

# Introduction to Mathematics for Data Science

## Personal Assignment 2

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### 1 Question 1

#### 1.1 Solve Matrix B

Consider the matrix  $A = \begin{pmatrix} 5 & 2 & 3 \\ -2 & 3 & 1 \\ 0 & 2 & 2 \\ 1 & 0 & 1 \end{pmatrix}$  Find a matrix  $B$  such that  $AB = \begin{pmatrix} 8 & 6 \\ -1 & 9 \\ 2 & 6 \\ 2 & 0 \end{pmatrix}$ .

#### 1.2 Analytics

Due to the shape of matrix  $A = [a_{ij}]$  shape is  $4 \times 3$ , and the shape of matrix  $AB = [a_{ik}]$  is  $4 \times 2$ , we can use the matrix multiplication equation to solve for  $B$  (we can also infer that matrix  $B$  is 3 rows and 2 columns).

To solve for  $B$ , we have the matrix multiplication equation:

$$\begin{pmatrix} 5 & 2 & 3 \\ -2 & 3 & 1 \\ 0 & 2 & 2 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} = \begin{pmatrix} 8 & 6 \\ -1 & 9 \\ 2 & 6 \\ 2 & 0 \end{pmatrix}$$

Equating the corresponding elements with the given matrix  $AB$ , we get the following system

$$\text{of equations} \Rightarrow \begin{cases} 5b_{11} + 2b_{21} + 3b_{31} = 8 \\ 5b_{12} + 2b_{22} + 3b_{32} = 6 \\ -2b_{11} + 3b_{21} + b_{31} = -1 \\ -2b_{12} + 3b_{22} + b_{32} = 9 \\ 2b_{21} + 2b_{31} = 2 \\ 2b_{22} + 2b_{32} = 6 \\ b_{11} + b_{31} = 2 \\ b_{12} + b_{32} = 0 \end{cases} \quad \text{Then solve the equations and get } B = \begin{pmatrix} 1 & 0 \\ 0 & 3 \\ 1 & 0 \end{pmatrix}.$$

## 2 Question 2

### 2.1 Find Linear Transformation

Consider the vectors  $\vec{a} = \begin{pmatrix} 7 \\ 1 \\ 2 \end{pmatrix}$ ,  $\vec{b} = \begin{pmatrix} 8 \\ 1 \\ 4 \end{pmatrix}$ ,  $\vec{c} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$ . Is there a linear transformation  $f$ :

$\mathbb{R}^3 \rightarrow \mathbb{R}^4$  such that the following conditions are satisfied?  $f(\vec{a}) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ ,  $f(\vec{b}) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$  and

$f(\vec{c}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ . If there is such an  $f$ , then find its matrix. If there is not, then explain why.

#### 2.1.1 Analytics

We can represent the matrix  $f$  as:

$$f = \begin{pmatrix} | & | & | \\ \vec{f}(\vec{a}) & \vec{f}(\vec{b}) & \vec{f}(\vec{c}) \\ | & | & | \end{pmatrix}$$

Then, set up the following system of equations:

$$\begin{pmatrix} | & | & | \\ \vec{f}(\vec{a}) & \vec{f}(\vec{b}) & \vec{f}(\vec{c}) \\ | & | & | \end{pmatrix} \begin{pmatrix} 7 & 8 & -1 \\ 1 & 1 & 0 \\ 2 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

we can find the matrix  $f$  by solving this system of equations.

```
1 import numpy as np
2
3 # Define the given vectors
4 a = np.array([7, 1, 2])
5 b = np.array([8, 1, 4])
6 c = np.array([-1, 0, 2])
7
8 # Define the target matrices
9 fa_target = np.array([1, 0, 0, 0])
10 fb_target = np.array([0, 1, 0, 0])
```

```

11 fc_target = np.array([0, 0, 0, 0])
12
13 # Form the matrix equation [f] * [a, b, c] = [fa_target,
14     fb_target, fc_target]
15 A = np.vstack((a, b, c)).T # Transpose to make it 3x3
16 B_target = np.vstack((fa_target, fb_target, fc_target)).T #
17     Transpose to make it 3x4
18
19 # Solve for the matrix [f]
20 try:
21     f_matrix, residuals, rank, s = np.linalg.lstsq(A, B_target,
22         rcond=None)
23     print("Matrix [f]:")
24     print(f_matrix)
25 except np.linalg.LinAlgError:
26     print("No solution exists for the given conditions.")

```

**Output Analytics:** 'No solution exists for the given conditions.' It implies that there is no matrix  $f$  satisfying the specified conditions.

## 2.2 Another Linear Transformation

Is there a linear transformation  $f: \mathbb{R}^3 \rightarrow \mathbb{R}^4$  such that the following conditions are satisfied?

$$f(\vec{a}) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, f(\vec{b}) = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \text{ and } f(\vec{c}) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

### 2.2.1 Analytics

Bring  $f(\vec{a})$ ,  $f(\vec{b})$ ,  $f(\vec{c})$  to the Python program we created. The output is still '**No solution exists for the given conditions.**'

### 3 Question 3

#### 3.1 Find Value for x and y

Consider the linear system of equations

$$\begin{pmatrix} 6 & -4 \\ 8 & -6 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Either find a solution or explain why it does not have any solutions.

##### 3.1.1 Analytics

To determine whether the linear system has a solution, we can check the consistency of the system using the augmented matrix. The augmented matrix for the given system is:

$$\left( \begin{array}{cc|c} 6 & -4 & 1 \\ 8 & -6 & 1 \\ 2 & 0 & 1 \end{array} \right) \xrightarrow{\text{Row2}=\text{Row2}-\frac{4}{3}\times\text{Row1} \quad \text{Row3}=\text{Row3}-\frac{1}{3}\times\text{Row1}} \left( \begin{array}{cc|c} 6 & -4 & 1 \\ 0 & -\frac{2}{3} & -\frac{1}{3} \\ 0 & \frac{4}{3} & \frac{2}{3} \end{array} \right)$$

$$\xrightarrow{\text{Row2}\times(-3)} \left( \begin{array}{cc|c} 6 & -4 & 1 \\ 0 & 2 & 1 \\ 0 & \frac{4}{3} & \frac{2}{3} \end{array} \right)$$
$$\xrightarrow{\text{Row3}\times(\frac{3}{2})} \left( \begin{array}{cc|c} 6 & -4 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{array} \right)$$
$$\xrightarrow{\text{Row3}=\text{Row3}-\text{Row2}} \left( \begin{array}{cc|c} 6 & -4 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{array} \right)$$

$$\Rightarrow \begin{cases} 6x - 4y = 1 \\ 2y = 1 \end{cases} \Rightarrow y = \frac{1}{2}, x = \frac{1}{2}$$

**Conclusion:** The system has only one determine solution.  $x = \frac{1}{2}$ ,  $y = \frac{1}{2}$

#### 3.2 Vector Inclusion in Column Span

Determine whether or not the vector  $\begin{pmatrix} 10 \\ 10 \\ 10 \end{pmatrix}$  in the range (i.e., column span) of the matrix

$$\begin{pmatrix} 1 & 2 \\ 0 & 2 \\ 1 & 3 \end{pmatrix}.$$

### 3.2.1 Analytics

We can perform row reduction or use the concept of matrix multiplication. The augmented matrix for the given system is:

$$\left( \begin{array}{cc|c} 1 & 2 & 10 \\ 0 & 2 & 10 \\ 1 & 3 & 10 \end{array} \right)$$

Performing row operations:

$$\xrightarrow{R_1=R_1-R_2} \left( \begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 2 & 10 \\ 0 & 1 & 10 \end{array} \right) \xrightarrow{R_3=R_3-\frac{1}{2}R_2} \left( \begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 2 & 10 \\ 0 & 0 & 5 \end{array} \right)$$

The last row indicates that the system is inconsistent (no solution), which means that the vector  $\mathbf{v} = \begin{pmatrix} 10 \\ 10 \\ 10 \end{pmatrix}$  is not in the column span of the matrix  $A$ . Therefore, the vector cannot be represented as a linear combination of the columns of matrix  $A$ .

## 4 Question 4

Consider a discrete dynamical system:  $\begin{pmatrix} x_{i+1} \\ y_{i+1} \\ z_{i+1} \end{pmatrix} = A \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix}.$

$$A = \begin{pmatrix} 1.03552632 & 0.01842105 & -0.16447368 \\ -0.00921053 & 1.10263158 & -0.00921053 \\ -0.13815789 & 0.03947368 & 1.06184211 \end{pmatrix}$$

### 4.1 Calculate the Next State

If  $(x_0, y_0, z_0) = (100, 100, 100)$ , calculate  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$

#### 4.1.1 Analytics

```
1 import numpy as np
2
3 # Given matrix A
4 A = np.array([
5     [1.03552632, 0.01842105, -0.16447368],
6     [-0.00921053, 1.10263158, -0.00921053],
7     [-0.13815789, 0.03947368, 1.06184211]
8 ])
9
10 # Initial state (x0, y0, z0)
11 initial_state = np.array([100, 100, 100])
12
13 # Calculate the next states (x1, y1, z1) and (x2, y2, z2)
14 state_1 = np.dot(A, initial_state)
15 state_2 = np.dot(A, state_1)
16
17 # Print the results
18 print("Initial State:", initial_state)
19 print("Next State (x1, y1, z1):", state_1)
20 print("Next State (x2, y2, z2):", state_2)
```

#### 4.1.2 Result

Initial State: [100 100 100]

Next State (x1, y1, z1): [88.947369 108.421052 96.31579]

Next State (x2, y2, z2): [78.26315889 117.84210399 94.26315877]

## 4.2 Combination of Eigenvectors

Given the initial state vector  $(x_0, y_0, z_0) = (100, 100, 100)$ , write this vector as a linear combination of eigenvectors of  $A$ .

### 4.2.1 Analytics

To express the initial state vector  $(x_0, y_0, z_0) = (100, 100, 100)$  as a linear combination of the eigenvectors of matrix  $A$ , we need to find the eigenvectors and eigenvalues of  $A$ .

Let  $v_1, v_2, v_3$  be the eigenvectors of  $A$  corresponding to the eigenvalues  $\lambda_1, \lambda_2, \lambda_3$ , respectively.

The expression for the initial state vector as a linear combination of eigenvectors is given by:

$$\mathbf{x}_0 = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3$$

where  $c_1, c_2, c_3$  are the coefficients to be determined.

Let's calculate the eigenvectors and eigenvalues using NumPy:

```
1 import numpy as np
2
3 # Given matrix A
4 A = np.array([
5     [1.03552632, 0.01842105, -0.16447368],
6     [-0.00921053, 1.10263158, -0.00921053],
7     [-0.13815789, 0.03947368, 1.06184211]
8 ])
9
10 # Calculate eigenvectors and eigenvalues
11 eigenvalues, eigenvectors = np.linalg.eig(A)
12
13 # Given initial state vector
14 x0 = np.array([100, 100, 100])
15
16 # Solve for coefficients c1, c2, c3
17 coefficients = np.linalg.solve(eigenvectors, x0)
18
```

```

19 # Print the results
20 print("Eigenvectors:")
21 print(eigenvectors)
22 print("\nEigenvalues:")
23 print(eigenvalues)
24 print("\nInitial state vector as a linear combination of
    eigenvectors:")
25 print(f"x0={coefficients[0]:.2f}*v1+{coefficients[1]:.2f}
    *v2+{coefficients[2]:.2f}*v3")

```

This code calculates the eigenvectors and eigenvalues of matrix  $A$  and then solves for the coefficients  $c_1, c_2, c_3$  in the expression  $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ .

Note: The solution for coefficients may involve complex numbers if the eigenvalues are complex. If that's the case, you would represent the eigenvectors in terms of complex numbers.



## 5 Question 5