THE LIGHT FIELD

By A. Gershun

Translated by Parry Moon and Gregory Timoshenko

Theoretical photometry constitutes a case of "arrested development", and has remained basically unchanged since 1760 while the rest of physics has swept triumphantly ahead. In recent years, however, the increasing needs of modern lighting technique have made the absurdly antiquated concepts of traditional photometric theory more and more untenable.

A vigorous attempt to bring the theory of light calculation into conformity with the spirit of physics is now in progress. Professor Gershun of the State Optical Institute, Leningrad, is one of the pioneers in this reform, some of his work being contained in the book, *"The Light Field"**, a translation of which follows.

The purpose of our translation is two-fold: first, to bring to the attention of engineers the new methods of radiometric and photometric calculation; and secondly, to interest mathematicians and physicists in the further development of this branch of field theory. In its present form, the concept of the light vector is of distinct practical value and its general use by illuminating engineers would tend to simplify calculations and clarify thinking.

Valuable as the methods are, however, they probably do not constitute the ultimate solution of the problem. The light field considered in this book is a classical three-dimensional vector field. But the physically important quantity is actually the *illumination*, which is a function of *five* independent variables, not three. Is it not possible that a more satisfactory theory of the light field could be evolved by use of modern tensor methods in a five-dimensional manifold? We must look to the mathematician for any such development.

* *Svetovoe pole*, Moscow, 1936. A brief treatment is also given by Gershun in a paper, Notions du champ lumineux et son application á la photométrie. R. G. E., **42**, 1937, p. 5. The paper is criticized by A. Blondel, Sur une prétendue théorie du champ lumineux. R. G. E., **42**, 1937, p. 579. We have weighed Blondel's criticisms and do not consider that they vitiate Gershun's results. The reader will find the elements of light-field theory also in Chap. X of Moon, Scientific Basis of Illuminating Engineering, New York, 1936.

The translation aims at an accurate reproduction of the spirit of the original, but it is not a literal translation. Sentences and even paragraphs have been omitted at times to fit the work for a somewhat different class of readers than that for which the book was originally written. No additional material has been added, with the exception of a few footnotes, which are inclosed in square brackets. We have taken the liberty, however, of numbering the equations and the references.

THE TRANSLATORS,
Massachusetts Institute of Technology.

CONTENTS

CHAPTER I

Introduction

1. Problems of Theoretical Photometry

Let us define first the meaning of the word *photometry*. Customarily, the term is given a rather narrow meaning corresponding to a literal translation from the Greek (φῶς—light, μέτρον—measure). Photometry, in this sense, is that part of optics dealing with the measurement of radiant energy, evaluated according to its visual effect and related therefore solely to the visible part of the spectrum. Such a definition has historical sanction, since for many years the eye was the sole instrument used in evaluating radiant energy. This purely physiological criterion, as applied directly to basic measurements of quantities characterizing the distribution of radiant energy, resulted in a peculiar set of photometric concepts, standards, and units which stood apart from the general system of physical units. The usual treatise on photometry consisted largely in descriptions of apparatus used in the visual comparison of radiations.

In the present treatment, however, we attach a rather broad meaning to the words *light* and *measurement* when we define photometry as a branch of science dealing with the measurement of light. In "light" we include all radiations, visible or invisible. This corresponds, for example, to the definition of photometry given by Charles Fabry, who says[1] that photometry is "la mesure de l'intensite des rayonnements, visibles ou non visibles, quelle que soît leur place dans le spectre et quel que soît l'appareil de mesure employé."

The word *measurement* is also defined in a broad sense, since it is not confined to the experimental technique but is used rather for the entirety of theoretical and experimental questions connected with quantitative comparisons. Thus photometry deals with energy relationships in emission processes and in the propagation and absorption of radiation. The radiation may be either visible or invisible. The quantity of radiation may be evaluated in units of energy or in its effect upon a receptor: the human eye, the photographic plate, the human skin, etc. Depending on the receptor, the result will be evaluated either in the usual physical units or in some special units, as light

[1] Charles Fabry: Introduction générale a la photométrie, Paris, 1927.

units, photographic units, or erythemal units. In these cases, to each monochromatic component of radiation is attributed a weight-number, corresponding to the response produced in the receptor by a unit quantity of radiant energy of the given wavelength.

All theorems of photometry are independent of the units in which the quantity of radiation is measured (energy units, for example, or photometric units). For the sake of convenience, *photometric terminology is used in this book.**

Before treating the more recent problems of theoretical photometry, we shall give a brief account of the history of its development. Photometry was founded as a scientific discipline by a French scientist, Pierre Bouguer (1698–1758),[2] who was the author also of researches in mathematics, astronomy, geodesy, physics, geophysics, and navigation. Johann Heinrich Lambert (1728–1777) gave a mathematical form to some of the ideas originated by Bouguer and elaborated upon methods of photometric computation.[3] In the modest history of photometry, Bouguer and Lambert deserve a position corresponding to that occupied in the history of electromagnetic phenomena by those Titans of science, Faraday and Maxwell. The importance of the work of Bouguer and Lambert has become evident only in recent times, when the questions considered by them have become of vital interest to engineers.†

In the one hundred and fifty years following the work of the founders of photometry, the questions of photometric computations were neglected almost completely.[4] The problems of theoretical photometry were pushed aside from the main path of the development of physics. This resulted in the present state of photometric theory, based upon eighteenth-century mathematics and devoid of generality. The con-

* [Though all the theorems can be interpreted equally well in radiometric terms.] The Translators.

[2] Traité d'optique sur la gradation de la lumière. Ouvrage posthume de M. Bouguer, de l'Academie Royale des Sciences, Paris, 1760. A less complete edition of work by Bouguer appeared under the title "L'Essai d'Optique" in 1729.

The first book on photometry was written by a Paris Capuchin, François Marie: Nouvelles découvertes sur la lumière, Paris, 1700.

[3] J. H. Lambert, Photometria sive de mensura et gradibus luminis, colorum et umbrae, 1760. German edition with annotations by E. Anding: Ostwald's Klassiker der exacten Wissenschaften, Nos. 31–33, Leipzig, 1892.

† Note that the problems of photometric computations are of interest not only in optics and illuminating engineering but also in other branches of science, as for example in heat engineering.

[4] See, however, August Beer, Grundriss des photometrischen Calcüles. Braunschweig, 1854.

cepts of the force field and the methods of vector analysis are absent. This condition can be explained by the fact that the structure of the theory has not been of special interest to physicists and has not been needed until recently by engineers.

Problems of theoretical photometry have come into prominence with the growth of illuminating engineering. This growth has, in turn, been caused by the demands of humanity for better artificial lighting—demands that could be satisfied for the first time by means of electrical light sources. For some time the engineer required from the theory only methods of calculating illumination from point sources, but the recent development of luminous architectural elements and of methods of calculating daylighting in buildings have necessitated the study of large surface sources. Problems of reflector design have been developed also. Knowledge of the reflecting properties of specular surfaces and diffusing surfaces was required, as well as knowledge of the transmission of light through absorbing and diffusing media. Thus appeared a series of practical problems requiring a large variety of photometric computations and giving the first impulse toward the construction of a *generalized theory of photometry.*

In the first quarter of this century, illuminating engineers were concerned primarily with the problem of obtaining the required illumination on the working surfaces. Experience showed this criterion to be quite inadequate, however, since the illumination of a working surface is not a universal measure of the lighting. In everyday design practice, consideration must be given to the magnitude of the illumination and also to the correct coordination of general and local illumination, to the direction of the light, and to shadows. In illuminating engineering literature one finds the expression "quality of lighting", by which is understood all the properties that the engineer cannot characterize as yet by definite numbers. The problem of the quality of lighting has been responsible for the introduction of several new concepts in illuminating engineering: the *space illumination* which characterizes the density of light at a point in space, independent of the light direction; and the *light vector*, which characterizes the predominant direction of the light. Illuminating engineering, which was originally merely a branch of electrical engineering, has become an independent and broad discipline dealing with theoretical methods and technical applications of radiant energy.

Illuminating engineering has need of a theoretical basis, analogous to the theory of the electromagnetic field which forms the foundation of electrical engineering. In the author's opinion, the *theory of the light field* must form one of the most important parts of this theoretical foundation.

2. The Light Field

Problems of theoretical photometry fit nicely into the frame of classical mathematical physics. Mathematical physics includes the analytical treatment of heat conduction, elasticity, electricity and magnetism. All these branches are united by a general theory of the physical field, which can be classified as physics or as geometry. We may define the physical field as a part of space, studied from the standpoint of a definite physical process happening within that space. Analogously, we shall introduce the concept of the *light field*, as a part of space studied from the standpoint of transmission of radiant energy within that space. Until recent times, photometry limited itself to concepts concerning the emission and absorption of light by bodies, while the transmitting medium was ignored. The older photometric science was a peculiar manifestation of the concept of *actio in distans*. The modern study of the light field consists of an investigation of the space-distribution of luminous flux. The separate photons are disregarded and the assumption is made that radiant energy is continuous in time and in space and that the flux of this radiant energy varies continuously from point to point. The familiar methods of vector analysis are used in the theoretical study of the light field.

A physicist may ask why the author distinguishes the light field from the well-known electromagnetic field. Obviously, the light field is caused by electromagnetic phenomena, but it is quite different in its qualities from the electromagnetic field. In studying the latter we are considering the electric and magnetic forces caused by an elementary emitter. In studying the light field, however, we deal with bodies of finite size consisting of great numbers of elementary emitters. In contrast with the elementary electromagnetic field, we have here a macrocosmos with respect to time as well as space. We assume that the differential of time exceeds by far the period of a single vibration and that the differential of distance is far greater than a single wavelength.[5] The theory, as presented in this book, corresponds to *geometrical optics* rather than physical optics. We shall deal with a beam of light which propagates in a straight line through a homogeneous

[5] Max Planck, Vorlesungen über die Theorie der Wärmestrahlung. Leipzig, 1923.

medium, and which is the carrier of radiant energy. Thus the treatment will be a geometrical one, to which is added the concept of energy. The development of a wave-theory of the light field may be considered as a step to follow. One feels the need of this extension, for example, in the absolute determination of the energy distribution in the optical image.

CHAPTER II

Fundamental Theory of the Light Field

1. Fundamental Photometric Quantities

We shall consider first the quantities used in the study of the light field. It is most convenient to choose as fundamental magnitude the amount of radiant energy transferred through a hypothetical surface per unit time. This quantity is called *radiant flux.* However, we shall use the term **luminous flux**, the photometric analog of radiant flux, in accordance with our convention of employing the names associated with photometric concepts rather than with radiometric concepts. The book is written for physicists, however, as well as for illuminating engineers, and the derivations presented here will apply to radiant flux as well as to luminous flux.

Let us first recall the definitions of the photometric quantities: *luminosity, illumination, intensity,* and *brightness.** Consider a point O of the surface S of a luminous body in space (Fig. 1). The element of surface dS containing the point O is chosen sufficiently small so that it may be considered uniformly bright over its surface. The total luminous flux emitted in all directions from the element dS is denoted by dF and the **luminosity** L of the surface S at the point O is defined as

$$L = \frac{dF}{dS} \tag{1}$$

Luminosity is surface density of emitted luminous flux. The total flux emitted by the source is

$$F = \int_S L dS \tag{1a}$$

Now consider an element ds, illuminated by the luminous surface S. The **illumination** E of the surface s at the point P is defined as

$$E = \frac{dF'}{ds} \tag{2}$$

where dF' is the total luminous flux on ds from all directions of space. *Illumination is surface density of received luminous flux.*

* [Svetimost, osveshchennost, sila sveta, and yarkost]. Translators.

It is obvious that the concept of illumination may be divorced from the concept of an actual illuminated body. One may define illumination at a point in the light field, on an arbitrarily oriented surface, by using the abstract concept of a geometric element of surface placed at the point in question. This concept, which is characteristic of physical field theory, may already be found in illuminating engineering, e.g. in the expression, "the horizontal illumination at the elevation of one meter from the floor". Since the orientation of the surface element is completely and uniquely defined by the direction of the surface normal, the illumination may be regarded as a function of two factors: position in space, and direction.

In computing the values of illumination, we must know the intensity* of each element of the surface of the luminous body in all directions. Consider the regions of the light field that are remote from the luminous body S, so that the dimensions of the source are negligible in comparison with the distance from it to the point P. Then the body S can be considered as a point source. For a study of these distant parts of the light field, it is sufficient to use the concept of the **luminous intensity** of the entire body in the given direction. This concept is derived in the following manner. Consider a definite elementary bundle of rays diverging from the source. As a measure of the multitude of these rays, we shall use the solid angle $d\Omega$ corresponding to the conical surface enclosing all the rays. The luminous flux in these directions has the value dF. The luminous intensity I in the given direction is defined as

$$I = \frac{dF}{d\Omega} \tag{3}$$

When the values of the intensity in various directions are known, the light field of a luminous body is defined at a sufficiently large distance from it. But the concept of intensity loses its meaning if the solid angle becomes large; and it is then necessary to divide the radiating surface into small elements, each of which is considered as a point source.

Let us recall the fundamental laws defining the light field produced by an element of luminous surface. These laws follow from purely geometrical considerations, together with the law of conservation of energy. If dI represents the intensity of the element dS in the direction of the illuminated element ds (Fig. 1), and if the distance between

* [Or candlepower]. *Trans.*

the two elements is far greater than the dimensions of either of them, the illuminated element ds receives from dS the flux d^2F, which is an infinitesimal part of the flux dF emitted by the element dS. The index 2 denotes an infinitesimal of higher order. The luminous flux is

$$d^2F = dI\, d\Omega \tag{4}$$

where $d\Omega$ is the solid angle defined by ds. Let r be the distance between ds and dS, and let θ be the angle between the incident rays and the normal n of the illuminated element ds. Then

$$d\Omega = \frac{ds \cos\theta}{r^2} \tag{5}$$

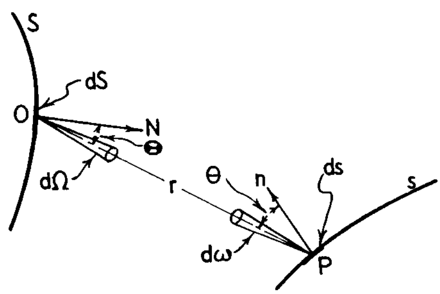

FIG. 1

and the illumination dE caused by radiation from the element dS is

$$dE = \frac{d^2F}{ds} = \frac{dI}{r^2}\cos\theta \tag{6}$$

This fundamental photometric formula expresses two relations: the inverse-square law and the cosine law of illumination. These elementary relations may be applied also in the case of a source of finite size at distances sufficiently large compared to the size of the body.

Let us now introduce the concept of **brightness** B of a luminous surface S at a point O, in the direction OP. Denote by Θ the angle between the direction OP and the normal N of the luminous element dS, and by $d\sigma$ the area of the projection of the element dS upon a plane normal to OP. Then

$$B = \frac{dI}{d\sigma} \tag{7}$$

where

$$d\sigma = dS\cos\Theta \cdot$$

The flux d^2F may be expressed as

$$d^2F = \frac{B \cos \theta \cos \Theta \, dS \, ds}{r^2}$$

But

$$d\omega = \frac{dS \cos \theta}{r^2} \tag{8}$$

Thus,

$$dE = B \cos \theta \, d\omega. \tag{9}$$

This relation represents another form of the basic law of photometric theory. Denoting by dE_n the normal component of the illumination at the point P, we obtain

$$dE_n = B \, d\omega \tag{10}$$

and

$$B = \frac{dE_n}{d\omega} \tag{10a}$$

Analogous to the definition of intensity, which is measured by the luminous flux per unit solid angle, the *brightness of a luminous source may be defined as the normal component of illumination produced by that source per unit solid angle.* This relation may be considered as a fundamental definition of brightness, which must be given preference over the usual definition of brightness as intensity per unit area. The latter can be used only in the special case of a light-emitting surface, and fails, for example, in evaluating the brightness of the sky. The concept of sky brightness is indeed a puzzling one when the classical idea of brightness (candlepower per square centimeter of the surface of the source) is used. It is difficult to decide where to locate these square centimeters which have to be divided into the equally illusive luminous intensity.

2. Brightness of a Ray

The concept of brightness must now be generalized to adapt it for use in the theory of the light field. Consider the light field produced by a number of luminous bodies S_1, S_2, $\cdots$ (Fig. 2), and let us characterize the structure of the light field at an arbitrary point P. Consider a plane s through the point P, and assume this plane to be opaque except for a small area ds containing the point P. This area is sufficiently small so that the light field may be considered uniform at all points of ds, but it is sufficiently large so that diffraction phenomena may be neglected. On this assumption, the fluxes dF_1 and dF_2 are obviously proportional to the size of the opening ds, or

$$\left.\begin{aligned} dF_1 &= E_1\,ds \\ dF_2 &= E_2\,ds \end{aligned}\right\} \qquad (11)$$

The coefficients of proportionality E_1 and E_2 represent the surface density of luminous flux on the given side of the aperture, or E_1 and E_2 represent the illuminations of the two sides of the plane s at the point P. E_1 is the illumination from side 1 and E_2 is the illumination from side 2. (Fig. 2.) For computation of the energy balance we shall assume

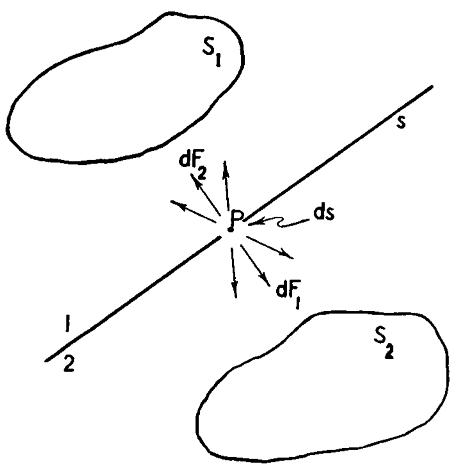

FIG. 2

$dF_1 > dF_2$, which means that the energy is transmitted through the diaphragm ds, entering through the side 1 and emerging through the side 2. The quantity of light transmitted per unit time is equal to $dF = dF_1 - dF_2$. The surface density of the luminous flux through the aperture is

$$D_n = \frac{dF}{ds} = E_1 - E_2. \qquad (12)$$

The quantity D_n determines the transfer (flow) of energy through unit area of the aperture. We shall call D_n the **flux density**.

The aperture ds, through which the light passes, may be considered as an independent source of light, and we may apply to it all the concepts used in the study of light sources. At great distances, the aperture may be considered as a point source with a definite intensity in each direction. We may also introduce the concept of brightness of the aperture in various directions, defining this quantity as was done in considering the brightness of real surfaces. Introducing this concept of brightness allows one to derive an energy balance at the aperture *ds* and also to characterize the directional distribution of the luminous flux through the aperture.

Consider a direction P_1P_2 making an angle θ_1 with the normal to the screen s_1 (Fig. 3), and consider a solid angle containing this direction P_1P_2. The solid angle may be constructed in the following manner: we make use of a second opaque screen s_2 with an infinitely small aperture ds_2 containing the point P_2. The elementary solid angle $d\omega_1$ is defined by a conical surface with its apex at P_1 and having a base ds_2.

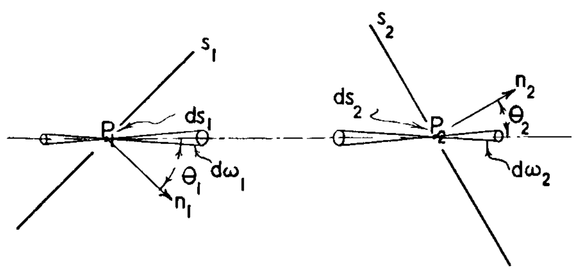

FIG. 3

This solid angle is chosen sufficiently small so that all directions contained within it may be considered equal. The luminous flux d^2F through the aperture ds_1 and enclosed by the angle $d\omega_1$ is

$$d^2F = dI\ d\omega_1 \tag{13}$$

The coefficient of proportionality dI represents the luminous intensity of the aperture ds_1 in the direction P_1P_2. This quantity depends on the size and on the orientation of the element ds_1 and is proportional to the area

$$d\sigma_1 = \cos\theta_1\, ds_1 . \tag{14}$$

Referring the intensity to a unit area of the projection; that is, determining the brightness of the aperture ds_1 in the direction P_1P_2, we obtain a fundamental photometric concept which depends solely on the position of the point P_1 and on the direction P_1P_2. This quantity is called *the brightness B at the point P_1 in the direction P_1P_2.* To each point in the light field and to each direction at that point corresponds a definite value of brightness. In other words, brightness is a function of position and of direction. One may say that B represents the brightness of the light ray P_1P_2 at the point P_1. The concept of the brightness of a light ray was first introduced by Planck.

In geometrical optics, the light ray is identical with a straight line and has no thickness. But obviously energy cannot be propagated through zero cross-sectional area. In photometry we must use a light beam composed of an infinite number of adjacent geometrical rays, and an infinitesimal amount of energy is carried by such a beam. The light beam is determined in this case by two infinitesimal apertures ds_1 and ds_2, which are at a finite distance from each other. The geometry of the light beam is characterized at the input aperture by the area $d\sigma_1$, Eq. (14), and the solid angle $d\omega_1$. Equivalent to these geometrical characteristics of a light beam are the magnitudes $d\sigma_2$ and $d\omega_2$ at the exit aperture of the beam. Here

$$d\sigma_2 = \cos\theta_2 \, ds_2 \tag{14a}$$

The divergence of the geometrical rays forming the light beam is determined at the point P_1 by the solid angle $d\omega_1$ and at the point P_2 by the solid angle $d\omega_2$. It is easily shown that

$$d\omega_2/d\omega_1 = d\sigma_1/d\sigma_2 \tag{15}$$

so

$$d\sigma_1 \, d\omega_1 = d\sigma_2 \, d\omega_2 \tag{16}$$

A more rigorous proof of Eq. (16) is given in Section 1, Chap. VII.

Let us consider the variation of brightness along the light beam. The brightness at the point P_1 is denoted by B_1, while that at P_2 is denoted by B_2. The flux d^2F through the aperture ds_1, in the direction ds_1–ds_2, is

$$d^2F_1 = B_1 \, d\sigma_1 \, d\omega_1 \tag{17}$$

which represents the luminous flux in the light beam at the point P_1. Similarly, at P_2,

$$d^2F_2 = B_2 \, d\sigma_2 \, d\omega_2 \tag{18}$$

Taking into account that $d\sigma \, d\omega$ is constant, we find that the brightness along the ray is directly proportional to the luminous flux, or

$$B_2 : B_1 = d^2F_2 : d^2F_1 \tag{19}$$

If the light beam emerges into empty space, then, by the principle of conservation of energy,

$$d^2F_2 = d^2F_1$$

and

$$B_1 = B_2 \tag{20}$$

or the brightness is the same at all points of the light beam. This conclusion is often formulated in text books of physics as the principle of independence of the brightness of a luminous body with distance, and the principle is often used in the solution of problems of illuminating engineering where absorption and dispersion of light can be neglected. This explains why the concept of the brightness in a beam is not widely used in illuminating engineering; for if brightness is the same at all points in a beam, it is simpler to talk about the brightness of the source in the given direction, rather than the brightness in the beam. Thus

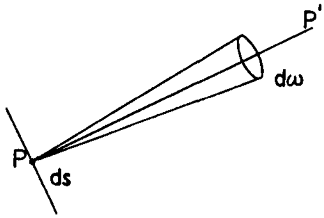

FIG. 4

the concept of brightness is customarily associated intimately with luminous sources. But if in the path of the light beam there are regions which emit, absorb, or disperse the light, then the luminous flux carried by the beam changes from point to point and the brightness varies correspondingly. For example, if the light beam passes through an absorbing medium, the luminous flux decreases because of the gradual change of radiant energy into heat energy. The luminous flux decreases exponentially and the brightness of the beam varies according to the same law.

The necessity of introducing the concept of brightness as a function of position and of direction may be illustrated by the question of the brightness of the sky, which depends on the direction in which it is observed as well as the position of the observer. Thus brightness should be considered as a function of position and direction in the light field. Let us place at a given point a small plane element *ds* which is perpendicular to the direction PP′ (Fig. 4). The brightness is considered as the coefficient by which the solid angle $d\omega$ must be multiplied to obtain the illumination dE_n produced on the element ds by the luminous flux in the angle $d\omega$, or

$$B = dE_n/d\omega \tag{21}$$

As in the measurement of illumination, brightness may be measured objectively by means of a photocell. The photocell is mounted on an opaque tube m (Fig. 5) which intercepts all the rays not contained within the solid angle ω. If the length of the tube is considerably larger than the area of the photocell the luminous flux on the photocell may be considered as proportional to the average value of the brightness within the solid angle ω.

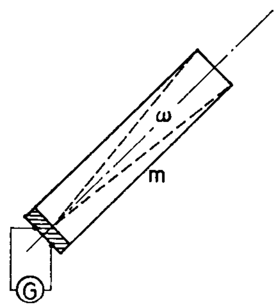

FIG. 5

3. The Brightness-distribution Solid

The structure of the light field at a point P may be studied in the following manner. Let us form an infinite number of infinitesimal solid angles adjoining each other and filling all space, all the solid angles having a common origin at P. The transfer of light within an arbitrary solid angle $d\omega$ is characterized by the illumination dE_n of a surface element placed at that point, with its normal collinear with the axis of the solid angle $d\omega$. The values dE_n depend not only on the character of the light field but also on the initial arbitrary subdivision of the space. Dividing the values of dE_n by the corresponding values $d\omega$, however, we obtain values of brightness which are functions of direction and which are independent of the original geometrical subdivision. The totality of these values of brightness determines completely the structure of the light field at the point P.

The distribution of brightness may be characterized geometrically in the following manner. Let us mark off in each direction from the point P a distance which represents the brightness in that direction. The geometric locus of the ends of these distances forms a closed surface. This surface may be called the *brightness-distribution surface* for the given point in the light field. The space enclosed by this surface is called the *brightness-distribution solid.* The section of the *brightness-distribution surface* by a plane passing through P gives in polar coordinates the *brightness-distribution curve* in that plane. Thus for completely diffused light, where brightness is the same in all directions, the brightness-distribution solid is represented by a sphere with center at P. For a parallel beam of light where the brightness is zero in all directions except one, the brightness-distribution solid degenerates into a line segment in the direction of the beam.

Depending on the values of brightness in various directions, the eye evaluates the level of illumination of an entire room and also evaluates the uniformity of the lighting, the degree of diffusion, and the presence or absence or glare. It seems peculiar that until recent times the quality and quantity of lighting have been judged by the distribution of illumination at various points on a horizontal plane. The visual evaluation of the uniformity of illumination is determined in the first place by the brightness distribution of the surfaces enclosing the room. This was first correctly pointed out by L. Bloch.[6]

For example, with daylight entering a room from the side, the horizontal illumination near the windows is many times as great as the illumination at points distant from the windows. The illumination of such a room is highly nonuniform, according to present methods of evaluation. We know, however, that if the room is not too deep, and is illuminated by windows in one of the walls, it appears to be illuminated sufficiently uniformly. This can be explained by the fact that there is almost equal illumination on the side and back walls; for although the back wall is further from the windows, the light rays are normal to it. Even if the eye is fixed on a definite work surface, the distribution of brightness in all other directions plays an important rôle.

4. Functions of Position and Direction

The brightness-distribution solid characterizes completely and uniquely the distribution of luminous flux at any point of the light field. Knowing the brightness-distribution solid, one can compute the

[6] L. Bloch, Die Leuchtdichteverteilung im Raum. Licht und Lampe, 1930, No. 13, p. 663.

values of all other magnitudes serving to describe the light field at the given point. Thus, for example, the brightness-distribution solid allows one to find the illumination of an arbitrary oriented element ds at a given point P (Fig. 1). Let us denote by B the brightness in the direction of $d\omega$. The illumination of the element ds by the rays contained within the solid angle $d\omega$ is

$$dE = \cos\theta \, dE_n \tag{22}$$

where

$$dE_n = B \, d\omega$$

is the normal component of illumination. The total illumination of the element ds is

$$E = \int_{(2\pi)} \cos\theta \, dE_n = \int_{(2\pi)} B \cos\theta \, d\omega \tag{23}$$

where the sign (2π) indicates that the integration is performed over a solid angle 2π, or all directions on one side of the element ds.

For convenience, let us introduce some terms and notations that will allow us to describe more briefly the characteristics of the light field. Thus, the expression "illumination of the element ds placed at the point P and having its normal in the direction P_n" may be shortened to "illumination at the point P in the direction P_n". The illumination in a given direction can be represented by an arrow oriented in that direction, the arrow tail being placed at the point in question and its length being equal to the numerical value of the illumination. If an element of surface is illuminated from both sides, it is necessary to agree on the *sense* of the normal. Some authors use the outward-drawn normal to the illuminated surface, while others use the inward-drawn normal. This question is considered by Yamauti in his paper on the theory of the light field.[7] In illuminating engineering practice it is customary to employ the outer normal to the illuminated surface; but in theoretical treatments it seems more natural to choose the opposite direction, for then the illumination is represented by an arrow directed in the sense of the incident light. A similar question arises in constructing a graph of brightness distribution in various directions. We shall not bind ourselves to definite rules but shall apply in each separate case the method that is most convenient.

[7] Ziro Yamauti, Theory of Field of Illumination. Researches of the Electrotechnical Laboratory, Tokyo, No. 339, 1932.

Luminous intensity, brightness, and illumination can be represented graphically by arrows at the point in question. One might conclude that intensity and brightness are vector quantities. However, this is wrong. Not every magnitude requiring for its determination a number and a direction represents a vector, though the vector is often defined in such a manner. The vector concept contains the principle of geometric addition. Intensity and brightness are not vectors, inasmuch as the vectorial addition of intensities or of brightnesses in different directions has no meaning. Two vectors representing the same kind of quantity at a point in space can usually be replaced by a single vector found by the parallelogram rule. But the brightnesses of beams arriving at a point from various directions are completely independent of each other.

One must approach the vector concept of photometric quantities in the following manner. Each physical field is characterized by a number of quantities. The totality of values of each of these magnitudes for all points of the field (and in some cases also for all directions at each of these points) may be defined as the field of that quantity. Let us note that the concepts of the physical field and of the field of each of these quantities are just as different in principle as the physical process itself and those mathematical relations that describe it. The quantities characterizing the light field may be subdivided in two fundamental classes:

(1) functions of position and direction
(2) functions of position

The latter class may again be subdivided into two groups:

(a) vector point-functions
(b) scalar point-functions.

A function of position and direction may be defined in the following manner. Let us take a region of space and assume that to each point in this region and to each direction at this point corresponds a definite numerical value of some quantity. We shall call this magnitude a function of position and direction. As an example of the first class of quantities we have brightness and illumination. Luminous intensity may also be put in this class. Such a quantity (more exactly speaking, its value for a given direction) may be represented by an arrow in space. However, the function of position and direction is not a vector in any sense. A vector at a given point is defined by a single arrow; a function of position and direction is defined by a bundle of arrows, one for each direction of space.

A quantity of prime importance in the study of the light field is the *flux density* D, that is, the difference in illumination of the two sides of an infinitesimal diaphragm. This quantity characterizes the transfer of light through a unit area. It is a function of the position of the diaphragm and of the direction (normal to the surface of the diaphragm) and it may be regarded, as it will be shown later, as a projection of a vector. This vector is an example of a vector point-function in the light field. As an example of scalar point-functions, we may mention the *space illumination* (or space-density of radiant energy, which differs from the former by a constant factor).

5. The Illumination-distribution Solid

The most complete description of the light field at a given point is provided by the brightness-distribution solid. When this body is known, the illumination may be computed on an arbitrarily oriented element of surface at the point in question. Marking off the value of illumination along the normal to the illuminated element and rotating that element about P, so that its normal occupies all possible positions in space, we obtain the illumination-distribution surface (solid). The section of the surface by a plane passing through P gives an illumination-distribution curve in polar coordinates. When the field possesses axial symmetry, the illumination-distribution solid is characterized by a single curve. The concepts of the illumination-distribution solid and the brightness-distribution solid were introduced by L. Weber,[8] but his paper was unknown to illumination engineers until 1928, when Weber's concepts were revived by Lingenfelser.[9]

Let us give a few examples of the illumination-distribution solid for various simple cases of lighting. The outer normal to the illuminated surface will be used, as is customary in illumination engineering, though for simplicity the inner normal would be preferable.

(*a*) *A single point source.* The illumination of an element at P, (Fig. 6) with its normal in the direction PI is

$$E = D_n = I/r^2 \tag{24}$$

[8] Leonhard Weber, Intensitätsmessungen des diffuses Tageslichtes. Ann. d. Phys., **26**, 1885, p. 374.

[9] H. Lingenfelser. Über den diffusen Anteil der Beleuchtung und ihre Schattigkeit. Licht und Lampe, 1928, No. 9, p. 313; Zur Messung und Beurteilung der räumlichen Beleuchtung. Licht und Lampe, 1930, No. 12, p. 619. (Also in Technisch-wissenschaftliche Abhandlungen aus dem Osram-Konzern, **2**, 1931, p. 143.)

When the element is inclined by the angle θ, the illumination is

$$E = D_n \cos \theta \tag{25}$$

The geometrical locus of the ends of the arrows representing the illumination E is a sphere with the diameter D_n. The illumination-distribution solid is a sphere which is tangent to a plane passing through P, the plane being normal to the incident light. The radius vectors on the other side of this plane are equal to zero. Eq. (25) holds only for those values of θ for which $\cos \theta > 0$ (for example, in the region $0 < \theta < \pi/2$). For values of θ for which $\cos \theta < 0$ $\left(\text{for example in the region } \pi/2 < \theta < \frac{3\pi}{2}\right)$ the illumination is zero. Graphically, the dependence

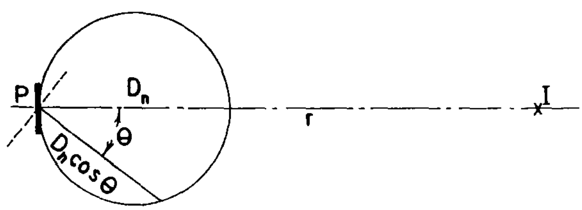

FIG. 6

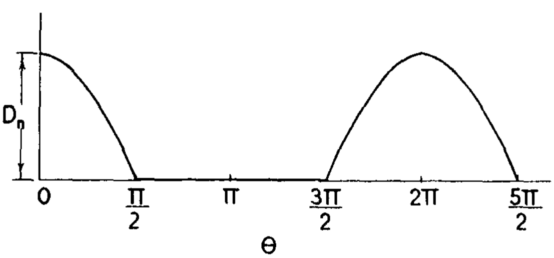

FIG. 7

of E on the angle θ is represented in rectangular coordinates by a periodic curve (Fig. 7), the illumination of the opposite side of the element being represented by a similar curve displaced by the angle π. A Fourier-series development of this function is

$$E = D_n \left[\frac{\cos \theta}{2} + \frac{1}{\pi} - \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k \cos 2k\theta}{(2k)^2 - 1} \right] \tag{26}$$

The expression in brackets is equal to zero for those values of θ for which $\cos \theta < 0$ and is equal to $\cos \theta$ if $\cos \theta > 0$. This law of variation of illumination, because it is valid for all values of θ replaces the cosine law in the study of illumination-distribution solids.

The cosine law holds also for the difference of illumination of the two sides of a surface, that is, for the flux density. Negative illumination is absurd, but when we obtain a negative value for the flux density it means that the illumination is higher on the side opposite to the one upon which the normal has been constructed. The sinusoidal character of the variation of the difference of illumination in the case of a single point source insures the existence of the same law for any number of sources. This circumstance results in extraordinary simplicity.

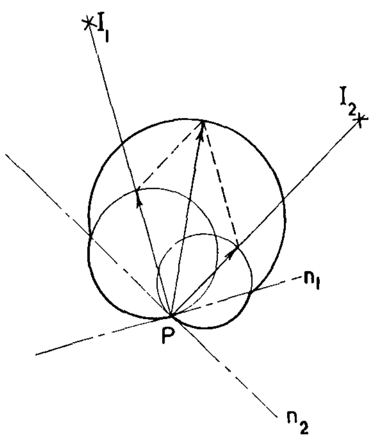

FIG. 8

(*b*) *Several point sources of light.* The illumination-distribution solid is obtained by constructing the illumination distribution for each of the sources separately, followed by an addition of the radius vectors. An example of two light sources is shown in Fig. 8. The illumination-distribution solid is bounded by three spherical surfaces. We recommend that the reader study in detail this important case, which we shall use in a number of further proofs.

(*c*) *The luminous hemisphere.* The distribution of illumination is determined at the point P, which is at the center of a uniformly luminous hemisphere of brightness B (Fig. 9). If the illuminated element is contained in the base-plane of the hemisphere, the illumination is

$$E_o = \pi B. \tag{27}$$

If the element is inclined by the angle ϕ, the illumination is

$$E = E_0 \frac{1 + \cos \phi}{2} = E_0 \cos^2 \phi/2 \tag{28}$$

The values $\phi < \pi/2$ correspond to the upper side of the illuminated element, the values $\phi > \pi/2$ to the lower. This formula was derived by Lambert for the illumination of a space shadowed by an infinitely long wall and lighted by diffused light from the sky.

The illumination-distribution curve is given in this case by a cardioid. The illumination-distribution solid is obtained by rotating the cardioid about its axis. An identical illumination distribution would be produced by a uniformly bright plane of infinite extent.

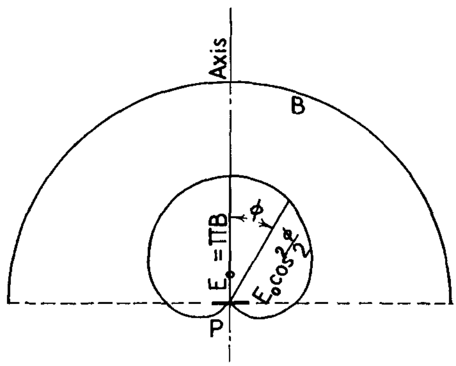

Fig. 9

The corresponding brightness-distribution solid is obtained in the following manner: In the directions within the solid angle 2π, the brightness has a constant value different from zero; in the other half the brightness is zero. Thus the brightness-distribution solid is represented by a hemisphere with its center at the point P.

After having discussed in his paper the case of the point-source and that of the luminous hemisphere, Weber proceeds to consider the illumination-distribution solid for the natural illumination from the open sky. In first approximation, the sun may be considered as a point source and the sky as a uniformly bright hemisphere. The section of the illumination-distribution solid by a vertical plane passing through the sun and through the location of the observer is obtained, as shown in Fig. 10, by addition of the radius vectors drawn to the circle (resulting from the sun) and those drawn to the cardioid (resulting from the sky).

The distribution of the illumination resulting from the sky and sunlight is shown by the heavy line.

(*d*) *The luminous sphere.* If the brightness of the inner surface of the sphere is B, the illumination-distribution surface at any point within the sphere is again a sphere with a radius πB and a center at the point considered. This may serve as an illustration of completely diffused light, where the brightness in all directions is the same. The brightness-distribution solid is a sphere with its center at P. This case is the opposite of the first case of parallel light, in which the brightness is zero in all directions but one. In both cases the illumination-distribution solid is a sphere, but in one case P is placed at the center of this sphere, and in the other case it is found on the surface of the sphere.

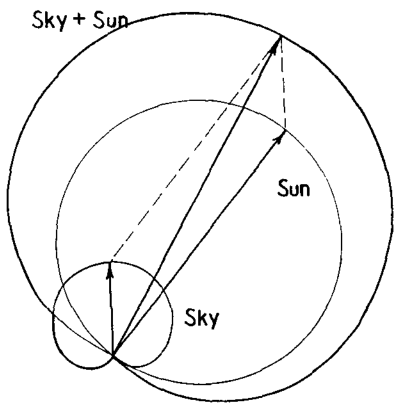

FIG. 10

Experiments on the determination of illumination-distribution solids for various important arrangements of artificial lighting were carried out by Lingenfelser.[9] He measured the illumination at the center and in the corner of an experimental room with direct, semi-indirect, and indirect lighting, with light and dark room walls, and with the receiving surface tilted at various angles. For the determination of the lighting conditions, he suspended two small white balls in the room. The shadows formed by the spheres on the walls and on the floor, compared with the shadows formed on the spheres themselves, showed the differences between the "hard" direct lighting and the "soft" indirect lighting. In Fig. 11 the solid line indicates the axial cross-section of the illumination-distribution solid at the center of the room for indirect lighting. For this point the illumination-distribution solid was found to be a body of revolution, to within an accuracy of ten or fifteen per cent. For a point in the corner of the room, this symmetry does not exist.

Lingenfelser remarks in his article, quite correctly, that the main purpose in constructing illumination-distribution solids is not the quantitative evaluation of illumination, but the consideration of qualitative properties of lighting (degree of diffusion, production of shadows, etc.). Space forbids further consideration of Lingenfelser's analysis of illumination-distribution solids, but the method is presented briefly in the treatise on illuminating engineering by Sirotinsky and Fedorov.[10]

The illumination-distribution solid does not completely define the lighting conditions at a given point of the field, as was assumed by Weber. Knowing only the illumination-distribution solid, we cannot

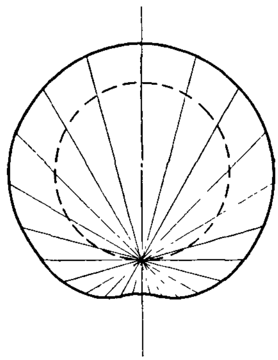

FIG. 11

compute the distribution of illumination on a surface of a real object because the various elements of the surface of the object are shading each other unless the body is convex.

If the effect of shadows of the object is to be obtained, the brightness-distribution solid must also be known. The distribution of brightness allows us to determine uniquely the distribution of illumination. The inverse problem of finding the brightness distribution, when the illumination-distribution solid is given, has no unique solution. Usually, the same illumination-distribution solid may be obtained for quite different brightness-distribution solids. This can be shown in a simple way in two-dimensional photometry, where brightness is measured as

[10] L. I. Sirotinsky and B. F. Fedorov, Principles of Electric Lighting. Energoizdat, 1923.

an intensity per unit length of the source, and illumination is inversely proportional to the first power of the distance. It may happen also that a distribution of sources cannot be found to satisfy arbitrary conditions of illumination. A question of fundamental interest and importance is this: May the illumination-distribution solid have an arbitrary form; may one choose an arbitrary illumination-distribution solid, assuming that for the point in question a corresponding brightness distribution can be found? Or the question may be formulated in the following manner: May one specify independent values of illumination upon the various inclined planes of a working surface, and assume that one can always find the lighting conditions which will result in this illumination distribution? The illuminating engineers answer this question in the affirmative. However, this is not quite right. The values of illumination at a given point for various orientations of the illuminated element are not independent of each other but are related, as we shall see, in a definite way. Not every kind of illumination distribution may exist! This seemingly unexpected conclusion is obvious *a priori*, for it is completely impossible to create at a given point such conditions as will give a finite value of illumination for one orientation of the illuminated element and zero illumination for all other orientations. It is impossible to create a horizontal illumination without illuminating also the inclined surfaces. We shall return later to this important problem.

SPACE ILLUMINATION

1. The Average Spherical Illumination

The analogy between the illumination-distribution solid and the curves of illumination distribution at a point in the light field, on the one hand, and the solid and curves of candlepower distribution for a point source, on the other hand, is obvious. One may also introduce the concept of *average illumination at a point*—a concept analogous to that of average intensity in a solid angle ω. The **average illumination** at P is defined as

$$E_\omega = \frac{1}{\omega} \int_\omega E \, d\omega \tag{29}$$

where E is the illumination of an element that is normal to the axis of the infinitesimal solid angle $d\omega$ having its apex at P. The integration is performed for all directions contained within the solid angle ω.

The **average spherical intensity** of a source may be defined as

$$I_{4\pi} = \frac{1}{4\pi} \int_{(4\pi)} I \, d\omega \tag{30}$$

where I is the intensity in the direction of the elementary solid angle $d\omega$ with its apex at the source, and where the integration is performed over all directions of space. Analogous to this, one may introduce the concept of **average spherical illumination** at P,

$$E_{4\pi} = \frac{1}{4\pi} \int_{(4\pi)} E \, d\omega \tag{31}$$

where $d\omega$ is the infinitesimal solid angle with its apex at P, and E is the illumination of an element normal to the axis of the solid angle.

The average spherical illumination is a fundamental concept for the light field. Indeed, the condition of the field at a point must be characterized by a quantity which has a single value at each point in space. The average spherical illumination satisfies this condition, whereas the common concept of illumination depends not only on position in space but also on the orientation of a plane element at P.

One may introduce also a concept of **average hemispherical illumination** at P:

$$E_{2\pi} = \frac{1}{2\pi} \int_{(2\pi)} E \, d\omega \tag{32}$$

or, for example, of the average illumination of a given cross section of the illumination-distribution solid (analogous to average horizontal candlepower of a source).

One may visualize in the following manner the concept of the illumination-distribution solid and of quantities which are defined by it. Let us describe a sphere about P, the radius being so small that the variation of illumination may be neglected from one point within the sphere to another. The surface of the sphere contains elements oriented in all possible ways in space. The illumination distribution at the point P may be considered as the distribution of illumination on the surface of the sphere.

This leads to a lucid interpretation of average values of illumination for a given ensemble of orientations of the illuminated element. Thus, for example, the average spherical illumination at a point P of the light field may be defined as the average illumination of the outer surface of a sphere of infinitesimal radius with its center at P. Let us denote by dF the luminous flux incident on an element of spherical surface,

$$ds = r^2 \, d\omega$$

where r is the radius of the sphere and $d\omega$ is the solid angle defined by the point P and the element ds. If E is the illumination of this element,

$$dF = E \, ds = r^2 E \, d\omega$$

and the flux incident on the entire surface of the sphere is

$$F = r^2 \int_{(4\pi)} E \, d\omega \tag{33}$$

Therefore the average illumination of the surface is equal to $E_{4\pi}$.

This proof may be generalized by stating that the average illumination at a point P in the light field, and within the boundaries of a definite solid angle, is the same as the average illumination of a part cut out by this solid angle from a sphere of infinitesimal radius with center at P. Thus we may introduce into illuminating engineering a new quantity, namely, the average hemispherical illumination, which was invented by the author to parallel the concept of the average spherical illumination.[11] As indicated by its name, this quantity represents the average of the values at the given point for an element, the normal to which assumes all possible directions within the solid angle 2π. Thus the average hemispherical illumination is the limit of the average outside illumination on a hemisphere, with its center at the point P, when the radius approaches zero. It is obvious that the average spherical illumination depends only on the position of the point in the light field, whereas the average hemispherical illumination is also a function of the direction of the normal to the base plane of the hemisphere. Usually, however, the average illumination of the upper hemisphere with horizontal plane base is used.

When the illumination-distribution solid is known, the average values of the illumination within a given solid angle may be found by computation. For example, when the illumination-distribution solid possesses axial symmetry, it is completely described by the illumination-distribution curve obtained by a meridional section of it. The illumination at a given point depends only on the angle ϕ between the normal of the illuminated element and the axis of symmetry. When the equation of the curve of distribution of $E = E(\phi)$ is given in polar coordinates, the average spherical illumination may be defined as

$$E_{4\pi} = \frac{1}{2} \int_0^{\pi} E(\phi) \sin \phi \, d\phi \tag{34}$$

For a symmetrical source, the average spherical intensity is obtained from the intensities in different directions by a similar formula. Analytical and graphical methods may be used to evaluate the integral, as in the computation of total flux from a Rousseau diagram.

2. Methods of Measurement

Instruments may be built for the direct determination of average illumination at a given point. For this purpose one may use a receptor of radiant energy which has a light-sensitive surface in the shape of a sphere or a hemisphere. A barrier-layer photocell of spherical shape suggests itself. The realization of such a photoelement, possessing absolutely uniform properties over its entire surface, is difficult; but one may use a polyhedron as is sometimes done in constructing apparatus for the measurement of luminous flux. This polyhedron may

[11] A. Gershun, Characteristics of Conditions of Illumination, Trans. Optical Institute, Leningrad, **6**, No. 59, 1931.

consist of similar, plane, barrier-layer cells. Such a quasi-spherical photoelement has been constructed by the research laboratory of Tungsram Company.[12] It consists of twelve selenium cells which form a regular icosahedron. The elements are connected in parallel, and the indication of the associated electrical instrument is proportional to the sum of the photoelectric currents produced by the individual cells.

Apparatus for measurement of average spherical and hemispherical illumination may be constructed by using a somewhat different principle. Let us place in the light field a small hollow sphere made of a diffusing material which follows Lambert's law for transmitted light as well as for reflected light. The illumination of the inner surface of the diffusing envelope will be uniform over the entire surface, and this illumination will differ only by a constant factor from the average illumination of the outer surface of the sphere. The first statement follows directly from the known photometric property of the sphere. To prove the second statement, let us determine the illumination E_{in} of the inner surface of the sphere. Let τ and ρ be the transmission factor and reflection factor of the spherical shell, and let s be the surface area of the entire sphere. Denoting by $E_{4\pi}$ the average spherical illumination at a given point, we have a flux

$$F = E_{4\pi}\, s$$

on the outside of the sphere. Part of this flux

$$\tau F = \tau E_{4\pi}\, s$$

enters the sphere and is distributed over the entire inner surface. Multiple reflections must be considered, but the balance for the fluxes is given by the expression

$$\underbrace{E_{in}\, s}_{\text{on inner surface}} = \underbrace{\tau E_{4\pi}\, s}_{\text{from outside}} + \underbrace{\rho E_{in}\, s}_{\text{reflected from within}} \tag{35}$$

Thus,

$$E_{in} = \frac{\tau}{1 - \rho} E_{4\pi} \tag{36}$$

Therefore the illumination within differs only by a constant factor from the average illumination on the outside of the sphere. Knowing this factor, and having measured the illumination E_{in} at any point of

[12] Über eine Sperrschichtphotozelle zur Messung der Raumhelligkeit. (Mitteilung aus dem Tungsram-Forschungslaboratorium). Das Licht, 4, 1934, p. 155.

the inner surface of the sphere, we may determine the value of average spherical illumination. If absorption of light is neglected,

$$E_{in} = E_{4\pi} \tag{36a}$$

Thus in the determination of average spherical illumination one may use a diffusing sphere as has been done in the measurement of average spherical candlepower; but in the former case one measures the light transmitted through the sphere rather than the light reflected from it. A number of simple photometric devices can be used for the measurement of average spherical illumination, or more exactly speaking, of space-illumination, which differs from the former by a constant factor and which will be discussed later. The diffusing shell of the sphere is made of opal glass. One should remember, however, that opal glass is not the ideal diffusing material of which the foregoing imaginary sphere was made. Opal glass does not follow Lambert's law, and possesses also selective properties. Moreover, any arrangement for the measurement of illumination within the sphere necessarily blocks off part of the total surface. All these factors, together with the finite dimensions of the sphere, impose a limit on the possible accuracy of the measurement of average spherical illumination.

For measurement of illumination on the inner surface of the sphere, one may use objective methods as well as visual methods. For objective measurements one uses a receptor (photoelement, photographic plate), the light-sensitive surface of which constitutes a small part of the inner surface of the spherical probe. A simple photometer containing a barrier-layer photocell has been constructed on this principle by Lange.[13]

Arrangements for measuring average spherical illumination may be combined with any visual photometer* used in measuring brightness. The neck of a spherical flask made of opal glass (the author used envelopes of low-wattage opal-glass lamps) is attached to a metal tube (Fig. 12) which leads to the photometric screen. The end of the tube that terminates at the spherical surface is closed by a piece of opal glass. On this opal glass window falls only the light diffusing from the flask.

[13] B. Lange, Über die Photometrische Anwendbarkeit der Halbleiterphotozellen. Zs. für Instrumentenkunde, **53**, 1933, p. 380.

* Laboratory photometers, as well as most illumination meters, such as the one designed by the State Optical Institute and used extensively in the U. S. S. R. Attachments for use in measuring space illumination are manufactured by Franz Schmidt & Haensch according to the design of Arndt (Licht u. Lampe, **7**, 1928, p. 247).

The illumination of the glass, and therefore its brightness, is proportional to the average illumination of the sphere. When the sphere is sufficiently small, this illumination may be assumed to be proportional to the average spherical illumination at the center of the sphere. In this manner the measurement of average spherical illumination is reduced to a single measurement of brightness.

The same apparatus may be used not only for the measurement of $E_{4\pi}$ but also for the measurement of such magnitudes as E_ω. For the latter purpose one must enclose the measuring sphere in an opaque hood, leaving open only part of its surface. The whole inner surface of the sphere will be uniformly illuminated as previously, and this illumination will be proportional to the average outside illumination

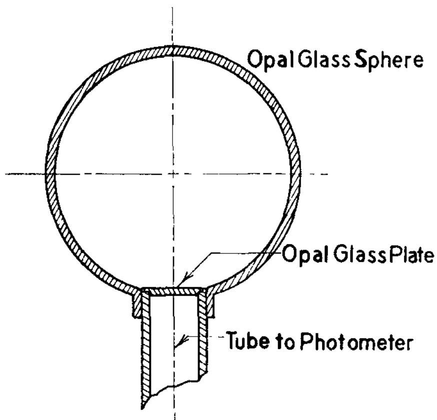

FIG. 12

of the unshielded part of the measuring sphere. Several such designs have been made by the author[11] in collaboration with V. A. Zelenkov and D. N. Lazareff. One of these attachments intended for measurement of average hemispherical illumination $E_{2\pi}$ is indicated in Fig. 13. A hemisphere of opal glass is attached to a cylindrical box, the height of which is equal to the radius of the hemisphere. The box is blackened inside. At the center of the bottom of the box there is an orifice closed by matt opal-glass plate. This window represents, let us say, an element of the sphere, the other part of which is the light-diffusing hemisphere. The observer looks through the tube and measures the brightness of the opal-glass plate. The reading of the photometer corresponding to the photometric balance is then multiplied by a constant which corresponds to the particular attachment, and which had been previously determined by calibration. For the solution of problems of illuminating engineering it may be useful to introduce fractional magnitudes such as $E_{3\pi/2}$, E_{π}, $E_{\pi/2}$, as was done by D. N. Lazareff. For this purpose the opal-glass hemisphere is screened by various attachments.

3. **Methods of Computing Space Illumination**

Let us consider now in greater detail the newly introduced fundamental magnitude called the average spherical illumination. One can give the following interesting definition of this magnitude from which will follow directly the method of its computation if the brightness-

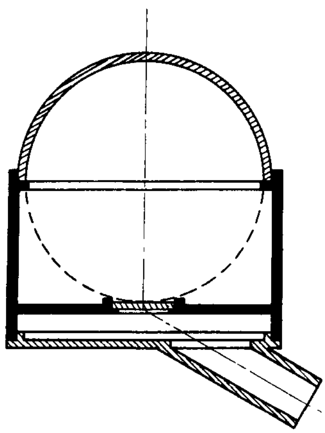

FIG. 13

distribution solid is known. Let us subdivide the entirety of all rays passing through a given point into a set of elementary solid angles $d\omega$. Let us denote by dD_n the normal illumination* produced by the rays that are contained within $d\omega$. The part of the flux received by the outer surface of a small sphere of radius r is

$$\pi r^2 \, dE_n$$

The entire flux received by the sphere from all directions of space is

$$F = \pi r^2 \int_{(4\pi)} dE_n \tag{37}$$

* [By "normal illumination" is meant the value of illumination when the light rays are perpendicular to the illuminated surface.] Trans.

The average spherical illumination for the point in question is

$$E_{4\pi} = \frac{F}{4\pi r^2} = \frac{1}{4}\int_{(4\pi)} dE_n \tag{38}$$

In this manner the average spherical illumination may be defined as a quarter of the sum of all elementary normal illuminations.

When the illumination is produced by a set of point sources,

$$E_n = \frac{I}{r^2} \tag{39}$$

where I is the intensity of the source in the direction considered, r is the distance from the point to the source, and E_n is the normal illumination* produced by each of the sources separately. The average spherical illumination is

$$E_{4\pi} = \tfrac{1}{4}\sum E_n \tag{40}$$

In general we have, according to our definition of brightness,

$$dE_n = B\, d\omega \tag{41}$$

and therefore,

$$E_{4\pi} = \frac{1}{4}\int_{(4\pi)} B\, d\omega \tag{42}$$

where B is the brightness in the direction of the elementary solid angle $d\omega$. It is customary to use the name **space illumination**[14] for the quantity

$$E_0 = 4E_{4\pi} = \int_{(4\pi)} dE_n = \int_{(4\pi)} B\, d\omega \tag{43}$$

Equation (43) allows us to determine the value of average spherical illumination when the brightness-distribution solid is known. The average hemispherical illumination $E_{2\pi}$ is

$$E_{2\pi} = E_{4\pi} + \frac{E_h - E_h'}{4} \tag{44}$$

as has been shown by the author.

[14] Introduced by Leonhard Weber, Die Albedo des Luftplanktons. Ann. d. Phys., **51**, 1916, p. 427. [The concept does not seem to have been used in the United States and no English name has been established for it. Russian, *prostranstvennaya osveshchennost*; German, *Raumbeleuchtung*; French, *èclairement spatial.*] Trans.

$E_{4\pi}$ is the average spherical illumination, while E_h and E'_h are the values of illumination of the two sides of the base of the hemisphere. If the upper hemisphere is considered, these values refer to the horizontal illumination from above and from below.

4. Space Density of Light*

Because light is propagated with a finite velocity, we find in each volume of the light field at any instant of time a definite quantity of energy and a definite number of light particles or *photons*. Therefore one may introduce a concept of *volume density of radiation,* as was done by Planck. Let us define the limit of the ratio of the quantity of light to the volume containing it, as this volume approaches zero, as the **space density of light** at the point in question. The space density of light differs from the space illumination only by a constant factor, a

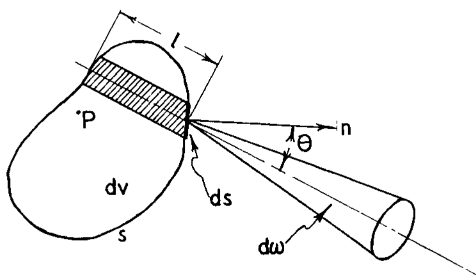

FIG. 14

constant of the medium, the medium being assumed to be isotropic.[11] Let us compute u, the space density of light, at the point P of the light field (Fig. 14). We assume for a moment that the light is removed from all the space with the exception of the elementary volume dv containing the point P; then we compute the energy which flows out of this volume through its surface s.[15]

The luminous flux through the element of the surface ds in the direc-

* ["Light energy" in the original. We are using the word "light" to mean *radiant energy evaluated with respect to the standard visibility function.* We use the photometric rather than the radiometric name in accordance with the policy stated in Chap. II. Throughout his book, Professor Gershun wishes to emphasize the intimate connection between illuminating engineering and radiation engineering and the fact that the same principles (conservation of energy, etc.) can be applied in both fields.] Trans.

[15] A. F. Joffe, Thermodynamics of Radiant Energy. Chapter 5 of the Treatise on Physics by O. D. Chwolson. (In Russian.)

tions contained within an elementary cone $d\omega$, the axis of which makes an angle θ with a normal n to the element ds, is equal to

$$B \cos \theta \, ds \, d\omega$$

This light will flow during the time l/c, where l is the distance along the light ray from the element ds to the opposite wall of the volume dv, and c is the velocity of propagation. Thus the beam will carry through ds the light equal to

$$\frac{Bl}{c} \cos \theta \, ds \, d\omega$$

The quantity $l \cos \theta \, ds$ represents the shaded element of volume shown in the figure. Integrating over the entire surface, we obtain the quantity of light that has emerged from dv in directions contained within $d\omega$, or

$$\frac{B \, d\omega \, dv}{c}$$

Integrating once more over ω within the limits of the solid angle 4π, we obtain the quantity of light contained in the volume dv, or

$$dQ = \frac{dv}{c} \int_{(4\pi)} B \, d\omega \tag{45}$$

Thus the space density of light is

$$u = \frac{dQ}{dv} = \frac{1}{c} \int_{(4\pi)} B \, d\omega = \frac{E_0}{c} \tag{46}$$

which differs by a constant factor from the space illumination. In the accepted system of units of illumination engineering, the space density of light is measured in *lumen sec* cm^{-3}.

5. The Concept of Space Illumination

As has been shown, the average spherical illumination $E_{4\pi}$, the space illumination E_0, and the space density of light u are equivalent concepts, and inasmuch as

$$E_0 = 4E_{4\pi} = cu \tag{47}$$

one need consider only one of these quantities. The space density of light is not a convenient quantity to deal with in engineering, one of the reasons being that its values are expressed by very small numbers in the accepted system of units and the concept is not easily visualized by an engineer, who customarily deals with power and with luminous flux.

The average spherical illumination is expressed in units of illumination, the same units being used also for space illumination. According to Arndt the unit of space illumination should be called "Raumlux" to distinguish it from the ordinary lux. To obtain the number of "space lux" at a point, one must add the values of normal illumination (expressed in lux) which are caused by all the light sources. Space illumination represents a fundamental quantity characterizing the light field at a point, and this concept must find a wide application in illuminating engineering practice. Space illumination characterizes the general "density of light" at a given point in a room, independent of the direction from which the light is coming; in other words, it characterizes the average of all values of brightness taken in all directions from the point in question. Indeed, the average spherical brightness $B_{4\pi}$ is defined as

$$B_{4\pi} = \frac{1}{4\pi} \int_{(4\pi)} B \, d\omega = \frac{E_0}{4\pi} \qquad (48)$$

Thus space illumination defines the average level of brightness seen from the point in question. The value of space illumination at each point in a room lighted by luminaires that throw the light in comparatively narrow beams is conveniently divided into two parts—the first produced by direct light from the luminaires, and the second produced by light reflected from surfaces in the room. The ratio of these two parts characterizes approximately the degree of direct lighting and the depth of the shadows. The value of the second part gives the average level of brightness of the enclosing surfaces and of the surfaces of objects contained within the room. Thus it determines the adaptation level of the eye and defines the visual evaluation of all brightness. The actual evaluation of the adequacy and uniformity of lighting as it appears to the eye does not correspond to the evaluation of lighting based on the illumination distribution on a fictitious working surface, but is more closely characterized by the space illumination.

The transition from standardization according to horizontal illumination to standardization according to space illumination is completely analogous to the change from candlepower rating to lumen rating—a change that is universally accepted in the specification of lamps. The transition to space illumination is desirable in the natural lighting of buildings. At the present time the adequacy of natural lighting is judged by the "coefficient of natural lighting," which is equal to the ratio of horizontal illumination within the room to the corresponding value of outside illumination. Such a choice of horizontal illumination as a measure produces an unjustifiable discrepancy in the comparison of lighting from windows and from luminaires. Much larger values are obtained for coefficients of lighting in the case of lighting from above, than in the case of lighting from the side. This is due to the fact that these coefficients are arbitrarily taken relative to the horizontal plane. The side light from the windows falls on the horizontal surfaces at large angles of incidence and thus produces low values of illumination, though it may well fill the entire room with light. This consideration suggests the desirability of using the concept of space illumination for the specification of daylighting, as has been pointed out in a number of foreign articles.* One must note especially the articles of Arndt, which are summarized in his book, *Raumbeleuchtungstechnik.*

Space illumination is an average of the values of illumination for all possible orientations of the illuminated surface. One usually looks upon objects from above, however, and thus it seems reasonable in many cases to judge the illumination by the value of the upper hemispherical illumination. In the choice of the characteristics of illumination one also has to consider the micro-relief of the surface in question, together with its macro-orientation and the space orientation of sepa-

* Wilhelm Arndt, Raumhelligkeit als neuer Grundbegriff der Beleuchtungstechnik. Licht u. Lampe, 1928, No. 7, p. 247.

Wilhelm Arndt, Beleuchtungsstärke oder Raumhelligkeit. Licht u. Lampe, 1928, No. 23, p. 833.

Wilhelm Arndt, Neue Grundzüge der Beleuchtungstechnik. Licht u. Lampe, 1930, No. 10, p. 537.

Wilhelm Arndt, Über den Stand der Arbeiten, Beleuchtung räumlich zu bewerfen. I. I. C. Proc., 1931, p. 197.

Wilhelm Arndt, Raumbeleuchtungstechnik, Berlin, 1931.

* A. Dresler and W. Arndt, Beleuchtungswertung mit Hilfe der Raumhelligkeit. Licht u. Lampe, 1930, No. 20, p. 997; No. 22, p. 1092; No. 23, p. 1142.

A. Dresler, Entwurf und Beurteilung von Beleuchtungsanlagen auf räumlicher Grundlage. Dissertation, Tech. Hochschule, Berlin, 1930.

H. Hellmann, Die Bewertung von Beleuchtungsanlagen auf messtechnischer Grundlage. Dissertation, Berlin, 1930.

Jean Dourgnon, Définition et calcul des grandeurs caracteristiques de l'éclairage d'un éspace clos. R. G. E., **33**, 1933, p. 579.

rate elements forming its structure.[16] In evaluating the illumination of an aeroplane landing field covered with grass, for instance, one must consider not only the horizontal illumination of the earth but also the vertical illumination of the blades of grass.

[16] Gershun and Lazareff, Light Engineering (Russian periodical), No. 4, 1935, p. 1.

CHAPTER IV

THE LIGHT VECTOR

1. Proof of Existence

The luminous flux through a surface may be considered as the flux of a vector point-function, the projection of this vector on a given direction being proportional to the difference in illumination of the two sides of a plane element which is normal to that direction.

Let us consider a light field in an arbitrary medium. This medium may diffuse the light that is passing through it, or it may emit and absorb light. Enclose a point of the light field by a small surface. Reduction in the size of the surface reduces the enclosed volume; and if the surface remains geometrically similar to its previous shape, then the luminous flux through the surface will decrease in proportion to the area of the surface. But the radiation absorbed or emitted within this surface will decrease much more rapidly, i.e. in proportion to the volume enclosed by the surface. When the linear dimensions are decreased to $1/n$, the surface will decrease to $1/n^2$ and the volume to $1/n^3$. Thus a sufficiently small volume can always be found to justify the assumption that all the flux entering the region is leaving the region.

The proof of the existence of a vector point-function defining the flux of radiant energy is completely analogous to the proof used in the analytic theory of heat.[17] The method was applied to photometry by Boldyreff.[18] Consider a point P with coordinates x, y, z, measured with respect to a rectangular system of coordinates X, Y, Z, (Fig. 15). Form the elementary tetrahedron having three planes meeting at the point P, these planes being parallel to the planes formed by the coordinate axes. The distances dx, dy, dz, are assumed to be sufficiently small so that the illumination is uniform over any one of the four sides of the tetrahedron. We assume also that absorption and emission of light *within* the elementary volume may be neglected.

[17] G. Kirchhoff, Vorlesungen über die Theorie der Wärme. Leipzig, 1894.
I. Boussinesq, Théorie analytique de la chaleur. Paris, 1901, (Vol. I, §§51–54).
H. Helmholtz, Vorlesungen über die Theorie der Wärme. Leipzig, 1903 (Vol. 6, Part 2, Chapter I).

[18] N. G. Boldyreff, The Light Field in Diffusing Media. Trans. Optical Institute, Leningrad, **6**, No. 59, 1931.

Let us denote by E_x, E_y, E_z, the illumination of the three sides that meet at the point P. The illumination of the fourth side is E_n, where n denotes the normal to that side. To distinguish between the two sides of each of the four surfaces, we shall add an index plus when we consider the side of the surface that is seen from the origin of coordinates, and an index minus for the opposite case. Thus the illumination of the sides of the tetrahedron from outside will be E_{+x}, E_{+y}, E_{+z}, E_{-n}. Correspondingly, we shall denote by E_{-x}, E_{-y}, E_{-z}, E_{+n}, the illuminations from within. Let us compute the fluxes entering and leaving the tetrahedron. If $d\sigma_x$, $d\sigma_y$, $d\sigma_z$, $d\sigma_n$ are the areas of the four sides, the entering flux is

$$dF_{ent} = E_{+x}\,d\sigma_x + E_{+y}\,d\sigma_y + E_{+z}\,d\sigma_z + E_{-n}\,d\sigma_n \tag{49}$$

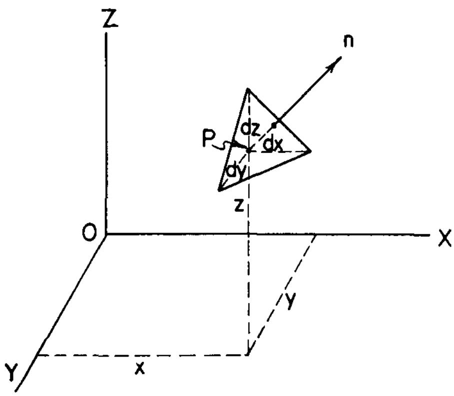

FIG. 15

and the flux that leaves is

$$dF_{leave} = E_{-x}\,d\sigma_x + E_{-y}\,d\sigma_y + E_{-z}\,d\sigma_z + E_{+n}\,d\sigma_n \tag{50}$$

According to our assumptions, dF_{ent} equals dF_{leave}. Therefore,

$$\begin{aligned}(E_{+n} - E_{-n})\,d\sigma_n = (E_{+x} - E_{-x})\,d\sigma_x + (E_{+y} - E_{-y})\,d\sigma_y \\ + (E_{+z} - E_{-z})\,d\sigma_z .\end{aligned} \tag{51}$$

This equation states that the luminous flux through an arbitrarily oriented element of surface is equal to the algebraic sum of the fluxes through the projection of this element on three planes which are perpendicular to each other and which pass through the point in question. The orientation of these planes is arbitrary. The flux through a surface is defined, as usual, by the difference of fluxes on the two sides of the surface. Dividing Equation (51) by $d\sigma_n$ and introducing the expression for the cosines of the angles between the normal n and the coordinate axes, we obtain

$$(E_{+n} + E_{-n}) = (E_{+x} - E_{-x}) \cos(n, x) + (E_{+y} - E_{-y}) \cos(n, y) + (E_{+z} - E_{-z}) \cos(n, z) \cdots . \tag{52}$$

Denoting the flux density through the surface (the difference of illumination) by D, we may write

$$D = E_+ - E_- .$$

Thus

$$D_n = D_x \cos(n, x) + D_y \cos(n, y) + D_z \cos(n, z). \tag{53}$$

The difference of illumination represents the projection of a vector. Let us mark off from P, along the directions of the coordinate axes, distances D_x, D_y, D_z, and let us construct upon these three a parallelopiped. The diagonal may be considered formally as a vector. Let us denote the vector by $\mathbf{D}$ and its numerical value by D.

The flux densities D_x, D_y, D_z may be considered as projections of the vector $\mathbf{D}$ on the coordinate axes, or

$$\left.\begin{aligned} D_x &= D \cos(\mathbf{D}, x) \\ D_y &= D \cos(\mathbf{D}, y) \\ D_z &= D \cos(\mathbf{D}, z) \end{aligned}\right\} \tag{54}$$

where

$$D = \sqrt{D_x^2 + D_y^2 + D_z^2} \tag{55}$$

is the absolute magnitude of the vector. The direction cosines of the vector are equal to

$$\frac{D_x}{D}, \qquad \frac{D_y}{D}, \qquad \frac{D_z}{D}$$

Substituting these expressions for D_x, D_y, D_z in Equation (53), and introducing the expression for the cosine of the angle between the vector $\mathbf{D}$ and the normal n, which is equal to

$$\cos(\mathbf{D}, n) = \cos(\mathbf{D}, x) \cos(n, x) + \cos(\mathbf{D}, y) \cos(n, y) + \cos(\mathbf{D}, z) \cos(n, z) \cdots \tag{56}$$

we obtain the following fundamental relationship:

$$D_n = D \cos(\mathbf{D}, n) \tag{57}$$

which is independent of the choice of coordinate axes.

Inasmuch as the normal n has been chosen arbitrarily, we may state the following: *At each point P of the light field there is a vector, which is independent of the choice of coordinate axes, and which possesses the property that its projection upon any direction is numerically equal to the difference of illumination of the two sides of a plane element placed at P and normal to that direction.* This vector and the space-illumination function represent fundamental functions of position in the light field. The space illumination is a scalar point function, while the vector is a vector point function. We shall call this vector the **light vector**.

The greatest difference of illumination of the two sides of an element at a given point is obtained when the element is perpendicular to the vector. This position corresponds to the maximum transfer of energy. When the illuminated element is placed in the plane of the vector, it is equally illuminated on both sides and the difference of illumination is zero. There is no transfer of energy and dynamic equilibrium obtains. At each point of the light field the illuminated element may be rotated about an arbitrary axis so as to obtain equal illuminations on the two sides. That is, the element will determine the plane of the light vector. Photometric equilibrium will not be destroyed if the element in such a position is rotated about an axis which coincides with the direction of the light vector.

2. Properties of the Light Vector

Let us consider a geometrical surface s in the light field (Fig. 16). By a geometrical surface is meant a surface that possesses no physical properties, and which therefore does not affect the light field. As a rule the surface s will be illuminated from both sides. We shall denote the two sides by subscripts 1 and 2. The illuminations of the two sides of an element ds are E_1 and E_2, and the light vector for the element ds is $\mathbf{D}$. The angle between $\mathbf{D}$ and n (the sense of the normal is indicated in Fig. 16 by an arrow) is denoted by θ. Then

$$E_1 - E_2 = D \cos \theta. \tag{58}$$

Multiplying the left and the right sides of this equation by ds, we obtain

$$E_1\, ds - E_2\, ds = D \cos \theta\, ds. \tag{59}$$

The expression on the left side of the equation represents the difference of the flux

$$dF_1 = E_1\, ds$$

on the side No. 1 of ds, and the flux

$$dF_2 = E_2\, ds$$

on the other side. Denoting this quantity by dF, we shall call it the luminous flux through the element ds.

Consider now the right side of Equation (59), representing the flux of the vector $\mathbf{D}$ through the surface element ds. Introducing the concept of a vectorial element of surface $d\mathbf{s}$, which is equal in magnitude to ds, and which is directed in the direction of the normal n,

$$dF = \mathbf{D} \cdot d\mathbf{s}.$$

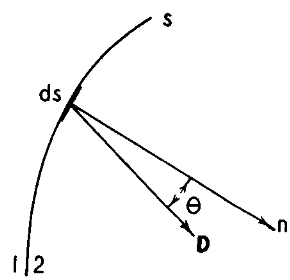

FIG. 16

Integrating this expression over the entire surface s,

$$F = \int_s \mathbf{D} \cdot d\mathbf{s} \tag{60}$$

In the terminology of vector analysis the luminous flux through the surface is equal to the flux of the light vector through that surface.[19]

This theorem may be considered as the fundamental definition of the light vector. The existence in the light field of a vector point-function, the integral of which over a surface represents the quantity of energy passing through the surface, must be considered as the initial and most fundamental theorem of theoretical photometry. But today it is still not universally accepted—a state of affairs that may astonish

[19] A. Gershun, The Light Field Resulting from Luminous Surfaces of Uniform and Non-uniform Brightness. Trans. Opt. Inst., Leningrad, **4,** No. 38, 1928.

a physicist who is not acquainted with the literature on photometry. We may recall that in his study of the electromagnetic field of an elementary vibrator, Poynting showed in 1885 that the increase of energy in an arbitrary volume is measured by the flux of a vector through the surface enclosing this volume. The light vector, introduced in the present treatise, may be considered as a macroscopic average (in space and in time) of the values of the Poynting vector. Obviously the light vector characterizes the magnitude and direction of the light pressure on a small spherical body placed at the point in question.

The light vector has been defined above as a vector, the projection of which upon a given direction is equal to the difference of illumination of the two sides of an element that is normal to that direction. In other words, the projection of the light vector determines the flux density, which characterizes the energy balance and which is of great importance in the theory of the physical field. However, the illuminating engineer is not interested in energy flow through imaginary apertures but in the illumination of real bodies, especially in the illumination of their outer surfaces. From this standpoint, the engineer may doubt the soundness of introducing the new concept of the light vector. The author hopes that if the reader has a similar prejudice it will disappear as a result of the study of this book. Before we take up the matter of photometric computation, let us illustrate several applications of the vector method to show its usefulness in illuminating engineering.

As the first example let us consider the following problem. Two point sources of light of intensity I_1 and I_2 are at distances r_1 and r_2 from a point P (Fig. 17). What is the maximum value of the illumination at P, and how should the illuminated element be placed to obtain on it the maximum illumination? Without the concept of the light vector we should have to derive an expression for the illumination of the element (arbitrarily oriented) due to each of the sources, to write an expression for the sum of two such effects, and to find by means of differential calculus the position of the element, for which its total illumination has a maximum value. All this is not very difficult, but that it is sufficiently involved is indicated by the absence of the treatment of this problem (which seems to be a fundamental one) in the scientific literature of photometry.

Let us solve the problem by the vector method. We denote the two light vectors by $\mathbf{D}_1$ and $\mathbf{D}_2$. Obviously the directions of these vectors coincide with the directions of the rays arriving at P from the sources.

The absolute values of these vectors are

$$D_1 = \frac{I_1}{r_1^2}, \qquad D_2 = \frac{I_2}{r_2^2}$$

By means of the parallelogram rule the resultant light vector **D** is equal to

$$\mathbf{D} = \mathbf{D}_1 + \mathbf{D}_2 . \tag{61}$$

For all orientations for which both I_1 and I_2 are on the same side of the illuminated element, the illumination of the other side is zero; and thus the projection of the vector on the normal to the surface is equal

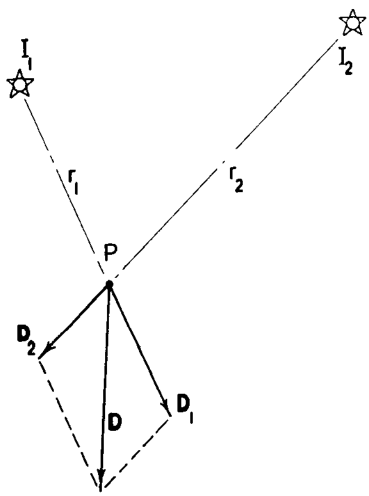

FIG. 17

to the illumination of the surface. The illumination resulting from each of the sources considered separately varies as the cosine of the angle between the vector $\mathbf{D}_1$ or $\mathbf{D}_2$ and the normal to the illuminated element. The sum (or difference, when both sides are illuminated) varies as the cosine of the angle between **D** and n. The total illumination has its maximum (which is equal to the length of the vector **D**) when n is collinear with **D**. Thus the problem is solved. Its solution consists of the construction of a parallelogram and the determination of the length of its diagonal, which may be performed either graphically or analytically. *The parallelogram of illuminations* may play in photometry a rôle similar to that played by the *parallelogram of forces* in mechanics.

The distribution of total illumination in this case is shown in Fig. 8 by the heavy line. The same figure shows the parallelogram of illuminations, but the vectors are marked off in directions opposite to those chosen in Fig. 17. The curve representing the distribution of illumination, resulting from each of the sources separately, is a circle. The total illumination is represented again by a circle (the heavy curve, with the exception of the part of it enclosed between n_1 and n_2; this latter part corresponds to those positions of the illuminated element where the sources are placed on both sides of it). The diameter of this circle is obtained as the vector sum of the diameters of the other two circles.

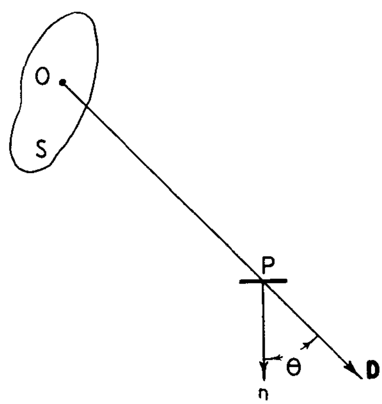

FIG. 18

As a second example, we shall consider the illumination resulting from large luminous surfaces. Let us determine the illumination at the point P (Fig. 18); that is, let us determine the illumination of an element s at the point P, as a function of orientation. The illumination E of the element s is directly proportional to cos θ, or

$$E = D \cos \theta.$$

Thus the computation of illumination is reduced to the computation of the light vector. The straight line OP may be called the light axis, and the point O, where the continuation of the vector intersects the surface S of the luminous body, may be called the *light center of the body S for the point P.* As a rule, the position of the light center varies as P moves in the light field.

If the illuminated element s is placed in such a manner that its plane intersects the luminous body S and divides it into two parts, S_1 and S_2, the projection of the vector is equal to the difference of illumination of the two sides of the element. Let E_1 be the illumination on side 1 (which is produced by S_1) and let E_2 be the illumination of side 2 (produced by S_2). Then

$$E_1 - E_2 = D \cos \theta. \tag{58}$$

This relationship does not allow the evaluation of E_1 and E_2, though it enables the illumination on one side to be determined when the illumination of the other side is known.

The cosine law may be applied not only when illumination is produced by a point source but also when the source is an extended surface, provided the illuminated surface is so oriented that all of the source or sources are found on one side of it. Let us state this important theorem in the following manner: *The illumination of an element varies as the cosine of the angle between its normal and the normal for which the illumination is a maximum. This result is valid for all those orientations of the illuminated element for which all the light sources are on one side of it.* From this statement the eleventh theorem of Lambert[20] follows as a special case.

3. Measurement of the Light Vector

Let us consider methods of measuring the light vector. Suppose we have a plane, two-sided receptor of radiant energy, combined with an instrument that allows us to determine equality of illumination on the two sides of the receptor. The sensitivity of the receptor must of course be the same on both sides. Such an arrangement may be accomplished by means of a plane set of thermocouples, with junctions on both sides, connected to a galvanometer; or we may use a visual photometer and judge the photometric balance by the disappearance of the brightness difference. Or we may use an objective photometer, such as a two-sided photoelement, connected to a galvanometer so as to read the difference between the illuminations of the two sides of the photoelement. These receptors may be classed as the *differential type*, since they respond to the difference in the illuminations of the two sides.

The apparatus for the determination of the light vector must be constructed in such a manner that the plane of its receptor may be oriented arbitrarily in space. Rotating the receptor about two axes,

[20] Ostwald's Klassiker d. exakten Wissen. No. 31, 1892, p. 53; No. 33, p. 84.

we find two different positions of the receptor plane (s_1 and s_2, Fig. 19), for which there is a photometric balance. The light vector must be contained in the plane s_1 as well as in the plane s_2, and thus it is directed along the intersection of the two planes. The direction of the vector may be characterized by two angles (for example, by the longitude and the latitude) and provision to read these angles must be made in the construction of the apparatus. Placing the plane of the receptor in a position s, which is normal to the direction of the light vector, and determining the difference of illumination of the two sides of it, we find the absolute magnitude of the vector, which we shall express in units of illumination.

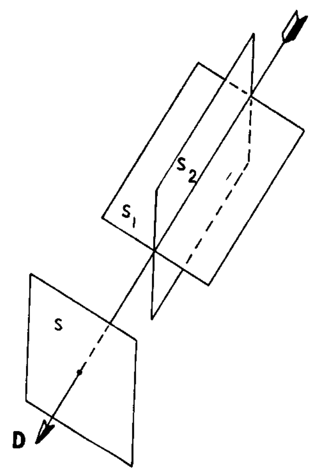

FIG. 19

One may also determine the light vector indirectly by measurement of the illumination on the two sides of the element for three different orientations of the element; i.e., by measurement of the components of the vector in the corresponding inclined system of coordinates. Most convenient is the choice of three mutually perpendicular directions. This may be interpreted as measurement of illumination on the six sides of a small cube, which is placed at the point in question. When the three components of the vector are determined in this way, one knows the magnitude and the direction of the vector.

The first apparatus for the determination of the light vector, called the *Vectorscope*, was constructed at the Optical Institute by M. M. Gurevitch. Another apparatus, of simpler and improved type, was constructed by V. A. Zelenkoff (Illuminating Engineering Laboratory of the State Optical Institute) at the request of the author.

4. Computation of the Light Vector

We consider now the methods of computation of the light vector. For a light field produced by a set of point sources, the resultant light vector $\mathbf{D}$ at a point P is defined as the sum of the light vectors $\mathbf{D}_n$ of each of the sources, or

$$\mathbf{D} = \sum \mathbf{D}_n$$

Each of these component vectors $\mathbf{D}_n$ is directed along the ray from the source to P, and its length is equal to the normal illumination produced by the source in question. Let us recall that the space illumination E_0 has been defined as

$$E_0 = \sum E_n$$

The space illumination is a *scalar sum* of normal illuminations, while the light vector is a *vector sum* of normal illuminations. If the light field is produced by luminous bodies whose angular size cannot be neglected, the light vector is computed from the known brightness-distribution solid.

Let us subdivide the entirety of rays passing through a point into elementary bundles, as was done in Chapter II. Let dE_n be the illumination produced by this elementary set of rays on a plane perpendicular to their axis. Introducing the vector $d\mathbf{D}_n$, which is equal in length to dD_n and which is in the direction of the propagation of light along the axis of the elementary cone, we obtain

$$\mathbf{D} = \int_{(4\pi)} d\mathbf{D}_n$$

where (4π) indicates that the integration is performed over all directions of space. Denoting by $d\omega$ the solid angle of the elementary cone, we obtain

$$dE_n = B\, d\omega.$$

Let us introduce a vector $d\boldsymbol{\omega}$, which is equal in magnitude to the angle $d\omega$ and which is directed along the axis of the elementary cone. Using this geometric concept, we obtain

$$d\mathbf{D}_n = B\, d\boldsymbol{\omega}$$

and

$$\mathbf{D} = \int_{(4\pi)} \mathbf{B}\, d\omega \tag{62}$$

Let us recall that the space illumination is defined by the relationship

$$E_0 = \int_{(4\pi)} B\, d\omega \tag{43}$$

Thus one may state that the space illumination represents a *scalar integral* of brightness, taken over the solid angle 4π, and that the light vector is a *vector integral* of brightness over the same solid angle. These two quantities are the fundamental functions of position in the light field.

5. The Light Vector and the Illumination-distribution Solid

Let us illustrate the relationship that exists between the light vector and the illumination-distribution solid. This will also give an answer to the question, whether the illumination-distribution solid may be of any arbitrary shape. The flux density represents the projection of the light vector. From this follows that the flux-density distribution solid (the entirety of radius vectors representing the differences of illumination on the two sides of an element which is perpendicular to them) is always a sphere; and the point P, for which this body was constructed, is found on the surface of the sphere. The diameter of the sphere gives the direction and magnitude of the light vector. The curve of illumination distribution always has the property that points corresponding to the difference of oppositely directed radius vectors are found on a circle passing through P. The distribution curve of the illumination difference is a circle. The largest radius vector of this curve (the diameter of the circle) represents the projection of the light vector on the plane which intersects the brightness-distribution solid. In Fig. 11 the dotted line shows the illumination-difference distribution. To construct the illumination-distribution solid we must determine the light vector and one-half of the illumination-distribution solid; i.e., that half which is on one side of the plane of the illuminated element. The other half of the illumination-distribution solid can be computed, since the differences of opposite radius vectors are known.

Thus *an arbitrary illumination-distribution solid is not always possible.* The difference of any two opposite radius vectors should always represent the projection of the light vector; i.e., it should follow the cosine law when we move from one direction to the next. The validity of the cosine law for the differences of any two opposite radius vectors is a *necessary* but not a *sufficient condition* that a curve represent the distribution of illumination.

To find the possible changes of illumination when the orientation of the element is varied, we shall need the illumination-distribution solid. The illumination of an element is determined by the brightness distribution for the rays passing through P. Let us consider a set of rays within a solid angle $d\omega$ and denote the normal illumination produced by them by dE_n. The resulting elementary illumination-distribution solid is a sphere with diameter dE_n. Constructing an illumination-distribution solid for each of the elementary solid angles, we obtain an infinite number of infinitesimal spheres of different diameters which have different orientations, which pass through the point P. The radius vector of the illumination-distribution solid is the sum of similarly directed radius vectors for these elementary spheres. *The limiting condition for the shape of the illumination-distribution solid is found in that each of the elementary solids is a sphere.* This must be the starting condition for the solution of the problem about the possible shape of the illumination-distribution solid. One must consider that each of the elementary illumination-distribution solids (the small spheres, each of which is produced by a single light ray) is found only on one side of the point (on one side from the plane, passing through the point, and normal to the considered ray). The radius vectors which are directed to the other side of this plane are equal to zero. *The cosine law may be applied only to a definite set of values of the angle between the ray and the normal to the illuminated element; for other values of this angle, the illumination is zero.* The resultant distribution of illumination is found by addition of curves which are similar to the one shown in Fig. 7. The curves have various amplitudes and are displaced with respect to each other in phase.

6. The Divergence of the Light Vector

The flux of the light vector,

$$F = \oint \mathbf{D} \cdot d\mathbf{s} \tag{63}$$

through a closed geometric surface is equal to the difference between the quantity of light entering the volume enclosed by the surface per unit time, and the quantity of light leaving the volume. The $d\mathbf{s}$

has the direction of the outer normal. Thus F represents the quantity of light produced (or absorbed) in unit time within the volume v. For a unit volume,

$$\frac{F}{v} = \frac{1}{v} \oint \mathbf{D} \cdot d\mathbf{s} \tag{64}$$

If the surface s is decreased so that the volume enclosed by it approaches zero, we obtain for the left side of Equation (64) the space density of light, produced per unit time, which we shall denote by f. If we deal with absorption rather than emission, the values of f will be negative. On the right side of Equation (64),

$$\lim_{v \to 0} \left[\frac{1}{v} \int_S \mathbf{D} \cdot d\mathbf{s} \right]$$

which is the *divergence of* **D**. Thus,

$$f = \operatorname{div} \mathbf{D} \tag{65}$$

The space density of light, produced (or absorbed) per unit time, is equal to the divergence of the light vector.

From this follows that for all the points in the light field at which there is no emission or absorption of light,

$$\operatorname{div} \mathbf{D} = 0. \tag{65a}$$

Thus in empty space the divergence of the light vector is equal to zero. The flux lines do not appear and do not disappear in that space, they may be grouped into flux tubes, and the field is solenoidal. Equation (65a) applies with extremely good approximation in the majority of problems in illuminating engineering, where the absorption of light in air may be neglected.

For empty space, therefore,

$$\operatorname{div} \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = 0 \tag{65b}$$

For a material medium, especially within light-emitting and light-absorbing bodies,

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = f \tag{65c}$$

where the function of position f is a measure of the generation or absorption of energy per unit volume surrounding the point in question. The quantity f may be either positive or negative.

Let us consider a finite volume v enclosed by the surface s. The flux of the light vector through the surface s is

$$\oint \mathbf{D} \cdot d\mathbf{s}$$

which, according to the law of conservation of energy, must be equal to the flux produced within the volume v, or

$$\int_v \operatorname{div} \mathbf{D} \, dv$$

The equation

$$\int_s \mathbf{D} \cdot d\mathbf{s} = \int_v \operatorname{div} \mathbf{D} \, dv \tag{66}$$

expresses Gauss's theorem: *The integral of a vector, taken over a closed surface, is equal to the integral of the divergence of this vector over the enclosed volume.*

7. Mechanical Analogies and the Geometric Interpretation

Let us consider some mechanical analogs of the light field produced by an arbitrary luminous body S (Fig. 20). The brightness B for rays that arrive at P from the body S may be different for different rays. Let us describe about the point P a sphere of unit radius. Rays from S cut out from the surface of this sphere the region ω. Let us ascribe to each element $d\omega$ of this area a weight number

$$dq = B \, d\omega$$

i.e., we shall consider the unit sphere as a material surface with the surface density equal at each place to the brightness B of the particular ray that intersects the surface of the sphere at that place. The space illumination is

$$E_0 = \int_\omega B \, d\omega = q$$

which may be considered as the weight of the shell of the sphere. The illumination of a surface s placed at the point P will be

$$E = \int_\omega B \cos\theta \, d\omega = \int h \, dq \tag{67}$$

where h is the distance along the normal from the element $d\omega$, which has the weight dq, to the surface s. Thus the illumination is measured as a sum of the moments of the forces dq taken with respect to the illuminated plane. This sum is equal to the moment of the resultant force q which is applied to the center of gravity Q of the spherical shell.

When the point Q is found at a height h_0 with respect to the surface s, then

$$E = h_0 q.$$

Denoting the distance from the center of gravity Q to the illuminated point P by $r_0 (r_0 \leqq 1)$, we obtain

$$E = r_0 E_0 \cos \theta_0 \tag{68}$$

where θ_0 is the angle between the ray passing through the center of

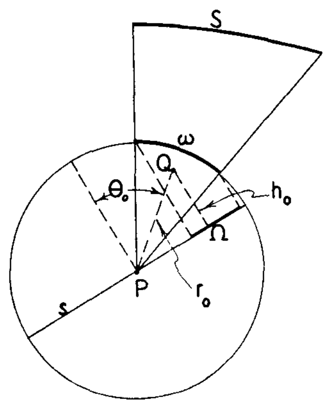

FIG. 20

gravity and the normal to the illuminated surface. Thus the light vector is equal to

$$\mathbf{D} = \mathbf{r}_0 E_0 \tag{69}$$

where $\mathbf{r}_0$ is the vector representing the distance from Q to P. The light vector $\mathbf{D}$ passes through the center of gravity, and its absolute value is

$$D = r_0 E_0 . \tag{70}$$

The quantity r_0, which is equal to the ratio of the numerical value of the light vector to the value of space illumination, characterizes the degree of diffusion. If the illumination is produced by a point source, $r_0 = 1$. Within a uniformly luminous body, $r_0 = 0$. This mechanical analogy was first introduced by Mehmke.[21] It allows the solution of a number of photometric problems by applying known formulas of mechanics. For example, knowing that the center of gravity of a hemispherical shell is found at the middle of the radius ($r_0 = \frac{1}{2}$), we can say that the illumination within a uniformly luminous hemisphere of brightness B is equal to $\frac{1}{2} \cdot 2\pi B = \pi B$. The value of this mechanical analogy is found also in that it leads directly to the concept of the light vector.

The following geometrical interpretation of the values of illumination for uniformly bright surfaces is given by Wiener.[22] The illumination E is equal to

$$E = B\Omega \tag{71}$$

where Ω represents the area of the orthogonal projection of the spherical region ω (Fig. 20). The quantity Ω is customarily called the projection of the solid angle. The above relationship is widely used in photometry, and reduces the computation of illumination to a determination of areas of plane figures. Obviously, if B is not uniform,

$$E = \int_{\Omega} B \, d\Omega \tag{72}$$

[21] R. Mehmke, Über die mathematische Bestimmung der Helligkeit in Räumen mit Tagesbeleuchtung, insbesondere Gemäldesälen mit Deckenlicht. Zs. für Math. u. Phys., **43**, 1898, p. 41.

[22] Christian Wiener, Lehrbuch der darstellenden Geometrie, Leipzig, 1884. (Vol. 1, Chapter on "Beleuchtungslehre mit ihrer Anwendung auf ebenflächige Körper.")

CHAPTER V

The Structure of the Light Field

1. Lines of flux

Generally to each point of the light field corresponds a definite value and direction of the light vector. Thus one may construct lines of the light vector or *flux lines*. A flux line is a space curve, the tangents to which coincide with the directions of the light vector.

Consider a point P to which belongs the light vector **D**, and assume that P is located on the surface of a luminous body. Then draw through P a straight line having the length PP_1 in the direction of the vector **D**. To this line segment PP_1 add another straight section P_1P_2 in the direction of the vector **D** at the point P_1. Repeating this process, we obtain a broken line. If now the size of the line segments is reduced, in the limiting case we obtain a curve, each element of which coincides in direction with the direction of the light vector at that point. This curve is called a *line of the light vector* or a *flux line*. The light vector is a single-valued point-function: Through each point of the field passes a single flux line.

For a region of the light field where div $\mathbf{D} = 0$, the flux line passes uninterrupted. At the source of the light field,

$$\operatorname{div} \mathbf{D} > 0$$

while at a sink of the light vector, where the light is absorbed,

$$\operatorname{div} \mathbf{D} < 0.$$

In empty space, the flux lines emerge from the surfaces of luminous bodies and pass to infinity or are interrupted by absorbing bodies.

Strictly speaking, a flux line is not interrupted exactly at the surface of the illuminated body, but at some depth in it. The line may be followed the deeper, the more transparent is the illuminated body. Similarly, a line is not generated on the surface of a luminous body, but is generated within its depths, where the process of light emission takes place. If the coefficient of absorption for its own radiation is very large, as for example in the case of incandescent metals, all the radiation that emerges is produced by surface layers. In the general case, however, an element of the outer surface of a luminous body should be considered as a window through which the inner radiation emerges. This becomes obvious for a light field produced by a gaseous-discharge lamp. The flux line originates not at the surface of the envelope, but in each element of the volume of the gas. The sources of the light vector are identical with the light-emitting centers. The sinks of the vector coincide with absorbing centers. Thus in a light-absorbing medium, flux lines emerging from a luminous body will be thinned out gradually as they proceed away from the source; for the

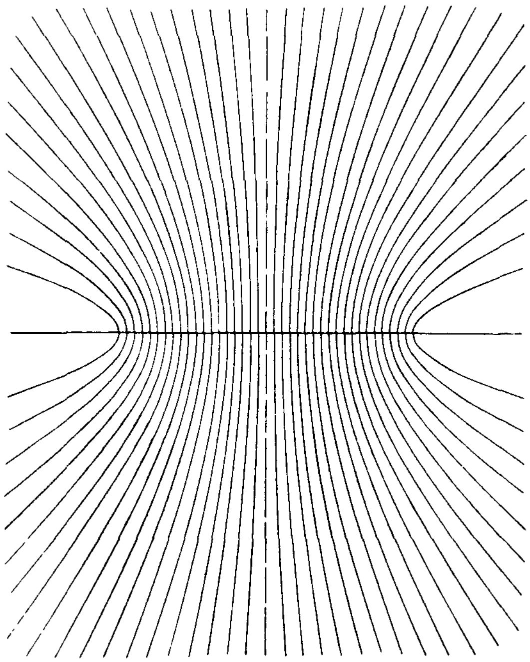

FIG. 21

light lines must terminate again and again as light is absorbed. If the medium emits as well as absorbs light, the sources and sinks of the light vector are distributed over the whole body.

A picture showing the flux lines illustrates the structure of the light field. Let us consider a number of examples. The light field produced by a point source is radial, corresponding to the field of a positive charge of electricity or of a positive magnetic pole. Fig. 21 shows the flux lines produced by an infinite, uniformly luminous strip.[23] Here the lines are confocal hyperbolas. Inasmuch as the concept of the line of luminous flux is a novel one, we may mention once more that a difference exists between a *flux line* and a *light ray*. Through each point of the field pass a multitude of light rays, each of which is a straight line. The concept of a ray does not require any explanation, but the new concept of the flux line is not so easily visualized.

In concluding this section, we note that the shape of the flux line may be identical for different light fields. This is always the case when the fields differ from each other only in the numerical values of the light vectors. Also, for example, for a uniformly luminous, infinite cylinder, as well as for a point source, the flux lines are radially diverging rays. The flux lines emerging from a circular disc, as well as those from a luminous, infinite strip (Fig. 21) are represented by confocal hyperbolas. Let us also recall that the light vector defines directly not the condition of illumination, but the transfer of energy. From the standpoint of energy transfer, the fields may be identical while at the same time *the brightness distribution solids and the corresponding illuminations may be quite different for the two cases.*

2. Tubes of Flux

We may construct an infinite number of flux lines in the light field; but it is natural to introduce the concept of *tubes of flux*, since the replacement of the infinity of lines by a finite number of tubes results in a convenient method of approximate study of the light field.

Let us consider a section of a light tube, bounded by two arbitrary surfaces s_1 and s_2 (Fig. 22), and write for this section the equation of energy balance. The sides of the tube are illuminated from the inside and from the outside and these illuminations are equal for each element of the side surface. Indeed, the sides are formed by flux lines, so the projection of the light vector on the normal to the element considered, which projection of the vector determines the difference of illumination on the two sides of the surface element, is zero for all the points of the side surface of the tube. Thus the light flowing in per unit time through the side surface of the tube is equal to the light flowing out through that surface.

Denoting by F_1 the flux through section s_1 and by F_2 the flux through the surface s_2, we have, according to the law of conservation of energy,

[23] Edward P. Hyde, Geometrical Theory of Radiating Surfaces with Discussion of Light Tubes. Bu. Stds., Bull. **3**, 1907, p. 81.

$F_2 = F_1 + F_v$. Here F_v is a measure of the intensity of the source (or the sink) within the volume v of this section of the tube. In case we consider a section of the light tube in empty space, the volume v does not contain either sinks or sources, and $F_v = 0$ and $F_2 = F_1$. Because luminous flux is the flux of a vector, the above equality follows immediately from a theorem of the vector field: *the flux of a vector through a vector tube containing no sources or sinks is constant.* This statement represents a special case of Gauss's theorem. Thus in empty space the luminous flux F through an arbitrary cross-section of the light tube is always the same, F = const. For any arbitrary cross-section of the tube, the difference between the fluxes, on this section from both of its sides, remains constant. *This constant characterizes the particular tube.*

If the light field is produced by a convex luminous body, the tube constant has a simple physical meaning. Obviously, the tube constant is equal to the flux F emitted in all directions by the surface

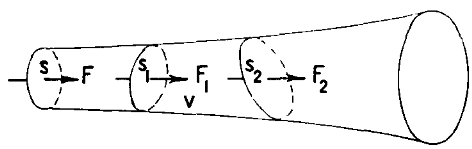

Fig. 22

element S of the luminous body, (Fig. 22). Thus in photometric computation it may be assumed that all energy emitted by a portion of a convex luminous surface flows along a light tube. This is, of course, a mathematical abstraction, as in reality the area S of the emitting surface sends light in all directions within a solid angle 2π. The light falls on section s of the tube, generally, from all possible directions and the surface s is illuminated as a rule from both sides. In a number of cases, however, it is convenient to use this abstraction and to assume that all the light emitted by the area S of the luminous surface flows along the tube, entering the volume v (Fig. 22) through the section s_1 and flowing out of it through the surface s_2.

On the basis of these formal concepts, it is easy to compute in a number of cases the luminous flux from one surface to another. For this purpose, we draw flux lines through the points of the boundary of the illuminated surface. These lines cut out from the luminous surface a definite area. The flux emitted by this area in all directions will be equal to the flux from the illuminated surface, or will be equal to the difference of the fluxes on the two sides of it if the surface is illuminated from both sides. This leads to a simple method of graphical or analytical computation of the flux from one surface to another. We shall use this method further when treating the light field produced by a luminous disc and by a luminous strip.

To illustrate the vector method, consider one of the photometric problems of Lambert: "Required the flux from a uniformly luminous sphere to a circular disc, the axis of the disc passing through the center of the sphere." (Fig. 23.) From considerations of symmetry, it is obvious that the flux lines are straight and diverge radially from the center of the sphere. Let us draw flux lines through the circular boundary of the illuminated disc. The light tube formed by these

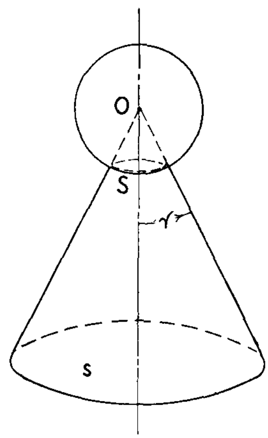

FIG. 23

lines has the shape of a circular cone and cuts an area S from the surface of the luminous sphere. According to what has been said, the flux F on the disc is numerically equal to the flux from the spherical surface S. This flux is obviously proportional to the total flux F_0 emitted by the sphere, the constant of proportionality being given by the ratio of the area S to the total area of the sphere. Thus,

$$F/F_0 = \omega/4\pi$$

or

$$F = F_0 \frac{\omega}{4\pi} = \frac{1 - \cos\gamma}{2} F_0 \tag{73}$$

The sphere may be replaced by a point source located at its center, which is permissible in all cases when the plane of the illuminated surface does not intersect the sphere. More difficult examples which we shall encounter later will convincingly illustrate the practical usefulness of the vector concepts in photometry.

Let us consider now in greater detail some properties of the tubes of the light vector. The part of space in which we study the light field may be subdivided into a number of light tubes. To each of these tubes will correspond a definite value of luminous flux. It is natural to subdivide the region in such a manner that for all the tubes this flux will be the same. It is obvious that the space may be subdivided into a given number of tubes in a large number of different ways. When the space is subdivided into an infinite number of elementary light tubes, which we shall call the *light threads,* they are sufficiently narrow so that the field within each of the thread cross-sections may be considered uniform.

Let us consider a cross-section ds of a light thread that carries the flux dF. Denoting by D_n the component of the light vector along the normal n of the element ds, we have

$$D_n = dF/ds. \tag{74}$$

The projection of the vector on the normal represents the flux density through the surface or the difference of illumination of the two sides of that surface. If all the rays fall on the section ds from one side of it, the flux density represents the illumination of the element ds.

We have seen that the picture of light tubes intersecting a surface not only yields a graphical illustration of the distribution of luminous flux, but allows us to determine numerically the *flux density* of different parts of that surface, as well as the *flux* through an arbitrary part of the surface. This computation will be the more accurate the larger is the number of light tubes considered. In the limiting case, when we change from light tubes to light threads, we can determine the flux density as a function of position on a surface, and can determine the exact value of the flux through any contour on this surface.

Let us replace each of the light tubes by a single line within the tube, and assume that the entire flux of the tube is along this line. A surface placed in the light field will be pierced at a number of points by the *axial flux lines.* The number of such points in a portion of the surface is proportional to the luminous flux through this area. The number of points per unit area determines the flux density. If the surface is illuminated from one side only, the distribution of illumination on the surface is given by the flux diagram and the determination of the flux is reduced to counting the number of points within the given contour.

In the following paragraphs we shall treat several examples of the vector method of computation and the graphical representation of the light field. These examples will illustrate the soundness of introducing Faraday's concept of a force-field into photometry. Before we consider these examples let us say a few words about the possibilities of graphical representation of the vector field.

As a rule, the flux lines are represented by space curves that cannot be shown graphically because they are not in the plane of the drawing. In a number of cases, however, the flux lines are represented by plane curves which may be subdivided into families, each of which is contained in a single plane. This situation occurs in

a) A two-dimensional field; for example, the field from an infinitely long, uniformly luminous strip. The flux lines are contained in planes that are normal to this strip. The intersections of the field by these planes are identical.

b) A field with axial symmetry; for example, a light field produced by a circular disc of uniform luminosity. The flux lines are contained in meridional planes and the diagram of flux lines is the same for all these planes. In the same class belongs a field produced by a luminous straight line.

c) A field resulting from a point source, where the flux lines coincide with the radius vectors.

We have indicated three possible classes of light fields which permit simple graphical representation. In some special cases, the light field may simultaneously belong to two classes. For example, the light field produced by a uniformly luminous circular cylinder of infinite length is a plane field, and on the other hand it is a field possessing axial symmetry. The light field at a sufficiently large distance from a symmetrical source possesses axial symmetry and may also be regarded as a field of a point source.

3. The Light Field of a Strip

Let us begin with the two-dimensional case of an infinitely long, perfectly diffusing, uniformly luminous strip (Fig. 24). The strip extends to infinity on both sides of the plane of the drawing. We find in the plane of the drawing a family of flux lines whose equation is easily found. Let us consider a point P_1 of the light field. The brightness on all the rays arriving at the point P_1 from the strip is equal. These rays form a symmetrical pencil, and the line of symmetry bisects the angle FP_1F', which we shall denote by α. It follows that the light vector **D** is directed along the bisectrix of α. Thus the line of the vector **D** (the flux line) must possess the property that its tangent at any point P_1 bisects the angle FP_1F'. This condition, as it is well known from analytic geometry, is satisfied by an hyperbola, the foci of which are at F and F′. Thus the flux lines in the plane of the drawing form a family of confocal hyperbolas. All the planes that are parallel to the plane of the drawing, at a finite distance from it, yield a similar picture. Thus the field may be subdivided into light tubes by a family of planes and by a family of confocal hyperbolic cylinders. Each pair of adjacent cylinders, intersecting a pair of adjacent parallel planes produces a light tube. In order that the luminous flux through a cross section of a tube be the same for all the tubes it is necessary and sufficient that the tubes originate from equal areas of the luminous surface. For this purpose, we subdivide FF′ into equal sections and pass the hyperbolas through these points. The diagrams of Figs. 24 and 21 allow us to judge not only the direction of the vector but also the flux density D_n at any point P of a surface S. To determine D_n we draw the intersection of this surface with the plane of the drawing and measure along the line of intersection the distance l between the two adjacent flux lines forming the tube containing the point P (Fig. 24). Denoting by l' the smallest distance between two neighboring lines at the place where they emerge from the luminous strip, and by L the luminosity of the strip,

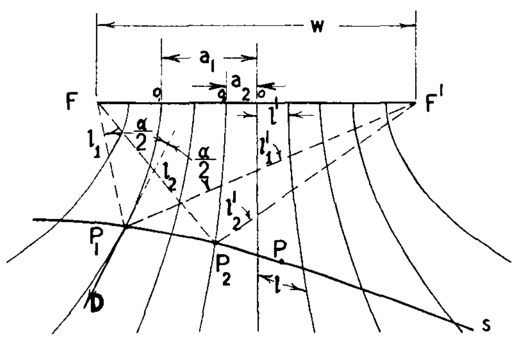

FIG. 24

$$D_n/L = l'/l.$$

If light falls on only one side of the illuminated surface, then D_n represents the illumination of that surface, while D_n/L represents the coefficient of illumination. This coefficient gives the ratio of the flux density on the illuminated surface to the flux density at the source. From Fig. 24 we can judge the magnitude of the coefficient of illumination at various points of any surface in the field.

The number of light tubes arriving at a given area of the surface gives the luminous flux on this area. Let us derive, on the basis of vector concepts, a formula for the value of the luminous flux F on the region P_1P_2 of the surfaces (Fig. 24). The total flux emitted by the strip from one side is denoted by F_0. The flux lines through the points P_1 and P_2 emerge from the points O_1 and O_2 of the luminous strip and have the distances a_1 and a_2 from the center line of the strip. Thus,

$$F/F_0 = \frac{a_1 - a_2}{w},$$

where w is the width of the luminous strip.

This formula reduces the computation of luminous flux to a construction of two hyperbolas and a determination of the shortest distance between them. Even this construction may be avoided, for

$$2a_1 = l_1' - l_1; \qquad 2a_2 = l_2' - l_2 .$$

Thus,

$$F/F_0 = \frac{1}{2w}\left[(l_1' - l_1) - (l_2' - l_2)\right] \tag{75}$$

This formula yields the simplest method of computing the luminous flux from one infinite strip to another.

The fact that the flux lines are hyperbolas allows the easy determination of the magnitude of the light vector. To this end, we consider two infinitely close confocal hyperbolas, the shortest distance between which is da (Fig. 25). It is easily shown that the length $d\lambda$ of the element of the orthogonal trajectory enclosed between the two hyperbolas is

$$d\lambda = \frac{d\alpha}{\sin \alpha/2} \tag{76}$$

It follows that the magnitude of the light vector $\mathbf{D}$ is

$$D = L\frac{da}{d\lambda} = L \sin\frac{\alpha}{2} \tag{77}$$

Thus the light vector bisects the angle subtended by the luminous strip, and the magnitude of the vector is proportional to sin $\alpha/2$. This may be shown also by starting from the geometrical interpretation given in Section 7 of Chapter IV. At large distances from the luminous strip, the field is a radial one with the asymptotes of the hyperbolas as flux lines. In these regions of the light field, the illumination is inversely proportional to the first power of the distance. The magnitude of the light vector is

$$D = \frac{Bw \cos \theta}{l} \tag{78}$$

where B is the brightness of the strip, l the distance from the strip,

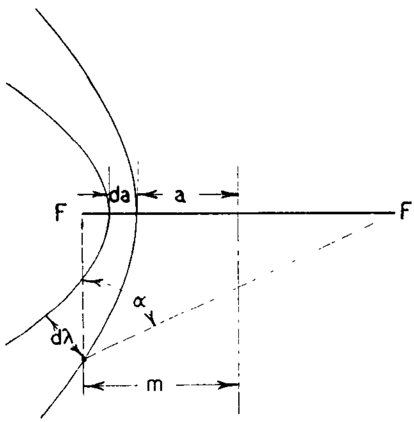

FIG. 25

and w cos θ is the projection of FF′ on a straight line normal to the direction considered, which in turn forms the angle θ with the axis.

The case of a uniformly luminous strip is of practical value in the computation of natural lighting.[24] As an example, let us consider the light field, produced by the diffused light of the sky, between the walls of equally tall houses bounding a street. We assume that the street is infinitely long, and that the sky is perfectly diffusing. Then the field between the walls of the houses will be the same as that produced by an infinitely long source located above the street and having the same luminosity as the sky. The flux lines give not only a qualitative but also a quantitative picture of the distribution of daylight on the walls

[24] A. Gershun, Computation of Natural Lighting. Trans. Opt. Inst., **5**, No. 44, 1929. Leningrad.

of the houses and on the surface of the ground.* A more detailed treatment of vector methods of computing natural lighting is beyond the scope of this book but is considered in a paper by the author.

4. The Light Field of a Luminous Disc

As an example of a field possessing axial symmetry, let us consider the illumination produced by a uniformly luminous, perfectly diffusing circular disc. The solid angle subtended by the disc at a point P has the shape of an elliptic cone. In Fig. 26, FF′ is the diameter of the disc and FPF′ is the axial cross-section of the cone. The axis of the cone, along which the light vector is directed, is the bisectrix ZP of the angle α; the point M in which ZP intersects the disc represents its "light-center" for the point P.

Thus the flux lines are again represented by confocal hyperbolas with the foci at FF′. Rotating these hyperbolas about the axis of the disc we obtain hyperboloids of revolution that subdivide the whole space into light tubes. In order that these tubes shall carry equal fluxes, the annular regions of the disc from which they originate must be equal in area. For this we subdivide the diameter of the disc FF′ into sections such that the squares of the distances from the center form an arithmetic progression.

In this example, the flux diagram does not characterize the variation of the light vector as simply as in the two-dimensional example. Indeed, for the two-dimensional field the magnitude of the light vector was fully determined by the length $d\lambda$ of a section of the orthogonal trajectory enclosed between two neighboring flux lines (Fig. 25). For the disc, the value of the light vector varies inversely as the area of the conical surface formed by $d\lambda$ when rotated about the axis of the disc; or D is inversely proportional to $m\,d\lambda$, where m is the distance from the point to the axis of the disc. The distance between neighboring lines characterizes directly the distribution of illumination only when m is constant, i.e. for all cylindrical surfaces having a common axis with the disc.

To determine the magnitude and direction of the light vector, let us consider a sphere determined by the boundary of the luminous disc and the point P (Fig. 26). A sphere possesses the property that for all its points the angle α is the same. Obviously, the inner surface of the sphere is evenly illuminated, since the light field of the disc S is identical

* [But not inside the rooms.] Trans.

with the light field produced by the cap σ. The area of the spherical cap is

$$\sigma = 2\pi R^2 (1 - \cos \alpha) \tag{79}$$

where R is the radius of the sphere. Attributing to the cap a luminosity L, which is equal to the luminosity of the disc, we find that the flux emitted by σ is $L\sigma$. This flux is uniformly distributed over the whole surface of the sphere* and produces an equal illumination E at all points, or

$$E = \frac{L\sigma}{4\pi R^2} = L \frac{1 - \cos \alpha}{2} = L \sin^2 \frac{\alpha}{2} \tag{80}$$

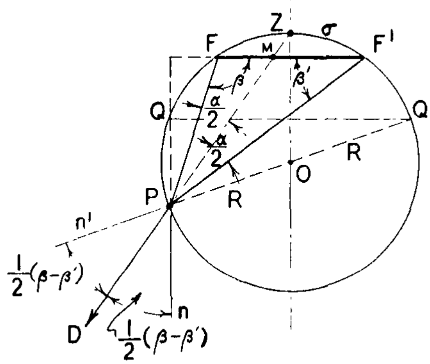

Fig. 26

It is of interest to note that the sphere described about the disc represents not only a uniformly illuminated surface but also the geometrical locus of all points in which the illumination of an element parallel to the plane of the disc is constant and equal to E. Indeed, the light vector at P is directed along the bisectrix of the angle α. This bisectrix forms equal angles with the normal to the illuminated element when it is an element of the sphere (normal n'), as well as when it is parallel to the plane of the disc (normal n). Thus the illumination E on a plane that is parallel to the disc is

$$E = E' = L \sin^2 \frac{\alpha}{2}. \tag{81}$$

* [Note that in the entire discussion the assumption is tacitly made that the sphere reflects no light. In an *actual* sphere, the illumination may be much greater than that given by Eq. (80) because of interreflections.] *Trans.*

Let us now introduce the angles β and β' between the outer rays FP and F′P and the plane of the disc. The angle between **D** and the normal n is

$$\frac{\beta - \beta'}{2}$$

Thus the absolute value of the light vector is

$$D = \frac{L \sin^2 \alpha/2}{\cos \frac{\beta - \beta'}{2}} \tag{82}$$

This formula, together with the condition that the light vector is directed along the bisectrix of the angle FPF′, completely defines the illumination produced by the disc. The illumination is given by the projection of **D** on the normal to the illuminated surface. Let us introduce an angle defined by the relationship

$$\sin \frac{\alpha'}{2} = \frac{\sin \alpha/2}{\cos \frac{\beta - \beta'}{2}} \tag{83}$$

Then the absolute value of the light vector is

$$D = L \sin \frac{\alpha}{2} \sin \frac{\alpha'}{2} \tag{84}$$

The angle α' is the angle subtended by the chord which passes through the light center M and which is normal to the diameter FF′. The conical surface enclosing the rays passing from the disc to the point P is an elliptical one, and it cuts a spherical ellipse from a unit sphere with center at P. The minor and major semi-axes, expressed in angular units, are $\alpha/2$ and $\alpha'/2$. The orthogonal projection of this spherical ellipse on a surface that is normal to the axis of the cone represents a plane ellipse with semi-axes of $\sin \alpha/2$ and $\sin \alpha'/2$. The area of the ellipse is

$$\Omega = \pi \sin \frac{\alpha}{2} \sin \frac{\alpha'}{2}. \tag{85}$$

Multiplying the expression for Ω by the brightness B, we obtain, according to Section 7 of Chapter IV, the absolute value of the light vector. For the points on the axis of the disc, $\alpha = \alpha'$, and

$$D = L \sin^2 \frac{\alpha}{2} \tag{86}$$

The illumination varies inversely as the square of the distance from the rim of the disc. This problem has been treated in detail by Steinmetz[25] and by Yamauti.[26]

At points that are close to the disc, the illumination does not depend on the distance from the disc. In this region, the flux lines are normal to the plane of the disc. The cross-section of the light tubes (and therefore, also the illumination) does not vary as we move from one point to another and the light field is homogeneous. At great distance from the disc, the light field is a radial one.

The circular disc has always been a favorite example in photometric computations and is considered in the books of Lambert and of Beer, as well as in a number of other publications. The formulas expressing the illumination are usually derived as a result of tedious integrations over the surface and are usually presented in a clumsy and complicated way. It is often forgotten that the law of similitude may be exactly applied in photometry. *The scale is arbitrary because the linear dimensions appear in all the formulas in the form of their quotients; and the photometric relationships become elegant, and thus convenient to remember and to use in computations, only when the whole geometry is characterized by angles.*

The problem of the illumination produced by a uniformly luminous circular disc also gives the inverse problem of finding the flux from an element of perfectly diffusing surface to a circular disc which is placed arbitrarily with respect to that element. This follows from the principle of reciprocity which states that if two bodies are equally bright, then the flux from the first body to the second is equal to the flux from the second body to the first. The correctness of this statement follows directly from the formula for the luminous flux from an element of one surface to an element of another surface (Section 1, Chapter 2). This statement also follows from the second law of thermodynamics.

Lambert, having proved the principle of reciprocity (Theorem 16 of "Photometry"), considers its use in photometric computations. For example, using the formula for the illumination of a plane from a disc which is parallel to it, we find according to the principle of reciprocity

[25] Chas. Steinmetz, Radiation, Light and Illumination. McGraw-Hill Book Co., New York, 1918.

[26] Ziro Yamauti, Geometrical Calculation of Illumination due to Light from Luminous Sources of Simple Forms. Researches of the Electrotechnical Laboratory, Ministry of Communications, Tokyo, No. 148, 1924.

that the perfectly diffusing element dS of brightness B sends the flux dF:

$$dF = B\pi \sin^2 \frac{\alpha}{2}\, dS \tag{87}$$

to a disc parallel to dS. Here α is the angle subtended by the disc from the element dS. This formula, which usually is proved only for the special case of the luminous element on the axis of the illuminated disc, is of considerable importance in optical apparatus. It yields an expression for the luminous flux entering an optical apparatus from an element normal to the axis of the apparatus.

Let us now compute Lambert's problem of the flux from a luminous circular disc FF′ to the coaxial disc PP′ of Fig. 27. (Theorems 17 and 19 of "Photometry"). This flux must be equal to the flux emitted in all directions by the inner part of the disc of radius a, where a is the

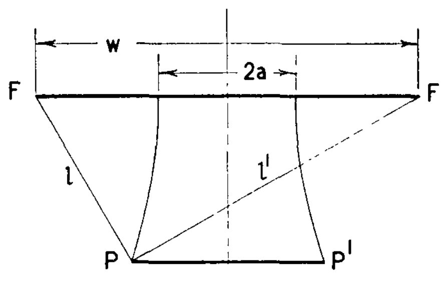

FIG. 27

real semi-axis of hyperbolas that pass through the rim of the illuminated disc. Denoting by F_0 the total luminous flux from the disc FF′, and by F the part on the illuminated disc PP′, we have

$$F/F_0 = \left(\frac{2a}{w}\right)^2 = \left(\frac{l' - l}{w}\right)^2 \tag{88}$$

In conclusion, we note that the light vector within a uniformly luminous hollow space with a circular opening may be reduced to the case of the circular disc. If the space were completely closed, we would have the case of "neutralization", and the light vector would be a null vector at all points within the enclosure. When an opening is present, the light vector is equal in magnitude to the vector which the opening would produce, were it a source with the same brightness as the walls of the enclosure. The direction of this vector is opposite to the one which would be produced by the opening.

5. The Light Field Produced by Point Sources

The flux lines of a point source are radially diverging straight lines. Subdividing the space into a set of cones, each with its apex at the source and each carrying equal luminous flux, we obtain tubes of equal flux. This construction can obviously be carried out in a large number of ways.

If the source has a symmetrical distribution about an axis, the space may be subdivided into tubes by conical surfaces, the axes of which correspond to the axis of symmetry. When n light tubes are constructed, the angle α_i between the i^{th} conical surface and the axis is determined by the relationship:

$$2\pi \int_0^{\alpha_i} I_\alpha \sin \alpha \, d\alpha = \frac{i}{n} F \tag{89}$$

where I_α is the intensity of the source at an angle α from the axis, and

$$F = 2\pi \int_0^{\pi} I_\alpha \sin \alpha \, d\alpha \tag{90}$$

is the total flux of the source. The determination of the angles α_i represents a usual problem in illuminating engineering computations. The rays, which when rotated form the side walls of the light tubes, give a map of the light field. Just as easy is the reverse path from the map of the field to the candlepower-distribution curve. Dividing the flux through a single tube by the solid angle of the tube, we obtain the average intensity in the directions contained within that tube.

The construction of the map of the light field simplifies illuminating-engineering computations in a large number of cases. An illustration is the vector method for computing the candlepower distribution from a symmetrical specular reflector, as suggested by the author and elaborated in collaboration with N. G. Boldyreff.[27] Almost as easy is the reverse problem of a design of a reflector to produce a required transformation of the light field of the lamp.

Let us now treat a somewhat different method of graphical representation of the light field from a point source. This method was discovered by the author in collaboration with N. G. Boldyreff and may be also used for sources with asymmetric distributions.

The construction of the flux lines in a plane S containing the source I (Fig. 28) is based on the following principle. Let us consider the

[27] A. A. Gershun and N. G. Boldyreff, The Vector Method of Computation of Symmetrical Reflectors Used in Illumination. "Svetotechnika", 1936, No. 1.

plane S′, which is infinitely close to S and which intersects the latter along the straight line C_o. To characterize the distribution of the flux emitted by the source within the angle formed by the planes S and S′, we subdivide this angle into tubes carrying equal fluxes by planes Q_i which are normal to S. The following relationship must be fulfilled:

$$2\pi \int_{\beta_i}^{\beta_{i+1}} I_\beta \sin\beta \, d\beta = \text{const.} \tag{91}$$

where I_β is the intensity of the source in the direction contained in the plane S and making the angle β with the straight line C_o. The diagram of rays obtained by the intersection of the plane S by the planes Q_i possesses the following important properties. Let us consider a plane s which intersects the plane S along a straight line C which is parallel

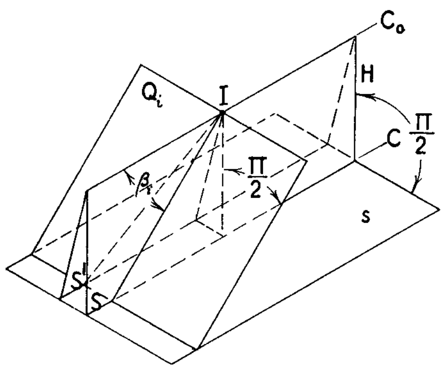

Fig. 28

to C_o and passes at a distance H from the latter. The light rays of the vector map will cut the straight line C into a number of sections, whose length will directly characterize the distribution of illumination in the plane s along the straight line C. Indeed, the illumination of the plane s at points on C is inversely proportional to H and inversely proportional to the distances intercepted on C by neighboring rays of the vector diagram.

6. Addition of Fields

When there are a number of luminous bodies in the field, each of the quantities characterizing the light field represents a sum of values corresponding to each of the luminous bodies separately. *The light field produced by a set of luminous bodies is the sum of the light fields produced by these bodies individually.*

Thus we arrive at the question of the addition of fields. This question is of great practical importance, inasmuch as in practice we always encounter lighting from a set of sources, and all objects in the field reflect to some extent. In some cases, the source is complex and must be divided into a number of simpler elements. When adding the fields of several sources, one must take into account the production of shadows caused by the screening action of one luminous body with respect to a light field produced by another source. Sometimes the effect of the shadow may be accounted for if we introduce not only the addition of fields but also their subtraction.

In a number of cases (a two-dimensional field, a field with axial symmetry, a field of a set of point sources on a straight line) the fields that are added, as well as the resultant field, may be represented graphically. Cross-sections are considered such that the vector has zero component normal to the plane of the diagram, so the flux lines are contained in these planes and the fields may be added graphically. A method of graphical addition of fields has been given by Maxwell.[28]

The problem of addition of fields of two point charges (in photometry, point sources with uniform intensity in all directions) is treated in a number of texts on electricity and magnetism. Denote the vectors to be added by $\mathbf{D}_1$ and $\mathbf{D}_2$, and the vector of the resultant field by $\mathbf{D}$. Let the vector lines of both fields be in a single plane; and, in addition, let the ratio of the absolute magnitude of the vector to the density of the lines be the same point function for both of the fields. Then the condition is fulfilled in the entire plane that the ratio of the absolute value of the vector $\mathbf{D}_1$ to the absolute value of the vector $\mathbf{D}_2$ is equal to the ratio of the density of vector lines of the first field to the density of vector lines of the second field.

The vector lines of both fields divide the plane into a number of regions (Fig. 29) and define two new sets of curves, which are directed along the diagonals. Maxwell showed that the fields of the diagonal curves completely determine the field of the resultant vector. Depending on the choice of one of the two systems of diagonals, we obtain the field of the vector $(\mathbf{D}_1 + \mathbf{D}_2)$ or of the vector $(\mathbf{D}_1 - \mathbf{D}_2)$. The correctness of Maxwell's theorem, upon which the graphical addition of vector fields is based, may be easily shown. Let us consider a very small region in the plane, such that within that region the lines of each

[28] James Clerk Maxwell. A Treatise on Electricity and Magnetism (Vol. I, Part I, Electrostatics. Ch. VII.) See also the footnote by A. Cornu to this chapter in the French edition.

of the two vectors $\mathbf{D}_1$ and $\mathbf{D}_2$ may be considered as equidistant parallel lines. These subdivide the regions of the plane into a number of parallelograms. We denote by l_1 and l_2 the distances between neighboring lines of the component fields, and by l the distance between the diagonal lines of the resultant field. Consider the parallelogram, one corner of which is at P, and denote by $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}$ the vectors corresponding to the sides of the parallelogram and its diagonal. A comparison of the three expressions for the area of the parallelogram gives

$$e_1 l_1 = e_2 l_2 = el \tag{92}$$

$$e_1 : e_2 : e_3 = \frac{1}{l_1} : \frac{1}{l_2} : \frac{1}{l} \tag{93}$$

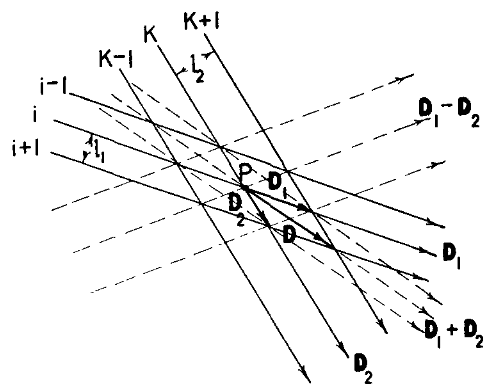

FIG. 29

According to our assumption,

$$D_1 : D_2 = \frac{1}{l_1} : \frac{1}{l_2} \tag{94}$$

Therefore, $\mathbf{e}_1$ and $\mathbf{e}_2$ represent on some scale the vectors $\mathbf{D}_1$ and $\mathbf{D}_2$. The vector $\mathbf{e} = \mathbf{e}_1 + \mathbf{e}_2$ represents on the same scale the vector $\mathbf{D} = \mathbf{D}_1 + \mathbf{D}_2$. The parallelogram formed by sections of vector lines meeting at the point P, represents the parallelogram of addition of vectors.

Vector fields produced by several sources of light may have a common axis of symmetry, may be two-dimensional fields, or may be fields of point-sources along a straight line. On one drawing we superpose the maps of two of the fields to be added, making use of the requirement that the number of flux lines of each field should correspond to the strength of its source. Having numbered the lines, we connect by curves the points of intersections of the lines of both fields in the following manner. Let i be the order number of a line of the first field and k a corresponding number in the second field (Fig. 29). Each point of intersection is characterized by two indices and may be denoted by the symbol (i, k). The transfer from the point (i, k) to the point $(i + 1, k + 1)$ is in the direction of the resultant vector. Connecting by a curve the points: $\cdots (i - 1, k - 1); (i, k); (i + 1, k + 1); \cdots$ we obtain a line of the resultant field. Adding a field obtained in this manner to the field of a third source, we obtain the field of the sum of three vectors, and so on.

In physics, the graphical representation and addition of fields is used commonly only to obtain illustrative pictures. The author believes, however, that in illuminating engineering the methods of graphical representation of the light field may also be applied in quantitative design work. This is the reason why the method of graphical addition, which has never been applied in photometry, is treated here in such detail.

CHAPTER VI

Photometric Computations

1. General Theorems

In the present chapter we shall consider photometric computations for large luminous surfaces. The assumption will be made that the surfaces have uniform luminosity and that they satisfy Lambert's law. The light field produced by such surfaces is subject to a number of simple relationships which serve as a basis for photometric computations. We shall treat the theory in general terms and shall avoid tedious computations which often have the character of exercises in integral calculus. Computation of the illumination and the flux from large luminous surfaces usually requires multiple integrations in each special case, but these integrations may be performed once for all in deriving the general formulas. In order to generalize and simplify photometric computations it is permissible and sound to create a special mathematical machinery by introducing, when necessary, new mathematical concepts. The history of the development of physics illustrates on a large scale the correctness of such a procedure.

The theory of the field produced by a uniformly luminous surface is primarily a problem of geometry and we shall approach it as such. Let us consider a point P of the light field produced by a uniformly luminous surface S bounded by the contour C, the brightness of this surface being denoted by B. The surface S subtends the solid angle ω at the point P. The angle ω is composed of elements $d\omega$ subtended by the elements dS of the luminous surface. To characterize the light field, we shall also introduce a vector solid angle $d\boldsymbol{\omega}$. The space illumination is

$$E_0 = \int_\omega B\, d\omega \tag{95}$$

and the light vector is

$$\mathbf{D} = \int_\omega B\, d\boldsymbol{\omega} \tag{96}$$

where the symbol ω below the integral sign indicates that the integration is performed within the solid angle ω.

For a uniformly luminous surface, B is the same for all the rays at the point P within the solid angle ω. Taking B outside the integral sign, we obtain

$$E_0 = B \int d\omega \tag{95a}$$

and

$$\mathbf{D} = B \int_\omega d\boldsymbol{\omega} \tag{96a}$$

The integral in Eq. (95a) represents the solid angle ω. Thus the computation of space illumination in this case is reduced to the geometrical problem of computing the solid angle. The expression obtained for the space illumination,

$$E_o = B\omega \tag{97}$$

is analogous to the expression for the potential produced by a magnetic sheet.

Analogously, the computation of the light vector may be reduced to a purely geometric problem. For this we shall introduce a new mathematical concept of the vector $\boldsymbol{\omega}'$ obtained by the summation of vectors $d\boldsymbol{\omega}$:

$$\boldsymbol{\omega}' = \int_\omega d\boldsymbol{\omega} \tag{98}$$

the integration being performed over all directions of space within the angle ω.+ To each conical surface corresponds a definite solid angle ω

+ [Note that $|\boldsymbol{\omega}'|$ does not give the value of the solid angle ω, since the elementary vectors $d\boldsymbol{\omega}$ are not collinear.] Trans.

as well as a definite vector $\boldsymbol{\omega}'$. The reverse statement is obviously not true, for to one value of ω or $\boldsymbol{\omega}'$ may correspond different conical surfaces.

The light vector is defined as

$$\mathbf{D} = B\,\boldsymbol{\omega}' \tag{99}$$

The absolute value of the vector $\boldsymbol{\omega}'$ is denoted by ω'. It obviously depends not only on the magnitude ω but also on the shape of the conical surface enclosing the solid angle; and $\omega' \leqq \omega$, the equality sign being for the case of an infinitely small solid angle. When the contour C of the luminous surfaces is given, then to each point of the field corresponds a definite cone. This cone has its apex at P and has as its base the contour C. To each conical surface, and thus to each point of the field, correspond definite and unique values of ω and $\boldsymbol{\omega}'$. The study of the light field is completely reduced to the study of two geometrical fields: the field of the *scalar solid angle* and the field of the *vector solid angle*. The space density of radiant energy is determined by the scalar solid angle at a given point of the field. The flux lines are the lines of the solid-angle vector, and a computation of the flux through a surface is reduced to a computation of a flux of the solid-angle vector.

The following interpretation of ω' may be given. Let us describe a sphere of a unit radius about the apex of the conical surface considered. The conical surface will cut from this sphere a region whose area is equal to ω. To each elementary solid angle $d\omega$ will correspond a definite element of the spherical surface. The vector corresponding to this element is $d\boldsymbol{\omega}'$, which is directed along the inner surface normal. Thus the vector $\boldsymbol{\omega}'$ may be considered as the vector sum of the elements of the surface cut from a unit sphere by a given cone, or, using the terminology of vector analysis, as the vector of this surface. It is easy to show that the vector $\boldsymbol{\omega}'$ passes through the center of gravity of the surface.

The projection of the solid-angle vector on a given direction is equal to the area of the projection, on a plane which is normal to the given direction, of the region cut from the unit sphere by the cone. Thus if the direction of the vector is known (for example, from considerations of symmetry) then its magnitude may easily be determined as the area of the projection of a spherical region on a plane that is normal to the direction of the vector.

The solid-angle vector represents a fundamental quantity in the computation of illumination produced by large luminous surfaces of uniform brightness. Projecting this vector upon the normal to the illuminated element, and multiplying the length of this projection by the brightness, we obtain the flux density through this element.

2. Transformation of the Expression for the Solid Angle

If the brightness of a luminous surface S is equal to unity, the space illumination is measured by the solid angle subtended by S. The solid angle is

$$\omega = \int_S \frac{\cos (n, \mathbf{r})\, dS}{r^2} \tag{100}$$

where **r** is the radius vector from the apex of the solid angle to the element dS; and n is the direction of the normal to this element. Let us transform this surface integral into a contour integral. We describe a unit sphere about P (Fig. 30). The intersection of the cone with the sphere defines a spherical figure whose area is equal to the solid angle ω. The opposite half of the conical surface will cut from the sphere a symmetrical figure, the area of which is also equal to ω. Along an element of the contour, the tangents to the contour undergo a definite change in direction. Let us draw tangent planes to the conical surface, these planes being separated by an angle $d\gamma$, which characterizes the elementary change of direction of the tangent to the contour as we move along the contour. The planes intersect the sphere in two great circles

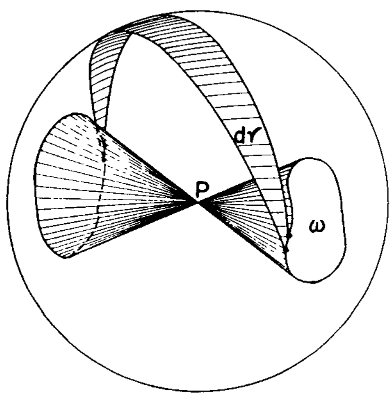

FIG. 30

and cut from it a lune, the area of which is equal to $2d\gamma$. Subdividing the contour into separate elements, and constructing at each of these points a plane that is tangent to the conical surface, we obtain an infinite number of elementary lunes formed by pairs of neighboring planes. These lunes cover the entire sphere, with the exception of the two areas cut from the sphere by the conical surface. In this manner

$$\oint 2\, d\gamma + 2\omega = 4\pi \tag{101}$$

or

$$\omega = 2\pi - \oint d\gamma \tag{102}$$

The integration is performed over the entire conical surface. For a multi-sided solid angle, the integral becomes the sum of outer angles formed by the planes of this solid angle.

The author realizes that the foregoing simple method is not mathematically rigorous. The transformation could also be carried out in the classical manner but this procedure would be more laborious. In the manner we have treated the subject, the pyramid may be considered as a special case of a cone. The pyramid will cut from the sphere a spherical polygon, the number of sides of which we shall denote by m. To each side of this polygon corresponds an index i which varies from 1 to m. Let us denote by $\gamma_{i,i+1}$ the outer angle between the sides i and $i + 1$ (Fig. 31). With this notation, $\gamma_{m,m+1}$ corresponds to the

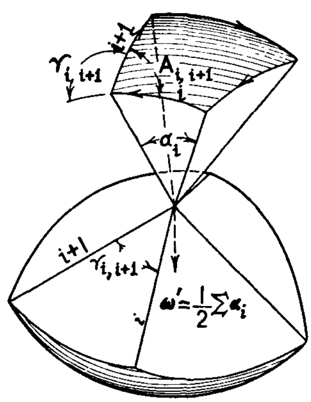

FIG. 31

angle $\gamma_{m,1}$ between the "last" and the "first" side of the polygon. The expression for the solid angle is

$$\omega = 2\pi - \sum_{i=1}^{m} \gamma_{i,i+1} \tag{103}$$

The amount to be added to the solid angle to obtain 2π is equal to the sum of the outer angles.

Denoting the angle between the inner sides of the pyramid by

$$A_{i,i+1} = \pi - \gamma_{i,i+1}$$

we may express the solid angle in the form

$$2\pi - \omega = \sum_{i=1}^{m} (\pi - A_{i,i+1}).$$

The amount to be added to the solid angle to obtain a hemisphere is equal to the sum of the supplements of the inner angles between the pyramid sides. For a three-sided angle ($m = 3$), we obtain as a special case the well known formula of spherical trigonometry,

$$\omega = \sum_{i=1}^{3} A_{i,i+1} - \pi \tag{104}$$

which states that the area of a spherical triangle is proportional to the spherical excess.

The following geometrical interpretation may be given to the contour integral expressing the solid angle. Consider a point within the conical surface and draw normals from it to the elements of the conical

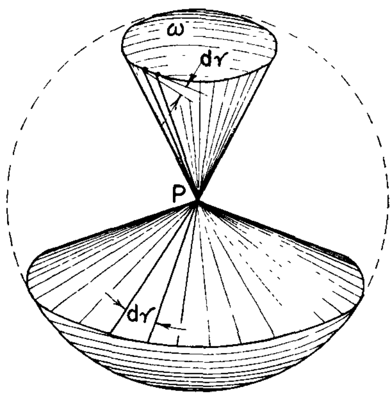

FIG. 32

surface. Drawing the outer normals as shown in Fig. 32, we obtain a new conical surface which may be called *polar* (generalizing the concept of polar triangles in spherical trigonometry) with respect to the original conical surface. To each tangential plane of the original cone corresponds a definite surface ray of the polar surface. The angle $d\gamma$ between two tangent planes is equal to the angle between the two normals to these planes, and these normals will be the two corresponding surface rays of the polar cone. Thus the contour integral of $d\gamma$ will be equal to the angle at the apex of the conical surface when the latter is cut along one of its rays and developed on a plane. The solid angle will be equal to the amount added to the apex angle to obtain 2π.

In the special case of a many-sided solid angle, the polar surface will also be represented by a many-sided angle. To each side of the original pyramid will correspond an edge of the polar pyramid (Fig. 31). The outer angle between two planes forming the original pyramid will be equal to the plane angle between two adjacent edges of the polar pyramid. As known from geometry, the sum of plane angles composing a many-sided angle is less than 2π and is, as we have seen, less by the solid angle of the polar pyramid.

As an example, let us determine the solid angle subtended by the rectangle s when P is on a normal to the plane of s and the normal is drawn from one of the corners of s (Fig. 33). We number the sides of the rectangle and the corresponding sides of the pyramid, as shown in

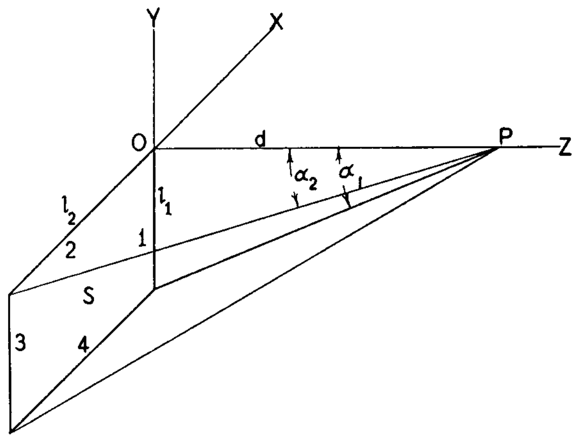

FIG. 33

the drawing. Three of the four angles, namely $\gamma_{1,2}$; $\gamma_{2,3}$; $\gamma_{4,1}$ are equal to $\pi/2$. Therefore,

$$\omega = \frac{\pi}{2} - \gamma_{3,4}$$

where ω is expressed in steradians and $\gamma_{3,4}$ is in radians. Since

$$\cos \gamma_{3,4} = \sin \alpha_1 \sin \alpha_2$$

we have

$$\omega = \sin^{-1} (\sin \alpha_1 \sin \alpha_2),$$

where α_1 and α_2 denote the plane angles subtended by the first and second sides of the rectangle. If α_1 and α_2 are small, $\omega \cong \alpha_1\alpha_2$.

Starting from linear dimensions, it is convenient to use the following expression for the solid angle:

$$\omega = \tan^{-1} \frac{\frac{l_1}{d} \cdot \frac{l_2}{d}}{\sqrt{1 + \left(\frac{l_1}{d}\right)^2 + \left(\frac{l_2}{d}\right)^2}} \tag{105}$$

In conclusion, let us note that the formulas may also be useful in the computation of the flux through a given contour and originating from a uniform point source.

3. Transformation of the Expression for the Vector of the Solid Angle

The solid-angle vector is represented by a *vector integral* over the part of a sphere cut out by the conical surface considered. Let us transform the surface integral into a contour integral. Consider the

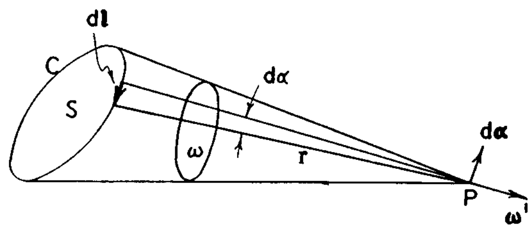

FIG. 34

closed surface formed by the conical surface and the sphere of unit radius having its center at the apex of the cone (Fig. 34). The vector of a closed surface is equal to zero. Indeed, the condition that a surface be closed is equivalent to the statement that the vectors of the elements of this surface form a closed figure. Thus the total vector obtained by the addition of the vector of the spherical surface and the vector of the conical surface is equal to zero, on the assumption that the vectors of separate elements are marked off along the outer normals to both the spherical and conical surfaces.

Subdivide the conical surface into its elements and consider the vector of the element of the conical surface enclosed between two rays which form the angle $d\alpha$. This element of the cone is a sector of a circle of unit radius with the angle $d\alpha$ at its apex, and its area is $d\alpha/2$. Each plane angle formed by two intersecting straight lines may be characterized by a vector directed along the normal to its plane and equal in length to the magnitude of the angle. Introducing the vector $d\boldsymbol{\alpha}$, we find that the vector of the element of the conical surface is equal to $d\boldsymbol{\alpha}/2$. Therefore the solid-angle vector is

$$\boldsymbol{\omega}' = \tfrac{1}{2} \oint d\boldsymbol{\alpha} \tag{106}$$

where the integration is performed over all the elements of the conical surface. When the base of the cone is a polygon and the cone degenerates into a pyramid, the solid-angle vector is

$$\boldsymbol{\omega}' = \tfrac{1}{2} \sum_{i=1}^{m} \boldsymbol{\alpha}_i \tag{106a}$$

where $\boldsymbol{\alpha}_i$ is the vector of the plane angle α_i that is subtended by the i^{th} side of the polygon (Fig. 31).

It is easy to show (Section 7, Chapter IV) that

$$\boldsymbol{\omega}' = \mathbf{r}_0\omega$$

where $\mathbf{r}_0$ is the vector from the center of gravity Q to the apex of the solid angle (Fig. 20). The projection of the vector $\boldsymbol{\omega}'$ on a direction n (which is in general a generatrix of a conical surface) is equal to

$$\omega_n' = \tfrac{1}{2} \oint \cos \beta \, d\alpha \tag{107}$$

where β is the angle between the vector $d\boldsymbol{\alpha}$ and the direction n. For an m-sided pyramid, the projection of the vector $\boldsymbol{\omega}'$ on a direction n is one half of the sum of projections of the vectors $\boldsymbol{\alpha}_i$ on this direction. In other words,

$$\omega_n' = \tfrac{1}{2} \sum_{i=1}^{m} \alpha_i \cos \beta_i \tag{107a}$$

where β_i is the angle between the vector $\boldsymbol{\alpha}_i$ and the direction n.

The above transformations are of great importance in theoretical photometry and they completely define the computation of the illumination produced by large uniformly luminous surfaces. The light vector of a luminous surface of brightness B is

$$\mathbf{D} = B\boldsymbol{\omega}' = \frac{B}{2} \oint d\boldsymbol{\alpha}. \tag{108}$$

Introducing the luminosity of the surface, we obtain $L = \pi B$, and

$$\mathbf{D} = L \oint \frac{d\boldsymbol{\alpha}}{2\pi}. \tag{108a}$$

If the luminous surface is an m-sided polygon,

$$\mathbf{D} = L \sum_{i=1}^{m} \frac{\boldsymbol{\alpha}_i}{2\pi} \tag{108b}$$

Thus the light vector produced by a luminous polygon is determined by constructing a pyramid with apex at P and with its base formed by the luminous polygon. We determine in degrees the angles $\alpha_1, \alpha_2 \cdots, \alpha_m$ in the sides of this pyramid, and consider vectors which are normal to the sides of the pyramid and whose lengths are equal to $\alpha_1/360$, $\alpha_2/360, \cdots, \alpha_m/360$. The sum of these vectors determines the light vector.

The flux density for any element of the surface is given by the projection of the light vector on the normal n to this element. Thus the flux density may be computed from the formula,

$$D_n = B\omega_n' = L \oint \frac{d\alpha}{2\pi} \cos\beta, \tag{108c}$$

For a polygon,

$$D_n = L \sum_{i=1}^{m} \frac{\alpha_i}{2\pi} \cos\beta_i, \tag{108d}$$

If the tangent plane does not intersect the luminous polygon, the quantity D_n represents the illumination. Thus the above formula gives a general expression for the illumination produced by an arbitrarily shaped luminous figure. These formulas make it unnecessary to perform an integration over a surface when considering special cases. Applied to the case of a luminous polygon, the computation of the illumination is reduced to the determination of the angles α_i and β_i.

Eq. (108d) was first derived by Lambert, who proceeded from the case of a spherical triangle, and was derived in a simpler manner by Wiener[22] and shown in its vectorial form for the first time by Mehmke.[21] For surfaces bounded by arbitrary contours, Eq. (108c) was derived in 1926 by Yamauti[29] and independently by the author of this book in collaboration with Gurevitch.[30] A somewhat different derivation is given in the paper by Genkin.[31] According to the principle of reci-

[29] Z. Yamauti, J. O. S. A., **13**, 1926, p. 561. Researches of the Electrotechnical Laboratory, Tokyo, No. 190 (in Japanese).

[30] A. Gershun and M. Gurevitch, The Light Field. Journal of the Russian Physico-Chemical Society, **60**, 1928, No. 4, p. 355.

[31] V. Genkin, Calcul de l'éclairement moyen en présence des surfaces diffusantes de brillance uniforme. R. G. E., **29**, 1931, p. 369.

procity, Eq. (108c) allows the determination of the flux through an arbitrary contour, the light being emitted according to Lambert's law by an element of the surface. The flux through this contour is equal to the product of the axial intensity of the luminous element and the projection of the vector of the solid angle on the axis. The computation is reduced to the determination of a contour integral or of a sum along the perimeter of a polygon.

4. Example: Computation of Illumination from a Rectangular Source

Let us determine the light vector at P produced by a uniformly luminous, perfectly diffusing rectangular source of brightness B (Fig. 33). A table of cosines of the angles between the vectors α_i and the coordinate axes follows.

Vector	X	Y	Z
$\boldsymbol{\alpha}_1$	1	0	0
$\boldsymbol{\alpha}_2$	0	1	0
$\boldsymbol{\alpha}_3$	$-\cos \alpha_2$	0	$\sin \alpha_2$
$\boldsymbol{\alpha}_4$	0	$-\cos \alpha_1$	$\sin \alpha_1$

The vector

$$\boldsymbol{\omega}' = \frac{1}{2} \sum_{i=1}^{4} \mathbf{a}_i$$

is determined from the table:

Vector	X	Y	Z
$\boldsymbol{\alpha}_1$	α_1	0	0
$\boldsymbol{\alpha}_2$	0	α_2	0
$\boldsymbol{\alpha}_3$	$-\alpha_3 \cos \alpha_2$	0	$\alpha_3 \sin \alpha_2$
$\boldsymbol{\alpha}_4$	0	$-\alpha_4 \cos \alpha_1$	$\alpha_4 \sin \alpha_1$
$\boldsymbol{\omega}' = \frac{1}{2} \sum \boldsymbol{\alpha}_i$	$\omega_x = \dfrac{\alpha_1 - \alpha_3 \cos \alpha_2}{2}$	$\omega_y = \dfrac{\alpha_2 - \alpha_4 \cos \alpha_1}{2}$	$\omega_z = \dfrac{\alpha_3 \sin \alpha_2 + \alpha_4 \sin \alpha_1}{2}$

From the diagram,

$$\tan \alpha_3 = \cos \alpha_2 \tan \alpha_1$$

and

$$\tan \alpha_4 = \cos \alpha_1 \tan \alpha_2 .$$

Thus,

$$
\begin{aligned}
D_x &= B\omega_x' = \frac{B}{2}\{\alpha_1 - \cos\alpha_2 \tan^{-1}(\cos\alpha_2 \tan\alpha_1)\} \\
D_y &= B\omega_y' = \frac{B}{2}\{\alpha_2 - \cos\alpha_1 \tan^{-1}(\cos\alpha_1 \tan\alpha_2)\} \qquad (109) \\
D_z &= B\omega_z' = \frac{B}{2}\{\sin\alpha_1 \tan^{-1}(\cos\alpha_1 \tan\alpha_2) + \sin\alpha_2 \tan^{-1}(\cos\alpha_2 \tan\alpha_1)\}
\end{aligned}
$$

The absolute value of the light vector is

$$D = \sqrt{D_x^2 + D_y^2 + D_z^2} \qquad (110)$$

where D_z is equal to the illumination at the point P in a plane parallel to the luminous rectangle, D_x is the illumination of the plane YZ, and D_y is the illumination of the plane XZ. When the illuminated plane is inclined but does not intersect the rectangle, its illumination is

$$E = D_x \cos(n, x) + D_y \cos(n, y) + D_z \cos(n, z). \qquad (111)$$

If the linear dimensions of the sides of the rectangle are l_1 and l_2, and d is its distance from P, we obtain

$$
\left.
\begin{aligned}
D_x &= \frac{B}{2}\left\{\tan^{-1}\left(\frac{l_1}{d}\right) - \frac{d}{\sqrt{d^2 + l_2^2}} \tan^{-1} \frac{l_1}{\sqrt{d^2 + l_2^2}}\right\} \\
D_y &= \frac{B}{2}\left\{\tan^{-1}\left(\frac{l_2}{d}\right) - \frac{d}{\sqrt{d^2 + l_1^2}} \tan^{-1} \frac{l_2}{\sqrt{d^2 + l_1^2}}\right\} \\
D_z &= \frac{B}{2}\left\{\frac{l_1}{\sqrt{d^2 + l_1^2}} \tan^{-1} \frac{l_2}{\sqrt{d^2 + l_1^2}} + \frac{l_2}{\sqrt{d^2 + l_2^2}} \tan^{-1} \frac{l_1}{\sqrt{d^2 + l_2^2}}\right\}
\end{aligned}
\right\} \qquad (112)
$$

These formulas, it seems, were first derived by R. A. Herman.[32] The direction and the absolute value of the light vector from a rectangle are easily determined by graphical methods. We shall illustrate this, following the method of Mehmke[21] for daylight illumination through a window (Fig. 35). We draw from P the straight lines PQ_1, PQ_2, PQ_3, PQ_4, which enclose a region of the sky. As a rule this region will be bounded on its lower side by the plane passing through the roof of the opposite building. If the brightness B of the sky is assumed to be equal

[32] R. A. Herman, A Treatise on Geometrical Optics. Cambridge, 1900. Also see H. H. Higbie, Prediction of Daylight from Vertical Windows, I. E. S. Trans., **20**, 1925.

H. H. Higbie and A. Levin, Prediction of Daylight from Sloping Windows, I. E. S. Trans., **21**, 1926.

in all directions, and if loss of light in the glass is neglected, the relative value of the light vector $\mathbf{e}$ at P will be

$$\mathbf{e} = \frac{\mathbf{D}}{B\pi} = \sum_{i=1}^{4} \mathbf{e}_i , \tag{113}$$

where

$$\mathbf{e}_i = \frac{\alpha_i}{2\pi} . \tag{114}$$

The summation is performed over the four sides of the light pyramid which encloses the light rays arriving at P.

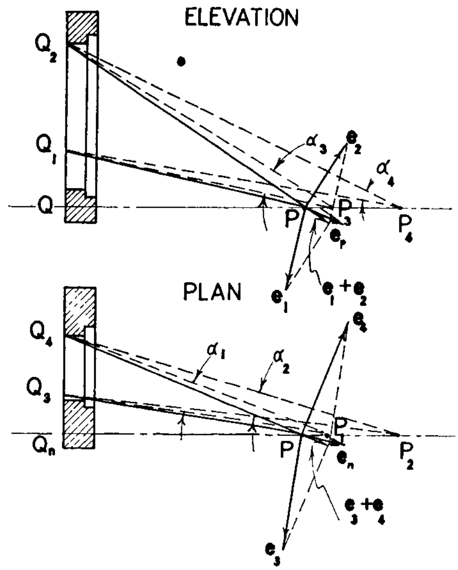

FIG. 35

For direct reading of the absolute values of the vectors $\mathbf{e}_i$ from the drawing, we may use a protractor suggested by Mehmke. An Archimedean spiral, i.e. a curve for which the radius vector is proportional to the angle, is drawn on tracing cloth. The radius vector is made of unit length for the angle 2π. To determine a length equal to $\alpha/2\pi$ we place the protractor on the drawing in such a manner that its center P coincides with the apex of the angle and the line PQ coincides with one of the sides of the angle. The spiral will cut from the other side of the angle the distance $\alpha/2\pi$, which is equal to the absolute value of the vector $\mathbf{e}_i$ corresponding to this angle α.

The solid angle has in this case the shape of a four-sided pyramid, two sides of which (1 and 2) are horizontal, while the other two sides (3 and 4) are vertical (Fig. 35). Thus the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ may be drawn and added on the vertical projection, and the vector $\mathbf{e}_3$ and $\mathbf{e}_4$ on the horizontal projection. To determine from the drawing the absolute values of these vectors, we imagine the planes of the angles α_1, α_2, α_3, α_4, to be placed parallel to the planes of the drawing. The angles α_1 and α_2 will be parallel to the vertical projection, and the angles α_3 and α_4 will be parallel to the plane of the horizontal projection. Then $P_1Q_n = PQ_1$; $P_2Q_n = PQ_2$; $P_3Q_p = PQ_3$; $P_4Q_p = PQ_4$ and $\angle Q_3P_1Q_4 = \alpha_1$; $\angle Q_3P_2Q_4 = \alpha_2$; $\angle Q_1P_3Q_2 = \alpha_3$; $\angle Q_1P_4Q_2 = \alpha_4$. Using the Mehmke protractor, we determine from these angles the absolute values of the vectors $\mathbf{e}_i$, which are equal to $\alpha_i/2\pi$. Adding the vector $\mathbf{e}_1 + \mathbf{e}_2$ to the component of the vector $\mathbf{e}_3 + \mathbf{e}_4$ that is parallel to the normal plane, and adding the vector $\mathbf{e}_3 + \mathbf{e}_4$ to the horizontal component of the vector $\mathbf{e}_1 + \mathbf{e}_2$, we obtain the vectors $\mathbf{e}_p$ and $\mathbf{e}_n$ which represent the projection of the light vector $\mathbf{e}$ on the vertical and on the horizontal planes.

The maximum illumination at P is obtained when its normal coincides with the light vector. In other words, the normal passes through the light center of the window. The light center is determined in the horizontal and vertical projection by the vectors $\mathbf{e}_p$ and $\mathbf{e}_n$.

CHAPTER VII

Mathematical Theory

1. The Light Beam and Its Brightness

In geometrical optics, the light ray is considered as a mathematical straight line. In photometry (see Chapter II, Section 2) the light ray is replaced by an elementary ensemble of directions along which the transfer of radiant energy proceeds. The geometry of the light beam is defined by two infinitesimal apertures ds_1 and ds_2 which are at a finite distance from each other (Fig. 3). It seems advisable to introduce a new, purely geometrical concept—*a measure of the multitude of directions composing the light beam.* One deals, not with the infinitesimal apertures, but with apertures s_1 and s_2 of finite size (Fig. 36). Of all straight lines in space we choose a definite set passing through s_1 and s_2. All these lines are concentrated in the region τ, whose boundaries are shown by dotted lines. To each point P corresponds a definite conical surface, within which are concentrated all the straight lines passing through P and belonging to the set of lines within τ. To this conical surface, in turn, correspond definite values of the solid angle ω and of its vector $\boldsymbol{\omega}'$. These quantities may be considered as point functions; and outside of the region τ, ω and $\boldsymbol{\omega}'$ are zero.

We construct an arbitrary surface S satisfying the condition that each straight line of τ intersects it in only one point. Consider the quantity

$$N = \int_S \boldsymbol{\omega}' \cdot d\mathbf{S} \tag{115}$$

where the integration is performed over the entire surface S, and where $\boldsymbol{\omega}'$ is the vector of the solid angle enclosing all the lines of τ that pass through the element dS of surface S.

The quantity N represents the flux of $\boldsymbol{\omega}'$ through the surface S. N depends only on the size and relative position of the apertures and is independent of the choice of the surface S. The flux of the solid angle vector through all surfaces that satisfy the above requirement is the same. The proof of this statement follows from the fact that the vector $\boldsymbol{\omega}'$ is solenoidal ($\operatorname{div} \boldsymbol{\omega}' = 0$).

Thus it seems natural to take as a *measure of the multitude of lines contained within* τ the quantity N which is an *invariant* with respect to all surfaces intersecting the set of lines. In place of S we may choose, as a special case, one of the apertures s_1 and s_2. If N is determined by two infinitesimal apertures, the measure of the multitude of the geometric rays composing the beam is

$$d^2N = d\sigma_1 \, d\omega_1 = d\sigma_2 \, d\omega_2 \tag{116}$$

where $d\sigma_1$ and $d\sigma_2$ are the areas of the cross-section of the beam at the two apertures and $d\omega_1$ and $d\omega_2$ are the corresponding values of the solid angle (Fig. 3). For other cross-sections of the light beam, we cannot give a definite value of the solid angle that measures the di-

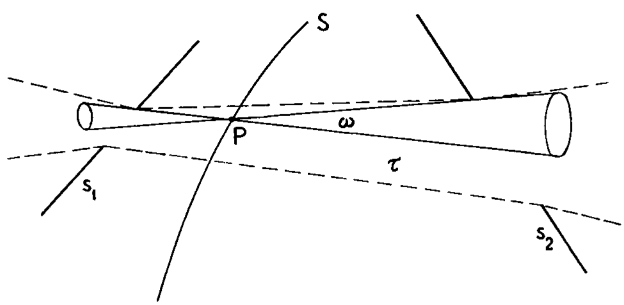

FIG. 36

vergence of the rays, because this quantity will have different values for different points of such a cross-section, and it will decrease as we move from the center to the edge of the cross-section of the beam. Thus the common statement that "the product of the cross-sectional area of the beam and the solid angle determining divergence of the rays is constant" is not true. What is constant is the flux of the solid angle vector having for all the cross-sections the value

$$d^2N = \frac{\cos\theta_1 \cos\theta_2 \, ds_1 \, ds_2}{l^2} \tag{117}$$

where l is the distance between the apertures ds_1 and ds_2, and θ_1 and θ_2 are the angles between the normals to the apertures and the direction of the beam.

The new concept of the solid-angle vector not only clarifies the somewhat foggy question of the geometry of the light beam, but may also be used in building up a system of photometric concepts. Thus the brightness B along the light beam may be defined as the coefficient of proportionality between the light flux d^2F and the quantity d^2N, which measures the multitude of the geometrical rays composing the light beam:

$$d^2F = B\,d^2N. \tag{118}$$

Inasmuch as N is an invariant, the brightness varies along the beam in proportion to the flux. In empty space, the flux is constant for all the cross-sections of the beam and the brightness is also constant. Let us consider now the question of the change of brightness when the beam passes from one transparent medium into another. If the light beam passes from a medium (subscript k) in another medium (subscript

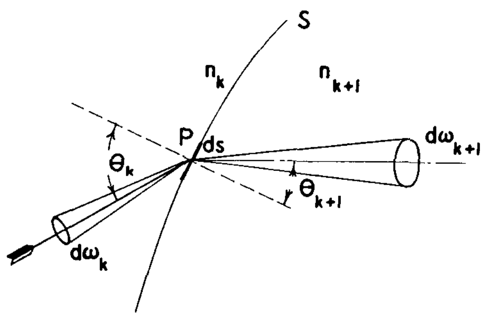

FIG. 37

$k + 1$, Fig. 37), the indices of refraction of these media being n_k and n_{k+1}, and if the boundary s of the two media is ideally polished, the change in direction of the beam at s is defined by the following:

$$n_k \sin \theta_k = n_{k+1} \sin \theta_{k+1}\,. \tag{119}$$

We shall now prove that $n^2\,d^2N$ is an invariant for all the points of the beam. *The invariability of this expression must be considered as one of the fundamental laws of geometrical optics.* The validity of this law for the special case of n = const was demonstrated in the foregoing discussion. We denote the area of the element of the refracting surface by ds and the point corresponding to it by P. The cross-sectional area of the light beam at this point will differ, depending upon the side from which we approach the refracting surface. If from the medium k, the area will be

$$d\sigma_k = ds \cos \theta_k\,.$$

If from the medium $k + 1$, it will be

$$d\sigma_{k+1} = ds \cos \theta_{k+1} .$$

Similarly, the solid angles defining the divergence of the light beam at P are denoted by $d\omega_k$ and by $d\omega_{k+1}$. The elementary solid angle $d\omega_k$ includes the directions for which the angle of incidence varies from θ_k to $\theta_k + d\theta_k$, and the position of the plane of incidence varies by an angle $d\psi_k$. Then

$$d\omega_k = \sin \theta_k \, d\theta_k \, d\psi_k \tag{120}$$

and

$$d\omega_{k+1} = \sin \theta_{k+1} \, d\theta_{k+1} \, d\psi_{k+1} . \tag{121}$$

We must consider that $d\psi_k = d\psi_{k+1}$ because the plane defined by the rays and the normal does not change its position when the rays are refracted. Differentiating the expression,

$$n_k^2 \sin^2 \theta_k = n_{k+1}^2 \sin^2 \theta_{k+1} \tag{122}$$

we obtain

$$n_k^2 \sin \theta_k \cos \theta_k \, d\theta_k = n_{k+1}^2 \sin \theta_{k+1} \cos \theta_{k+1} \, d\theta_{k+1} . \tag{123}$$

Multiplying the expressions for $d\omega_k$ and $d\omega_{k+1}$ by $n_k^2 \, d\sigma_k$ and by $n_{k+1}^2 \, d\sigma_{k+1}$ and taking into account the above relationships, we obtain

$$n_k^2 \, d\sigma_k \, d\omega_k = n_{k+1}^2 \, d\sigma_{k+1} \, d\omega_{k+1} . \tag{124}$$

For convenience of the proof we proceeded from the value of the quantity d^2N at the boundary point P. As has been shown, this quantity is the same for all points of the light beam in a given medium. From this it follows that the magnitude $n_k^2 \, d^2N_k$ is the same for all values of k. Along the entire path of the light beam, independent on how many times it has been refracted, the product of n^2 and the measure of the multitude of the rays remain constant. The invariability of the expression $n^2 \, d\sigma \, d\omega$ was first demonstrated by R. Straubel, and represents one of the most important laws of geometrical optics. Many papers deal with this question.[33]

[33] M. Herzberger, Über Sinusbedingung, Isoplanasie und Homöoplanasiebedingung, ihren Zusammengang mit energetischen Überlegungen und ihre Ableitung aus dem Fermatschen Gesetz. Zs. für Instrumentenkunde, **48,** 1928, pp. 313, 465, 524. (Contains a bibliography up to the year 1928.)

André Dargenton, Théorème sur la réfraction des pinceaux de rayons lumineux. Application au calcul de la brillance. Revue d'Optique, **8,** 1929, p. 4. Note

Analogous relationships may be written for the rays confined by the diaphragms s_1 and s_2 of finite size (Fig. 36). If the rays are refracted, the invariant is no longer represented by $N = \int_s \omega' \cdot d\mathbf{S}$, but by

$$n^2 \int_s \omega' \cdot d\mathbf{S}$$

where n is the index of refraction of the medium in which the surface S is placed. It is better to introduce as an invariant the quantity

$$\frac{1}{\lambda^2} \int_s \omega' \cdot d\mathbf{S}$$

where λ is the wavelength. This invariant represents a pure number which may play an important rôle in the study of the phenomena of diffraction.

We turn now from the questions of geometrical optics to the problems of photometry, and consider the change of brightness of a light beam as it passes from one medium to another. Denote the flux by d^2F_k for a point P_k in the medium k and by d^2F_{k+1} for a point P_{k+1} in the medium $k + 1$. In general,

$$d^2F_{k+1} < d^2F_k .$$

Denoting the values of brightness by B_k and B_{k+1}, we obtain

$$d^2F_k = B_k \, d^2N_k$$

and

$$d^2F_{k+1} = B_{k+1} \, d^2N_{k+1} .$$

Since $n_k^2 \, d^2N_k = n_{k+1}^2 \, d^2N_{k+1}$, we have

$$\frac{B_{k+1}}{n_{k+1}^2} : \frac{B_k}{n_k^2} = d^2 F_{k+1} : d^2 F_k .$$

The brightness along the light beam is directly proportional to the flux and to the square of the index of refraction, when passing from one medium into another. Disregarding the question of absorption of light, we obtain the invariability of the ratio B/n^2. From this

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sur une propriété de la réfraction des pinceaux de rayons lumineux. Application au calcul de la brillance. Revue d'Optique, **12,** 1933, p. 172.

Pierre Copel, Sur un nouvel invariant relatif a l'ensemble de deux pinceaux lumineux ayant même rayon moyen. Rev. d'Optique, **13,** 1934, p. 193.

follows that the brightness of beams arriving at an image, produced by an ideal lens that does not absorb any light, is equal to the brightness of rays emitted by the object, provided that the image is in the same medium as the object.

The question of the change of brightness of a light beam when it is reflected from a boundary of two regions does not call for a separate analysis. The quantity d^2N remains unchanged in the process of reflection of a given light beam, and thus the ratio of the brightness of the reflected beam to the brightness of incident beam is equal to the ratio of the reflected flux to the incident flux. From this follows that the reflection factor of specular surfaces, which is equal by definition to the ratio of reflected flux to incident flux, may also be defined as the ratio of the brightness of the image to the brightness of the object. All these relationships are fundamental in the study of photometry of the optical image.[34]

2. Some Mathematical Relationships

Let us consider some of the most important theorems of the mathematical theory of the light field in empty space. We shall treat them briefly, inasmuch as the purpose of this text is to present the physical basis of the theory of the light field and its application in radiation engineering. The mathematical theory of the light field is discussed in papers by Fock,[35] Gershun,[19] Gurevitch[30] and others. A review of the literature appears in a paper by Yamauti.[7]

At first sight it would seem that the theory of the light field in its systematic treatment must be very like the theory of the electrostatic field. The light vector produced by a point source varies inversely as the square of the distance, as does the electric force produced by a point charge. With a linear source, the light vector varies inversely as the first power of the distance, as in the electric field of a charged wire. With an infinite plane source, the light vector is constant in direction and magnitude at any finite distance from the plane, as in the homogeneous field of a charged plane. However, the light field possesses a

[34] A. I. Tudorovsky, Elements of the General Theory of Optical Apparatus. Edited by "The Military Technical Academy" PKKA of Dzerzinsky, 1932. M. M. Gurevitch, Light Measurements. Edited by the Educational Dept. of the Civil Air Fleet, Leningrad, 1934.

[35] V. A. Fock, Illumination Produced by Surfaces of Arbitrary Shape. Proc. of State Opt. Inst., **3**, No. 28, 1924.

V. Fock, Zur Berechnung der Beleuchtungsstärke. Zs. f. Phys., **38**, 1924, p. 102.

number of curious properties, the study of which may be of general interest from the standpoint of field theory.

The electric charge, or the strength of a magnetic pole, does not depend on the direction in which its action is considered. The value of the force vector is completely defined by the distance from the source. The field produced by uniform point sources of light, the intensity of light of which is the same in all directions, is identical with the field produced by point charges. In this special case, the light vector possesses a scalar potential. It is possible that the historical error of Euler and Laplace, who assumed that the luminous intensity of each element of a luminous surface is the same in all directions, was caused by a subconscious desire to introduce an analogy to the theory of gravitation, where the mass does not depend on direction. This error was discovered by Lambert, who assumed that the luminous intensity varies according to the cosine law.

Usually the luminous intensity of a point source is different in different directions, and the value of the light vector depends on both distance from the source and direction. Thus electrostatic and mechanical analogs cannot be used. The light field usually has vortices and the light vector does not possess a potential. In fields produced by several sources, in general, a potential does not exist and it is impossible to construct surfaces orthogonal to the flux lines.

Let us consider now some vectorial operations and relationships characterizing the light field in empty space. First, let us introduce the concept of the vector potential. If the light vector **D** is solenoidal (div **D** = 0), the flux of this vector through a surface s may be expressed, according to Stokes' theorem, as a line-integral of another vector **A** over a contour C embracing this surface, or

$$\int_s \mathbf{D} \cdot d\mathrm{s} = \oint \mathbf{A} \cdot d\mathrm{l}.$$

The vector **D** is the curl of the vector **A**, which is called the *vector potential*, or

$$\mathbf{D} = \mathrm{curl}\ \mathbf{A}. \tag{125}$$

The field of the vector **A** defines the field of the light vector **D**. The value of the concept of vector potential in photometry resides in the fact that it allows a simplified computation of flux from one surface to another. The integral of the light vector over a surface is reduced to the integral of the vector potential along a contour. Let us recall, first, the well-known relationships between the components of the vectors **D** and **A** in rectangular system of coordinates, or

$$\left.\begin{aligned} D_x &= \text{curl}_x \mathbf{A} = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \\ D_y &= \text{curl}_y \mathbf{A} = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \\ D_z &= \text{curl}_z \mathbf{A} = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \end{aligned}\right\} \tag{126}$$

These relationships determine the components D_x, D_y, D_z of curl **A** when the values A_x, A_y, A_z are known at all points of the field.

We consider now the inverse problem of finding the vector potential. For a uniformly luminous surface which radiates according to Lambert's law, the light vector is

$$\mathbf{D} = \frac{B}{2} \oint d\boldsymbol{\alpha} \tag{127}$$

Denoting by r the distance from the element dl to the point P, and introducing a unit vector $\mathbf{r}_1$,

$$d\boldsymbol{\alpha} = \frac{\mathbf{r}_1}{r} \times d\mathbf{l} \tag{128}$$

The components of the vector $\frac{\mathbf{r}_1}{r}$ along the coordinate axis may be considered as derivatives of a scalar function $ln\, r$ along the corresponding axis, or

$$\frac{\mathbf{r}_1}{r} = \text{grad ln } r \tag{129}$$

Therefore,

$$d\boldsymbol{\alpha} = \text{grad ln } r \times d\mathbf{l} = \text{curl } (ln\, r\, d\mathbf{l}) \tag{130}$$

The identity of the two above expressions for $d\boldsymbol{\alpha}$ may easily be verified by substitution.

Thus the solid-angle vector is

$$\boldsymbol{\omega}' = \frac{1}{2} \oint d\boldsymbol{\alpha} = \frac{1}{2} \oint \text{curl } (\ln r\, d\mathbf{l}) = \text{curl} \frac{1}{2} \oint \ln r\, d\mathbf{l} \tag{131}$$

The light vector is

$$\mathbf{D} = B\boldsymbol{\omega}' = \text{curl } \mathbf{A}$$

and its vector potential $\mathbf{A}$ is

$$\mathbf{A} = \frac{B}{2} \oint \ln r \, d\mathbf{l} \tag{132}$$

The luminous flux F from the luminous surface S of brightness B (enclosed by the contour C) to the illuminated surface s (enclosed by the contour c) is

$$\begin{aligned} F &= \int_s \mathbf{D} \cdot d\mathbf{s} = \int_c \mathbf{A} \cdot d\mathbf{l}' = \frac{B}{2} \int_c \int_C \ln r \, d\mathbf{l} \cdot d\mathbf{l}' \\ &= \frac{B}{2} \int_c \int_C \ln r \cos (dl, dl') \, dl \, dl' \end{aligned} \tag{133}$$

where r is the distance between dl and dl'. The computation of the flux is reduced to a two-fold integration in place of a four-fold integration. From the principle of reciprocity, the expression for F is symmetrical with respect to the elements of the contours of the luminous as well as of the illuminated surfaces.

Photometric computation using this formula has been carried out. V. Fock[35] computed the illumination produced by a luminous ellipse, using the foregoing method, while V. Genkin[31] considered the computation of the flux from one rectangle to another. The concept of the vector potential was introduced into photometric theory by R. Herman.[32]

To conclude, let us consider in greater detail the question of the existence of a scalar potential. Each luminous body may be subdivided into elements which may be considered as point sources. Thus we shall start with a light field produced by a point source which has an arbitrary intensity distribution. Let us denote by r the distance from the source to a point of the field, by $\mathbf{r}_1$ a unit vector, and by I the intensity of the source in the direction of $\mathbf{r}_1$. Then the light vector is

$$\mathbf{D} = \frac{I}{r^2} \mathbf{r}_1$$

The divergence is $\text{div } \mathbf{D} = \dfrac{\text{grad } I \cdot \mathbf{r}_1}{r^2}$. Inasmuch as *grad I* is normal to $\mathbf{r}_1$,

$$\text{div } \mathbf{D} = 0. \tag{134}$$

This relationship will be true for the light vector produced by any set of such sources and will therefore apply for an arbitrary luminous body. *The light vector is solenoidal.* This property was also found in Section 6, Chapter IV, on the basis of the law of conservation of energy.

The curl of the light vector is

$$\text{curl}\,\mathbf{D} = \frac{\text{grad}\,I \times \mathbf{r}_1}{r^2} \tag{135}$$

The curl is normal to $\mathbf{r}_1$ and to *grad I*. Inasmuch as these two vectors are in their turn normal to each other, the absolute value of the curl is

$$|\,\text{curl}\,\mathbf{D}\,| = \left|\frac{\text{grad}\,I}{r^2}\right| \tag{136}$$

Curl **D** is different from zero, and therefore the field of a point source does not, in general, possess a potential. The curl of the light vector is normal to that vector, so

$$\mathbf{D}\cdot\text{curl}\,\mathbf{D} = 0. \tag{137}$$

This relationship reveals the existence of surfaces orthogonal to **D**. The flux lines represent the geodesics of the function $\phi = 1/r$ which may be called the *quasi-potential*, because

$$\mathbf{D} = -I\,\text{grad}\,\phi. \tag{138}$$

When the light vector is produced by a set of non-uniform point sources,

$$\mathbf{D}\cdot\text{curl}\,\mathbf{D} \neq 0$$

so the light vector possesses neither a potential nor a quasi-potential. It is impossible in this case to construct surfaces orthogonal to the flux lines. The exceptions to this rule are the two-dimensional field, the field possessing axial symmetry, the field produced by a luminous plane source that is perfectly diffusing.

In all these cases, the light vector may be represented by two variable scalar functions:

$$\mathbf{D} = -\psi\,\text{grad}\,\phi \tag{139}$$

and the light lines may be constructed as orthogonal trajectories to surfaces upon which the values of the quasi-potential ϕ are constant. An example is the field produced by a plane, perfectly diffusing source.

Then ψ represents the distance along the normal from the luminous surface S to an arbitrary point of the light field, and

$$\phi = \frac{1}{2} \int_s \frac{B\,dS}{r^2} \tag{140}$$

where r is the distance from the point to the element dS whose brightness B is equal in all directions (A. Gershun. See footnote 19). Of unquestionable interest from the physical as well as from the engineering standpoint is the theory of the light field in diffusing, absorbing, and luminous media.[36] Space forbids a consideration of this question and a large number of other photometric problems.

[36] N. Boldyreff, see footnote 18.

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