WORKSHOP 1 — IMDS, DURHAM — MICHAELMAS 2023

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1. Key concepts for Week 1

You will have encountered some of these in the videos, and some of these will be developed in the exercises below, which you should discuss and tackle **as a group.**

(1) The mathematical modelling flow chart.
(2) Setting up a model.
(3) Interpreting the output of a model.
(4) Thinking about the range of validity of a model.
(5) Models given by a state and an update rule.
(6) Models involving randomness.
(7) Models built from cellular automata.
(8) Working with units and dimensionful quantities.
(9) Building functions that have a desired behaviour.

1.1. **Working with units.** Numerical data comes from measuring things, and when we measure things in the world they are measured with appropriate units. Paying attention to units is essential.

In September of 1999, after almost 10 months of travel to Mars, NASA's Mars Climate Orbiter hit the Martian atmosphere, burned and broke into pieces. The $327.6 million mission was a complete failure. An investigation eventually found that the failure was due to one piece of software using different units from the rest.

A **dimensionful quantity** is something that that must be specified in terms of some kind of unit. Examples of dimensionful quantities include lengths and distances, time intervals, masses and weights, temperatures, and so on. It doesn't make sense to add things of different types. For example, we cannot add 10kg + 5km. On the other hand, we can add $10km + 5m$ because these two have the same dimensionality; however, before adding we must convert them to be in terms of the same units: 10km = 10000m, so 10km + 5m = 10005m.

A **dimensionless quantity** is something that has a numerical value independent of any particular choice of units. Ratios of two things that have the same units is dimensionless. I can say that this distance is twice as long as that distance, and this statement makes sense independently of what units I might have been using.

We have units for:

(1) length/distance: meters, km, miles, light-years, etc.
(2) time: seconds, minutes, hours, days, weeks, years, etc.
(3) mass: grams, kg, etc.

(4) temperature: degrees Celsius/Kelvin, or Farenheit
(5) ... and various other physical quantities.

There are a couple of rules for working with dimensionful quantities.

(1) We can only add or subtract things that have the same dimensionality.
(2) In an equation, the things on either side of the equals sign must have the same dimensionality.

Exercise 1. This exercise will help you get used to working with units.

(1) Decide what units you would like to measure distance and time with. Now, what are the units of velocity, and acceleration?
(2) Newton's 2nd Law of Motion says
$$\text{force} = \text{mass} \times \text{acceleration}.$$
What units does force have?
(3) Newton's Law of Gravitation says that the gravitational attraction force between two masses m_1 and m_2 separated by a distance d is given by the formula
$$G\frac{m_1 m_2}{d^2}.$$
The letter G stands for Newton's coefficient of gravitation. What units does G have?

Exercise 2. Suppose p is pressure, ρ is density, u and v are velocities, F is force, m is a mass, A is an area, and z and ℓ are lengths. Which equations are dimensionally correct, and which appear to contain an error?

(1) $v^2 = u^2 + 2gz$ — (g is the acceleration of gravity)
(2) $p = \rho \ell u / z$
(3) $F = -pA$
(4) $p + \frac{1}{2}\rho u^2 = -mgz$

1.2. **Working with formulae.** An important skill in mathematical modelling, and mathematics more generally, is being able to invent mathematical functions that have certain types of behaviour. These exercises will help you begin to develop this skill. Think about making use of the following functions as your building blocks:

- Powers: x^2, x^3, $\ldots$
- And don't forget about negative powers, $x^{-1} = 1/x$, and fractional powers $x^{1/2} = \sqrt{x}$
- Exponential: e^x (growing) and e^{-x} (decaying)
- Logarithm: $\log(x)$
- Trig functions: $\sin(x)$, $\cos(x)$

Starting from these basic pieces, you want to make some adjustments:

- In a formula, $f(x) = \ldots$, changing x to $-x$ reflects the graph across the vertical axis.
- Changing x to $2x$ Compresses the graph horizontally by a factor of 2. For a different compression factor, use a different number. To stretch the graph horizontally, change x to $x/2$.

- Changing x to $x - a$ (for some number a) will shift the graph to the right by a. Use a negative number to shift to the left.
- $-f(x)$ flips the graph of $f(x)$ upside down.
- To stretch the graph vertically, use $af(x)$, where a is the stretching factor.
- $f(x) + a$ (for some number a) shifts the graph up by a.

As you are learning to work with functions, I encourage you to experiment and plot lots of different functions. You can use a graphing calculator, or an app like **Desmos** (`www.desmos.com`).

As an optional additional challenge: Your answers to these questions might involve some parameters/coefficients. What are the units of these? What measurements might you need to make to determine the values of these parameters?

Exercise 3. Consider a large forested island. Let F denote the total area of forest cover. Year by year the forest is being cut down to make space for farming. The deforestation proceeds quickly at first, but it slows down as there is less and less forest available to cut down. Write a reasonable formula for $F(t)$.

Optional additional challenge: Now suppose we look at some data and find that the island started with 1000 km^2 of forest, and after 10 years only half of this was left standing. Can you adjust your formula to fit this?

Exercise 4. The air temperature just above the ground at a particular point on the Earth often varies in a periodic manner over a 24 hour cycle. The daily mean value also varies with the seasons over an annual cycle. If we measure time t in hours, what would be an appropriate mathematical model for the temperature as a function of t?

Exercise 5. For a certain type of tree the rate of growth is low in the winter and greatest during the summer months. The rate of growth also lessens every year from a maximum in the first year until there is virtually no growth at all after 10 years. What would be a suitable mathematical model for the rate of growth?

1.3. **Simulations with a spreadsheet.** Starting in Week 2 we'll be doing a lot of work in Python since this is the computer language and a very large part of the data science world uses. However, before we get into Python, there is still a lot we can do with a simple spreadsheet (MS Excel, Google Sheets, or Apple Numbers).

If you have a computer at your table, open a spreadsheet and try some of these exercises.

Exercise 6. First a simple warm-up. You have a bank account. The balance in your account is $B(t)$, where t is measured in months. Put the initial balance in a cell at the top of a column, e.g., $A1$.

(1) Suppose your account earns 6% interest annual interest, compounded monthly, so 0.5% per month. For the next cell down, enter a formula for the balance one month later as a function of the previous balance

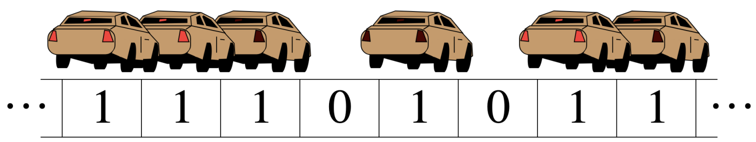
(the cell just above). Then copy this formula downwards so that the n^{th} cell down shows the balance after n months.

(2) Now suppose you spend some amount S per month. Modify the formula accordingly.
(3) Next suppose you have a part time job, earning E per month.
(4) Now suppose that the number of hours you get to work each month is a bit variable, and so your monthly earnings fluctuate randomly between E_{min} and E_{max}.
(5) Plot your balance over the course of a year.

Exercise 7. Suppose there are islands A and B populated by seagulls. Conditions on A are slightly better, so A's population increases by 5% each year, whereas B's population only increases by 3% per year. The gulls also move between the two islands. Each year, 1000 gulls move from A to B, and 800 move from B to A.

(1) Let A_n and B_n be the populations of the two islands after n years. Write down a *difference equation model*
$$A_{n+1} = \cdots \qquad B_{n+1} = \cdots$$
to describe this situation.
(2) Make a spreadsheet simulation of this model.
(3) Play with your model. Depending on the initial conditions, will the population on A or B eventually hit zero?

Exercise 8. Here is a very simple model for the movement of traffic along a road (the same mathematical rules have a few other interesting interpretations). Start with a 1d row of cells. At each time step, each cell can be either 0 or 1.



We think of 1 as representing a bit of road that has a car in it, and 0 as representing a bit of road that is clear.

The update rule that gets the road from one time step to the next is simply that a car moves forward (to the right) by one cell if the cell ahead is currently empty, so whenever we see a 10 at one time step, in the next time step these cells will be 01. This is known as *Rule 184* in the world of cellular automata.

If you want to simulate this model in a spreadsheet like Excel, then it is useful to express the rule in a slightly different form. The table below shows the rule for each cell in terms of the values of previous values of it and its two neighbours.

previous cells	111	110	101	100	011	010	001	000
new cell	1	0	1	1	1	0	0	0

Your task: Try setting this model up in a spreadsheet and see what happens.