Fintech545 Homework 3

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Question 1. Expectation and Standard Deviation about Price

For the problemset 1, we assume that $r_t \sim N(0, \sigma^2)$. We only concern the specific time t and t-1 period. Therefore, for each method, our calculation will be based on a conditional probability density function $f(P_t|P_{t-1})$.

Classical Brownian Motion

$$P_t = P_{t-1} + r_t \quad r_t \sim N(0, \sigma^2)$$

$$E(P_t|P_{t-1}) = E(P_{t-1} + r_t|P_{t-1}) = \int (P_{t-1} + r_t) f(r_t) dr_t = P_{t-1} + \int r_t f(r_t) dr_t = P_{t-1}$$

$$Var(P_t|P_{t-1}) = Var(P_{t-1} + r_t|P_{t-1}) = E_{p_{t-1}}(P_{t-1} + r_t - E_{p_{t-1}}(P_{t-1} + r_t))^2 = E(r_t)^2 = \sigma^2$$

$$Std(P_t|P_{t-1}) = \sigma$$

Arithmetic Return System

$$P_t = P_{t-1}(1 + r_t) r_t \sim N(0, \sigma^2)$$

$$E(P_t|P_{t-1}) = E(P_{t-1} + P_{t-1}r_t|P_{t-1}) = \int (P_{t-1}(1 + r_t)) f(r_t) dr_t = P_{t-1} + P_{t-1} \int r_t f(r_t) dr_t = P_{t-1}$$

$$Var(P_t|P_{t-1}) = Var(P_{t-1} + P_{t-1}r_t|P_{t-1}) = E_{p_{t-1}}(P_{t-1} + P_{t-1}r_t - P_{t-1})^2 = E_{p_{t-1}}(P_{t-1}r_t)^2$$
$$= P_{t-1}^2 E(r_t)^2 = P_{t-1}^2 \sigma^2$$

$$Std(P_t|P_{t-1}) = P_{t-1}\sigma$$

Geometric Brownian Motion

$$P_t = P_{t-1}e^{r_t}$$

$$E(P_t|P_{t-1}) = E(P_{t-1}e^{r_t}|P_{t-1}) = \int (P_{t-1}e^{r_t}) f(r_t) dr_t = P_{t-1} \int e^{r_t} f(r_t) dr_t = P_{t-1}E(e^{r_t})$$
$$= P_{t-1}e^{\frac{\sigma^2}{2}}$$

$$Var(P_t|P_{t-1}) = Var(P_{t-1}e^{r_t}|P_{t-1}) = E_{p_{t-1}}(P_{t-1}e^{r_t} - E_{p_{t-1}}(P_{t-1}e^{r_t}))^2$$
$$= P_{t-1}^2 E(e^{r_t} - e^{\frac{\sigma^2}{2}})^2 = P_{t-1}^2 E(e^{2r_t} - 2e^{r_t}e^{\frac{\sigma^2}{2}} + e^{\sigma^2})$$
$$= P_{t-1}^2 (e^{2\sigma^2} - 2e^{\sigma^2} + e^{\sigma^2}) = P_{t-1}^2 (e^{2\sigma^2} - e^{\sigma^2})$$

$$Std(P_t|P_{t-1}) = \sqrt{P_{t-1}^2 (e^{2\sigma^2} - e^{\sigma^2})}$$

Here, the process of applying the moment generating function$[E(e^{tX}) = e^{\mu t + \frac{\sigma^2}{2}t^2}, X \sim N(\mu, \sigma^2)]$ has been simplified.

Simulation Results

For the simulation part, we simulated 90,000,000 times for each price formula. The initial setings are $P_{t-1} = 5, \sigma = 2$.

Price Formula	Simulated Mean	Expectation	Simulated Std	Std
Classical Brownian Motion	4.999	5	2.002	2
Arithmetic Return System	4.999	5	10.012	10
Geometric Brownian Motion	36.926	36.945	264.017	270.479

Question 2. META VaR Simulation

In order to calculate the VaR, we calculate the returns of META and deduct the mean from META. Here is Result Table:

Simulation Method	VaR
Normal Distribution	0.06546
Normal Distribution with EW	0.09139
T Distribution	0.05726
AR(1) Model	0.03556
Historic Simulation	0.05125

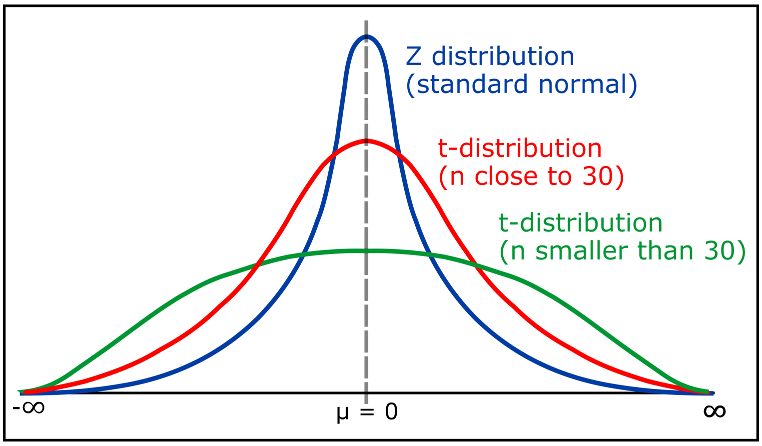
The Fitting Result Note:

- **AR(1) Model:** $y_t = 0.00723 \times y_{t-1} + w_t$ where $w_t \sim N(0, 0.0356)$;
- **T Distribution under MLE:** $\sigma = 0.026, \quad df = 3.92$

Question 3. Portfolio VaR Calculation

There, we choose Normal Monte Carlo VaR to calculate Portfolio risk separately and together. Here is the result table:

VaR($)	Portfolio A	Portfolio B	Portfolio C	Total
Normal Distribution	6404	4744	4023	14751
T Distribution:df = 1	11257	8562	7320	26499
T Distribution:df = 100	6404	4773	4087	14853



Summary:

- From the normal distribution simulation, we can find that the Portfolio A is high, compared with B and C. According to the table, generally VaR: A > B > C;
- From the T distribution simulation, we tried different settings degree of freedom settings for the VaR model, we can find that it affects the results seriously:
  - According to the above graph, when degree of freedom is low, it means that we set a high probability in the tails. When the alpha is fixed, then the distance between mean and corresponding points will be higher. Finally, our VaR assessment will be higher.
  - when degree of freedom is high, the result will approach to the result of normal distribution.