

Introduction to Math for DS Group Task 3

IMDS Group 24

Zehao Qian, Mohammad Jamshaid Iqbal, Chloe Mendez

November 17, 2023

1 Question 1

Are the vectors $(1, 3, 4)$, $(4, -1, 5)$ and $(5, 2, 9)$ linearly independent?

1.1 AI Bot's Answer

ChatGPT: To determine if the vectors $\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$, $\begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix}$, and $\begin{bmatrix} 5 \\ 2 \\ 9 \end{bmatrix}$ are linearly independent, you can create a matrix using these vectors and perform row operations to see if the matrix reduces to the row-echelon form with non-zero rows.

Let A be the matrix with the given vectors as its columns:

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 3 & -1 & 2 \\ 4 & 5 & 9 \end{bmatrix}$$

To determine whether the vectors $(1, 3, 4)$, $(4, -1, 5)$, and $(5, 2, 9)$ are linearly independent, ChatGPT sets up a matrix with these vectors as its columns and then perform row reduction to check for linear independence.

Now, perform row operations to get the row-echelon form:

$$\begin{bmatrix} 1 & 4 & 5 \\ 3 & -1 & 2 \\ 4 & 5 & 9 \end{bmatrix}$$

$$\begin{aligned}
 R_2 &\xrightarrow{R_2 - 3R_1} \begin{bmatrix} 1 & 4 & 5 \\ 0 & -13 & -13 \\ 4 & 5 & 9 \end{bmatrix} \\
 R_3 &\xrightarrow{R_3 - 4R_1} \begin{bmatrix} 1 & 4 & 5 \\ 0 & -13 & -13 \\ 0 & -11 & -11 \end{bmatrix} \\
 R_3 &\xrightarrow{R_3 - \frac{11}{13}R_2} \begin{bmatrix} 1 & 4 & 5 \\ 0 & -13 & -13 \\ 0 & 0 & 0 \end{bmatrix}
 \end{aligned}$$

Now, we have the row-echelon form. Since there is a row of zeros, the system is linearly dependent. Specifically, the third vector can be expressed as a linear combination of the first two vectors.

Therefore, the vectors $\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$, $\begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix}$, and $\begin{bmatrix} 5 \\ 2 \\ 9 \end{bmatrix}$ are linearly dependent.

ChatGPT's output is correct.

1.2 Analytics

Although the answer output of the AI bot is correct, we can use other methods to determine whether the vector is linearly dependent in addition to converting the matrix into a stepped type.

Another way is to calculate the value of the determinant. If these three vectors are linearly independent, then the determinant of the matrix they form should not be equal to zero.

The determinant of $\det(A)$, if $\det(A) \neq 0$, then the three vectors are linearly independent. The code is below:

```

1 import numpy as np
2
3 # Define the matrix
4 matrix = np.array([[1, 4, 5],
5                    [3, -1, 2],
6                    [4, 5, 9]])
7

```

```

8 # Calculate the determinant of the matrix
9 determinant = np.linalg.det(matrix)
10
11 # Check for linear independence
12 if determinant != 0:
13     print("Vectors are linearly independent.")
14 else:
15     print("Vectors are linearly dependent.")

```

Output: Vectors are linearly dependent. The conclusion is the same as that generated by ChatGPT, although we used a different method.

2 Question 2

Is the vector $(1, 0, -1)$ in the plane spanned by the vectors $(4, 11, 7)$ and $(3, 11, 8)$?

2.1 AI Bot's Answer

ChatGPT: To determine whether the vector $(1, 0, -1)$ is in the plane spanned by the vectors $(4, 11, 7)$ and $(3, 11, 8)$, we can check if $(1, 0, -1)$ can be expressed as a linear combination of the other two vectors.

Let A be the matrix formed by the vectors $(4, 11, 7)$ and $(3, 11, 8)$, and \mathbf{v} be the vector $(1, 0, -1)$:

$$A = \begin{bmatrix} 4 & 3 \\ 11 & 11 \\ 7 & 8 \end{bmatrix}$$

We want to check if there exist scalars x and y such that $A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

Solving the system of equations:

$$\begin{bmatrix} 4 & 3 \\ 11 & 11 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

This system is inconsistent (no solution) because the augmented matrix $[A|\mathbf{v}]$ does not reduce to the identity matrix. Therefore, $(1, 0, -1)$ is not in the plane spanned

by $(4, 11, 7)$ and $(3, 11, 8)$.

Output: Vectors are linearly dependent. ChatGPT gave the idea of using a system of linear equations to make a decision if $(1, 0, -1)$ is in the plane spanned by $(4, 11, 7)$ and $(3, 11, 8)$. However, it only gives the method, not the result. So we have to calculate it by Jupyter Notebook.

2.2 Analytics

```
1 import numpy as np
2
3 # Define the vectors
4 v = np.array([1, 0, -1])
5 u1 = np.array([4, 11, 7])
6 u2 = np.array([3, 11, 8])
7
8 # Create the coefficient matrix and the constant vector
9 coeff_matrix = np.array([[4, 3], [11, 11], [7, 8]])
10 constants = np.array([1, 0, -1])
11
12 # Solve the system of equations
13 coefficients = np.linalg.lstsq(coeff_matrix, constants, rcond=
    None)[0]
14
15 # Check if a solution exists
16 if np.allclose(np.dot(coeff_matrix, coefficients), constants):
17     print("The vector (1, 0, -1) is in the plane spanned by (4,
        11, 7) and (3, 11, 8).")
18 else:
19     print("The vector (1, 0, -1) is not in the plane spanned by
        (4, 11, 7) and (3, 11, 8).")
20
21 print("x =", coefficients[0])
22 print("y =", coefficients[1])
```

Output:

Vectors are linearly dependent. The vector $(1, 0, -1)$ is in the plane spanned by $(4, 11, 7)$ and $(3, 11, 8)$.

$x = 0.9999999999999999$

$y = -0.9999999999999994$

Solving a system of linear equations with NumPy gives that the vector $(1, 0, -1)$ can be spanned by two other vectors. However, the calculated x and y numbers were very strange, so we solved them manually again, and the result was as follows:

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 11 \\ 7 \end{bmatrix} - \begin{bmatrix} 3 \\ 11 \\ 8 \end{bmatrix} \Rightarrow x = 1, y = -1$$

When IEEE 754-2008: IEEE Standard for Floating-Point Arithmetic in a computer, small errors can occur due to a limited number of digits. This is due to the finite precision nature of floating-point representations.

That's why the x, y coefficient we calculate via NumPy is a floating-point number. But the final conclusion is correct:

The vector $(1, 0, -1)$ is in the plane spanned by $(4, 11, 7)$ and $(3, 11, 8)$.