Multilevel Modelling Practical 6 (Week 7)

Instructions - start here!

Exercise 1 considers long and wide data shapes, illustrated by the Oxford boys data set. **Exercise 2** involves a simulation exercise (simulating from a two-level longitudinal model). You may also use this lab to work on the formative assignment.

Let's begin by loading the necessary packages and data:

```r
require(lme4)
require(lmerTest)
require(ggplot2)
```

Exercise 1 (Long and wide data shape)

Let's return to the Oxboys data, discussed in the lecture, which is a built-in R data set:

```r
require(nlme)
data(Oxboys)
```

This data set was by default given in **long** format. But assume we wanted this data in **wide** format, for instance in order to carry out a multivariate analysis.

Let's have a brief look at the data frame:

```r
head(Oxboys)
```

```
## Grouped Data: height ~ age | Subject
##   Subject     age height Occasion
## 1       1 -1.0000  140.5        1
## 2       1 -0.7479  143.4        2
## 3       1 -0.4630  144.8        3
## 4       1 -0.1643  147.1        4
## 5       1 -0.0027  147.7        5
## 6       1  0.2466  150.2        6
```

```r
dim(Oxboys)
```

```
## [1] 234   4
```

We see that there are two variables capturing the "time" component:

- the `age` variable
- the `Occasion` variable

There is a one-to-one relationship between these two variables, so for the purpose of creating a "wide" data frame, we can choose any of these. In this case, we have decided to drop `age`:

```r
Oxboys.wide <- reshape(Oxboys, direction="wide", idvar="Subject", timevar="Occasion", drop="age")
```

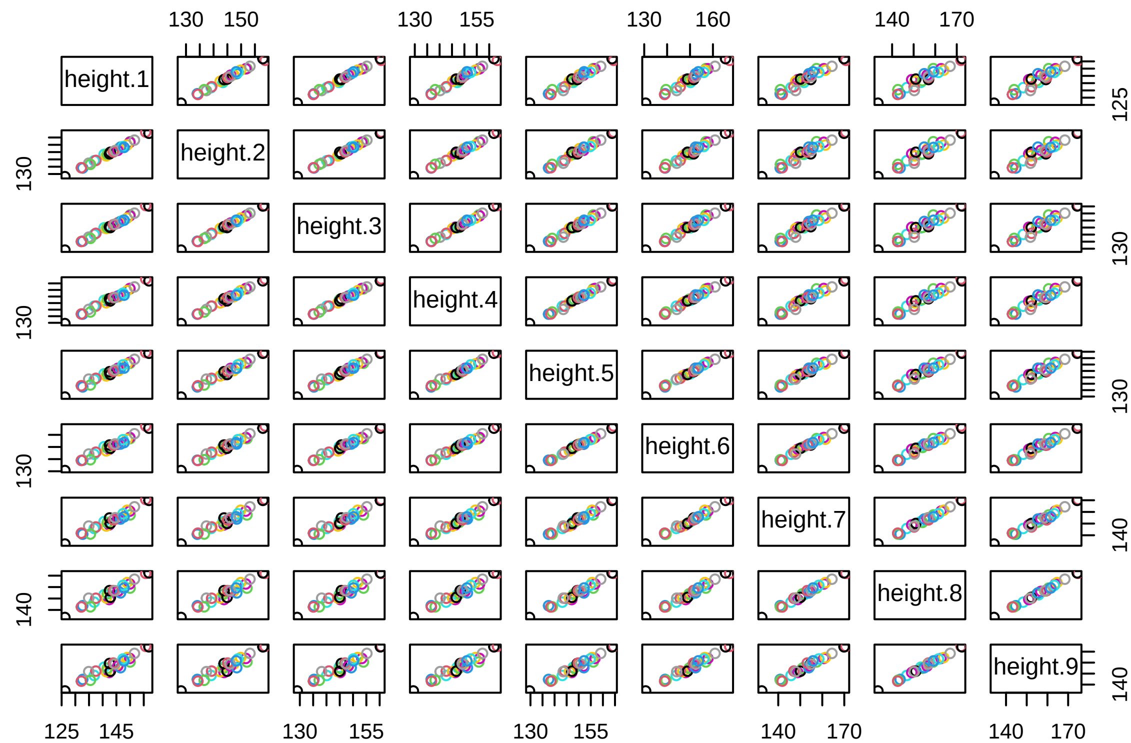
```r
head(Oxboys.wide)
```

```
##    Subject height.1 height.2 height.3 height.4 height.5 height.6 height.7
## 1        1    140.5    143.4    144.8   147.10   147.70    150.2    151.7
```

```
## 10        2    136.9    139.1    140.1    142.60    143.20    144.0    145.8
## 19        3    150.0    152.1    153.9    155.80    156.00    156.9    157.4
## 28        4    155.7    158.7    160.6    163.30    164.40    167.3    170.7
## 37        5    145.8    147.3    148.7    149.78    150.22    152.5    154.8
## 46        6    142.4    143.8    145.2    146.30    147.10    148.1    148.9
##    height.8 height.9
## 1     153.3    155.8
## 10    146.8    148.3
## 19    159.1    160.6
## 28    172.0    174.8
## 37    156.4    158.7
## 46    149.1    151.0
```

We can produce pairwise scatterplots of the data, which gives a sense of the correlation between heights measured at different times:

```
plot(Oxboys.wide[,-1], col=Oxboys.wide$Subject)
```



Let's re-transform the data back into a long data format:

```
Oxboys.long<- reshape(Oxboys.wide, direction="long",
        idvar="Subject", varying=list(2:10),v.names="height" )
head(Oxboys.long)
```

```
##     Subject time height
## 1.1       1    1  140.5
## 2.1       2    1  136.9
## 3.1       3    1  150.0
```

```
## 4.1       4    1  155.7
## 5.1       5    1  145.8
## 6.1       6    1  142.4
```

TASK: Convince yourself that the `Oxboys.long` data frame is equivalent to the original `Oxboys` frame. Check also the help file for `reshape` and make sure you're happy with the syntax.

Regression vs multilevel analysis

By ignoring group structure, we can fit a regression model of the form

$$y_{ti} = a + bT_{ti} + \epsilon_{ti}$$

```
model.lm <- lm(height ~ age, data=Oxboys)
```

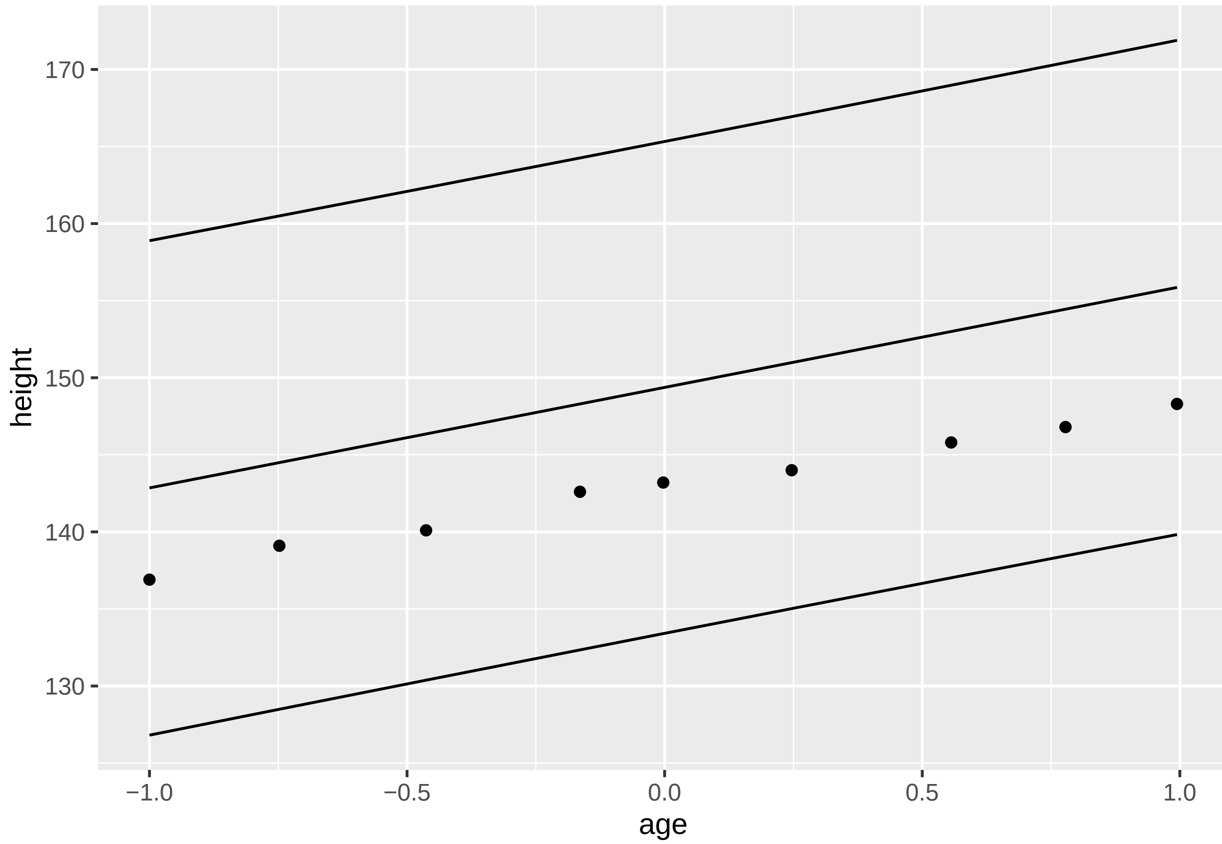
Let's also obtain prediction intervals for each response value:

```
lmpred <- data.frame(predict(model.lm,interval="prediction"))
lmpred$age <- Oxboys$age
lmpred$height <- Oxboys$height
lmpred$Subject <- Oxboys$Subject
head(lmpred)
```

```
##        fit      lwr      upr     age height Subject
## 1 142.8508 126.8115 158.8901 -1.0000  140.5       1
## 2 144.4947 128.4920 160.4975 -0.7479  143.4       1
## 3 146.3526 130.3788 162.3263 -0.4630  144.8       1
## 4 148.3004 132.3430 164.2578 -0.1643  147.1       1
## 5 149.3542 133.3995 165.3088 -0.0027  147.7       1
## 6 150.9799 135.0212 166.9386  0.2466  150.2       1
```

We see the fitted value $\hat{y}_{ti} = \hat{a} + \hat{b}T_{ti}$, lower and upper limits of a 95% prediction interval, age, height and the subject label. Let's plot the prediction interval for subject 2:

```
lmpredS2 <- lmpred[lmpred$Subject==2,]
ggplot(lmpredS2,aes(age,height))+
  geom_point()+
  geom_line(aes(y=fit))+
  geom_line(aes(y=lwr))+
  geom_line(aes(y=upr))
```



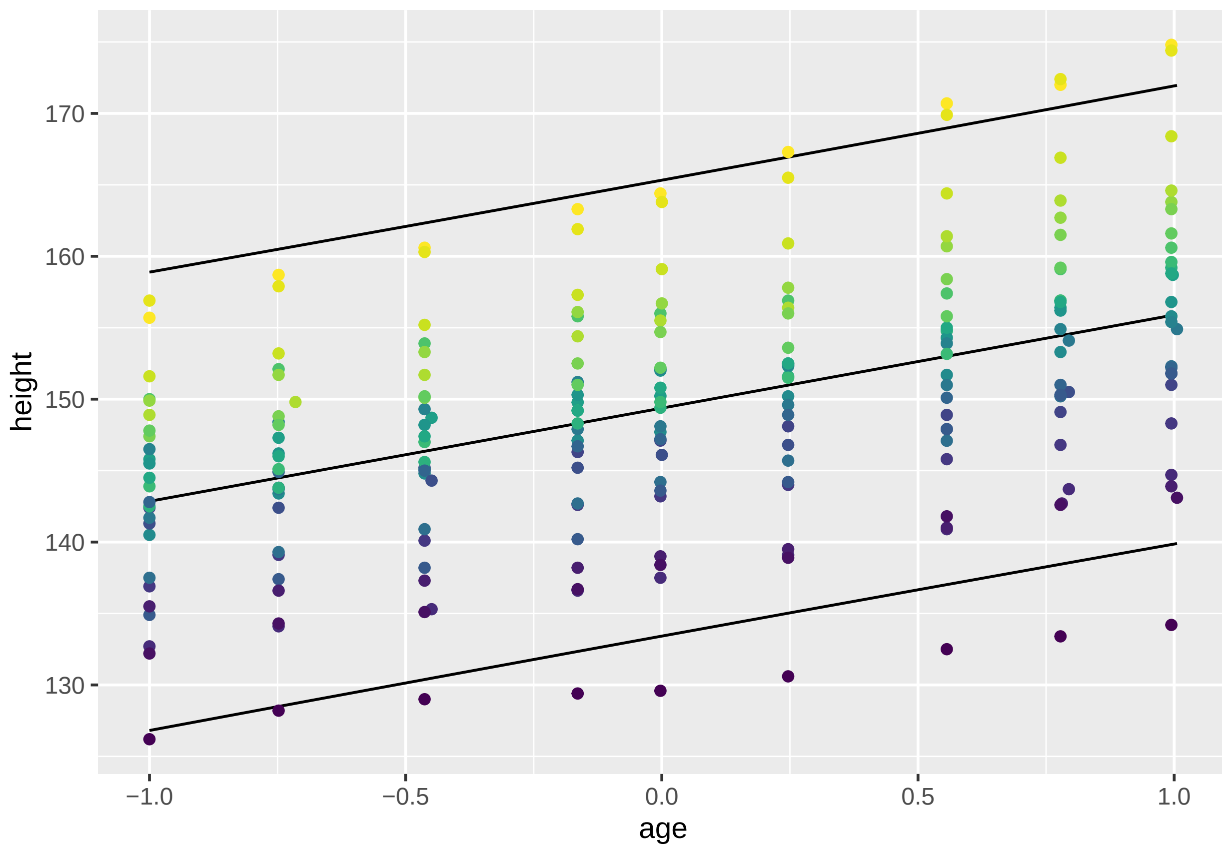
This doesn't look terrible, although the fit tracks somewhat above the actual data.

TASK: Try generating the above plots for different subjects (in particular, try subjects 1 and 10). Does the prediction interval change? If not, why not?

Click for solution

The prediction interval does not change. The model ignores group structure and can only explain one source of variation (within individuals) but not the variation between individuals. Consequently, the prediction intervals use the estimated error variance based on the entire data set, irrespective of which individual (subject) we're looking at. We can see this by plotting the full data set and overlaying the prediction interval:

```
ggplot(lmpred,aes(age,height))+
  geom_line(aes(y=fit))+
  geom_line(aes(y=lwr))+
  geom_line(aes(y=upr))+
  geom_point(aes(col=Subject),show.legend=FALSE)
```



Now, let's fit a random intercept model of the form

$$y_{ti} = a + u_i + bT_{ti} + \epsilon_{ti}$$

where the random intercept terms are $u_i \sim N(0, \sigma_u^2)$. We use the following code to fit the model:

```
model.lmer <- lmer(height ~ 1+age+(1|Subject),data=Oxboys)
```

To obtain prediction intervals based on the `lmer` output, we need the following package:

```
require(merTools)
```

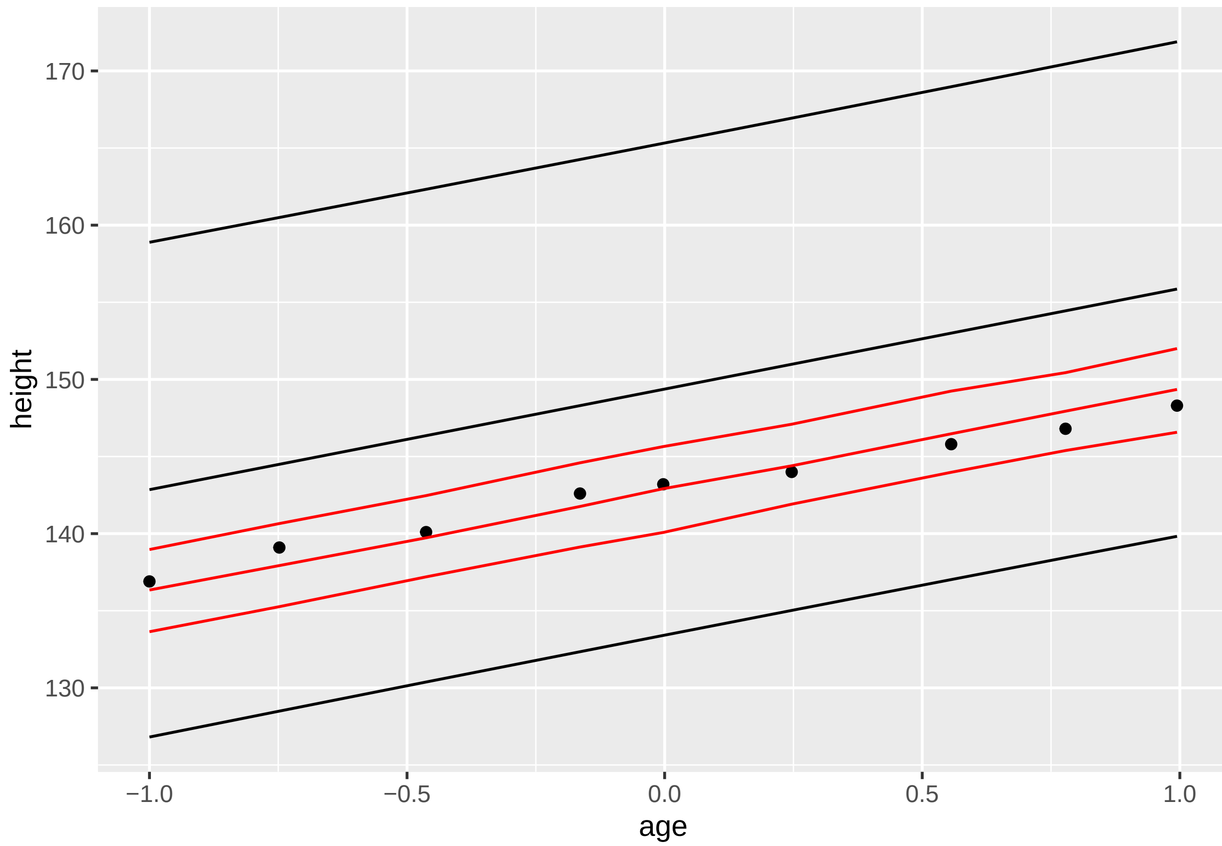
Now obtain prediction intervals for each response value and append to the `lmpred` data frame to give a new data frame `pred`:

```
lmerpred <- data.frame(predictInterval(model.lmer))
names(lmerpred) <- c("fit2","lwr2","upr2")
pred <- cbind(lmerpred,lmpred)
head(pred)
```

```
##       fit2     lwr2     upr2      fit      lwr      upr     age height Subject
## 1 141.7069 144.2978 138.8063 142.8508 126.8115 158.8901 -1.0000  140.5       1
## 2 143.2867 146.0491 140.6010 144.4947 128.4920 160.4975 -0.7479  143.4       1
## 3 145.1203 147.8607 142.2963 146.3526 130.3788 162.3263 -0.4630  144.8       1
## 4 147.0543 149.6969 144.2220 148.3004 132.3430 164.2578 -0.1643  147.1       1
## 5 148.1349 150.8230 145.5061 149.3542 133.3995 165.3088 -0.0027  147.7       1
## 6 149.8686 152.5781 147.0376 150.9799 135.0212 166.9386  0.2466  150.2       1
```

Finally, we can overlay prediction intervals for subject 2:

```
predS2 <- pred[pred$Subject==2,]
ggplot(predS2,aes(age,height))+
  geom_point()+
  geom_line(aes(y=fit))+
  geom_line(aes(y=lwr))+
  geom_line(aes(y=upr))+
  geom_line(aes(y=fit2),col="red")+
  geom_line(aes(y=lwr2),col="red")+
  geom_line(aes(y=upr2),col="red")
```



The prediction interval based on the random intercept model is much tighter (since the within subjects variation is considerably lower than for the simple linear regression model which ignores group structure).

To see this, consider first the summary of the linear regression model:

```
summary(model.lm)
```

```
##
## Call:
## lm(formula = height ~ age, data = Oxboys)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -21.6570  -5.1403   0.4872   4.7514  18.9430
##
## Coefficients:
```

```
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept) 149.3718     0.5286 282.599  < 2e-16 ***
## age           6.5210     0.8170   7.982 6.64e-14 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 8.081 on 232 degrees of freedom
## Multiple R-squared:  0.2154, Adjusted R-squared:  0.2121
## F-statistic: 63.71 on 1 and 232 DF,  p-value: 6.635e-14
```

The square of the residual standard error gives an estimate of the residual error variance σ^2.

Now consider the summary of the random intercept model:

```r
summary(model.lmer)
```

```
## Linear mixed model fit by REML. t-tests use Satterthwaite's method [
## lmerModLmerTest]
## Formula: height ~ 1 + age + (1 | Subject)
##    Data: Oxboys
##
## REML criterion at convergence: 940
##
## Scaled residuals:
##     Min      1Q  Median      3Q     Max
## -2.1857 -0.6350 -0.1339  0.6252  2.5357
##
## Random effects:
##  Groups   Name        Variance Std.Dev.
##  Subject  (Intercept) 65.555   8.097
##  Residual              1.718   1.311
## Number of obs: 234, groups:  Subject, 26
##
## Fixed effects:
##             Estimate Std. Error       df t value Pr(>|t|)
## (Intercept) 149.3717     1.5902  25.0002   93.93   <2e-16 ***
## age           6.5239     0.1325 207.0000   49.23   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Correlation of Fixed Effects:
##     (Intr)
## age -0.002
```

Look at the random effect variance estimate σ_u^2 versus the estimate of residual variance σ^2. Most of the variation is between subjects! Hence, after accounting for group structure, the within subjects variation is relatively small.

Exercise 2 (simulation from the two-level longitudinal model)

Suppose that we want to set up and simulate from a hypothetical model of MATH43515 student stress levels (for 30 students) over a 6 week period with the following variables:

- `stress` - response variable on [0,100] with 100 representing max stress.
- `week` - time covariate taking values 1 to 6.

- `ML` - a binary (upper level) covariate taking the value 1 if a student has taken the Machine Learning module and 0 otherwise.
- `ID` - a unique student identifier.

Imagine that the model we want to simulate from takes the form

$$y_{ti} = a + u_i + bT_{ti} + v_i T_{ti} + cz_i + \epsilon_{ti}, \quad i = 1, \ldots, 30, \quad t = 1, \ldots, 6$$

where $T_{ti} = t - 1$ represents week number, z_i represents the binary `ML` variable, $u_i \sim N(0, \sigma_u^2)$, $v_i \sim N(0, \sigma_v^2)$ and $\epsilon_{ti} \sim N(0, \sigma^2)$.

Let's set up a data frame within which to store the simulated data:

```
set.seed(43515)
ID <- rep(seq(1,30),6)
ML <- rep(sample(0:1,30,replace=TRUE),6)
week <- rep(seq(0,5),each=30)
stress <- rep(0,180) #overwrite this later
data <- data.frame(ID,stress,week,ML)
```

We will need to pick some parameter values. How about:

```
a <- 40 #baseline stress level
b <- 5 #stress increases 5 units with every week
c <- 5 #if you're taking ML, expected stress increases by 5 units!
sigu <- 1 # Random intercept standard deviation
sigv <- 1 # Random slope standard deviation
sig <- 1 #error standard deviation
```

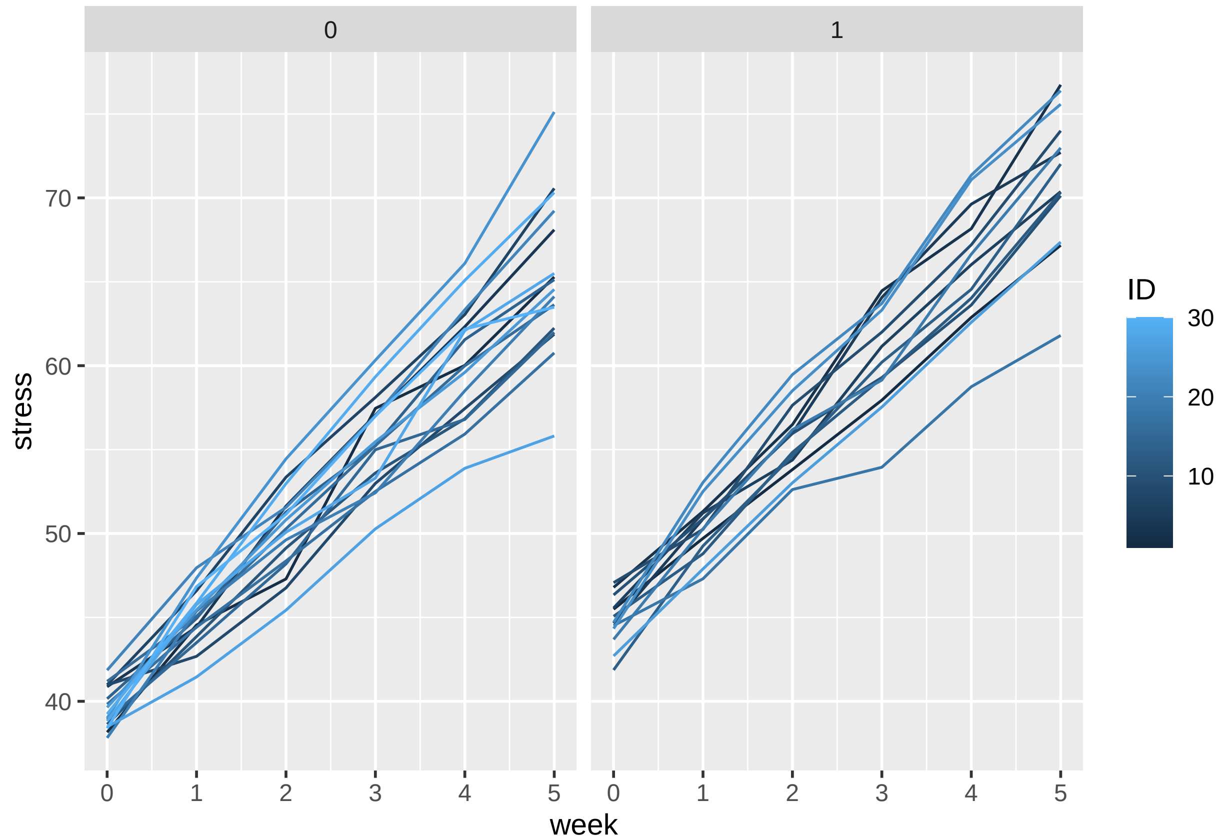
Now simulate from the model. We will do this by looping over time inside a loop over individuals:

```
set.seed(43515) #for reproducibility
#Simulate individual random effects
ui <- rnorm(30,0,sigu)
vi <- rnorm(30,0,sigv)
for(t in 1:6)
{
 for(i in 1:30)
 {
  #simulate response at each time within individuals
  data$stress[(t-1)*30+i] <-
    a+ui[i]+(b+vi[i])*(t-1)+c*data$ML[(t-1)*30+i]+rnorm(1,0,sig)
  }
}
head(data)
```

```
##   ID   stress week ML
## 1  1 45.48698    0  1
## 2  2 38.15454    0  0
## 3  3 46.78894    0  1
## 4  4 40.86827    0  0
## 5  5 44.64929    0  1
## 6  6 45.55972    0  1
```

Notice how the random effects only change from individual to individual. Let's visualise the data we've generated:

```r
ggplot(data,aes(x=week,y=stress,group=ID,col=ID))+
  geom_line()+
  facet_wrap(~ML)
```



Fit the model from which the data were simulated and check that the parameter estimates are consistent with the ground truth:

```r
model.synth <- lmer(stress ~ 1+week+ML+(1+week|ID),data=data)
summary(model.synth)
```

```
## Linear mixed model fit by REML. t-tests use Satterthwaite's method [
## lmerModLmerTest]
## Formula: stress ~ 1 + week + ML + (1 + week | ID)
##    Data: data
##
## REML criterion at convergence: 639.4
##
## Scaled residuals:
##      Min       1Q   Median       3Q      Max
## -2.50911 -0.59396  0.02096  0.62229  2.45261
##
## Random effects:
##  Groups   Name        Variance Std.Dev. Corr
##  ID       (Intercept) 0.5436   0.7373
##           week        0.5565   0.7460   0.50
##  Residual             1.0127   1.0063
## Number of obs: 180, groups:  ID, 30
```

```
##
## Fixed effects:
##             Estimate Std. Error      df t value Pr(>|t|)
## (Intercept)  39.8509     0.2499 28.3719  159.44  < 2e-16 ***
## week          5.1770     0.1431 29.0001   36.17  < 2e-16 ***
## ML            5.0378     0.3769 28.0001   13.37 1.12e-13 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Correlation of Fixed Effects:
##      (Intr) week
## week  0.122
## ML   -0.653  0.000
```

All looks well - the fixed effect estimates and random effect variances are consistent with the ground truth values that generated the data. There are various ways in which you could perform further analysis and check against what you expect to see e.g.

- Check to see that the interaction between `ML` and `week` is insignificant.

Click for solution

Add in the cross level interaction as follows:

```r
model.synth2 <- lmer(stress ~ 1+week+ML+week:ML+(1+week|ID),data=data)
summary(model.synth2)
```

```
## Linear mixed model fit by REML. t-tests use Satterthwaite's method [
## lmerModLmerTest]
## Formula: stress ~ 1 + week + ML + week:ML + (1 + week | ID)
##    Data: data
##
## REML criterion at convergence: 639.7
##
## Scaled residuals:
##      Min       1Q   Median       3Q      Max
## -2.49292 -0.59874  0.02114  0.62647  2.45536
##
## Random effects:
##  Groups   Name        Variance Std.Dev. Corr
##  ID       (Intercept) 0.5443   0.7378
##           week        0.5720   0.7563   0.50
##  Residual             1.0127   1.0063
## Number of obs: 180, groups:  ID, 30
##
## Fixed effects:
##             Estimate Std. Error      df t value Pr(>|t|)
## (Intercept)  39.8363     0.2514 27.9998 158.433  < 2e-16 ***
## week          5.1088     0.1925 28.0001  26.542  < 2e-16 ***
## ML            5.0713     0.3820 27.9998  13.277 1.32e-13 ***
## week:ML       0.1573     0.2924 28.0001   0.538    0.595
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Correlation of Fixed Effects:
##         (Intr) week   ML
## week     0.163
```

```
## ML      -0.658 -0.107
## week:ML -0.107 -0.658  0.163
```

The p-value is well above the 5% threshold indicating insufficient evidence to reject the null hypothesis that the fixed effect for the cross level intereaction is zero. We therefore conclude that this term is not needed.

- Check that the random time slope is needed (we know it is!)

Click for solution

Perform a likelihood ratio test of the null hypothesis $H_0 : \sigma_v = 0$ as follows:

```
ranova(model.synth)
```

```
## ANOVA-like table for random-effects: Single term deletions
##
## Model:
## stress ~ week + ML + (1 + week | ID)
##                         npar  logLik    AIC    LRT Df Pr(>Chisq)
## <none>                     7 -319.69 653.38
## week in (1 + week | ID)    5 -387.72 785.43 136.06  2  < 2.2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

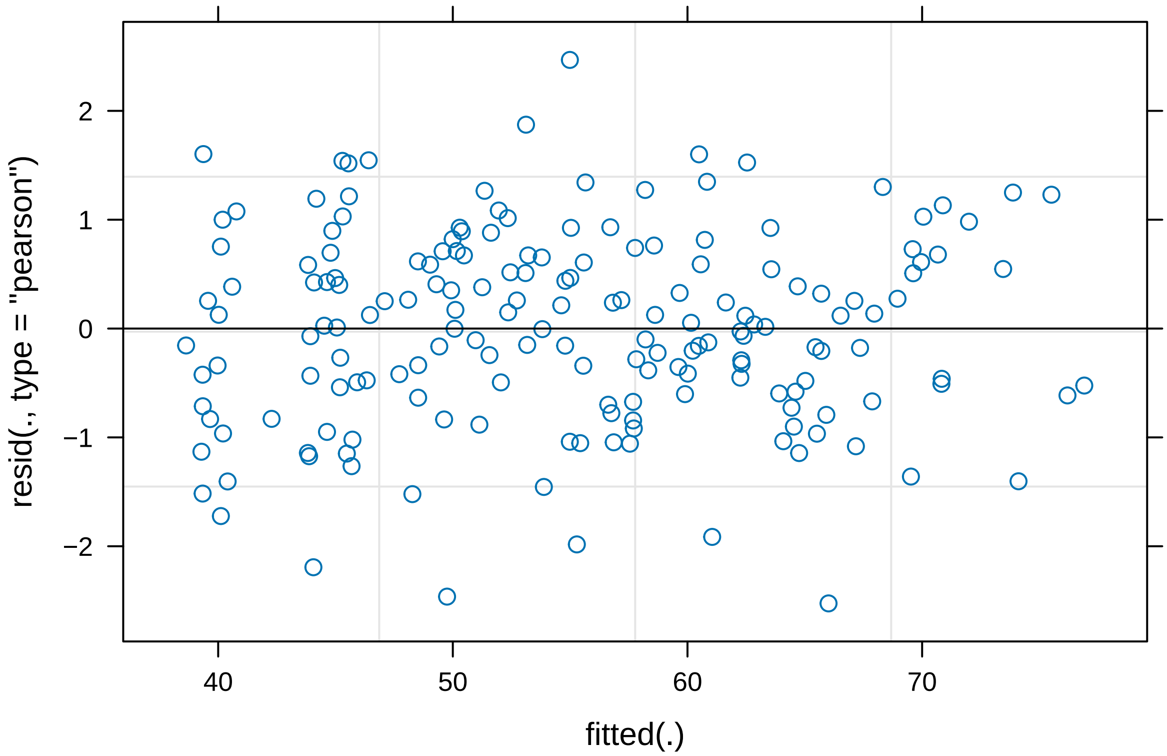
We see a very small p-value suggesting strong evidence against the null hypothesis. We conlude that the random slope on `week` is needed.

- Check the residual diagnostics.

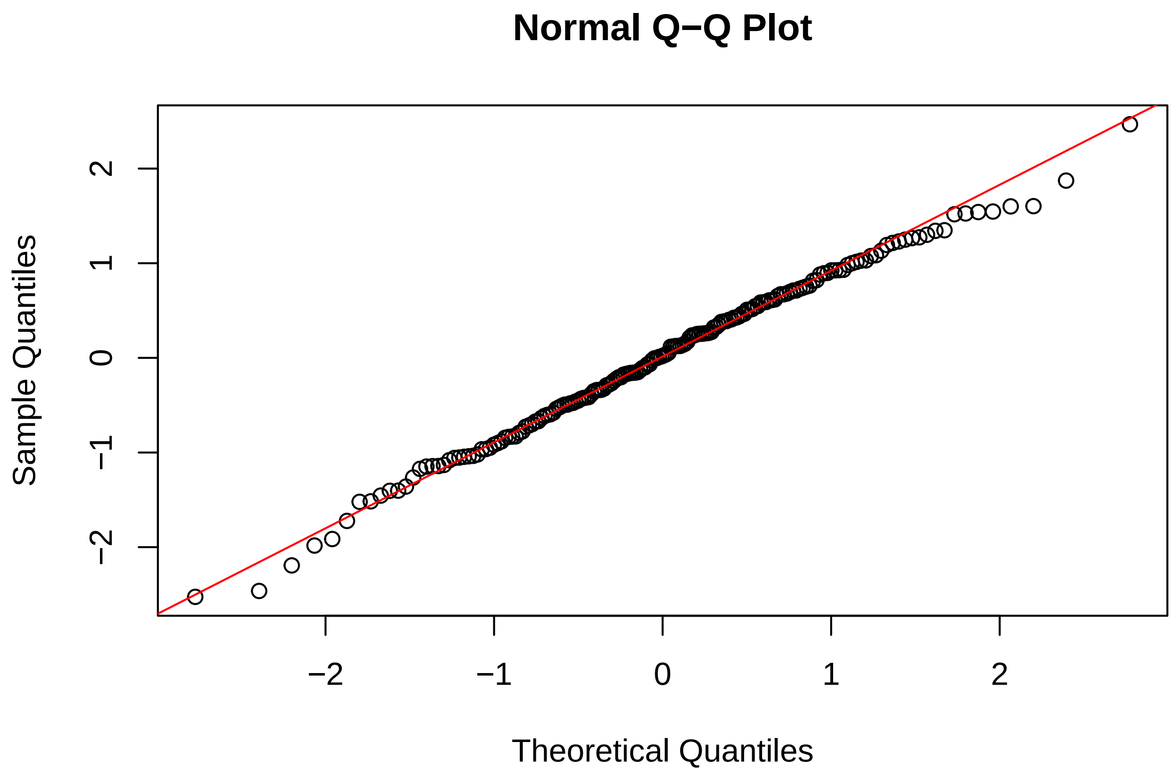
Click for solution

Check residuals versus fitted values and normality of residuals and random effects:

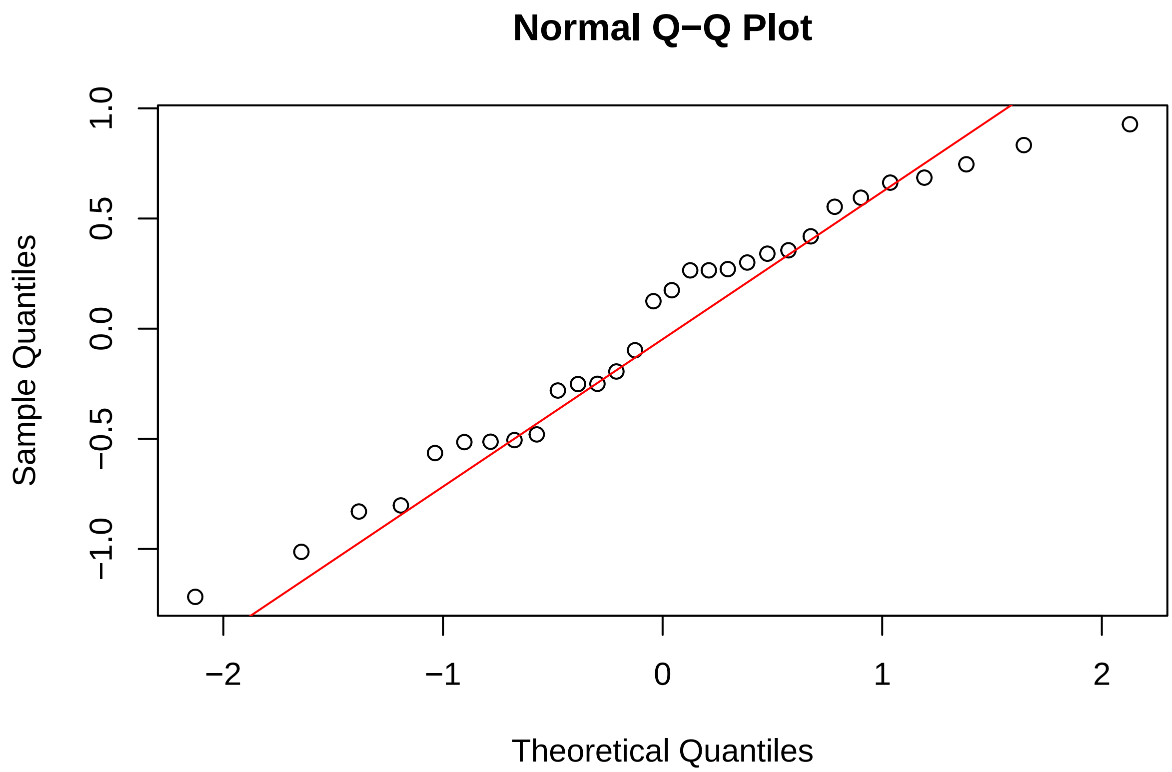
```
plot(model.synth)
```



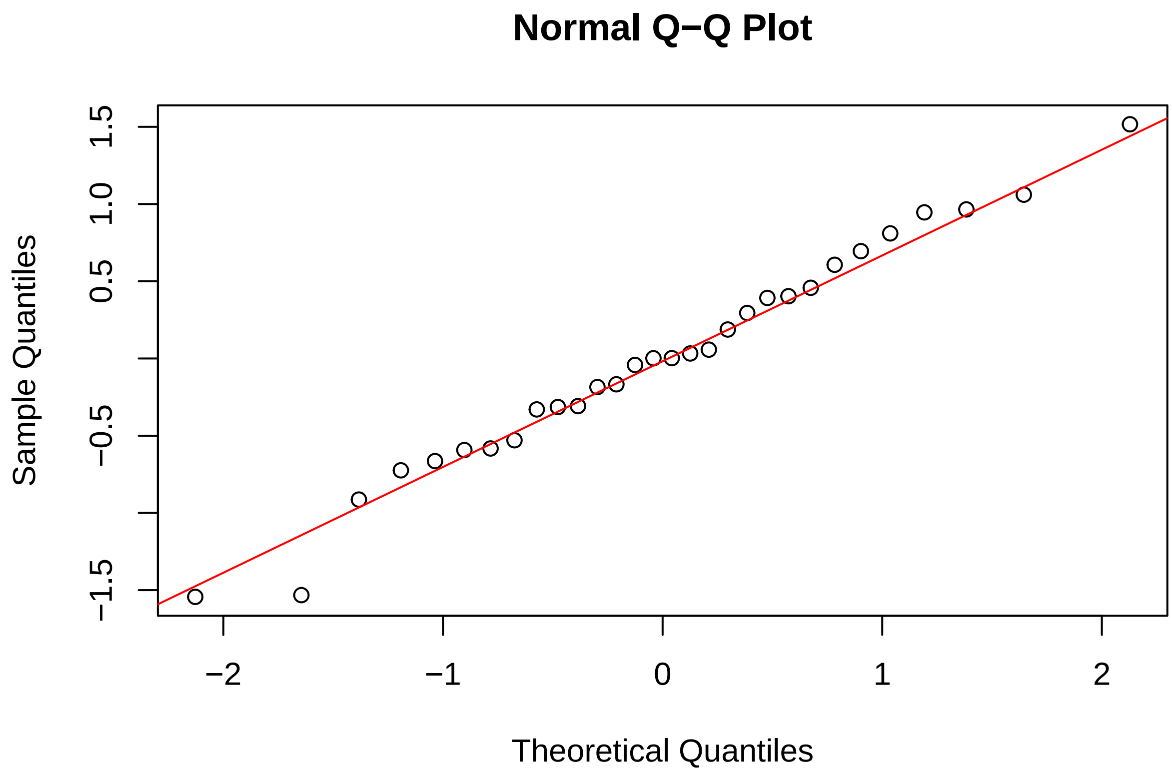

```r
qqnorm(resid(model.synth))
qqline(resid(model.synth),
       col = "red")
```




```
qqnorm(ranef(model.synth)$ID[,1])
qqline(ranef(model.synth)$ID[,1],
       col = "red")
```




```r
qqnorm(ranef(model.synth)$ID[,2])
qqline(ranef(model.synth)$ID[,2],
       col = "red")
```



Unsurprisingly (given how the data were generated), the model assumptions look reasonable.

You could also consider adding in an additional covariate at the lower level!

If you've got this far and have time to spare, feel free to work on the formative assignment.

End of lab!