MAT292
Tutorial 8 Solution

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1. i.

$$\begin{aligned}
\mathcal{L}\{Ce^{at}\} &= \int_0^\infty Ce^{(a-s)t}\,\mathrm{d}t \qquad (1)\\
&= \frac{C}{s-a} \qquad (2)
\end{aligned}$$

ii.

$$\begin{aligned}
\mathcal{L}\{e^t \sin t\} &= \int_0^\infty e^{(1-s)t}\sin t\,\mathrm{d}t \qquad (3)\\
&= \frac{1}{2i}\int_0^\infty e^{(1-s+i)t} - e^{(1-s-i)t}\,\mathrm{d}t \qquad (4)\\
&= \frac{1}{2i}\frac{1}{s-1-i} - \frac{1}{2i}\frac{1}{s-1+i} \qquad (5)\\
&= \frac{1}{2i}\left(\frac{2i}{(s-1)^2+1}\right) \qquad (6)\\
&= \frac{1}{(s-1)^2+1} \qquad (7)
\end{aligned}$$

iii.

$$\begin{aligned}
\mathcal{L}\{e^t \cos t\} &= \int_0^\infty e^{(1-s)t}\cos t\,\mathrm{d}t \qquad (8)\\
&= \frac{1}{2}\int_0^\infty e^{(1-s+i)t} + e^{(1-s-i)t}\,\mathrm{d}t \qquad (9)\\
&= \frac{1}{2}\frac{1}{s-1-i} + \frac{1}{2i}\frac{1}{s-1+i} \qquad (10)\\
&= \frac{1}{2}\left(\frac{2(s-1)}{(s-1)^2+1}\right) \qquad (11)\\
&= \frac{s-1}{(s-1)^2+1} \qquad (12)
\end{aligned}$$

Taking the Laplace transform of both sides, we are left with

$$sF(s) - f(0) = aF(s) \qquad (13)$$

and solving for $F(s)$ gives

$$F(s) = \frac{y_0}{s-a} \qquad (14)$$

so

$$f(t) = y_0 e^{-at} \qquad (15)$$

(b) The characteristic formula is

$$r^2 - 2r + 2 = 0 \qquad (16)$$

and the roots are $r = 1 \pm i$

(c) Taking the Laplace Transform of both sides, we get

$$s^2F(s) - sf(0) - f'(0) - 2sF(s) + 2f(0) + 2F(s) = 0$$

using $f(0) = 1$, we get

$$(s^2 - 2s + 2)F(s) - s + 2 = 0$$

and solving for $F(s)$ gives

$$F(s) = -\frac{s-2}{s^2 - 2s + 2}$$

(d) Completing the square, we get

$$F(s) = -\frac{(s-1)-1}{(s-1)^2+1} = \frac{-(s-1)}{(s-1)^2+1} + \frac{1}{(s-1)^2+1} \tag{17}$$

so using the table, we get

$$f(t) = e^{at}(\sin t + \cos t) \tag{18}$$

2. (a) i. e^{-t}

ii. Rewrite as $\dfrac{s}{s^2+4} + \dfrac{3}{s^2+4}$ which gives $\dfrac{1}{4}\cos(2t) + \dfrac{3}{4}\sin(2t)$

iii. Rewrite as $\dfrac{1!}{(s-(-1))^{1+1}}$ so the inverse transform is te^{-t}

iv. Note that $\dfrac{d}{ds}\left(-2\dfrac{1}{(s^2+4)}\right) = \dfrac{4s}{(s^2+4)^2}$ so the inverse Laplace Transform is $-t\sin(2t)$. We want to find the inverse transform of $\dfrac{4s-4}{(s^2+4)^2}$, so we are left with $-t\sin(2t) - 2\sin(2t)$

We want inverse transform of $\dfrac{4(s-4)}{(s^2+4)^2}$. Let $s' = s - 4$. Then we have

$$\frac{4s'}{(s'^2 + 8s + 16 + 4)^2} = \frac{4s'}{(s'^2 + 8s' + 20)^2} = \frac{4s'}{((s'+4)^2 + 4)^2} \tag{19}$$