IMDS 2023 — Individual Assessment 2
Due: Friday, 1 December at noon

Instructions

You will upload your work electronically via *Gradescope* (accessed through Blackboard). You may do some or all of your work by hand on paper and then scan or photograph your pages.

You may discuss these questions with classmates, but you should write out your solutions individually. Your solutions should be explained step-by-step in complete sentences.

You may work these problems by hand, and you may additionally use any of the following computer tools:

- Python code based on what we have covered in workshops;
- Desmos or Python for plotting functions.

Your submission should include any code that you use, with sufficient explanation of the code.

Please ask Prof. Giansiracusa if you need clarification of anything on this assignment.

Question 1 [20 points]

Consider the matrix $A = \begin{pmatrix} 5 & 2 & 3 \\ -2 & 3 & 1 \\ 0 & 2 & 2 \\ 1 & 0 & 1 \end{pmatrix}$ Find a matrix B such that $AB = \begin{pmatrix} 8 & 6 \\ -1 & 9 \\ 2 & 6 \\ 2 & 0 \end{pmatrix}$.
Be sure to explain how you found this matrix.

Question 2 [20 points]

a) Consider the vectors

$$\vec{a} = \begin{pmatrix} 7 \\ 1 \\ 2 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 8 \\ 1 \\ 4 \end{pmatrix}, \quad \vec{c} = \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix}.$$

Is there a linear transformation $f : \mathbb{R}^3 \to \mathbb{R}^4$ such that the following conditions are satisfied?

$$f(\vec{a}) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad f(\vec{b}) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \text{and } f(\vec{c}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

If there is such an f, then find its matrix. If there is not, then explain why.

b) Is there a linear transformation $f : \mathbb{R}^3 \to \mathbb{R}^4$ such that the following conditions are satisfied?

$$f(\vec{a}) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad f(\vec{b}) = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad \text{and } f(\vec{c}) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

If there is such an f, then find its matrix. If there is not, then explain why.

Question 3 [20 points]

a) Consider the linear system of equations

$$\begin{pmatrix} 6 & -4 \\ 8 & -6 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Either find a solution or explain why it does not have any solutions.

b) Determine whether or not the vector $\begin{pmatrix} 10 \\ 10 \\ 10 \end{pmatrix}$ in the range (i.e., column span) of the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 2 \\ 1 & 3 \end{pmatrix}$.

Question 4 [20 points]

Consider a discrete dynamical system

$$\begin{pmatrix} x_{i+1} \\ y_{i+1} \\ z_{i+1} \end{pmatrix} = A \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix}.$$

with $A = \begin{pmatrix} 1.03552632 & 0.01842105 & -0.16447368 \\ -0.00921053 & 1.10263158 & -0.00921053 \\ -0.13815789 & 0.03947368 & 1.06184211 \end{pmatrix}$.

1. If $(x_0, y_0, z_0) = (100, 100, 100)$, calculate (x_1, y_1, z_1) and (x_2, y_2, z_2).
2. Given the initial state vector $(x_0, y_0, z_0) = (100, 100, 100)$, write this vector as a linear combination of eigenvectors of A.
3. Find an initial state vector (x_0, y_0, z_0) of norm 10 such that the norm of (x_i, y_i, z_i) gets smaller and smaller as $i \to \infty$.

Question 4 [20 points] Find a matrix that has eigenvalues $3, 4, 5$ with corresponding eigenvectors

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Explain how you found this matrix.