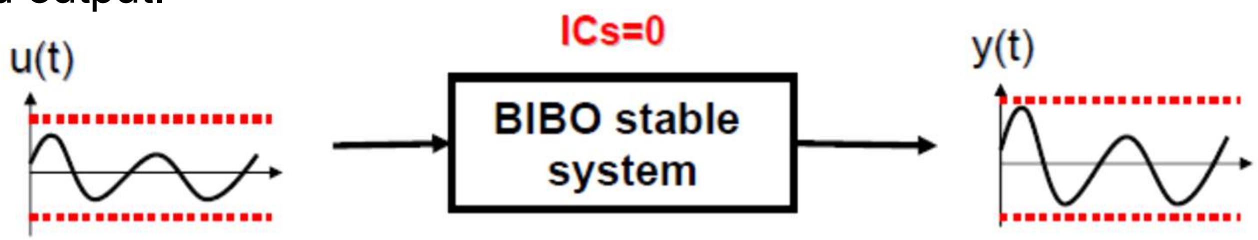
Stability of LTI Systems

$$H(s) = \frac{Y(s)}{U(s)} = K_H \frac{s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n} = K_H \frac{\prod_{i=1}^{m}(s - z_i)}{\prod_{j=1}^{n}(s - p_j)} \quad (m \leq n)$$

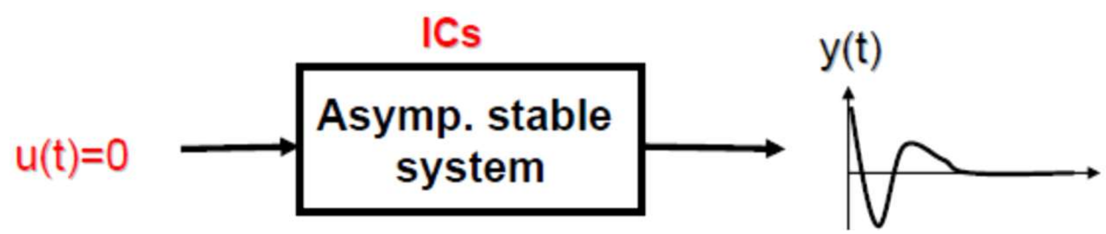
- Assuming poles are distinct (real or complex), the unit impulse response is:

$$h(t) = \sum_{i=1}^{n} C_i e^{p_i t}$$

- Coefficients C_i depend on the initial conditions and locations of zeros.
- If there are repeated poles, one additional exponential is added to the response for each repetition *j* with a factor of t^j (and its own coeficient).
- Bounded-Input-Bounded-Output (BIBO) Stability: any bounded input generates a bounded output.



- Asymptotic Stability: Any initial condition (IC) generates an output which converges to zero.



Stability of LTI Systems

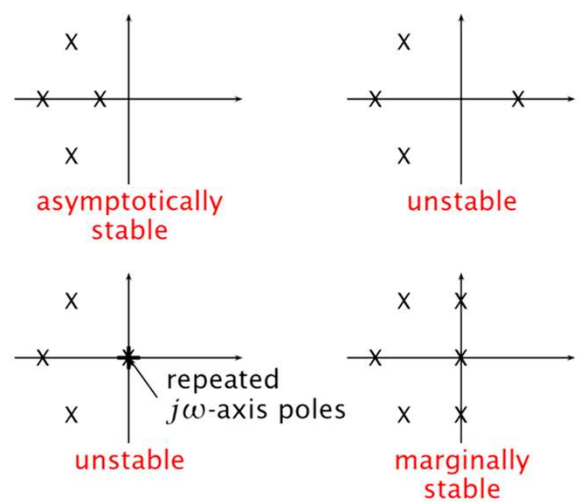
$$H(s) = \frac{Y(s)}{U(s)} = K_H \frac{s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n} = K_H \frac{\prod_{i=1}^{m}(s - z_i)}{\prod_{j=1}^{n}(s - p_j)} \quad (m \le n)$$

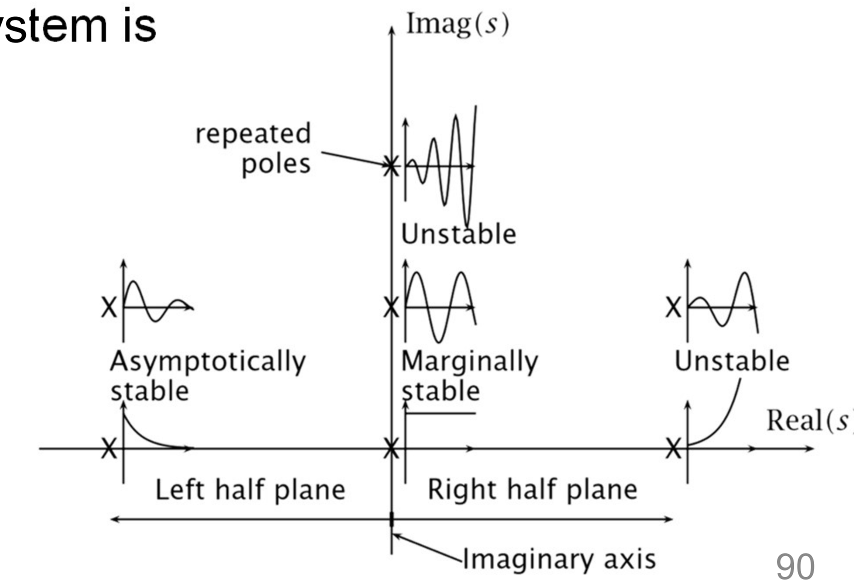
- For LTI systems the conditions for BIBO and asymptotic stability are the same.
- An LTI system is stable if and only if every exponential in the unit impulse response goes to zero at $t \rightarrow \infty$.

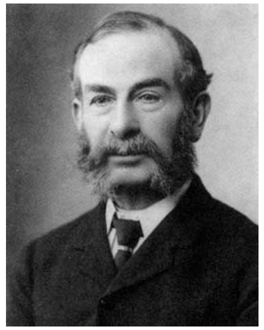
$$e^{p_i t} \rightarrow 0 \quad for\ all\ p_i \quad \Longleftrightarrow \quad Re\{p_i\} < 0$$

- The system is stable if all poles are in the LHP. If any pole is in the RHP, the system is unstable.

- If there are non-repeated poles on the $j\omega$ axis, the system is neutrally (marginally) stable. If there are repeated poles on the $j\omega$ axis, the system is unstable.







Edward J. Routh
(1831-1907)

Stability of LTI Systems

Routh's (Routh-Hurwitz) Stability Criterion

Adolf Hurwitz
(1859-1919)

- Method for checking the location of the poles without having to solve for or perform partial fractioning of characteristic equation.

$$a(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n$$

- If all the coefficients of the polynomial are real numbers, the roots will be real and/or pairs of conjugate complex numbers.
- If all the roots of the characteristic equation (poles) are in the LHP, i.e., the system is stable, then all the coefficients a_i of the characteristic polynomial are positive real numbers.
- If any of the coefficients a_i is zero (missing) or negative, then the system has some poles located out of LHP, hence the system is unstable.
- If all the coefficients a_i are positive real numbers, we need further examinations to check for the location of the poles.
- One examination method is done by creating an array from the characteristic polynomial coefficients, called Routh array.

Stability of LTI Systems

Routh's Stability Criterion

$$a(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n = 0$$

Ruth Array:

Row n	s^n:	1	a_2	a_4	$\cdots$
Row n - 1	s^{n-1}:	a_1	a_3	a_5	$\cdots$
Row n - 2	s^{n-2}:	b_1	b_2	b_3	$\cdots$
Row n - 3	s^{n-3}:	c_1	c_2	c_3	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
Row 2	s^2:	$*$	$*$		
Row 1	s^1:	$*$			
Row 0	s^0:	$*$			

where

$$b_1 = -\frac{det\begin{bmatrix}1 & a_2\\ a_1 & a_3\end{bmatrix}}{a_1} = \frac{a_1 a_2 - a_3}{a_1}\ ;\ b_2 = -\frac{det\begin{bmatrix}1 & a_4\\ a_1 & a_5\end{bmatrix}}{a_1} = \frac{a_1 a_4 - a_5}{a_1}\ ;\ b_3 = -\frac{det\begin{bmatrix}1 & a_6\\ a_1 & a_7\end{bmatrix}}{a_1} = \frac{a_1 a_6 - a_7}{a_1}$$

$$c_1 = -\frac{det\begin{bmatrix}a_1 & a_3\\ b_1 & b_2\end{bmatrix}}{b_1} = \frac{b_1 a_3 - a_1 b_2}{b_1}\ ;\ c_2 = -\frac{det\begin{bmatrix}a_1 & a_5\\ b_1 & b_3\end{bmatrix}}{b_1} = \frac{b_1 a_5 - a_1 b_3}{b_1}\ ;\ c_3 = -\frac{det\begin{bmatrix}a_1 & a_7\\ b_1 & b_4\end{bmatrix}}{b_1} = \frac{b_1 a_7 - a_1 b_4}{b_1}$$

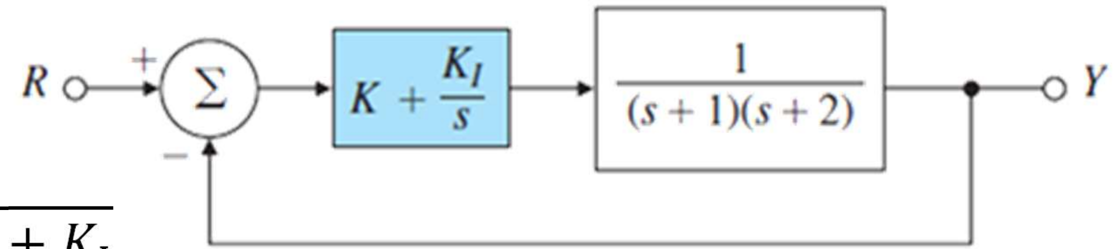
Note: To simplify the calculations, all elements of each row can be divided by a common factor, if there is any, before proceeding to the next row.

- A system is stable, i.e., all roots of the characteristic polynomial are in the LHP, if and only if all the elements in the first column of the Ruth array are positive.
- The number of poles in the RHP is equal to the number of sign changes in the first column of the Ruth array, e.g., for +, -, + there are two poles in the RHP.

Stability of LTI Systems

Routh's Stability Criterion

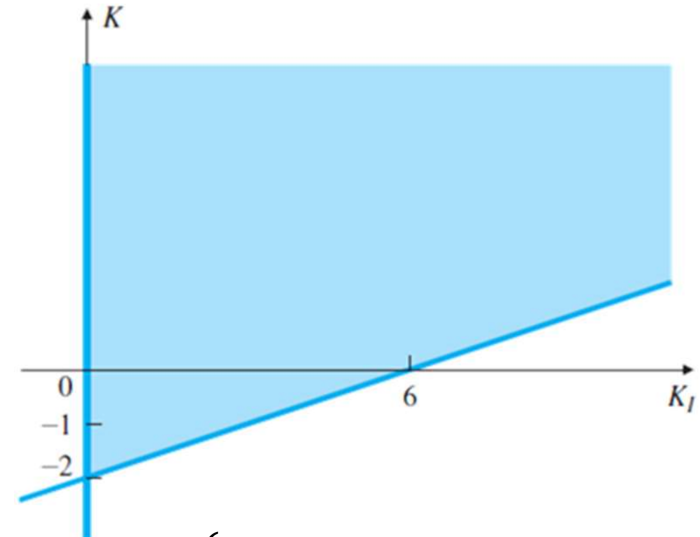
Example:

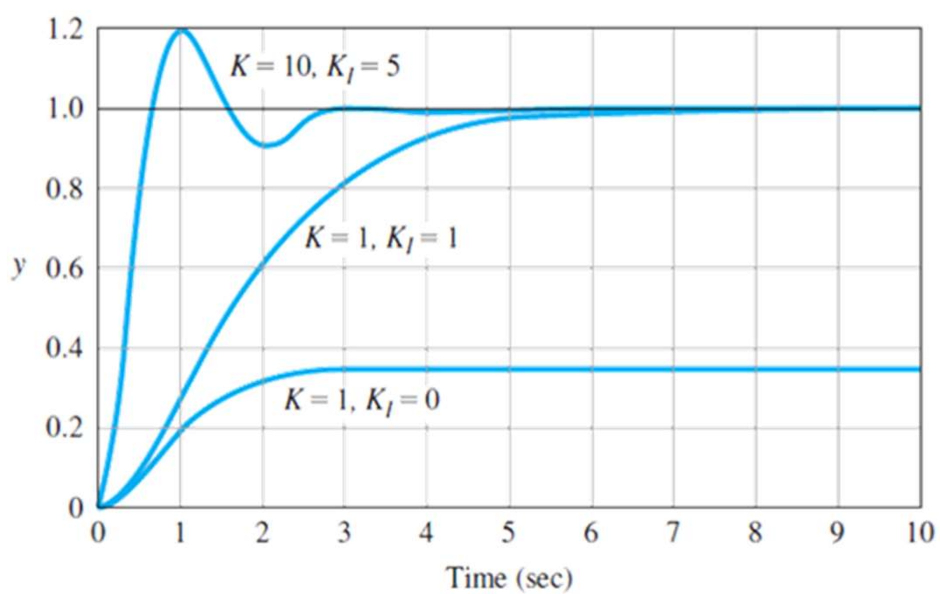


$$\frac{Y(s)}{R(s)} = \frac{Ks + K_I}{s^3 + 3s^2 + (2+K)s + K_I}$$

Row	3	s^3:	1	$2+K$
Row	2	s^2:	3	K_I
Row	1	s^1:	$(6+3K-K_I)/3$	
Row	0	s^0:	K_I	

Stability Condition $K_I > 0$ and $K > \frac{1}{3}K_I - 2$





$$K=1, K_I=0: \begin{cases} z = 0 \\ p_1 = 0 \\ p_{2,3} = -1.5 \pm j0.86 \end{cases}$$

$$K=1, K_I=1: \begin{cases} z = -1 \\ p_{1,2,3} = -1 \end{cases}$$

$$K=10, K_I=5: \begin{cases} z = -0.5 \\ p_1 = -0.46 \\ p_{2,3} = -1.26 \pm j3.3 \end{cases}$$

Stability of LTI Systems

Routh's Stability Criterion

Special Case 1: The first element of one row is zero.

- assume that the element is a small positive value $\varepsilon > 0$, and apply the test by taking the limit as $\varepsilon \to 0$.

Example: $a(s) = s^5 + 3s^4 + 2s^3 + 6s^2 + 6s + 9 = 0$

Row	5	s^5:	1	2	6
Row	4	s^4:	3	6	9
Row	4′	s^4:	1	2	3
Row	3	s^3:	0	3	
Row	3′	s^3:	ε	3	
Row	2	s^2:	$\frac{2\varepsilon-3}{\varepsilon}$	3	
Row	1	s^1:	$3 - \frac{3\varepsilon^2}{2\varepsilon-3}$		
Row	0	s^0:	3		

- Taking $\varepsilon \to 0$, the first element of Row 2 is negative, whereas the first element of Row 1 is positive. So, there are two sign changes in the first column of the array.

- Roots of $a(s)$ are: -2.9042, $-0.7046 \pm j9929$, and $0.6567 \pm j1.2881$.

Stability of LTI Systems

Routh's Stability Criterion

Special Case 2: All elements of one row (Row i) are zero.

a) Create an auxiliary polynomial $p(s)$ from Row $(i+1)$ with even powers.

b) Take derivative of $p(s)$, and use the resulting coefficients as elements of Row i.

- Auxiliary polynomial $p(s)$ always has even degree. If the degree is $2n$, there are n pairs of roots of equal magnitude and opposite sign.
- Roots of the auxiliary polynomial $p(s)$ are also included in the roots of the original polynomial.

Example: $a(s) = s^5 + 5s^4 + 11s^3 + 23s^2 + 28s + 12 = 0$

Row	5	s^5:	1	11	28
Row	4	s^4:	5	23	12
Row	3	s^3:	6.4	25.6	
Row	2	s^2:	3	12	$\longrightarrow$ $p(s) = 3s^2 + 12$
Row	1	s^1:	0	0	
Row	$1'$	s^1:	6	0	$\longleftarrow$ $\dfrac{dp(s)}{ds} = 6s$
Row	0	s^0:	12		

- No sign change in the first column of the array, thus no root in the RHP.
- Roots of $a(s)$ include two conjugate imaginary roots of $p(s)$, i.e., $\pm j2$.
- The other roots of $a(s)$ are: -3, -1, and -1 (repeated roots).