APM426: General Relativity
Wormhole Notes

QiLin Xue

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1 Wormhole Metric

1.1 Inter-universe Wormhole

- The simplest wormhole to describe is one that connects two universes. This allows for symmetry arguments to be made, simplifying the mathematics. Later on, we can generalize this to a wormhole that connects two regions of the same universe.
- Assume that wormhole are static, nonrotating, and spherically symmetric. The most general metric is,

$$\mathrm{d}s^2 = -e^{2\phi(\ell)}\,\mathrm{d}t^2 + \mathrm{d}\ell^2 + r^2(\ell)\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right) \tag{1.1}$$

- Here, ℓ is the **proper radial distance** (distance between two regions in space at a constant cosmological time)
- To ensure it is traversable, we must be able to obtain the standard metric far away from the wormhole,

$$\lim_{\ell\to\pm\infty} r(\ell) = \ell \tag{1.2}$$

  and that

$$\lim_{\ell\to\pm\infty} \phi(\ell) = \phi_\pm \tag{1.3}$$

  is finite.
- We can identify the radius of the throat of the wormhole to be

$$r_0 = \mathsf{min}(r(\ell)) = r(0), \tag{1.4}$$

  where WLOG we are letting the throat occur at $\ell = 0$.
- Often, to make computations simpler, the metric is written in Schwarzschild coordinates, where

$$\mathrm{d}s^2 = -e^{2\phi_\pm(r)}\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - b_\pm(r)/r} + r^2\left[\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right]. \tag{1.5}$$

  We can use two coordinate patches $[r_0, \infty)$ to identify the two universes whose intersection is at r_0.
- Note that $b(r)$ is the **shape function** and $\phi_\pm(r)$ is the **redshift function,** where $\lim_{r\to\infty} b_\pm(r) = b_\pm$ and $\lim_{r\to\infty} \phi_\pm(r) = \phi_\pm$ are finite.

  Note that we do not need to assume that $\phi_+ = \phi_-$, so time can travel at different rates between the two universes. However, for simplicity we will assume that $\phi_+ = \phi_-$ and $b_+ = b_-$. The $+$ and $-$ identify which universe we are in.
- Proper radial distance is related to the r coordinate by

$$l(r) = \pm\int_{r_0}^{r} \frac{dr'}{\sqrt{1 + b_\pm(r')/r'}} \tag{1.6}$$

  Proof: TBA

- The shape function scales such that $b_\pm(r_0) = r_0$ and $b_\pm(r) < r$ for $r > r_0$.

  Proof: TBA

- The Einstein tensors at the throat are given by

$$G_{tt} = \frac{b'(r_0)}{r_0^2}$$
$$G_{rr} = -\frac{1}{r_0^2}$$
$$G_{\theta\theta} = \frac{1 - b'(r_0)}{2r_0}\left(\phi' + \frac{1}{r_0}\right)$$

Proof:

- If we have $T_{tt} = \rho, T_{rr} = -\tau, T_{\theta\theta} = T_{\varphi\varphi} = p$ where ρ is energy density, τ is radial tension, and p is the transverse pressure, then we get the differential equations

$$\rho = \frac{b'}{8\pi G r^2}$$
$$\tau = \frac{1}{8\pi G}\left[\frac{b}{r^3} - 2\left(1 - \frac{b}{r}\right)\frac{\phi'}{r}\right]$$
$$p = \frac{1}{8\pi G}\left\{\left(1 - \frac{b}{r}\right)\left(\phi'' + \phi'\left[\phi' + \frac{1}{r}\right]\right) - \frac{1}{2r^2}(b'r - b)\left(\phi' + \frac{1}{r}\right)\right\}$$

Proof:

- The first equation gives
$$b(r) = b(r_0) + \int_{r_0}^{r} 8\pi G\rho(r')r^2 \,\mathrm{d}r = 2Gm(r),$$
where
$$m(r) = \frac{r_0}{2G} + \int_{r_0}^{r} 4\pi\rho r^2 \,\mathrm{d}r\,,$$
which can be interpreted as the effective mass inside some radius r. Therefore, the shape function $b(r)$ describes the distribution of mass.

- There are some important inequalities,
$$\exists r_* | \forall r \in (r_0, r_*), \qquad \rho < \tau$$
$$\rho(r_0) \leq \tau(r_0)$$