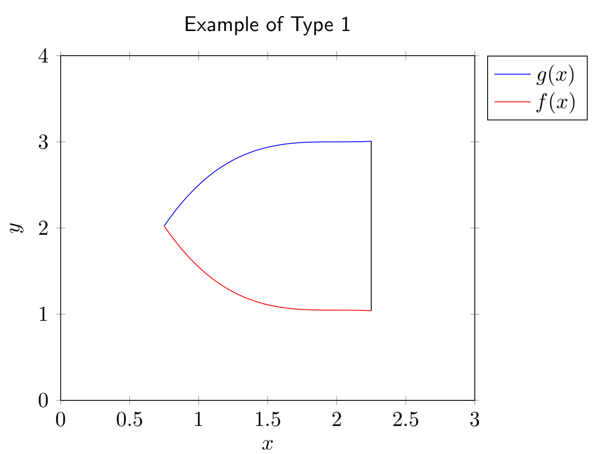
  This gives us a shortcut of evaluating double integrals in this form.

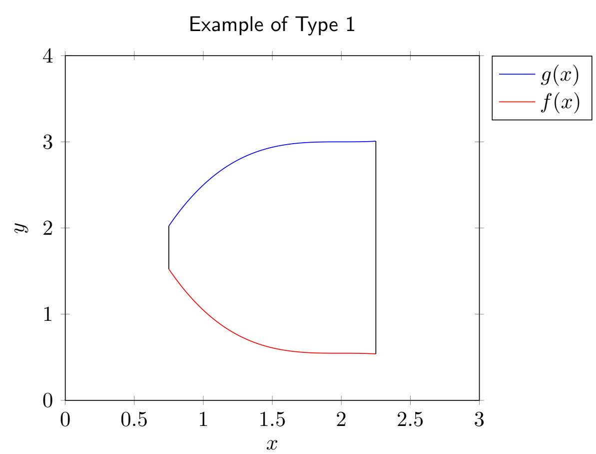
- Double integrals over general regions (What if region is non-rectangular?)

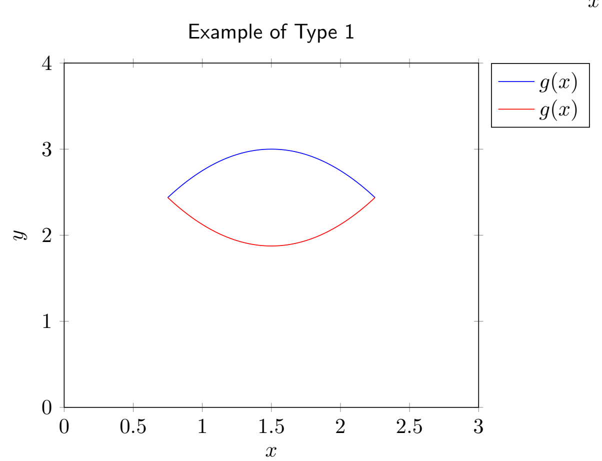
- **Type 1 Region** is in the form of

$$R = \{(x, y)|a \le x \le b, g_1(x) \le y \le g_2(x)\} \tag{23}$$

  Here are some examples

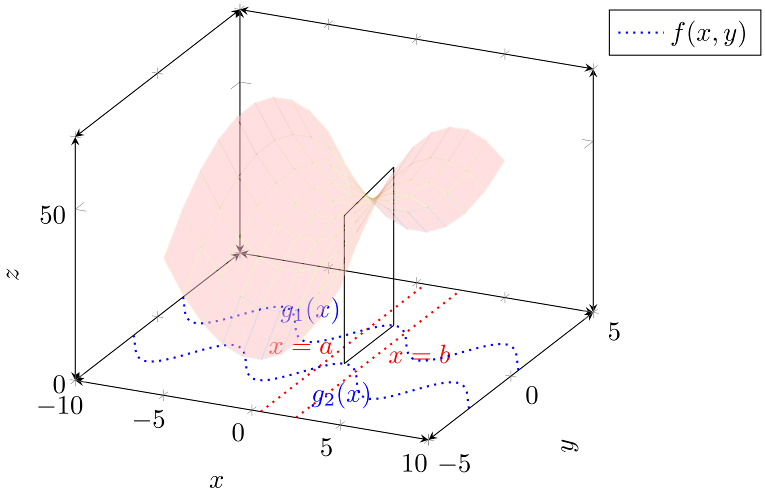






- Let's think about the case where $f(x, y) \ge 0$ on a type-1 region. Suppose we have the following illustration

Setup for Type 1



We find the area of slices, so

$$A(x) = \int_{g_1(x)}^{g_2(x)} f(x,y)\,\mathrm{d}y \tag{24}$$

and the volume is thus

$$V = \int_a^b A(x)\,\mathrm{d}X = \int_a^b \int_{g_1(x)}^{g_2(X)} f(x,y)\,\mathrm{d}y\,\mathrm{d}x \tag{25}$$

- Type-2 regions have the form

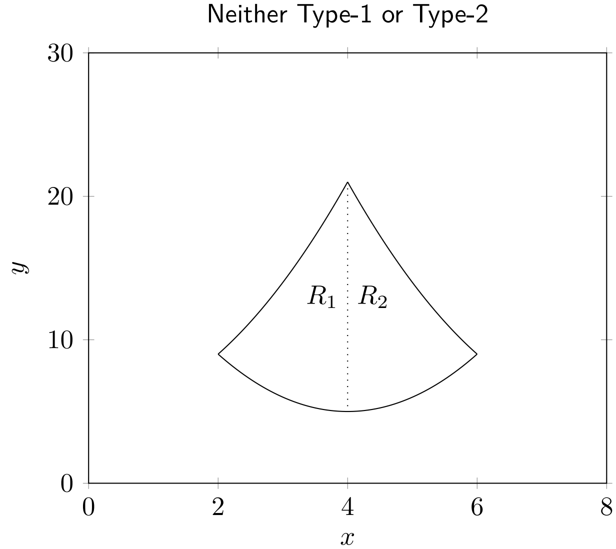
$$R = \{(x,y) | c \le y \le d \text{ and } h_1(y) \le x \le h_2(y)\} \tag{26}$$

In a similar way, the volume bounded by this region is

$$V = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,\mathrm{d}x\,\mathrm{d}y \tag{27}$$

- Type-3 regions are neither type-1 nor type-2. It is possible to break up the region into parts that can be classified as either type-1 or type-2:

Neither Type-1 or Type-2



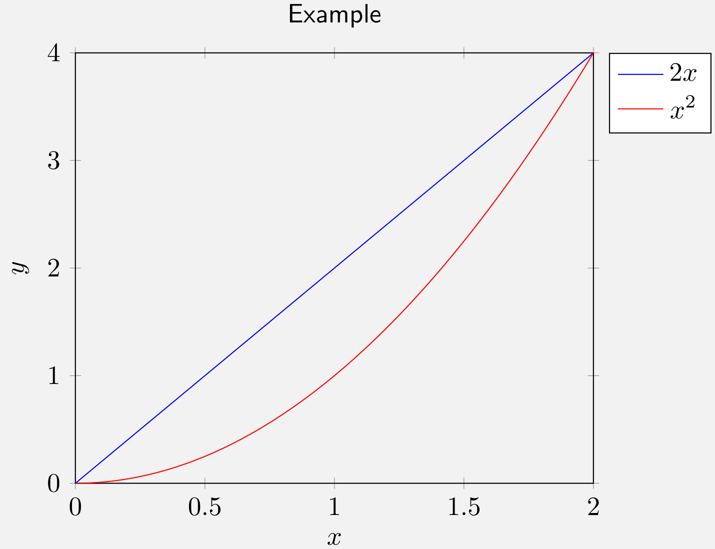
Idea: While these formulas are derived by assuming a positive volume (and thus cannot work if $f < 0$), they still work in general.

Example 5: Find the volume of the solid that lies under the surface

$$z = f(x, y) = x^2 + y^2 \tag{28}$$

and above the region R in the xy-plane. The region R is bounded by the straight line $y = 2x$ and the parabola $y = x^2$.

1. First we draw a diagram of the planar region R over which the surface is defined.



2. We then draw a line parallel to the axis of first integration (i.e. vertical lines for integrating in the y-direction first)
3. This gives us

$$V = \int_{x=0}^{x=2} \int_{y=x^2}^{y=2x} f(x, y)\, \mathrm{d}y\, \mathrm{d}x \tag{29}$$

$$= \int_0^2 \int_{x^2}^{2x} (x^2 + y^2)\, \mathrm{d}y\, \mathrm{d}x \tag{30}$$

$$= \frac{216}{35} \tag{31}$$

Alternatively, we can find the volume by integrating in the x direction first. In this case, we need to obtain boundary curves in the $x = x(y)$ form:

$$y = x^2 \implies x = \sqrt{y} \tag{32}$$

$$y = 2x \implies x = y/2 \tag{33}$$

This then gives us

$$V = \int_{y=0}^{y=4} \int_{x=y/2}^{x=\sqrt{y}} f(x, y)\, \mathrm{d}x\, \mathrm{d}y \tag{34}$$

$$= \frac{216}{35} \tag{35}$$

Warning: Do not just pick the minimum and maximum points. For example, the following is *incorrect*

$$\int_{y=0}^{y=4} \int_{x=0}^{x=2} f(x, y)\, \mathrm{d}x\, \mathrm{d}y \tag{36}$$

as that corresponds with a rectangular region.

Example 6: Integrate the surface given by $z = e^{x^2}$ over the following region:

We can first integrate wrt x

$$V = \in_{y=0}^{y=1} \int_{x=y}^{x=1} e^{x^2} \,\mathrm{d}x \,\mathrm{d}y \tag{37}$$

This is a hard problem since we don't know the anti-derivative of e^{x^2}. To solve this, we can first integrate wrt y, which gives us

$$V = \int_{x=0}^{x=1} \int_{y=0}^{y=x} e^{x^2} \,\mathrm{d}y \,\mathrm{d}x \qquad = \int_{x=0}^{1} e^{x^2} y\big|_{y=0}^{y=x} \,\mathrm{d}x \tag{38}$$

$$= \int_0^1 e^{x^2} x \,\mathrm{d}x \tag{39}$$

This integral can be more easily solved using the u-sub $u = x^2$, $\mathrm{d}u = 2x\,\mathrm{d}x$ to get

$$V = \frac{1}{2}(e - 1) \tag{40}$$