Introduction to Mathematics for Data Science
Personal Assignnment 2

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1 Question 1

1.1 Solve Matrix B

Consider the matrix $A = \begin{pmatrix} 5 & 2 & 3 \\ -2 & 3 & 1 \\ 0 & 2 & 2 \\ 1 & 0 & 1 \end{pmatrix}$ Find a matrix B such that $AB = \begin{pmatrix} 8 & 6 \\ -1 & 9 \\ 2 & 6 \\ 2 & 0 \end{pmatrix}$.

1.2 Analytics

Due to the shape of matrix $A = [a_{ij}]$ shape is 4×3, and the shape of matrix $AB = [a_{ik}]$ is 4×2, we can use the matrix multiplication equation to solve for B (we can also infer that matrix B is 3 rows and 2 columns).

To solve for B, we have the matrix multiplication equation:

$$\begin{pmatrix} 5 & 2 & 3 \\ -2 & 3 & 1 \\ 0 & 2 & 2 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} = \begin{pmatrix} 8 & 6 \\ -1 & 9 \\ 2 & 6 \\ 2 & 0 \end{pmatrix}$$

Equating the corresponding elements with the given matrix AB, we get the following system of equations $\Rightarrow \begin{cases} 5b_{11} + 2b_{21} + 3b_{31} = 8 \\ 5b_{12} + 2b_{22} + 3b_{32} = 6 \\ -2b_{11} + 3b_{21} + b_{31} = -1 \\ -2b_{12} + 3b_{22} + b_{32} = 9 \\ 2b_{21} + 2b_{31} = 2 \\ 2b_{22} + 2b_{32} = 6 \\ b_{11} + b_{31} = 2 \\ b_{12} + b_{32} = 0 \end{cases}$ Then solve the equations and get $B = \begin{pmatrix} 1 & 0 \\ 0 & 3 \\ 1 & 0 \end{pmatrix}$.

2 Question 2

2.1 Find Linear Transformation

Consider the vectors $\vec{a} = \begin{pmatrix} 7 \\ 1 \\ 2 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 8 \\ 1 \\ 4 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$. Is there a linear transformation f: $\mathbb{R}^3 \to \mathbb{R}^4$ such that the following conditions are satisfied? $f(\vec{a}) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $f(\vec{b}) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $f(\vec{c}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. If there is such an f, then find its matrix. If there is not, then explain why.

2.1.1 Analytics

We can represent the matrix f as:

$$f = \begin{pmatrix} | & | & | \\ \vec{f}(\vec{a}) & \vec{f}(\vec{b}) & \vec{f}(\vec{c}) \\ | & | & | \end{pmatrix}$$

Then, set up the following system of equations:

$$\begin{pmatrix} | & | & | \\ \vec{f}(\vec{a}) & \vec{f}(\vec{b}) & \vec{f}(\vec{c}) \\ | & | & | \end{pmatrix} \begin{pmatrix} 7 & 8 & -1 \\ 1 & 1 & 0 \\ 2 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

we can find the matrix f by solving this system of equations.

```python
import numpy as np

# Define the given vectors
a = np.array([7, 1, 2])
b = np.array([8, 1, 4])
c = np.array([-1, 0, 2])

# Define the target matrices
fa_target = np.array([1, 0, 0, 0])
fb_target = np.array([0, 1, 0, 0])
```

```
fc_target = np.array([0, 0, 0, 0])

# Form the matrix equation [f] * [a, b, c] = [fa_target,
    fb_target, fc_target]
A = np.vstack((a, b, c)).T  # Transpose to make it 3x3
B_target = np.vstack((fa_target, fb_target, fc_target)).T  #
    Transpose to make it 3x4

# Solve for the matrix [f]
try:
    f_matrix, residuals, rank, s = np.linalg.lstsq(A, B_target,
        rcond=None)
    print("Matrix [f]:")
    print(f_matrix)
except np.linalg.LinAlgError:
    print("No solution exists for the given conditions.")
```

Output Analytics: 'No solution exists for the given conditions.' It implies that there is no matrix f satisfying the specified conditions.

2.2 Another Linear Transformation

Is there a linear transformation $f\colon \mathbb{R}^3 \to \mathbb{R}^4$ such that the following conditions are satisfied?

$f(\vec{a}) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $f(\vec{b}) = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$ and $f(\vec{c}) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$.

2.2.1 Analytics

Bring $f(\vec{a})$, $f(\vec{b})$, $f(\vec{c})$ to the Python program we created. The output is still **'No solution exists for the given conditions.'**

3 Question 3

3.1 Find Value for x and y

Consider the linear system of equations

$$\begin{pmatrix} 6 & -4 \\ 8 & -6 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Either find a solution or explain why it does not have any solutions.

3.1.1 Analyics

To determine whether the linear system has a solution, we can check the consistency of the system using the augmented matrix. The augmented matrix for the given system is:

$$\begin{pmatrix} 6 & -4 & 1 \\ 8 & -6 & 1 \\ 2 & 0 & 1 \end{pmatrix} \xrightarrow{Row2=Row2-\frac{4}{3}\times Row1} \xrightarrow{Row3=Row3-\frac{1}{3}\times Row1} \begin{pmatrix} 6 & -4 & 1 \\ 0 & -\frac{2}{3} & -\frac{1}{3} \\ 0 & \frac{4}{3} & \frac{2}{3} \end{pmatrix}$$

$$\xrightarrow{Row2\times(-3)} \begin{pmatrix} 6 & -4 & 1 \\ 0 & 2 & 1 \\ 0 & \frac{4}{3} & \frac{2}{3} \end{pmatrix}$$

$$\xrightarrow{Row3\times(\frac{3}{2})} \begin{pmatrix} 6 & -4 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{pmatrix}$$

$$\xrightarrow{Row3=Row3-Row2} \begin{pmatrix} 6 & -4 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} 6x - 4y = 1 \\ 2y = 1 \end{cases} \Rightarrow y = \frac{1}{2},\ x = \frac{1}{2}$$

Conclusion: The system has no solution, since there exist two rows of zeros in the augmented matrix. This means that under a given constraint, the values of variables that satisfy all equations can not be found. Mathematically, this usually means that in vector space, the corresponding hyperplane (as defined by the equations) has no point of intersection with it.

At the same time, this system is underdetermined system. If the number of equations in the system is less than the number of variables, then the system may be underdetermined, there are infinitely many solutions. But if there are contradictions between the equations, then the system may still be unsolvable.

3.2 Vector Inclusion in Column Span

Determine whether or not the vector $\begin{pmatrix} 10 \\ 10 \\ 10 \end{pmatrix}$ in the range (i.e., column span) of the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 2 \\ 1 & 3 \end{pmatrix}$.

3.2.1 Analytics

We can perform row reduction or use the concept of matrix multiplication. The augmented matrix for the given system is:

$$\left(\begin{array}{cc|c} 1 & 2 & 10 \\ 0 & 2 & 10 \\ 1 & 3 & 10 \end{array}\right)$$

Performing row operations: 1. $R_3 = R_3 - R_1$ 2. $R_3 = R_3 - R_2$

Results in:

$$\left[\begin{array}{cc|c} 1 & 2 & 10 \\ 0 & 2 & 10 \\ 0 & 1 & 0 \end{array}\right]$$

Now, further row operations:
3. $R_1 = R_1 - 2R_2$ 4. $R_3 = R_3 - \frac{1}{2}R_2$
Result in:

$$\left[\begin{array}{cc|c} 1 & 0 & -10 \\ 0 & 2 & 10 \\ 0 & 0 & -5 \end{array}\right]$$

The last row indicates that the system is inconsistent (no solution), which means that the vector $\mathbf{v} = \begin{pmatrix} 10 \\ 10 \\ 10 \end{pmatrix}$ is not in the column span of the matrix A. Therefore, the vector cannot be represented as a linear combination of the columns of matrix A.

4 Question 4

5 Question 5