

Warping and Partitioning for Low Error Shadow Maps

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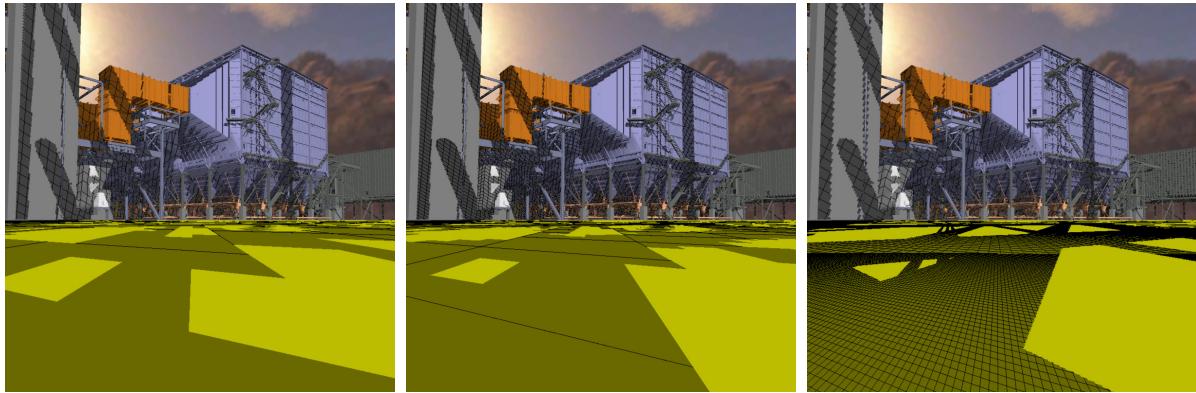


Figure 1: These 800×800 resolution images show the benefit of combining shadow map warping and frustum partitioning algorithms on a powerplant model with a high depth range. Left: A $2K \times 2K$ shadow map generated with only a warping algorithm (LSPSM) has high aliasing error concentrated near the viewer. Middle: The largest allocatable resolution of $4K \times 4K$ still has severe aliasing. Right: Warping combined with four frustum partitions produces low aliasing error with a total resolution of $2K \times 2K$ with only a 30% drop in frame rate. The aliasing error is distributed more uniformly over the scene.

Abstract

We evaluate several shadow map algorithms based on warping and partitioning using the maximum perspective aliasing error over the entire view frustum. With respect to our error metric, we show that a range of warping parameters corresponding to several previous warping algorithms have the same error. We also analyze several partitioning schemes to determine which produces the least maximum error using the least number of partitions. Finally, we show how warping and partitioning can be combined for interactive rendering of low error shadows in scenes with a high depth range.

Categories and Subject Descriptors (according to ACM CCS): I.3.7 [Computer Graphics]: Three-Dimensional Graphics and Realism – Color, Shading, Shadowing and Texture

1. Introduction

Shadows are an important component of an interactive rendering system. Shadow maps are one popular technique for rendering shadows. The standard shadow map algorithm as proposed by Williams [Wil78] is a two pass algorithm that first creates a depth map by rendering the scene from the light's view. In the second pass, the depth map is used to determine which surfaces lie in shadow. Shadow maps are a particularly attractive algorithm because they are easy to implement, they support a wide variety of geometry representations, and there exists wide support for shadow maps in cur-

rent graphics hardware. The main drawback of shadow maps is aliasing errors at shadow edges. Aliasing occurs when the local sampling density in the shadow map is too low. The aliasing errors are worst for scenes with a high depth range because samples in the shadow map must cover larger regions.

Two main approaches are used to address the sampling problem: warping and partitioning. *Warping algorithms* render a reparameterized shadow map that leads to increased sampling resolution where it is needed [SD02, WSP04, MT04, CG04]. Since warping algorithms simply change the

4×4 matrix used to render a standard shadow map, they incur almost no performance penalty and can be easily implemented on current GPUs. *Partitioning algorithms* take a different approach. These algorithms partition the scene and use a separate shadow map for each partition [TQJN99, FFBG01, Arv04, LKS^{*}06]. For example, one shadow map may be used for areas close to the viewer and another for the rest of the scene. While partitioning can reduce aliasing error, rendering shadow maps for too many partitions may be expensive. Some algorithms combine warping and partitioning [Koz04, CG04].

It is often difficult to determine which algorithm is best for a given situation. Moreover, it is not clear how and when to switch between different techniques. We seek a single shadow map algorithm that has low aliasing error and maintains high performance for complex models with high depth range.

Main Results: In this paper we present an error metric for evaluating shadow map algorithms based on the maximum perspective aliasing error over the entire view frustum. Aliasing error can be decomposed into two parts [SD02]: *perspective aliasing*, which depends only on the position of the light relative to the camera, and *projection aliasing*, which depends on the orientation of surfaces in the scene. We base our error metric on perspective aliasing because it is scene independent. Though we deal only with directional light sources in this paper, the error metric analysis can be extended to point lights.

Using our error metric we investigate how to combine warping and partitioning to obtain a low error shadow map solution with good performance and guarantees on the aliasing error. Warping algorithms based on perspective projections, such as perspective shadow maps (PSMs) [SD02], light-space perspective shadow maps (LSPSMs) [WSP04], and trapezoidal shadow maps (TSMs) [MT04] differ primarily in the way the perspective parameter is chosen. We show that when the aliasing errors in both shadow map dimensions are combined, that a range of parameter values corresponding to these algorithms have the same total error.

We also consider two kinds of view frustum partitioning:

- **Face partitioning** splits the view frustum at the edges of its faces as seen from the light’s point of view. Face partitioning allows warping to be used when it could not be used otherwise (e.g. when the light direction is parallel to the view direction) leading to reduced error.
- **z-partitioning** subdivides the view frustum or its face partitions along their length. z- partitioning provides error reductions for all light directions.

Frustum partitioning and z-partition can also be combined. We show that for a given number of partitions, z-partitioning alone combined with warping delivers the least maximum error over the entire view frustum. We demonstrate the performance of this hybrid algorithm on a small model, typi-

cally used in a game-like environment, and on massive models using a view-dependent rendering algorithm.

The rest of this paper is organized as follows. In Section 2 we briefly discuss work related to shadow map computation. In Section 3, we discuss how aliasing error should be measured and justify our choice of error metric. We analyze shadow map warping algorithms in Section 4 and frustum partitioning schemes in Section 5. We describe various implementation details for partitioned shadow maps in Section 6. In Section 7, we show some experimental results for combinations of partitioning and warping that leads to low aliasing error. Finally, we conclude with some ideas for future work.

2. Previous Work

Many techniques have been proposed for shadow generation. In this section, we limit ourselves to shadow maps and some hybrid combinations with object-space techniques. Shadow maps were first introduced by Williams [Wil78]. Segal et al. [SKV^{*}92] later implemented them on standard graphics hardware. In order to hide shadow map aliasing Reeves et al. [RSC87] filtered depth values to blur shadow map edges. Recently Donnelly and Lauritzen [DL06] introduced a way to use depth variance to facilitate better filtering of shadow depth maps.

Other algorithms seek to remove aliasing by locally increasing the shadow map resolution where it is needed either through warping or partitioning or both:

- **Partitioning algorithms.** Tadamura et al. [TQJN99] use z-partitioning for rendering scenes illuminated with sunlight. Adaptive shadow maps [FFBG01] use a quadtree that is refined in areas with high aliasing error. Increased programmability of GPUs has facilitated implementations of adaptive shadow maps for hardware rendering [LKS^{*}06], but performance can be slow. Tiled shadow maps [Arv04] partition a shadow map into tiles of different sizes guided by an aliasing measurement heuristic.
- **Warping algorithms.** Shadow map warping was introduced with perspective shadow maps (PSMs) [SD02]. PSMs use the camera’s perspective transform to warp the shadow map. A singularity may arise with PSMs that requires special handling [Koz04]. Light-space perspective shadow maps (LSPSMs) [WSP04] are a generalization of PSMs that do not have the singularity problem because they use a perspective projection that is oriented perpendicular to the light direction. Trapezoidal shadow maps (TSMs) [MT04] are similar to LSPSMs, except that they use a different formulation for the perspective parameter.
- **Combined algorithms.** Chong and Gortler [CG04] use a general projective transform to ensure that there is a one-to-one correspondence between pixels in the image and the texels in the shadow map on a single plane within the scene. They use a small number of shadow maps to cover a few large surfaces. Kozlov [Koz04] proposed using a

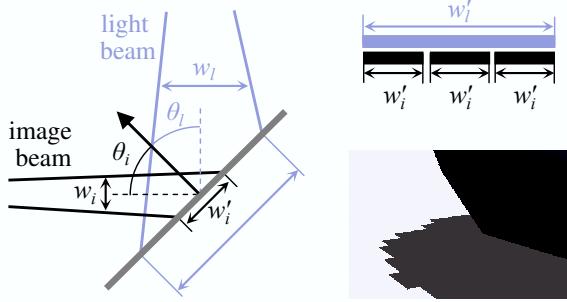


Figure 1: Shadow map aliasing. An image beam through a pixel and a light beam through a shadow map texel project onto a surface (left). When the light beam footprint is larger than the image beam footprint (upper-right), the light beam footprints can be distinguished as a jagged shadow edge (lower-right).

cube map in the post-perspective space of the camera. This corresponds to combining warping with face partitioning.

Irregular shadow maps [JMB04, AL04] avoid the aliasing problem altogether by storing shadow map samples that correspond exactly to the image samples for the eye. However, irregular shadow maps are difficult to implement on current graphics hardware.

Pure object-space shadow algorithms, such as shadow volumes, do not have aliasing problems. Some hybrid algorithms combine object-space techniques with shadow maps to reduce aliasing. McCool et al. [McC00] construct shadow volumes from a shadow map. Sen et al. [SCH03] create a shadow map that more accurately represents shadow edges. Both of these techniques, while generating better looking shadow edges, may miss small features if the shadow map resolution is inadequate. Chan and Durand [CD04] use shadow maps to restrict shadow volume rendering to the shadow edges. Govindaraju et al. [GLY*03] use shadow polygons for the most aliased areas and a shadow map everywhere else.

3. Measuring aliasing error

This section provides an overview of shadow map aliasing and introduces our error metric. We first review how shadow map aliasing occurs. Then we justify why we ignore projection aliasing and discuss the use of maximum perspective aliasing error over the whole frustum for evaluating shadow map algorithms.

3.1. Shadow map aliasing

Figure 1 offers geometric intuition of how shadow map aliasing occurs. A beam emanates from the eye through a pixel on the image plane and projects onto a surface in the scene with a footprint of width w'_i at the intersection point. A beam from

the light through a shadow map texel projects onto the same location with a footprint of width w'_l . When $w'_l > w'_i$, the light beam footprint is covered by multiple image beams and becomes distinguishable as a jagged, aliased edge at shadow boundaries.

Following [SD02], the aliasing error can be quantified as the mismatch ratio of the beam footprint widths:

$$m = \frac{w'_l}{w'_i} \approx \frac{w_l}{w_i} \frac{\cos \theta_i}{\cos \theta_l}, \quad (1)$$

where w_i and w_l are the widths of the image and light beams at the point of intersection and θ_i and θ_l are the angles between the surface normal and the beam directions. The w_l/w_i term is referred to as *perspective aliasing*. Perspective aliasing depends solely on the relative positions of the light and camera. It is independent of the scene geometry. The $\cos \theta_i / \cos \theta_l$ term is referred to as *projection aliasing*. This term depends on the orientation of the surfaces in the scene relative to the camera and the light. Perspective aliasing vanishes when the beam widths are the same, i.e. $w_i = w_l$. Projection aliasing vanishes when the surface is oriented with its normal parallel or perpendicular to the half-way vector between the beam directions, i.e. $\theta_i = \theta_l$.

3.2. Ignoring projection aliasing

Ideally, a shadow map algorithm should ensure that $m = 1$ everywhere in the scene. When $m > 1$ shadow map aliasing can appear at shadow boundaries. When $m < 1$, no aliasing appears, but the shadow map is oversampled and resolution is wasted. In practice, an ideal shadow map is difficult to compute due to the projection aliasing factor. Because of projection aliasing, the local resolution needed for different parts of the scene may vary dramatically depending on the orientations of the surfaces in the scene. Computing the resolution needed for each part of the scene requires a potentially expensive scene analysis, and storing an ideal shadow map requires data structures more complex than a regular grid. Adaptive shadow maps (ASMs) approach the ideal by storing the shadow map in a quad-tree and refining where more resolution is needed. But on current hardware, ASMs are too slow to provide all but a fairly coarse level of subdivision at high-frame rates in a complex environment. Chong et al. [CG04] also compute an optimal shadow map for a few surfaces in the scene, but for other surfaces there are no guarantees on the aliasing error.

We choose to ignore projection aliasing and to minimize perspective aliasing. This means that we can use a shadow map parameterization that is both independent of scene complexity and is simple and efficient to compute. In practice, projection aliasing error might not ever be completely eliminated because it is potentially unbounded. However if perspective aliasing error is small, the projection aliasing that does remain is much less visible for at least two reasons. First, when projection aliasing stretches light beam footprints across a surface, the sampling resolution is reduced

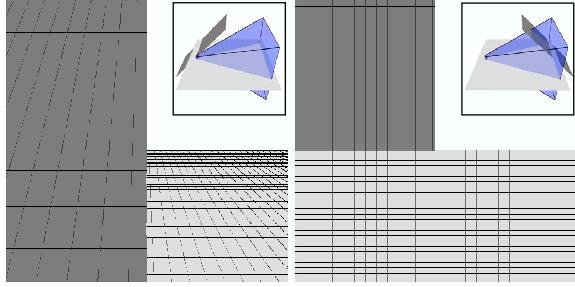


Figure 2: Visualizing aliasing error. These images show shadow map texels projected onto a scene consisting of a simple ground plane and an overhead directional light. The LSPSM algorithm (left) appears to be inferior to the PSM algorithm (right) due to projection effects. To see perspective aliasing more clearly, a plane is inserted on the left side of each image into the area of maximum perspective aliasing for each algorithm and is oriented such that projection effects are mostly removed. LSPSMs show error distributed evenly in both directions, while the error for PSMs is concentrated in a single direction. Both images, in fact, have the same total error.

only in the stretched direction. Second, surfaces which exhibit high projection aliasing error are nearly parallel to the light. For many surfaces, when the light angle is low, little light is reflected, so the shadows are not as noticeable anyway.

3.3. Maximum perspective aliasing error

For our error metric we minimize the L_∞ norm of perspective aliasing error. Specifically, we seek to minimize the maximum value of the w_l/w_i term of m in Eq. (1) over the entire view frustum. Other norms could be used such as the L_1 or the L_2 norms. These norms tend to ensure that the "average" error is low, but high error outliers may occur. (For a more in-depth discussion of error measures in the context of shadow map rendering see [Cho03].) For specific views, where there are no surfaces or shadows in an area with high error, it may be possible for one shadow map to *appear* to have lower error than another, even if *quantitatively* it is inferior (see Figure 2). But in an interactive application where the view is unconstrained or the scene geometry is arbitrary, there is no guarantee that the "bad areas" will not become visible. Our metric gives guarantees on the worst case error independent of the scene.

4. Shadow map warping with perspective projections

Perspective projections are used by prior warping techniques to reduce aliasing. The aliasing error is affected by both the warping parameter and the dimensions of the shadow map relative to the image. In this section we show how the area of the shadow map (in texels) can be used to measure error independent of specific shadow map dimensions. This leads

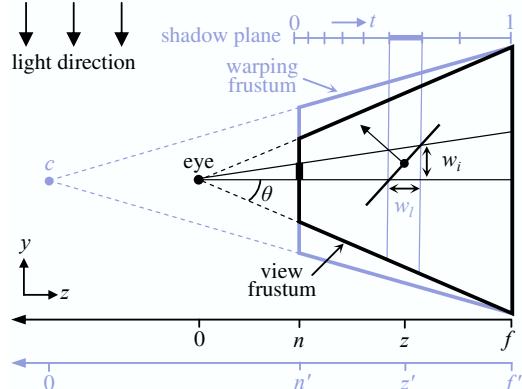


Figure 3: Perspective projection parameterization. Light space is defined with the y -axis aligned with the light direction and the z -axis in the plane of y -axis and view direction. The t -axis of the shadow map is aligned with the z -axis. The x and s axes point out of the page. The shadow map is warped by placing a warping frustum along the z -axis around the camera frustum. The warp is controlled by varying the parameter n' .

to the surprising result that for a shadow map occupying a fixed amount of memory, the warping parameters for PSMs, LSPSMs, and some TSMs all yield the same maximum perspective aliasing error.

We first consider the specific configuration shown in Figure 3 with a directional light overhead. The coordinate system for this Figure is the *light space* defined by Wimmer et al. [WSP04], except that we align the s and t directions of the shadow map with x and z , respectively, instead of vice versa as they do.

4.1. Maximum error for overhead light

A shadow map for an overhead directional light can be parameterized with low error using a perspective projection. The projection is parameterized by n' , the distance from the center of projection, c , to the view frustum near plane, n . For this configuration, PSMs, LSPSMs, and TSMs all use a perspective projection which differs only by the value of n' . PSMs use $n' = n$, LSPSMs use $n' = z + \sqrt{fn}$, and TSMs use a value of n' that maps a user selected focus point to the line 80% of the way from the bottom of the shadow map. Standard unwarped shadow maps use an orthogonal projection with $n' = \infty$.

Figure 4 shows how the parameterization changes with n' . The error in both x and z changes with n' but cannot be controlled independently. In this section, we extend the analysis of Wimmer et al [WSP04] to compute maximum error in x and z for all values of n' . The perspective aliasing error in each direction is given by the ratio of beam widths w_{lx}/w_i and w_{lz}/w_i . We assume that the image is square so that $w_{ix} = w_{iy} = w_i$. From Figure 3 we can see that the width

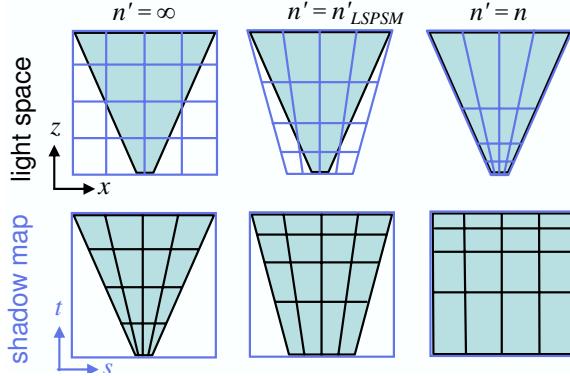


Figure 4: Perspective projection warping. From left to right the parameter n' decreases from ∞ to n . Top: In light space, the projected shadow map grid is compressed to match the sides of the view frustum. Bottom: In the view of the scene rendered into the shadow map, a tapered grid on the view frustum is stretched to fill the shadow map.

of an image beam through a single pixel is:

$$w_i(z) = \frac{2 \tan \theta}{res_i} z, \quad (2)$$

where n is the distance to the near plane, 2θ is the field of view of the camera, and res_i is the resolution of the image. The light beams are defined by texels in the shadow map. For a $res_s \times res_t$ resolution shadow map, the size of each texel is $1/res_s \times 1/res_t$. A texel sized step in the shadow map is related to a step in world space by the derivatives dx/ds and dz/dt for the x and z directions, respectively. Thus the width of the shadow beams in both directions can be written as:

$$w_{lx} = \frac{1}{res_s} \frac{dx}{ds}, \quad (3)$$

$$w_{lz} = \frac{1}{res_t} \frac{dz}{dt}. \quad (4)$$

Expressions for the s and t are given by the perspective projection. Using a standard OpenGL frustum matrix and transforming the result to the range $[0, 1] \times [0, 1]$ we have:

$$s(x, z') = \frac{x}{z' \tan \theta} + \frac{1}{2}, \quad (5)$$

$$t(z') = \frac{(f' + n')}{2(f' - n')} + \frac{f'n'}{z'(f' - n')} + \frac{1}{2}. \quad (6)$$

Since the derivatives of s and t are monotonic over the view frustum, the derivatives in Eqs. 3 and 4 can be evaluated as:

$$\frac{dx}{ds} = \frac{1}{ds/dx} \quad \text{and} \quad \frac{dz}{dt} = \frac{1}{dt/dz}.$$

Putting all of this together and substituting $z' = n' + z - n$ and $f' = n' + f - n$, we obtain the equations for error in

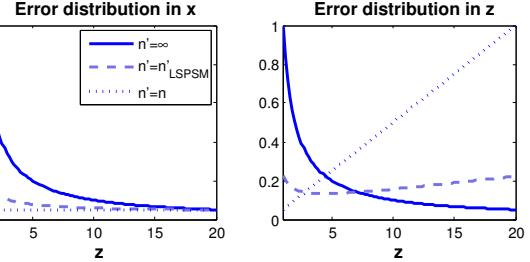


Figure 5: Error distribution. These plots show how perspective aliasing error is distributed along the view frustum for various values of n' . ($n = 1$ and $f = 20$.)

both directions:

$$m_x(z, n') = \frac{w_{lx}}{w_i} = \frac{res_i}{res_s} f \left(\frac{(n' + z - n)}{z(n' + f - n)} \right), \quad (7)$$

$$m_z(z, n') = \frac{w_{lz}}{w_i} = \frac{res_i}{res_t} \frac{(f - n)}{2 \tan \theta} \left(\frac{(n' + z - n)^2}{zn'(n' + f - n)} \right). \quad (8)$$

The last term of each of these equations determines the overall distribution of error over the length of the frustum. Plots of these terms are shown for several values of n' in Figure 5.

The maximum error for x always occurs at $z = n$. For z , the maximum error is at $z = n$ for $n' > n'_{LSPSM}$ and at $z = f$ for $n' \leq n'_{LSPSM}$. Plugging these values into Eqs. 7 and 8 we get the equations for maximum error for all z over the whole frustum which we denote as M_x and M_z :

$$M_x(n') = \frac{res_i}{res_s} \frac{f}{n} \frac{n'}{(n' + f - n)}, \quad (9)$$

$$M_z(n') = \frac{res_i}{res_t} \frac{(f - n)}{2 \tan \theta} \begin{cases} \frac{(n' + f - n)}{n' f} & n' \leq n'_{LSPSM}, \\ \frac{n'}{n(n' + f - n)} & n' > n'_{LSPSM}. \end{cases} \quad (10)$$

Parameterizing n' . The semi-infinite range of $n' \in [n, \infty)$ is inconvenient for analysis of these equations. We introduce a new parameter $\eta \in [-1, 1]$ in place of n' :

$$n' = n \begin{cases} \frac{\sqrt{\alpha+1}-\eta(\alpha-1)}{\eta+1}, & -1 \leq \eta \leq 0, \\ \frac{\sqrt{\alpha+1}}{\eta\sqrt{\alpha+1}}, & 0 < \eta \leq 1. \end{cases} \quad (11)$$

$$\alpha = f/n.$$

(see Appendix B for derivation.) Over the range $\eta \in [-1, 0]$, n' moves from $n' = \infty$ to $n' = n'_{LSPSM}$. Over the range $\eta \in [0, 1]$, n' continues decreasing down to n . Plugging this equation for n' into Eqs. 9 and 10 we can now more easily plot the behavior of the maximum error in x and z over the entire range of warping parameters (see Figure 6).

4.2. Using storage to measure error

From Eqs. 9 and 10 we can see that for a given view frustum there are only two quantities that are used to control the perspective aliasing error: the resolution of the shadow map and

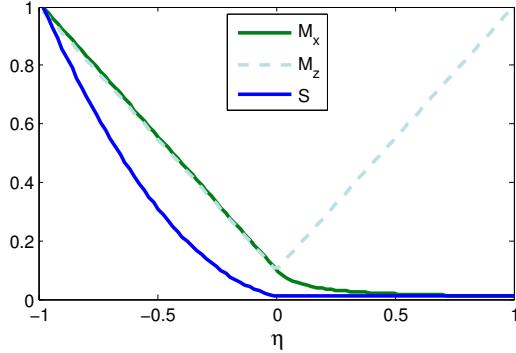


Figure 6: Varying n' . This plot shows the maximum error in x and z (M_x and M_z) and the shadow map storage (S) over all values of n' , parameterized in terms of η . $n'(-1) = \infty$, $n'(0) = n'_{LSPSM}$, and $n'(1) = n$. The plots have been normalized to fit on the same scale (Frustum parameters: $n = 1$, and $f = 100$).

the n' parameter. Perspective aliasing error vanishes when the resolution is chosen such that $M_x = M_z = 1$. We call this the *critical resolution*, $res_s^* \times res_t^*$. The total storage in texels required for a critical resolution shadow map is:

$$S^* = res_s^* \times res_t^* = res_i^2 \bar{S},$$

$$\bar{S} = \frac{(f/n - 1)}{2 \tan \theta} \begin{cases} 1 & n' \leq n'_{LSPSM}, \\ \frac{n'^2 f}{n(n' + f - n)^2} & n' > n'_{LSPSM}. \end{cases} \quad (12)$$

Typically we have a fixed budget of texture memory S_0 . In this case, we should choose the resolution subject to the constraints:

$$res_s \times res_t = S_0 \quad \text{and} \quad \frac{res_s}{res_t} = \frac{res_s^*}{res_t^*}.$$

The second equation ensures that error is equally divided between x and z . Solving these equations we get:

$$res_s = \sqrt{S_0 \frac{res_s^*}{res_t^*}}, \quad (13)$$

$$res_t = \sqrt{S_0 \frac{res_t^*}{res_s^*}}. \quad (14)$$

Storage factor. We call \bar{S} the *storage factor* for a critical resolution shadow map. It represents how many times larger than the image the shadow map must be (in texels) in order to eliminate perspective aliasing. \bar{S} is useful as an aggregate measure of error in both x and z that is independent of specific shadow map and image resolutions. We will use \bar{S} for the analysis in the rest of this paper.

4.3. Equivalence of PSMs, LSPSMs, and TSMs

We note that for values of $n' \leq n'_{LSPSM}$ in Eq. 12, \bar{S} is minimal and does not depend on n' . The value of n' chosen by PSMs, LSPSMs, and some TSMs all fall within this range.

This means that from the stand-point of maximum perspective aliasing error, which n' we choose makes little difference. The choice of n' primarily affects *where* the maximum error occurs within the view frustum and the relative dimensions of the critical resolution shadow map.

The equivalence of warping parameters means that the heuristic of "maximizing usage of the shadow map" that is often used in shadow map warping algorithms is perhaps too restrictive. For example, from Figure 4 it is clear that LSPSMs do not use the entire area of the shadow map while PSMs do. Yet \bar{S} for both the algorithms is the same.

We choose the warping parameter $n' = n'_{LSPSM}$ for three reasons. First, unlike the parameter computed by TSMs, n'_{LSPSM} is guaranteed to always lie within the minimal range. Second, n'_{LSPSM} distributes error more evenly between x and z . This is important because GPUs currently impose limits on the dimensions of a shadow map texture, and a squarish texture is more likely to fit within those limits than a long rectangular one with equal area. Finally, at n'_{LSPSM} the maximum error in z occurs at both the near and far planes. This is important for reasons which will be explained in Section 5.2.

4.4. Maximum error for general light directions

For a light in general position, not all of the equations we have derived for perspective aliasing error can be used directly because the light and eye space coordinate systems are no longer aligned. PSMs in particular require a new set of equations because the warping frustum chosen by that algorithm is no longer a simple one-point perspective projection.

We derive \bar{S} for general light directions from the beam widths w_i , w_{lx} , and w_{lz} computed directly at the vertices of the view frustum. It is sufficient to check just the vertices because the beam widths increase monotonically over the convex view frustum. The maxima must lie at the vertices. For a point p in the view frustum, we compute w_i by replacing z in Eq. 2 with $p \cdot v$, where v is the view vector. We set $res_i = 1$. For the LSPSM or TSM algorithms, we transform p into light space to get x and z and compute w_{lx} and w_{lz} from Eqs. 3 and 4 with the resolution terms set to 1. We then compute m_x and m_z at the vertices and take the maximums over the vertices, M_x and M_z . From these we get $\bar{S} = M_x M_z$.

Figures 7 shows \bar{S} over the entire hemisphere of light directions above a viewer with and without warping ($n' = \infty$ and $n' = n'_{LSPSM}$, respectively). Without warping, the error is high over all light directions. With warping the error is lowest for the overhead position at the center of the plot. It is highest when the light comes from directly behind or in front of the viewer. From these light directions, the view frustum appears to be square. Since it does not have a trapezoidal shape, no warping can be performed. For this reason, PSMs, LSPSMs, and TSMs all revert back to an orthogonal projection with $n' = \infty$ for these light directions.

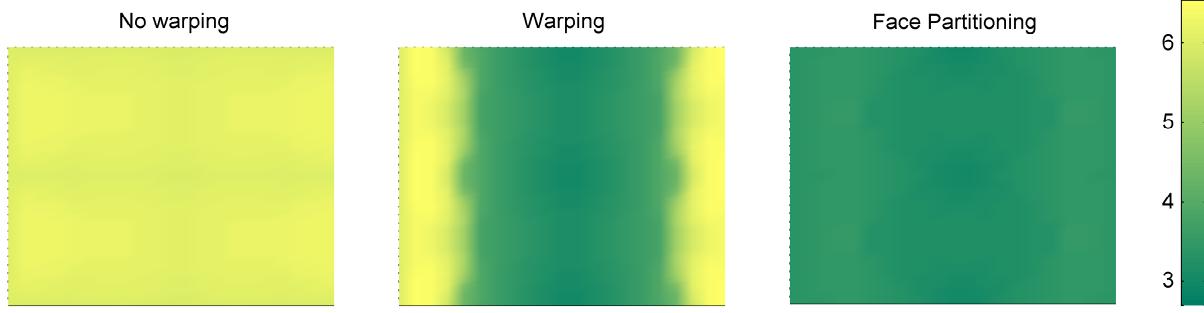


Figure 7: Storage factor. These plots show the storage factor over the hemisphere of light directions above the viewer. The storage factor is directly related to maximum perspective aliasing error over the view frustum. The overhead direction is at the center of the plot and behind and in front of the viewer are on the left and right sides, respectively. The plots use a \log_{10} scale.

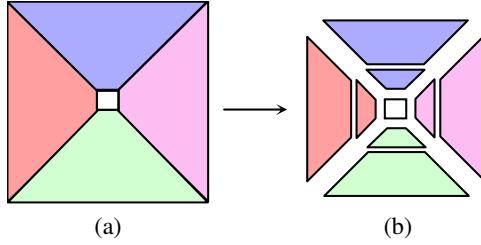


Figure 8: Face partitioning. (a) From behind, the view frustum is square and cannot be warped. (b) Partitioning along the faces allows warping to be used. z -partitioning may also be applied to face partitions.

5. Frustum partitioning

In this section we show how partitioning the view frustum and applying a separate shadow map to each partition can reduce perspective aliasing error. We consider two types of partitioning: *face partitioning*, which splits the frustum according to its faces, and *z-partitioning*, which splits the view frustum along its length.

5.1. Face partitioning

Face partitioning has been suggested as a way to reduce error for a light directions that are nearly aligned with the view direction [For03, Ald04]. From these directions, the view frustum has a square shape that is not amenable to warping with a perspective projection. The solution is simply to partition the frustum according to its faces (see Figure 8). The partitions are defined by the planes passing through the edges of the faces and the light (which is at infinity for a directional light). Each of the resulting trapezoidal partitions can then be warped independently, greatly reducing the error. Figure 7 shows how face partitioning reduces the error for the problematic light/camera configurations and leads to a more uniform error distribution over all light directions.

We use the LSPSM algorithm to fit a warping frustum to

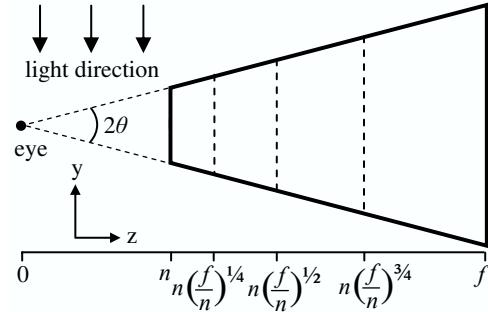


Figure 9: z-partitioning. Self-similar partitions along the z -axis. Choosing self-similar partition points makes the error the same in each partition and minimal over all possible partitions.

face partitions. The normal algorithm uses the view vector to align the light space z axis. For face partitions we first project the two side edges of the face e_0 and e_1 into a plane perpendicular to the light direction to obtain e'_0 and e'_1 . We use the bisector of the projected edges $e'_0 + e'_1$ to align the z axis. This ensures that the light beams have a cross-section that is as square as possible.

Which faces to use for partitioning depends on the direction of the light. The goal of warping is to eliminate perspective aliasing by ensuring that light beams are as wide as possible, but no wider than the narrowest image beams they intersect as they traverse the view frustum. The narrowest image beams are those first encountered by a light beam when $v \cdot y < 0$, where v is the view vector. Therefore the front faces of the view frustum with respect to the light should be used in this case. Likewise, when $v \cdot y > 0$ the narrowest image beams are encountered when the light beam exits the view frustum, so the back faces should be used.

5.2. z -partitioning

z -partitioning schemes [TQJN99], sometimes referred to as *cascaded shadow maps*, split the view frustum into smaller frusta along the eye space z -axis. z -partitioning is motivated by the fact that projective transforms, like the perspective transformation, can only approximate the optimal shadow map parameterization. The optimal parameterization for an overhead directional light should produce light beams with widths w_{lz} proportional to z . Projective transforms can only generate light beam widths that are proportional to $(z + c_0)^2$, where c_0 is a constant (see Appendix A). Since $(z + c_0)^2 \approx z$, the best we can do is a piecewise approximation.

The choice of partition locations affects the errors in each partition. We can see from Eq. 12 that the storage (and thus the error) grows with f/n . To minimize the maximum error overall all partitions, we should therefore minimize f/n for each partition and ensure that the maximum error of each partition is the same. This can be accomplished by making the partitions self-similar as shown in Figure 9. The near and far planes of each partition $i \in \{1, 2, \dots, k\}$ are given by:

$$n_i = n \left(\frac{f}{n} \right)^{(i-1)/k}, \quad (15)$$

$$f_i = n_{(i+1)} = n \left(\frac{f}{n} \right)^{i/k}. \quad (16)$$

A warping frustum is then fit to each partition separately.

Seams. If we render the image using a shadow map with sub-critical resolution, some perspective aliasing may be visible. The more abrupt the change in local aliasing error is between adjacent partitions, the more noticeable the seams between them will become. Using $n' = n$ there is no change in x error at a seam, but the change in z error is very large. With $n' = n'_{LSPSM}$ there is no change in z error at a seam, and the change in x error is typically less drastic than that of $n' = n$. For this reason we use $n' = n'_{LSPSM}$.

Combining with face partitioning. z -partitioning can be performed on face partitions for the frustum sides as shown in Figure 8. There is no need to partition the near plane because the image beam widths are constant along this face. In fact, for high depth ratios, the near plane is very small and can be left out altogether. By stretching the sides slightly the near plane be covered with only a slight increase in error.

If we are using a sub-critical resolution shadow map, changing the partitioning scheme from frame to frame can cause disturbing popping. For example, if we increase the number of z -partitions for light directions with fewer face partitions, there will be an abrupt shift in the distribution of aliasing error. In general, it is best to use the same partitioning scheme for all light directions to avoid popping.

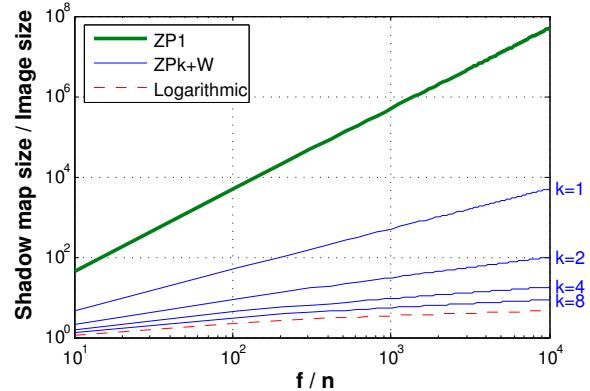


Figure 10: Storage factor for varying number of z -partitions for light overhead. The storage factor is an aggregate measure of x and z error. This plot shows the storage required for a varying number of z -partitions k . As k increases, the storage factor approaches that of the optimal, logarithmic parameterization.

5.3. Analyzing frustum partitioning

To analyze the effects of each type of partitioning on aliasing error we consider two light directions relative to the viewer: light overhead, and light behind.

Light overhead. The storage factor \bar{S} for z -partitioning (ZP) as a function of the number of partitions, k , is computed by plugging the partition locations from Eqs 15 and 16 into Eq. 12. There are k shadow maps for k partitions, so the storage factor is also multiplied by k . With no warping ($n' = \infty$) the storage factor is:

$$\bar{S}_{\text{overhead}}^{\text{ZP}} = k(f/n)^{1/k} \frac{((f/n)^{1/k} - 1)}{2 \tan \theta}. \quad (17)$$

Warping (W) with $n' \leq n'_{LSPSM}$ removes the $(f/n)^{1/k}$ factor:

$$\bar{S}_{\text{overhead}}^{\text{ZP+W}} = k \frac{((f/n)^{1/k} - 1)}{2 \tan \theta}. \quad (18)$$

Face partitioning (FP) gives no benefit over warping alone for a light overhead, since only one face is visible to the light:

$$\bar{S}_{\text{overhead}}^{\text{FP+ZP+W}} = \bar{S}_{\text{overhead}}^{\text{ZP+W}}. \quad (19)$$

Wimmer et al. [WSP04] showed that the optimal parameterization for an overhead light is logarithmic. Extending their analysis yields the optimal storage factor:

$$\bar{S}_{\text{optimal}} = \frac{\ln(f/n)}{2 \tan \theta}.$$

(see Appendix C for derivation). Figure 10 shows that as k increases, $\bar{S}_{\text{overhead}}^{\text{ZP+W}}$ approaches the optimal storage factor.

Light behind. When the light is behind the viewer, a ZP scheme cannot use warping because the view frustum is square. Figure 11 shows the rectangular warping frustum placed around the top frustum faces. The f/n ratio in light

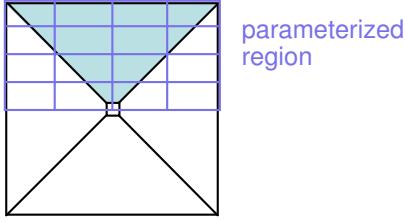


Figure 11: Parameterizing a single face with no warping and light behind the viewer. Half of two other faces are covered as well.

space for this face is the same as that for the entire frustum in the overhead case. θ becomes 45° . Therefore we can use Eq. 17 for this single face. We note, however, that the shadow map for this face covers half of the left and right faces. Likewise, the rectangle for the bottom face covers the other halves of the left and right faces. The storage factor for a single shadow map covering the entire frustum is the sum of the two. This holds true with $k > 1$ as well:

$$\bar{S}_{\text{behind}}^{\text{ZP}} = 2\bar{S}_{\text{overhead}}^{\text{ZP}}, \quad \text{with } \theta = 45^\circ. \quad (20)$$

If we add frustum partitioning, we can use warping, but we must use a shadow map for each of the side faces (ignoring the near face). The storage factor becomes:

$$\bar{S}_{\text{behind}}^{\text{FP+ZP+W}} = 4\bar{S}_{\text{overhead}}^{\text{ZP+W}}, \quad \text{with } \theta = 45^\circ. \quad (21)$$

When using $\text{FP} + \text{ZP}$, each level of z -partitioning results in up to four additional shadow maps. Rendering too many shadow maps can reduce performance. We therefore want to choose a partitioning scheme that will give us the greatest error reduction for the fewest number of shadow maps. Figure 12 shows ZP , $\text{ZP} + \text{W}$, and $\text{FP} + \text{ZP} + \text{W}$ for a varying number of shadow maps. $\text{ZP} + \text{W}$ is clearly the superior scheme. Even ZP without warping does better than $\text{FP} + \text{ZP} + \text{W}$. The reason for this is that for every four z -partitions with ZP , the face partitions of $\text{FP} + \text{ZP} + \text{W}$ only get one level of z -partitioning. For large values of (f/n) we have:

$$\bar{S}_{\text{overhead}}^{\text{ZP}} \sim (f/n)^{2/j} \quad (22)$$

$$\bar{S}_{\text{overhead}}^{\text{FP+ZP+W}} \sim (f/n)^{4/j}, \quad (23)$$

where j is the number of shadow maps. The storage factor for ZP decreases more rapidly with as the number of shadow maps increases.

We chose a small θ value for the view frustum used in Figure 12 to highlight the effect of the $\tan \theta$ factor. For a light overhead, a small θ requires more shadow map resolution because the image beams remain narrower along the length of the each partition. For a light behind the viewer, θ is always 45° . As the number of partitions increases, $\tan \theta$ begins to dominate. Thus even with no warping the storage factor for ZP with the light behind to dip below $\text{ZP} + \text{W}$ with light overhead.

Based on our analysis, we believe that z -partitioning with

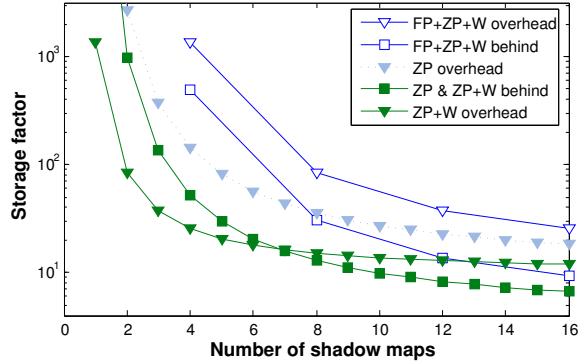


Figure 12: Storage factor for varying number of shadow maps. The storage factor is shown for the light overhead and behind the viewer for various combinations of z -partitioning (ZP), face partitioning (FP), and warping (W). (View frustum parameters: $f/n = 1000$ and $\theta = 20^\circ$)

warping ($\text{ZP} + \text{W}$) is the best scheme to use for rendering shadows with a low number of shadow maps in scenes with a high depth range. Most of the benefit comes from the partitioning. The maximum error over all light directions is not affected much by warping because it cannot be used for all light directions. However, warping does reduce the average maximum error over all light directions. This is similar to the difference between warping and no warping seen in Figure 7. Also the effect of warping is diminished with an increased number of partitions because the depth ratio of each partition decreases.

The analysis in this section is for only two light directions. Closed form expressions for the error in the general case are difficult to formulate because of the complex operation of fitting a warping frustum with varying parameters to an arbitrarily oriented view frustum. To get an idea of how the general case compares to the special cases we have treated here we compute the maximum \bar{S} over all light directions numerically using a dense sampling of light directions on the hemisphere. We found that for all combinations of warping, partitioning, and number of partitions that the worst case \bar{S} was within a factor of 2–3 times that which we computed analytically for the light behind case.

6. Implementation

This section addresses a few implementation details for partitioned shadow maps.

6.1. Shadow map texture layout

As the light moves relative to the camera, the number of faces used for frustum partitioning will change. The sizes of the partitions will also shrink and grow. The dimensions of the corresponding shadow maps should change accordingly. Some graphics hardware may not be optimized to handle texture dimensions that change every frame. In this case,

the shadow maps can be packed into a fewer number of fixed size textures.

6.2. Rendering multiple shadow maps

Partitioning requires that multiple shadow maps be rendered. For applications where shadow map rendering is fill bound, performance should not be impacted much. Partitioning will consume the about same amount of fill-rate as a single, warped shadow map.

If the entire scene is rendered for each shadow map and the application is geometry bound, then rendering k shadow maps will be k times slower than rendering only one. If instead we cull portions of the scene that fall outside of each shadow map's partition, then the overall performance will not change as much. Geometry bound applications typically perform view-frustum culling already, so the same mechanism used for that can be extended for use with partition culling.

6.3. Rendering the image with multiple shadow maps

The final image can be rendered one partition at a time, with all partitions in a single pass, or in multiple batches of partitions. The multi-pass algorithm can use clip planes or the stencil buffer to restrict rendering to a single partition while rendering with a single shadow map. Our current implementation of partitioned shadow maps performs the rendering in a single pass. Though dynamic branching in a fragment program could be used to select the proper shadow map, we use an approach that works on older GPUs. We track a set of texture coordinates for each partition. We pack all of the shadow maps into a single texture and use a fragment program to choose the appropriate set of texture coordinates for each fragment. For z -partitioning with four partitions we store the location of the partitions in two variables $ni = (n_1, n_2, n_3, n_4)$ and $fi = (f_1, f_2, f_3, f_4)$. For each fragment we compute a mask that is 0 in every component but the one which corresponds to the partition in which the fragment lies:

```

z = dot(fragment.pos, cameraZAxis);
mask = (ni < z) & (z < fi);
texCoord = mask.x * texCoord0 +
           mask.y * texCoord1 +
           mask.z * texCoord2 +
           mask.w * texCoord3;

```

The `texCoord` variable can then be used to sample the appropriate shadow map. For face partitioning we use a similar method as described by Aldridge [Ald04].

6.4. Depth clamping for increased depth resolution

A common problem with shadow map warping is the loss of depth precision. When the warping frustum is expanded to include all objects that occlude the view frustum it can become very elongated, leading to a loss of depth precision. We

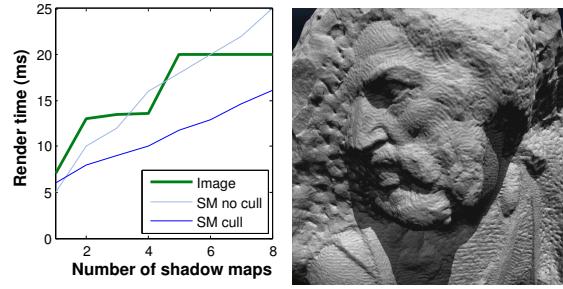


Figure 13: Render times for St. Matthew model. Culling occluders that do not lie in each partition (SM cull) leads to faster shadow map render times than with no culling (SM no cull). The steps in image render time at 2 and 5 shadow maps are due to changes in the fragment program. We get from 25–75 FPS for this view, depending on the number of shadow maps.

note that depth values are only needed for occluders within the view frustum. It is sufficient to clamp the depth of occluders between the view frustum and the light to 0. The warping frustum need only fit the view frustum. In practice, warping frustum must be expanded slightly for depth biasing to work correctly.

7. Results and discussion

We have implemented several warping and z -partitioning algorithms on a GeForce 7800GTX. We tested our system on a game-like scene consisting of 15 airplanes (Figure 14), each of which consists of 18K triangles. We also integrated our system with a view dependent renderer [YSGM04] and tested it with a power plant model consisting of 13M triangles (see Figure 1). These models have a high depth range and are therefore very susceptible to perspective aliasing error.

Figure 13 shows the time to render the image and a varying number of shadow maps for a St. Matthew model consisting of 350M triangles. The LOD renderer reduces this to about 1M triangles per frame. As expected the shadow map rendering increases linearly with the number of shadow maps. Partition culling improves shadow map rendering performance.

One disadvantage of warping algorithms is that the shadow map alignment depends on the view and the light. In an animation this can cause the shadow edges to crawl. One version of cascaded shadow maps solves this problem by using a ZP scheme and orienting the shadow map with respect to a fixed vector in world space [Blo04]. This fixes the location of the texels boundaries for a particular light direction. As the view frustum changes, the shadow map is permitted to move only in increments of a shadow texel, eliminating the crawling.

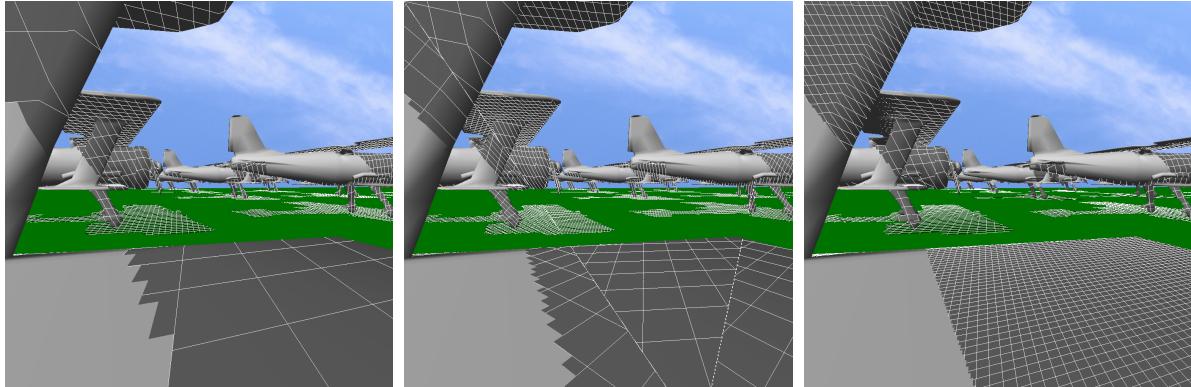


Figure 14: Various warping and partitioning schemes. These images show the difference in quality using warping (W) with a single shadow map (left), face partitioning ($FP+W$) (middle), and z -partitioning ($ZP4+W$) (right). The shadow map texel grid is projected onto the scene with grid lines 5 texels apart. The image is $1K \times 1K$. All shadow maps use $1K \times 1K$ total storage. $FP+W$ uses 3 shadow maps for this view while $ZP4+W$ uses 4. The frame rates from left to right are 143, 115, and 107 fps. ($f/n = 500$)

Conclusion and Future Work

We have presented a technique for analyzing shadow map warping and partitioning algorithms. We show that for a warping frustum oriented perpendicular to the light, that a range of warping parameters corresponding to several previous warping algorithms are equivalent. We also show that a combination of z -partitioning and warping can deliver low aliasing error with a small number of shadow maps.

We have shown that face partitioning is not as useful for rendering shadows with a small number of shadow maps. If we could use the optimal logarithmic parameterization, however, we would only need 4 shadow maps. We would like to investigate further the use of the logarithmic parameterization. We would also like to extend our analysis to point lights.

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Appendix A

A parameterization $t(z)$ using a general projective transform on z is given by:

$$s = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix} = \begin{bmatrix} az+b \\ cz+d \end{bmatrix}$$

After the perspective divide we have:

$$\begin{aligned} t &= \frac{az+b}{cz+d} \\ \frac{dt}{dz} &= \frac{ad-bc}{(cz+d)^2} \end{aligned}$$

A texel sized step in t results in a step in world space that is proportional to $dz/dt = 1/(dt/dz)$. Thus the light beams generated by a projective transform have spacing proportional to $(z + c_0)^2$, where c_0 is a constant.

Appendix B

We define η based on the behavior of z error shown in Figure 5. We note that as n' approaches $n' = n'_{LSPSM}$ from ∞ , the maximum error over the whole frustum (M_z) occurs at the near plane, moving from its highest value towards its lowest value. As n' continues from $n' = n'_{LSPSM}$ down to n , the maximum error switches to the far plane and moves back up to its highest value again. We map $\eta = -1$ to maximum M_z on the near plane, $\eta = 0$ to minimum M_z , and $\eta = 1$ to maximum M_z on the far plane. M_z is linearly interpolated between these values:

$$\eta = \begin{cases} \frac{M_z(\infty) - M_z(n')}{M_z(\infty) - M_z(n'_{LSPSM})} & n' > n'_{LSPSM} \\ \frac{M_z(n') - M_z(n)}{M_z(n) - M_z(n'_{LSPSM})} & n \leq n' \leq n'_{LSPSM} \end{cases}$$

We arrive at Eq. 11 by solving this equation for n' .

Appendix C

The warping frustum of the optimal shadow map parameterization for an overhead directional light is identical to the view frustum, like PSMs. Therefore $m_x = 1/res_s$. In the z direction the light beam widths w_{lz} should be proportional to z . From Eq. 4 we see that this implies that $dz/dt \sim z$. We solve for t by integrating $dt/dz = 1/z$ over the view frustum and normalizing the result to the range $[0, 1]$:

$$\begin{aligned} \tilde{t} &= \int_0^1 dt = \int_n^z \frac{dt}{dz} dz = \int_n^z \frac{1}{z} dz = \ln(z/n) \\ t &= \frac{\tilde{t}(z)}{\ln(f) - \ln(n)} = \frac{\ln(z/n)}{\ln(f/n)}. \end{aligned}$$

From this we can compute w_{lz} and m_z :

$$\begin{aligned} w_{lz} &= \frac{1}{res_t} \frac{dz}{dt} = \frac{1}{res_t} z \frac{\log(f/n)}{n} \\ m_z &= \frac{w_{lz}}{w_i} = \frac{1}{res_t} \frac{\ln(f/n)}{2 \tan \theta} \end{aligned}$$

Both m_x and m_z are constant over the view frustum. The value of \bar{S} is found by ignoring the resolution terms:

$$\bar{S} = \frac{\ln(f/n)}{2 \tan \theta}.$$