Holographic First-Order Phase Transition

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This is the first version of numerical code for the papers [arXiv:2209.12789 [hep-th]] and [arXiv:2211.11291 [hep-th]], and it will be improved and extended in the future.

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Let's build an open-source community together, cheers!

1 Redefined Fields

$$
\begin{aligned}
\phi &= \varphi_1(t) z + z^2 \widehat{\phi} \\
G &= 1 + z^3 \widehat{G} \\
\Sigma &= z^{-1} + \lambda - \frac{\varphi_1^2(t)}{8} z + z^2 \widehat{\Sigma} = z^{-1} \widetilde{\Sigma} \\
F &= -\partial_x \lambda + z \widehat{F} \\
d_+ \Sigma &= \frac{1}{2} \left(z^{-1} + \lambda \right)^2 - \frac{\varphi_1^2(t)}{16} + z \widehat{d_+ \Sigma} \\
d_+ G &= -\frac{3}{2} g_3 z^2 + z^3 \widehat{d_+ G} \\
d_+ \phi &= -\frac{\varphi_1(t)}{2} - z \left(\lambda \varphi_1(t) + \varphi_2(t,x) - \partial_t \varphi_1(t) \right) + z^2 \widehat{d_+ \phi} \\
A &= \frac{1}{2} \left(z^{-1} + \lambda \right)^2 - \frac{\varphi_1^2(t)}{8} + \widehat{A} \\
V(\phi) &= -6 - \phi^2 - 6 z^4 \widetilde{V}_{cosh} \\
\frac{d}{d\phi} V(\phi) &= -2\phi - 6 z^3 \widetilde{V}_{sinh}
\end{aligned}
\tag{1.1}
$$

2 Characteristic Formulation

2.1 For Σ

$$
\begin{aligned}
0 = & \left[z^2 \partial_{\mathbf{z}}^2 + 6 z \partial_{\mathbf{z}} + \frac{z^2}{4} \left((\partial_z \phi)^2 + \left(\frac{\partial_z G}{G} \right)^2 \right) + 6 \right] \widehat{\Sigma} \\
& + \left(\frac{\lambda(t,x)}{4} - \frac{\varphi_1^2(t)}{32} z \right) \left((\partial_z \phi)^2 + \left(\frac{\partial_z G}{G} \right)^2 \right) \\
& + \frac{z^3}{4} \left[\left(z^2 \frac{\partial_z \widehat{G}}{G} + 6 z \frac{\widehat{G}}{G} \right) \frac{\partial_z \widehat{G}}{G} + \left(\partial_z \widehat{\phi} \right)^2 + 9 \left(\frac{\widehat{G}}{G} \right)^2 \right] + z^2 \widehat{\phi} \partial_z \widehat{\phi} + z \left(\frac{\varphi_1(t)}{2} \partial_z \widehat{\phi} + \widehat{\phi}^2 \right) + \varphi_1(t) \widehat{\phi}
\end{aligned}
\tag{2.1}
$$

2.2 For F

$$
\begin{aligned}
0 = &\left[z\partial_{\mathbf{z}}^{\mathbf{2}} + \left(4 - z\frac{\partial_z G}{G}\right)\partial_{\mathbf{z}} + z\left(\frac{3}{2}\left(\frac{\partial_z G}{G}\right)^2 + \frac{1}{2}\left(\partial_z\phi\right)^2 - 2\frac{\partial_z\widetilde{\Sigma}}{\widetilde{\Sigma}}\left(\frac{\partial_z\widetilde{\Sigma}}{\widetilde{\Sigma}} + \frac{\partial_z G}{G}\right) - \frac{\partial_z^2 G}{G}\right) + 4\frac{\partial_z\widetilde{\Sigma}}{\widetilde{\Sigma}} - \frac{\partial_z G}{G}\right]\widehat{F} \\
&+ \left[2\frac{\partial_z\widetilde{\Sigma}}{\widetilde{\Sigma}}\left(\frac{\partial_z\widetilde{\Sigma}}{\widetilde{\Sigma}} + \frac{\partial_z G}{G}\right) + \frac{\partial_z^2 G}{G} - \frac{3}{2}\left(\frac{\partial_z G}{G}\right)^2 - \frac{1}{2}\left(\partial_z\phi\right)^2\right]\partial_x\lambda(t,x) \\
&+ z^3\left[2\left(z\frac{\partial_z\widehat{G}}{G} + 3\frac{\widehat{G}}{G}\right)\left(\frac{\partial_x\widehat{\Sigma}}{\widetilde{\Sigma}} - \frac{\partial_x\widehat{G}}{G}\right) + 2z\left(\partial_x\lambda(t,x)\left(\frac{\widehat{\Sigma}}{\widetilde{\Sigma}}\right)^2 + \frac{\partial_x\widehat{\Sigma}}{\widetilde{\Sigma}}\frac{\partial_z\widehat{\Sigma}}{\widetilde{\Sigma}}\right) - \frac{\varphi_1^2(t)\partial_x\lambda(t,x)}{2\widetilde{\Sigma}}\frac{\widehat{\Sigma}}{\widetilde{\Sigma}} + 6\frac{\widehat{\Sigma}}{\widetilde{\Sigma}}\frac{\partial_x\widehat{\Sigma}}{\widetilde{\Sigma}}\right] \\
&+ z^2\left[\frac{1}{\widetilde{\Sigma}}\left(2\partial_x\lambda(t,x)\left(\frac{\partial_z\widehat{G}}{G} + \frac{\partial_z\widehat{\Sigma}}{\widetilde{\Sigma}}\right) - \frac{\varphi_1^2(t)}{2}\frac{\partial_x\widehat{\Sigma}}{\widetilde{\Sigma}}\right) + \frac{\varphi_1^4(t)\partial_x\lambda(t,x)}{32}\frac{1}{\widetilde{\Sigma}^2} - 4\partial_x\lambda(t,x)\frac{\partial_z\widehat{\Sigma}}{\widetilde{\Sigma}} - \partial_x\widehat{\phi}\partial_z\widehat{\phi}\right] \\
&+ 2z\left[\frac{1}{\widetilde{\Sigma}}\left(\lambda(t,x)\frac{\partial_x\widehat{\Sigma}}{\widetilde{\Sigma}} + \partial_x\lambda(t,x)\left(4\frac{\widehat{\Sigma}}{\widetilde{\Sigma}} + 3\frac{\widehat{G}}{G}\right)\right) - 6\partial_x\lambda(t,x)\frac{\widehat{\Sigma}}{\widetilde{\Sigma}} - \frac{\partial_x\partial_z\widehat{\Sigma}}{\widetilde{\Sigma}} - \widehat{\phi}\partial_x\widehat{\phi} + \frac{1}{2}\frac{\partial_x\partial_z\widehat{G}}{G}\right] \\
&+ \varphi_1^2(t)\partial_x\lambda(t,x)\frac{1}{\widetilde{\Sigma}} - \left(2\lambda^2(t,x) + \frac{3\varphi_1^2(t)}{4}\right)\partial_x\lambda(t,x)\frac{1}{\widetilde{\Sigma}^2} - \varphi_1(t)\partial_x\widehat{\phi} + 3\frac{\partial_x\widehat{G}}{G} - 6\frac{\partial_x\widehat{\Sigma}}{\widetilde{\Sigma}}
\end{aligned}
\tag{2.2}
$$

2.3 For $d_+\Sigma$

$$
\begin{aligned}
0 = &\partial_{\mathbf{z}}\left(\widetilde{\Sigma}\widehat{d_+\Sigma}\right) \\
&+ \frac{1}{2G}\left[z^2\left(\frac{1}{4}F^2\left(\partial_z\phi\right)^2 + F\partial_z F\left(\frac{1}{2}\frac{\partial_z G}{G} - \frac{\partial_z\widetilde{\Sigma}}{\widetilde{\Sigma}}\right) - \frac{1}{4}\left(\partial_z F\right)^2\right) + zF\partial_z F + \partial_x F\left(\frac{\partial_z\widetilde{\Sigma}}{\widetilde{\Sigma}} - \frac{1}{2}\frac{\partial_z G}{G}\right) - \frac{1}{2}\partial_z F\frac{\partial_x G}{G} + \frac{1}{2}\partial_x\partial_z F\right. \\
&\left. + \left(\frac{\partial_x\partial_z\widetilde{\Sigma}}{\widetilde{\Sigma}} - \frac{1}{2}\frac{\partial_x\partial_z G}{G} + \frac{\partial_x G}{G}\frac{\partial_z G}{G} - \frac{\partial_x\widetilde{\Sigma}}{\widetilde{\Sigma}}\frac{\partial_z\widetilde{\Sigma}}{\widetilde{\Sigma}} - \frac{\partial_x G}{G}\frac{\partial_z\widetilde{\Sigma}}{\widetilde{\Sigma}}\right)F\right] + \frac{3}{2}\widetilde{V}_{cosh}\left(z,\widehat{\phi}\right)\widetilde{\Sigma}^2 \\
&+ \frac{1}{2}z^4\left[\frac{1}{G}\left(\left(\frac{\partial_x\widehat{\Sigma}}{\widetilde{\Sigma}}\right)^2 + \frac{\partial_x\widehat{G}}{G}\frac{\partial_x\widehat{\Sigma}}{\widetilde{\Sigma}} - \left(\frac{\partial_x\widehat{G}}{G}\right)^2\right) + \left(z\varphi_1(t)\widehat{\phi} + \frac{\varphi_1^2(t)}{2}\right)\widehat{\Sigma}^2 - \frac{\varphi_1^3(t)}{4}\widehat{\Sigma}\widehat{\phi}\right] \\
&+ z^3\left[\left(\lambda(t,x)\varphi_1(t)\widehat{\phi} - \frac{\varphi_1^4(t)}{16}\right)\widehat{\Sigma} + \frac{\varphi_1^5(t)}{128}\widehat{\phi} + \frac{1}{2}\frac{\widehat{F}}{G}\frac{\partial_x\widehat{G}}{G}\right] \\
&+ \frac{1}{2}z^2\left[\frac{1}{G}\left(\partial_x^2\lambda(t,x)\frac{\widehat{\Sigma}}{\widetilde{\Sigma}} + \frac{\partial_x\lambda(t,x)}{\widetilde{\Sigma}}\left(\frac{\partial_x\widehat{G}}{G} + 2\frac{\partial_x\widehat{\Sigma}}{\widetilde{\Sigma}}\right) - \frac{1}{4}\left(\partial_x\widehat{\phi}\right)^2 - \partial_x\lambda(t,x)\frac{\partial_x\widehat{G}}{G}\right)\right. \\
&\left. + \left(\lambda^2(t,x) - \frac{\varphi_1^2(t)}{8}\right)\partial_z\widehat{\Sigma} + 3\widehat{\Sigma}^2 + \left(\lambda(t,x)\varphi_1^2(t) + 2\varphi_1(t)\widehat{\phi}\right)\widehat{\Sigma} + \frac{\varphi_1^6(t)}{128} - \frac{\lambda(t,x)\varphi_1^3(t)}{4}\widehat{\phi}\right] \\
&+ z\left[\frac{1}{2G}\left(\frac{1}{2}\frac{\partial_x^2\widehat{G}}{G} - \frac{\partial_x^2\widehat{\Sigma}}{\widetilde{\Sigma}} - \frac{\varphi_1^2(t)\partial_x^2\lambda(t,x)}{8}\frac{1}{\widetilde{\Sigma}}\right) + \lambda^2(t,x)\widehat{\Sigma} - \frac{\lambda(t,x)\varphi_1^4(t)}{16} + \lambda(t,x)\partial_z\widehat{\Sigma} + \left(\frac{\lambda^2(t,x)\varphi_1(t)}{2} - \frac{\varphi_1^3(t)}{8}\right)\widehat{\phi}\right] \\
&+ \frac{1}{2G}\left[\frac{\lambda(t,x)\partial_x^2\lambda(t,x)}{\widetilde{\Sigma}} + \frac{\left(\partial_x\lambda(t,x)\right)^2}{\widetilde{\Sigma}^2} - \partial_x\widehat{F}\right] + \frac{3\lambda^2(t,x)\varphi_1^2(t)}{16} + 4\lambda(t,x)\widehat{\Sigma} + \lambda(t,x)\varphi_1(t)\widehat{\phi} - \frac{\varphi_1^4(t)}{32} + \frac{\partial_z\widehat{\Sigma}}{2} + \frac{1}{4}\widetilde{\Sigma}^2\widehat{\phi}^2 \\
&+ z^{-1}\left(\frac{\lambda(t,x)\varphi_1^2(t)}{8} + 3\widehat{\Sigma} + \frac{\varphi_1(t)\widehat{\phi}}{2}\right)
\end{aligned}
\tag{2.3}
$$

2.4 For d_+G

$$
\begin{aligned}
0 =& \left[z\partial_{\mathbf{z}} + z\left(\frac{\partial_z\widetilde{\Sigma}}{\widetilde{\Sigma}} - \frac{\partial_z G}{G}\right) + 2\right]\widehat{d_+G} \\
&+ \frac{1}{\widetilde{\Sigma}}\widehat{d_+\Sigma}\partial_z G + \frac{3g_3(t,x)}{2}\left(\frac{\partial_z G}{G} - \frac{\partial_z\widetilde{\Sigma}}{\widetilde{\Sigma}}\right) \\
&+ \frac{1}{\widetilde{\Sigma}^2}\left\{z^2\left[\left(\frac{\partial_z\widetilde{\Sigma}}{\widetilde{\Sigma}} - \frac{1}{2}\frac{\partial_z G}{G}\right)F\partial_z F + \left(\frac{3}{4}\left(\frac{\partial_z G}{G}\right)^2 - \left(\frac{\partial_z\widetilde{\Sigma}}{\widetilde{\Sigma}}\right)^2 - \frac{1}{2}\frac{\partial_z^2 G}{G} - \frac{\partial_z G}{G}\frac{\partial_z\widetilde{\Sigma}}{\widetilde{\Sigma}}\right)F^2 - \frac{1}{4}\left(\partial_z F\right)^2\right]\right. \\
&\left. + z\left(2F^2\frac{\partial_z\widetilde{\Sigma}}{\widetilde{\Sigma}} - F\partial_z F\right) + \left(\frac{1}{2}\frac{\partial_x\partial_z G}{G} - \frac{\partial_x\partial_z\widetilde{\Sigma}}{\widetilde{\Sigma}} + \frac{\partial_x\widetilde{\Sigma}}{\widetilde{\Sigma}}\frac{\partial_z\widetilde{\Sigma}}{\widetilde{\Sigma}} + \frac{\partial_z G}{G}\frac{\partial_x\widetilde{\Sigma}}{\widetilde{\Sigma}} - \frac{\partial_x G}{G}\frac{\partial_z G}{G}\right)F - F^2 + \frac{1}{2}\partial_x\partial_z F - \partial_z F\frac{\partial_x\widetilde{\Sigma}}{\widetilde{\Sigma}}\right\} \\
&+ \frac{1}{\widetilde{\Sigma}}\left[z^2\left(\frac{\lambda^2(t,x)}{2}\partial_z\widehat{G} - \frac{\varphi_1^2(t)}{16}\partial_z\widehat{G} - \frac{\left(\partial_x\widehat{\phi}\right)^2}{4\widetilde{\Sigma}}\right) + z\left(\frac{3\lambda^2(t,x)}{2}\widehat{G} - \frac{3\varphi_1^2(t)}{16}\widehat{G} + \lambda(t,x)\partial_z\widehat{G}\right) + 3\lambda(t,x)\widehat{G} + \frac{1}{2}\partial_z\widehat{G}\right] \\
&+ \frac{3}{2}\frac{1}{\widetilde{\Sigma}}z^{-1}\left(\widehat{G} - g_3(t,x)\widetilde{\Sigma}\right)
\end{aligned}
\tag{2.4}
$$

2.5 For $d_+\phi$

$$
\begin{aligned}
0 =& \widetilde{\Sigma}\left(z\partial_{\mathbf{z}} + \frac{z\partial_z\widetilde{\Sigma}}{\widetilde{\Sigma}} + 1\right)\widehat{d_+\phi} \\
&+ \frac{1}{G\widetilde{\Sigma}}\left[z^3\left(\frac{1}{2}\left(\frac{\partial_z G}{G}\partial_z\phi - \partial_z^2\phi\right)F^2 - F\partial_z F\partial_z\phi\right) - z^2F^2\partial_z\phi + \frac{1}{2}z\left(\left(2\partial_x\partial_z\phi - \frac{\partial_x G}{G}\partial_z\phi - \frac{\partial_z G}{G}\partial_x\phi\right)F + \partial_x F\partial_z\phi + \partial_z F\partial_x\phi\right)\right] \\
&+ \left(z\widehat{d_+\Sigma} + \frac{\lambda^2(t,x)}{2} - \frac{\varphi_1^2(t)}{16}\right)\partial_z\phi - \left(\lambda(t,x)\varphi_1(t) + \varphi_2(t,x) - \frac{d}{dt}\varphi_1(t)\right)\partial_z\widetilde{\Sigma} - 3\widetilde{\Sigma}\widetilde{V}_{sinh}\left(z,\widehat{\phi}\right) \\
&+ z^2\left(\frac{1}{2}z^2\partial_x\widehat{\phi}\frac{\partial_x\widehat{G}}{G}\frac{1}{G\widetilde{\Sigma}} - \widehat{\Sigma}\widehat{\phi} - \frac{\varphi_1(t)}{2}\partial_z\widehat{\Sigma}\right) + z\left(\lambda(t,x)\partial_z\widehat{\phi} - 2\widehat{\Sigma}\varphi_1(t) + \frac{\varphi_1^2(t)}{8}\widehat{\phi} - \frac{1}{2}\partial_x^2\widehat{\phi}\frac{1}{G\widetilde{\Sigma}}\right) + \lambda(t,x)\widehat{\phi} + \frac{3\varphi_1^3(t)}{16} + \frac{1}{2}\partial_z\widehat{\phi}
\end{aligned}
\tag{2.5}
$$

2.6 For A

$$
\begin{aligned}
0 =& \left(z\partial_{\mathbf{z}}^{\mathbf{2}} + 2\partial_{\mathbf{z}}\right)\widehat{A} \\
&+ \frac{z^3}{2G\widetilde{\Sigma}^2}\left[\left(\partial_z^2 F - \partial_z F\frac{\partial_z G}{G}\right)F + \left(\partial_z F\right)^2\right] + z^2\left(\frac{F\partial_z F}{G\widetilde{\Sigma}^2} - \frac{1}{2}\frac{\widehat{d_+G}}{G}\frac{\partial_z G}{G}\right) \\
&+ z\left[\frac{1}{2G\widetilde{\Sigma}^2}\left(-\partial_x\partial_z F + \partial_z F\frac{\partial_x G}{G}\right) - 3\widetilde{V}_{cosh}\left(z,\widehat{\phi}\right) - \frac{1}{2}\widehat{d_+\phi}\partial_z\phi - 2\frac{\partial_z\widehat{d_+\Sigma}}{\widetilde{\Sigma}} + \frac{3}{4}\frac{g_3(t,x)}{G}\frac{\partial_z G}{G}\right] \\
&+ \frac{1}{2}\left(\lambda(t,x)\varphi_1(t) + \varphi_2(t,x) - \frac{d}{dt}\varphi_1(t)\right)\partial_z\phi - 2\frac{\widehat{d_+\Sigma}}{\widetilde{\Sigma}} - \frac{z^2\varphi_1^2(t)}{4}\frac{\widehat{\Sigma}}{\widetilde{\Sigma}} + z\left(\frac{\varphi_1^4(t)}{32\widetilde{\Sigma}} + \frac{\varphi_1(t)}{4}\partial_z\widehat{\phi} - \frac{1}{2}\widehat{\phi}^2\right) - \frac{\lambda(t,x)\varphi_1^2(t)}{4\widetilde{\Sigma}} \\
&- 2\frac{\widehat{\Sigma}}{\widetilde{\Sigma}} - \frac{\varphi_1(t)}{2}\widehat{\phi}
\end{aligned}
\tag{2.6}
$$

3 Evolution Equations

3.1 For $\partial_t\phi$

$$\begin{aligned}\partial_t\widehat{\phi} = z^2\left(\frac{\lambda^2(t,x)}{2} - \frac{\varphi_1^2(t)}{8} + \widehat{A}\right)\partial_z\widehat{\phi} + z\left(\left(\lambda^2(t,x) - \frac{\varphi_1^2(t)}{4} + 2\widehat{A}\right)\widehat{\phi} + \lambda(t,x)\partial_z\widehat{\phi}\right)\\ + \frac{\lambda^2(t,x)\varphi_1(t)}{2} + 2\lambda(t,x)\widehat{\phi} - \frac{\varphi_1^3(t)}{8} + \varphi_1(t)\widehat{A} + \widehat{d_+\phi} + \frac{1}{2}\partial_z\widehat{\phi} + z^{-1}\left(\widehat{\phi} - \varphi_2(t,x)\right)\end{aligned} \tag{3.1}$$

3.2 For $\partial_t G$

$$\partial_t\widehat{G} = z^2\left(\frac{\lambda^2(t,x)}{2} - \frac{\varphi_1^2(t)}{8} + \widehat{A}\right)\partial_z\widehat{G} + z\left(\left(\frac{3\lambda^2(t,x)}{2} - \frac{3\varphi_1^2(t)}{8} + 3\widehat{A}\right)\widehat{G} + \lambda(t,x)\partial_z\widehat{G}\right) + 3\lambda(t,x)\widehat{G} + \widehat{d_+G} + \frac{1}{2}\partial_z\widehat{G} + \frac{3}{2}z^{-1}\left(\widehat{G} - g_3\right) \tag{3.2}$$

3.3 For $\partial_t\lambda$

$$\partial_t\lambda = -\lim_{z\to 0}\widehat{A} \tag{3.3}$$

3.4 For $\partial_t f_1$

$$\partial_t f_1 = \frac{2}{3}\partial_x a_1 - \partial_x g_3 \tag{3.4}$$

4 Horizon Conditions

4.1 Apparent Horizon Condition

$$\widehat{d_+\Sigma}\Big|_{z=1} = \left[\frac{1}{2G\widetilde{\Sigma}}\left(F^2\left(\frac{\partial_z\widetilde{\Sigma}}{\widetilde{\Sigma}} - 1\right) - \partial_x F + F\frac{\partial_x G}{G}\right) - \frac{\lambda^2(t,x)}{2} - \lambda(t,x) + \frac{\varphi_1^2(t)}{16} - \frac{1}{2}\right]_{z=1} \tag{4.1}$$

4.2 Stationary Horizon Condition

$$\left[\partial_x^2 A + \mathbf{P}\partial_x A + \mathbf{Q}A + \mathbf{S}\right]_{z=1} = 0 \tag{4.2}$$

$$\mathbf{P} = \partial_z F - \left(2\frac{\partial_z\Sigma}{\Sigma} + \frac{\partial_z G}{G}\right)F - \frac{\partial_x G}{G}$$

$$\begin{aligned}\mathbf{Q} = &\left(\left(\frac{\partial_z\Sigma}{\Sigma}\right)^2 + \frac{\partial_z G}{G}\frac{\partial_z\Sigma}{\Sigma} + \frac{1}{4}\left(\frac{\partial_z G}{G}\right)^2\right)F^2 - \left(\frac{1}{2}\frac{\partial_z G}{G} + \frac{\partial_z\Sigma}{\Sigma}\right)F\partial_z F\\ &+ \frac{1}{2}\left(\frac{1}{2}\partial_z F - \frac{\partial_x G}{G}\right)\partial_z F - \left(\frac{1}{2}\frac{\partial_z G}{G} + \frac{\partial_z\Sigma}{\Sigma}\right)\partial_x F + \frac{1}{2}\partial_z\partial_x F + \left(\frac{\partial_z\Sigma}{\Sigma}\frac{\partial_x G}{G} + \frac{\partial_x\Sigma}{\Sigma}\frac{\partial_z\Sigma}{\Sigma} + \frac{\partial_x G}{G}\frac{\partial_z G}{G} - \frac{1}{2}\frac{\partial_z\partial_x G}{G} - \frac{\partial_z\partial_x\Sigma}{\Sigma}\right)F\\ &+ \frac{1}{4}\left(\partial_x\phi\right)^2 + \frac{\partial_x^2\Sigma}{\Sigma} - \left(\frac{\partial_x\Sigma}{\Sigma}\right)^2 + \left(\frac{\partial_x G}{G}\right)^2 - \frac{1}{2}\frac{\partial_x^2 G}{G} - \frac{\partial_x\Sigma}{\Sigma}\frac{\partial_x G}{G} + \frac{1}{2}GV(\phi)\Sigma^2\end{aligned}$$

$$\begin{aligned}\mathbf{S} = &\left(\frac{1}{2}\partial_z\phi d_+\phi + \frac{1}{2}\frac{\partial_z G}{G}\frac{d_+G}{G} + \frac{\partial_z\Sigma}{\Sigma}\frac{d_+\Sigma}{\Sigma} + \frac{\partial_z d_+\Sigma}{\Sigma}\right)F^2 + \left(2\frac{d_+G}{G}\frac{\partial_x\Sigma}{\Sigma} + 2\frac{\partial_x G}{G}\frac{d_+\Sigma}{\Sigma} - d_+\phi\partial_x\phi\right)F - \left(2\frac{d_+\Sigma}{\Sigma} + \frac{d_+G}{G}\right)\partial_x F\\ &+ \frac{1}{G\Sigma^2}\left[F^4\frac{\partial_z\Sigma}{\Sigma}\left(\frac{1}{2}\frac{\partial_z G}{G} + \frac{\partial_z\Sigma}{\Sigma}\right) + F^3\left(\frac{\partial_x G}{G}\frac{\partial_z\Sigma}{\Sigma} - \partial_z F\frac{\partial_z\Sigma}{\Sigma} + 2\frac{\partial_x\Sigma}{\Sigma}\frac{\partial_z\Sigma}{\Sigma} + \frac{\partial_z\phi\partial_x\phi}{2} - \frac{\partial_z\partial_x G}{2G} + \frac{3\partial_z G\partial_x G}{2G^2}\right)\right.\\ &+ F^2\left(\frac{1}{2}\partial_z\partial_x F - \frac{1}{2}\left(\partial_x\phi\right)^2 - \partial_z F\left(\frac{\partial_x G}{G} + \frac{\partial_x\Sigma}{\Sigma}\right) - \partial_x F\left(\frac{\partial_z G}{G} + 2\frac{\partial_z\Sigma}{\Sigma}\right) - \frac{1}{2}\frac{\partial_x^2 G}{G} - \frac{\partial_x^2\Sigma}{\Sigma} + \left(\frac{\partial_x G}{G}\right)^2 + \left(\frac{\partial_x\Sigma}{\Sigma}\right)^2 + 3\frac{\partial_x G}{G}\frac{\partial_x\Sigma}{\Sigma}\right)\\ &\left. + F\left(\partial_z F\partial_x F + \partial_x^2 F - \frac{2\partial_x F\partial_x G}{G} - 2\partial_x F\frac{\partial_x\Sigma}{\Sigma}\right)\right] + \left(-\frac{G(d_+\phi)^2}{2} - \frac{(d_+G)^2}{2G}\right)\Sigma^2 - 2G(d_+\Sigma)^2\end{aligned} \tag{4.3}$$