Fast Wavelet Transform (FWT)

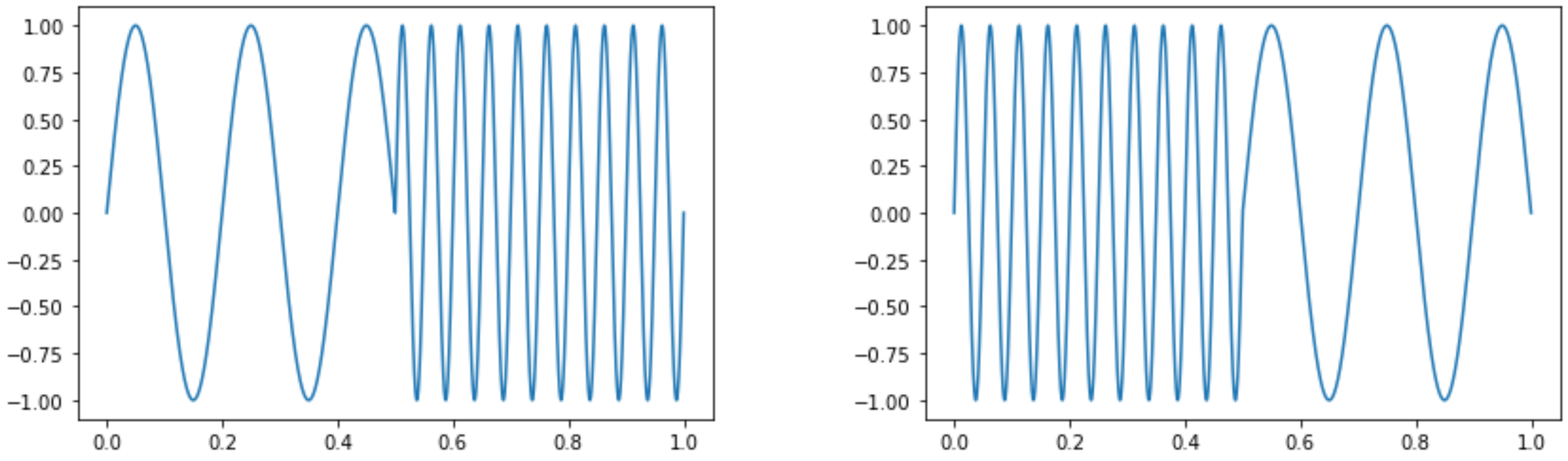
With their Applications

Qianbo Zang

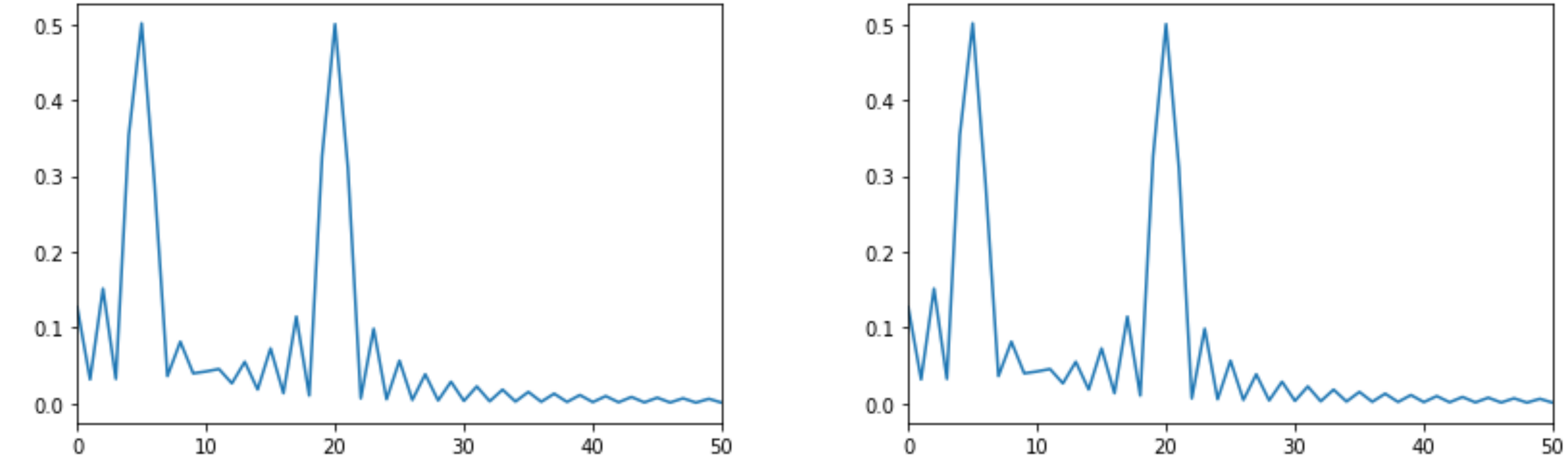
Why do we need Wavelet Transform

Bottlenecks of Fourier Transform (1): Uncertainty Principle

- Two signals



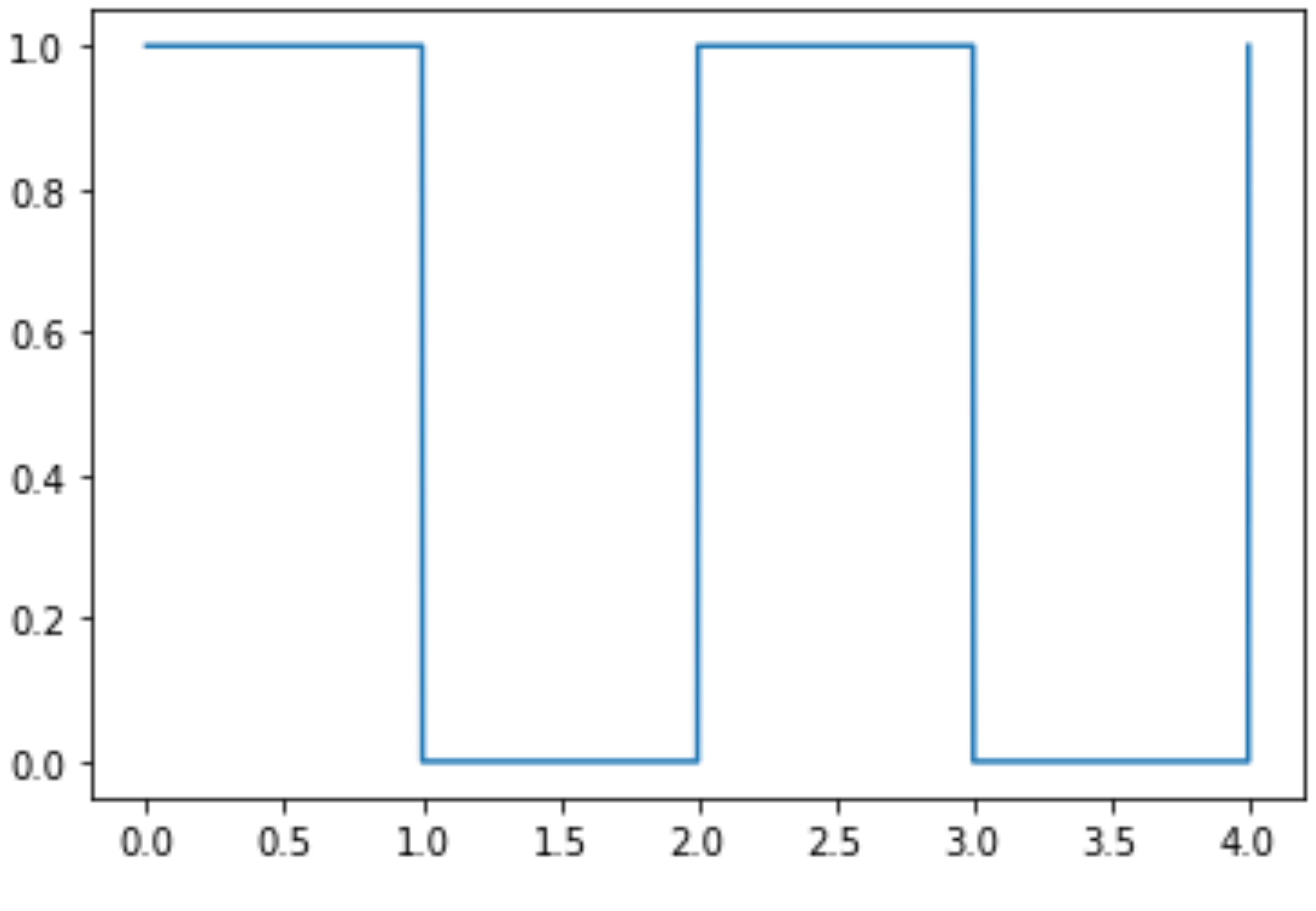
- Their FT



Why do we need WT

Bottlenecks of FT (2): Gibbs Phenomenon

- Example



$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = 1$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} f(x)\cdot\cos nx dx = 0$$

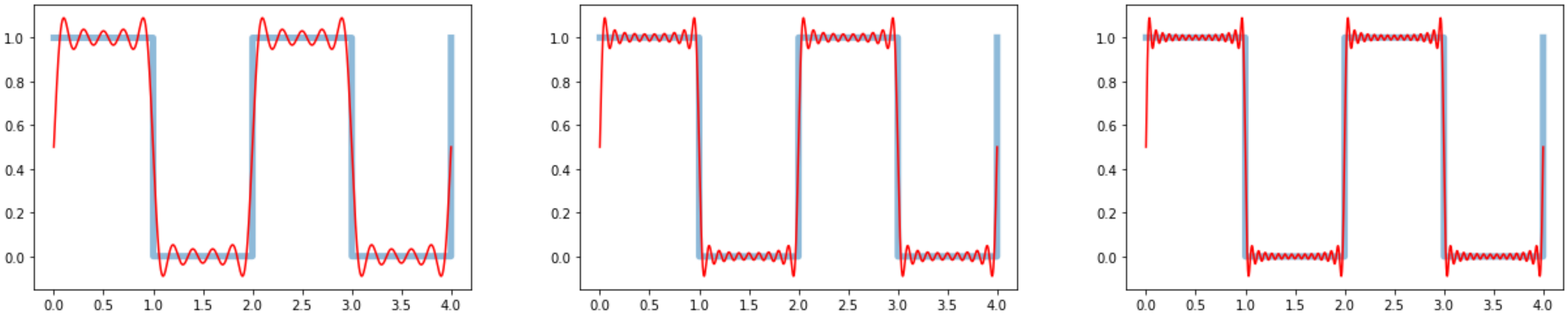
$$b_n = \frac{1}{\pi}\int_0^{\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_0^{\pi} \sin nx dx = -\frac{1}{n\pi}\cos nx|_0^{\pi}$$

$$= \begin{cases} -\frac{1}{n\pi}(1-1) = 0, n = 2k \\ -\frac{1}{n\pi}(-1-1) = \frac{2}{n\pi}, n = 2k+1 \end{cases}$$

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{sin5x}{5} + \dots\right)$$

Why do we need WT

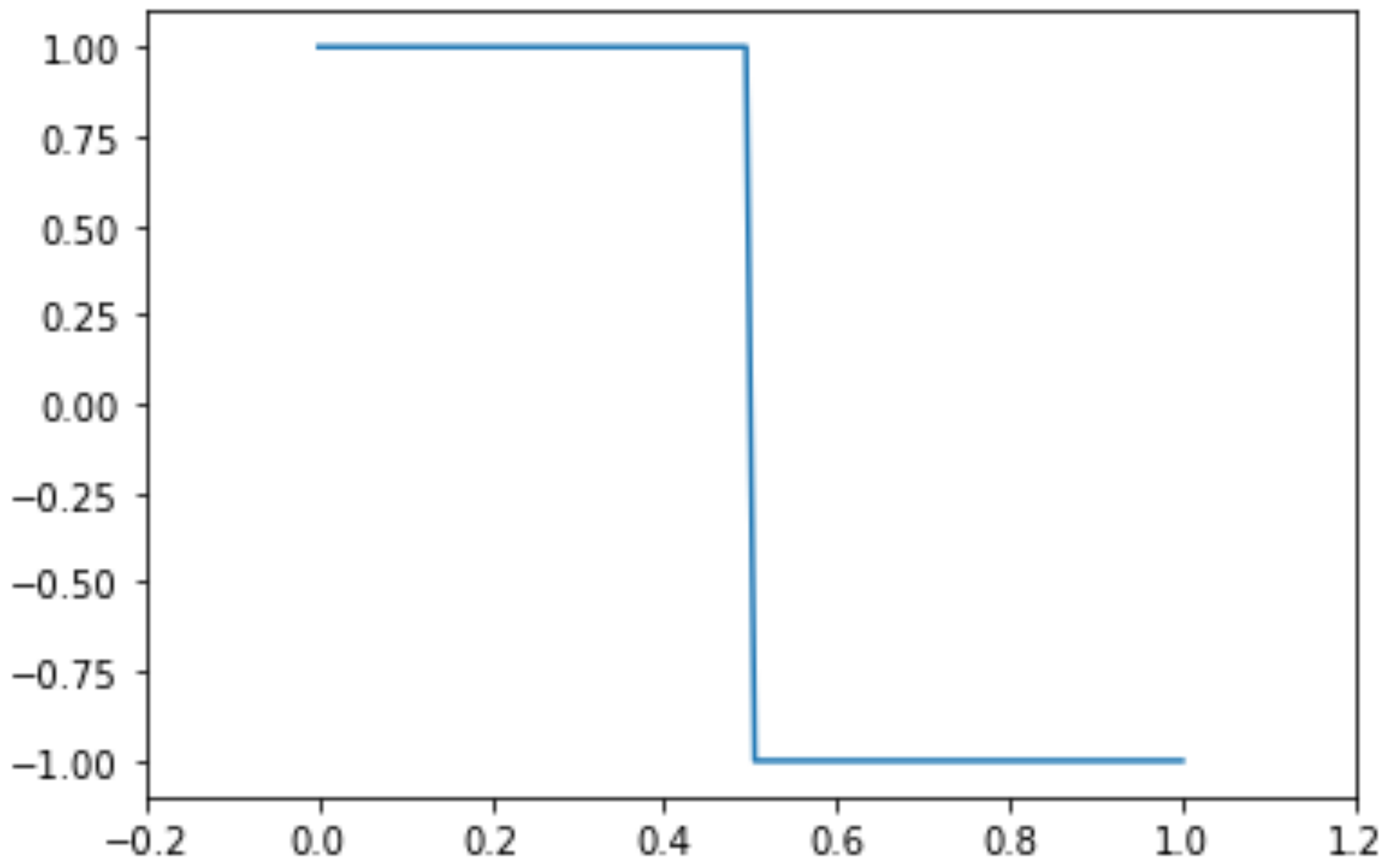
Bottlenecks of FT (2): Gibbs Phenomenon



What is Wavelet
Definition and Example

- A waveform of effectively limited duration that has an average value of zero.
- Example: Haar Wavelet (mother wavelet)

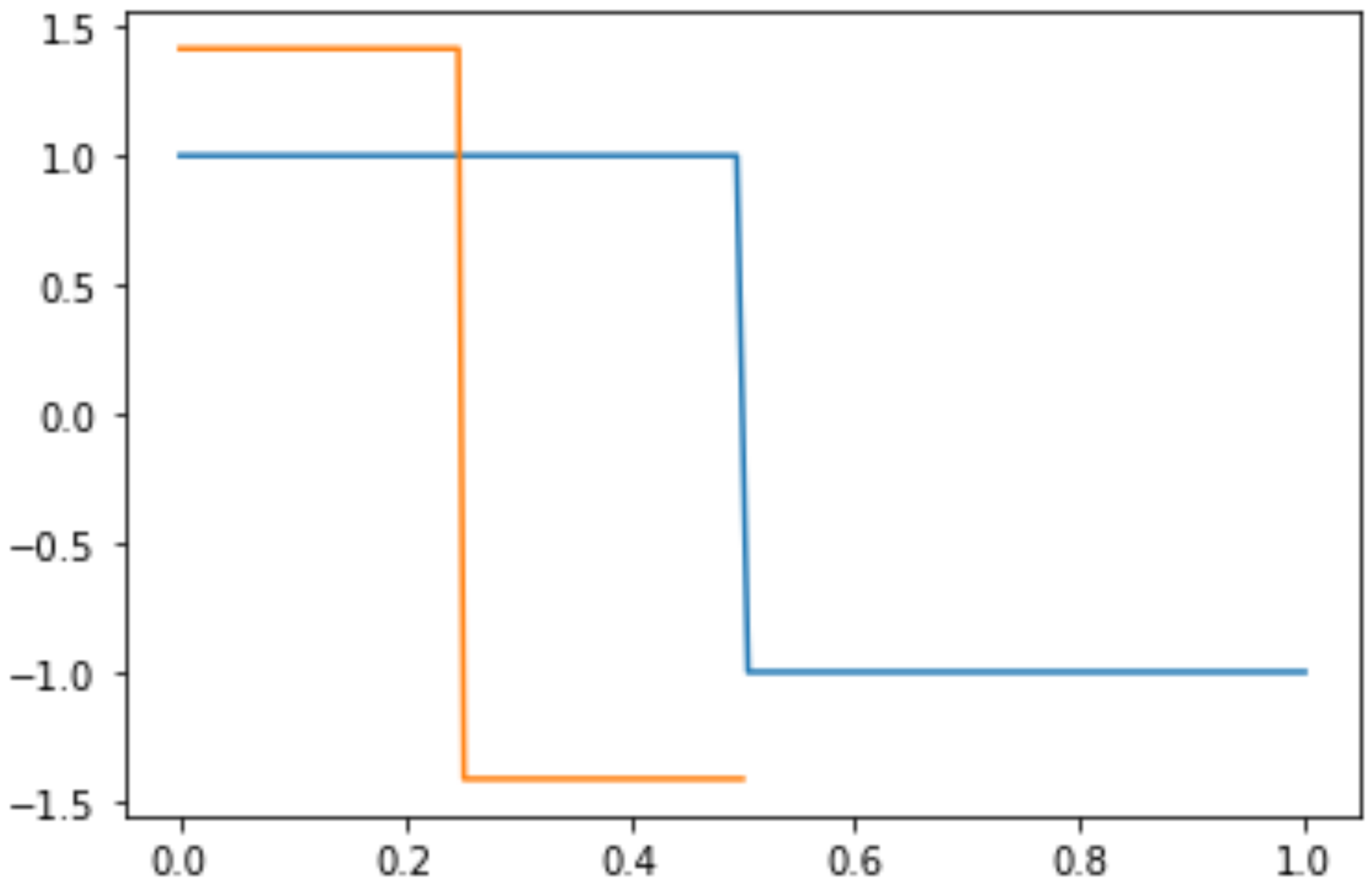
$$\psi(t) = \begin{cases} 1 & 0 \le t < \frac{1}{2}, \\ -1 & \frac{1}{2} \le t < 1, \\ 0 & \text{otherwise.} \end{cases}$$

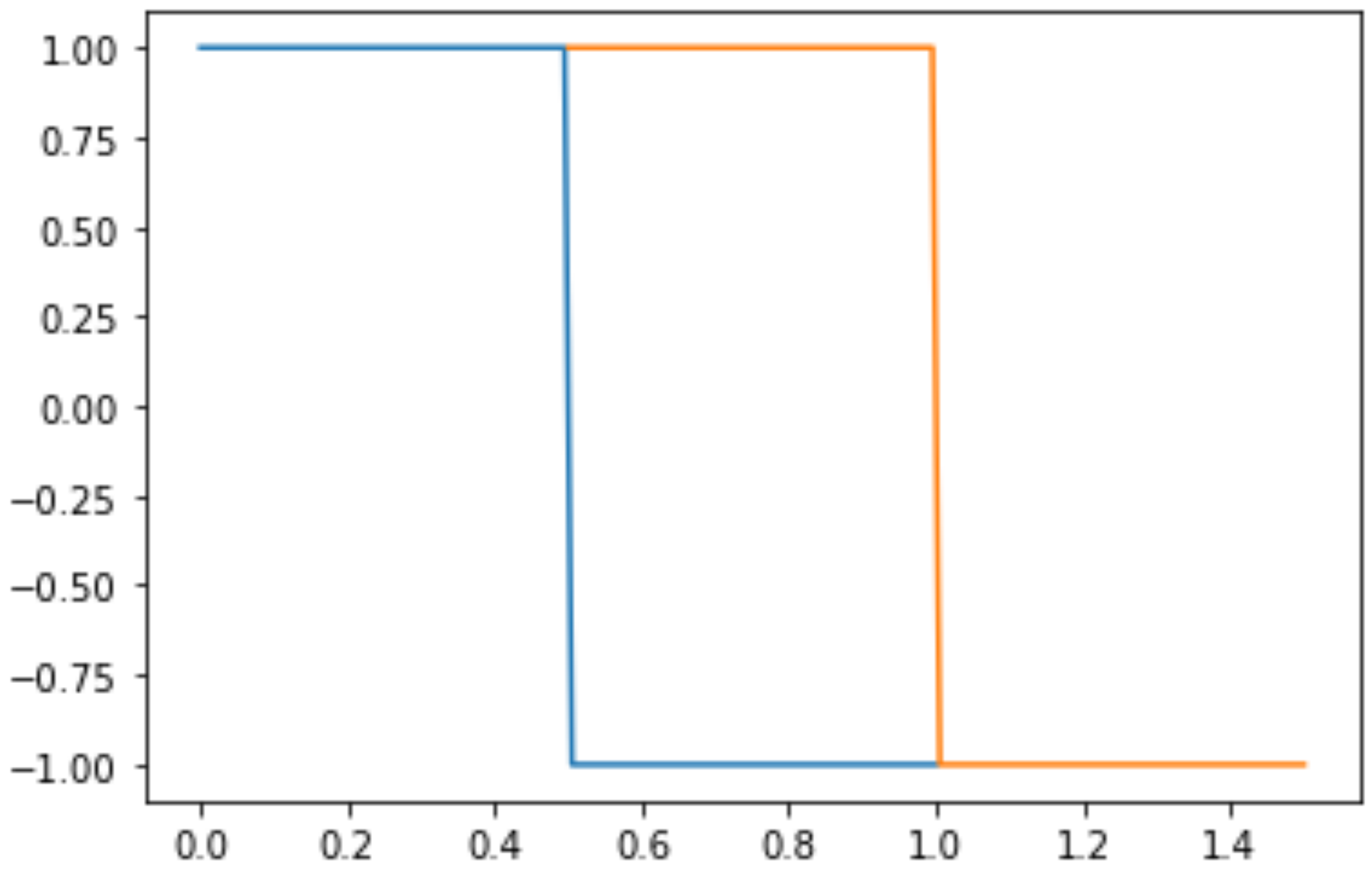


What is Wavelet

Definition and Example

- $\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right)$, a and b are called scale and position





What is Wavelet

Definition and Example

- Choose scale and position based on powers of 2

$$\psi_{j,k}(x) = \frac{1}{\sqrt{2^j}}\psi(\frac{t-k2^j}{2^j})$$

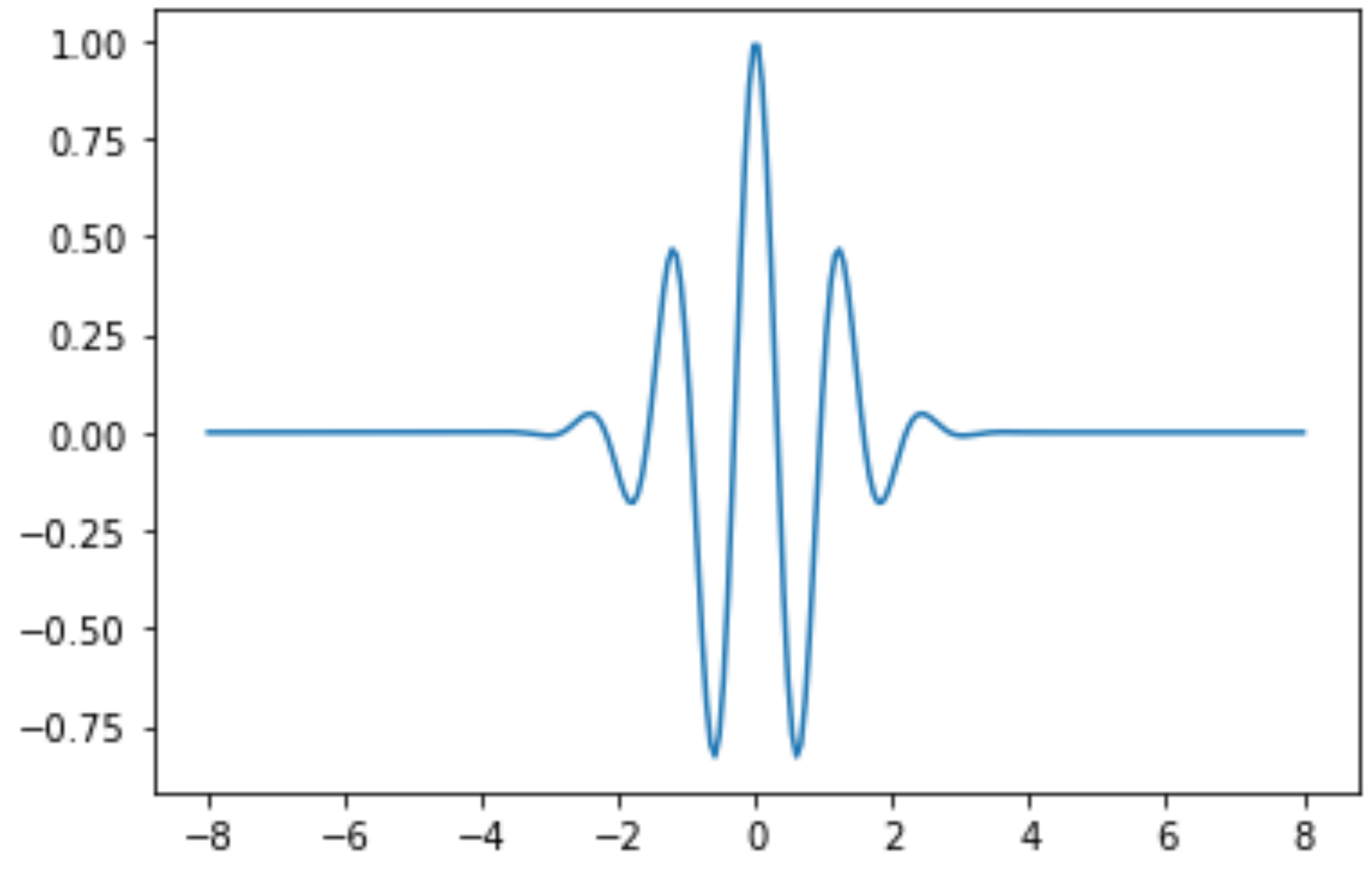
- Orthogonal wavelets

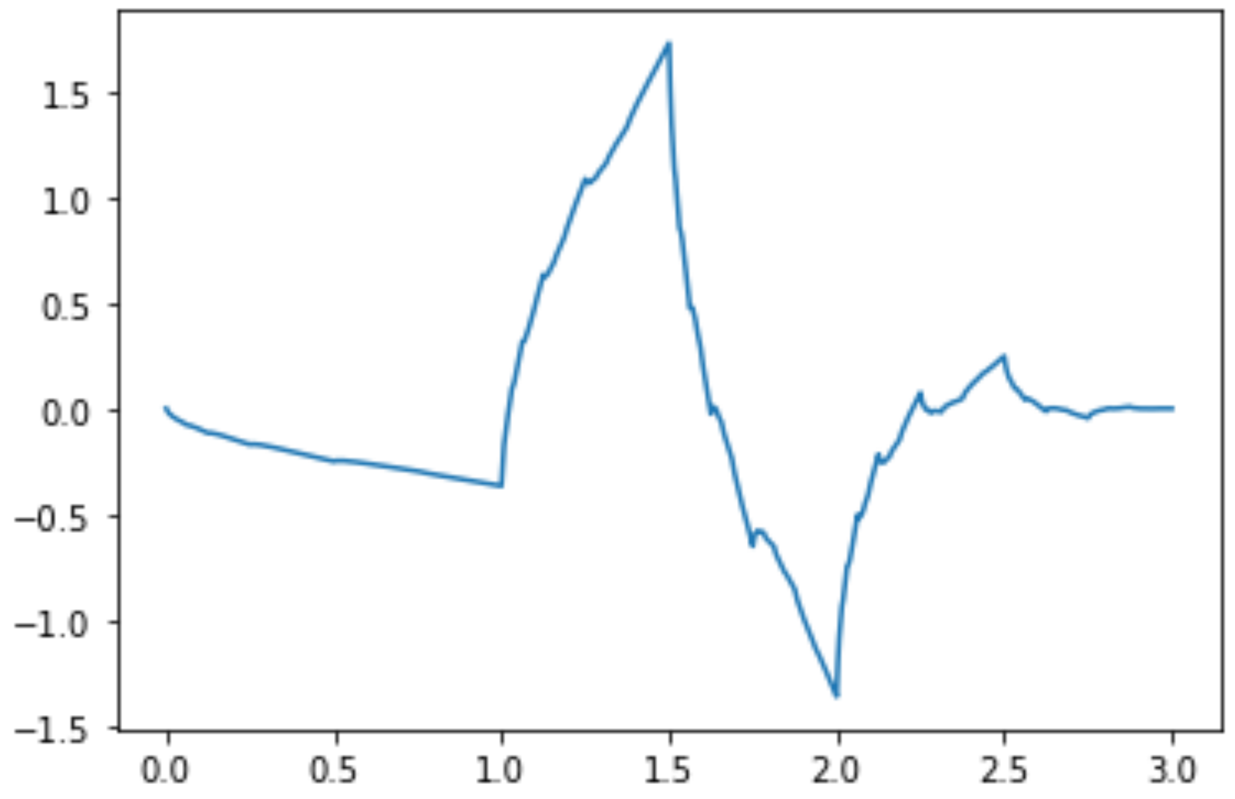
$$\psi_{j,k}(x) = 2^{-\frac{j}{2}}\psi(2^{-j}x - k)$$

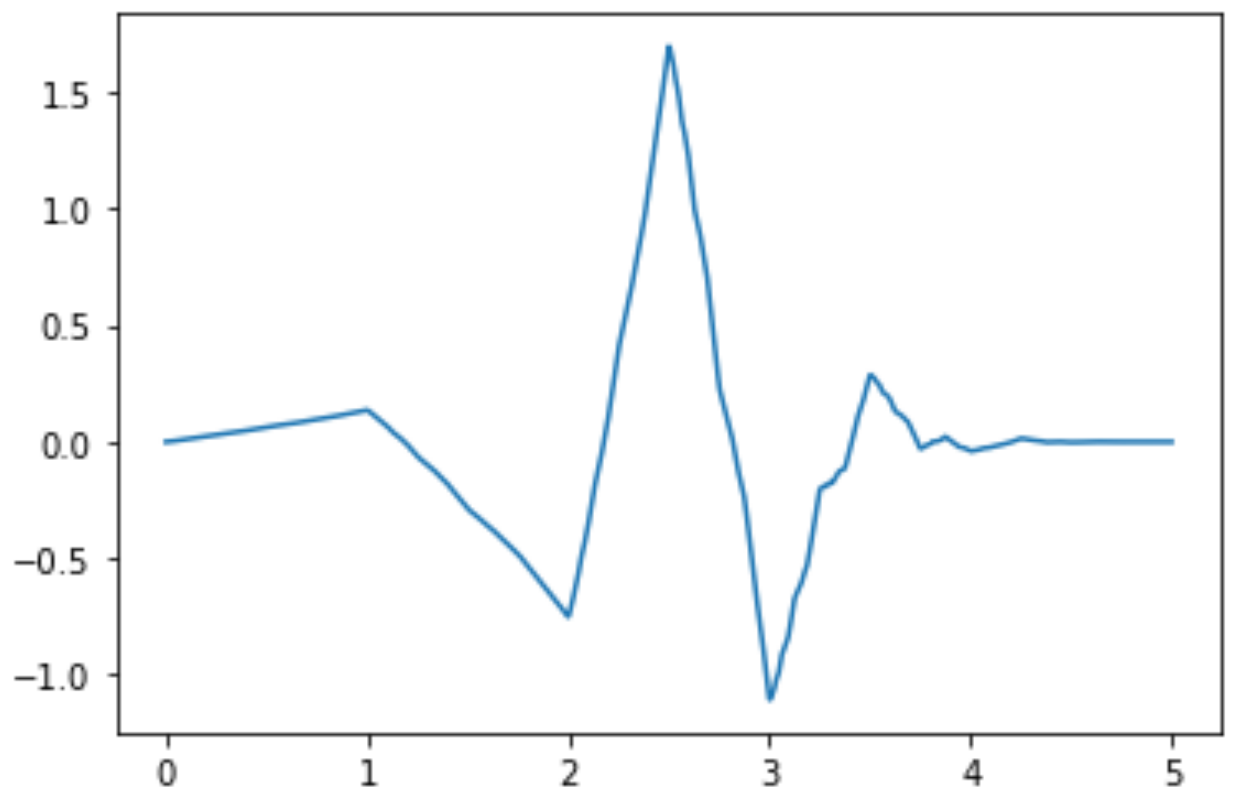
What is Wavelet

Definition and Example

- Another widely used mother wavelets (Morlet and Daubechies)







Wavelet Transform

Continuous Functions

- Mathematical — Changing Base Vectors (Projection)

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx$$

$$\left[W_\psi f\right](a, b) = \frac{1}{\sqrt{|a|}} \int_{-\infty}^{\infty} \overline{\psi\left(\frac{x-b}{a}\right)} f(x)dx$$

Wavelet Transform

Continuous Functions

- Fix scale a and modify position b.
- Modify scale a.

Wavelet Transform

Continuous Functions

- If now, we have the result of Wavelet Transform $W_g(a,b)$, how can we get the original function f(x)

- Parseval Equation: Norm will not change after orthogonal transformations.

$$C_\psi \int_R f(x)\bar{g}(x)dx = \int_R W_f(a,b)\bar{W}_g(a,b)\frac{dadb}{a^2}$$

$$C_\psi = \int_f \frac{\Psi(w)\bar{\Psi}(w)}{w}dw$$

Wavelet Transform

Continuous Functions

- Inverse Wavelet Transform

$$f(x) = \frac{1}{C_\psi} \iint_{R\times R*} W_f(a,b)\psi_{(a,b)}(x)\frac{dadb}{a^2}$$

Wavelet Transform

Continuous Functions Example: Pattern Recognition

- Dataset: 6 kinds of activities

Activity	Label
walking	0
walking upstairs	1
walking downstairs	2
sitting	3
staying	4
laying	5

Wavelet Transform

Continuous Functions Example: Pattern Recognition

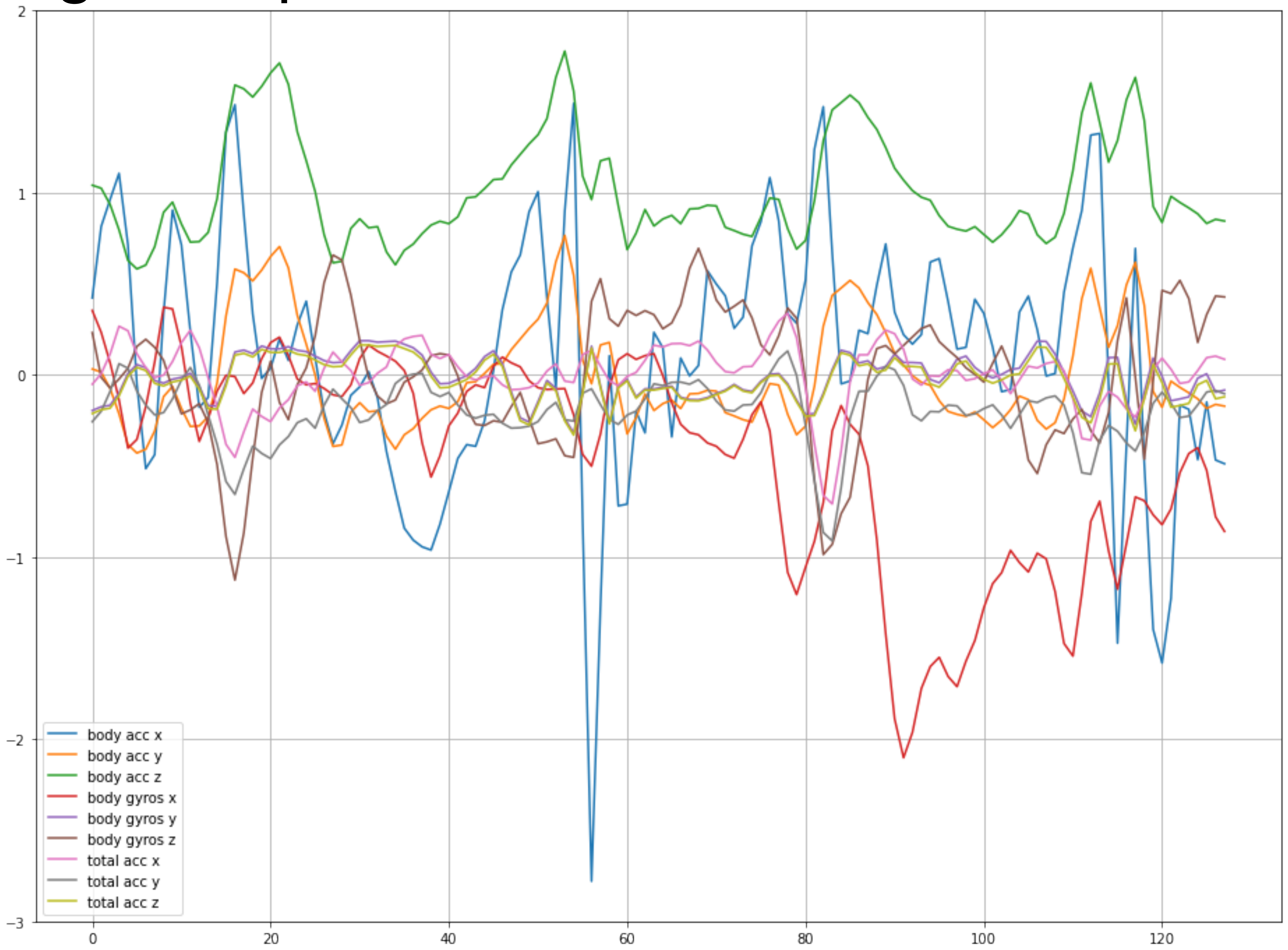
- Dataset: one activities measured by 9 time sequence variables

Signal type	Index
body acc x	0
body acc y	1
body acc z	2
body gyro x	3
body gyro y	4
body gyro z	5
total acc x	6
total acc y	7
total acc z	8

Wavelet Transform

Continuous Functions Example: Pattern Recognition

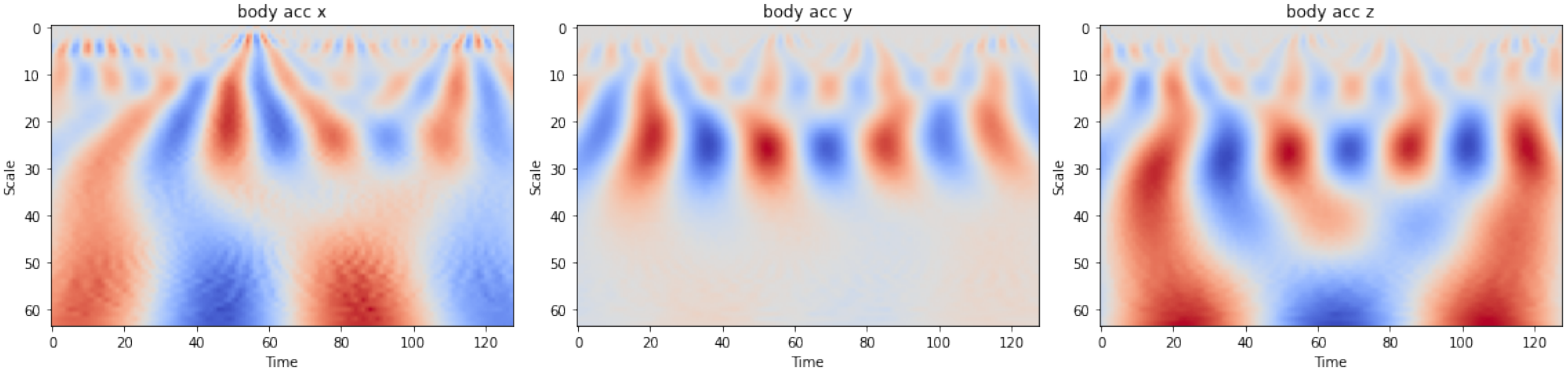
- Difficulties: convergence, parallelisation



Wavelet Transform

Continuous Functions Example: Pattern Recognition

- Idea: transform time sequence data to scalograms with CWT



Wavelet Transform

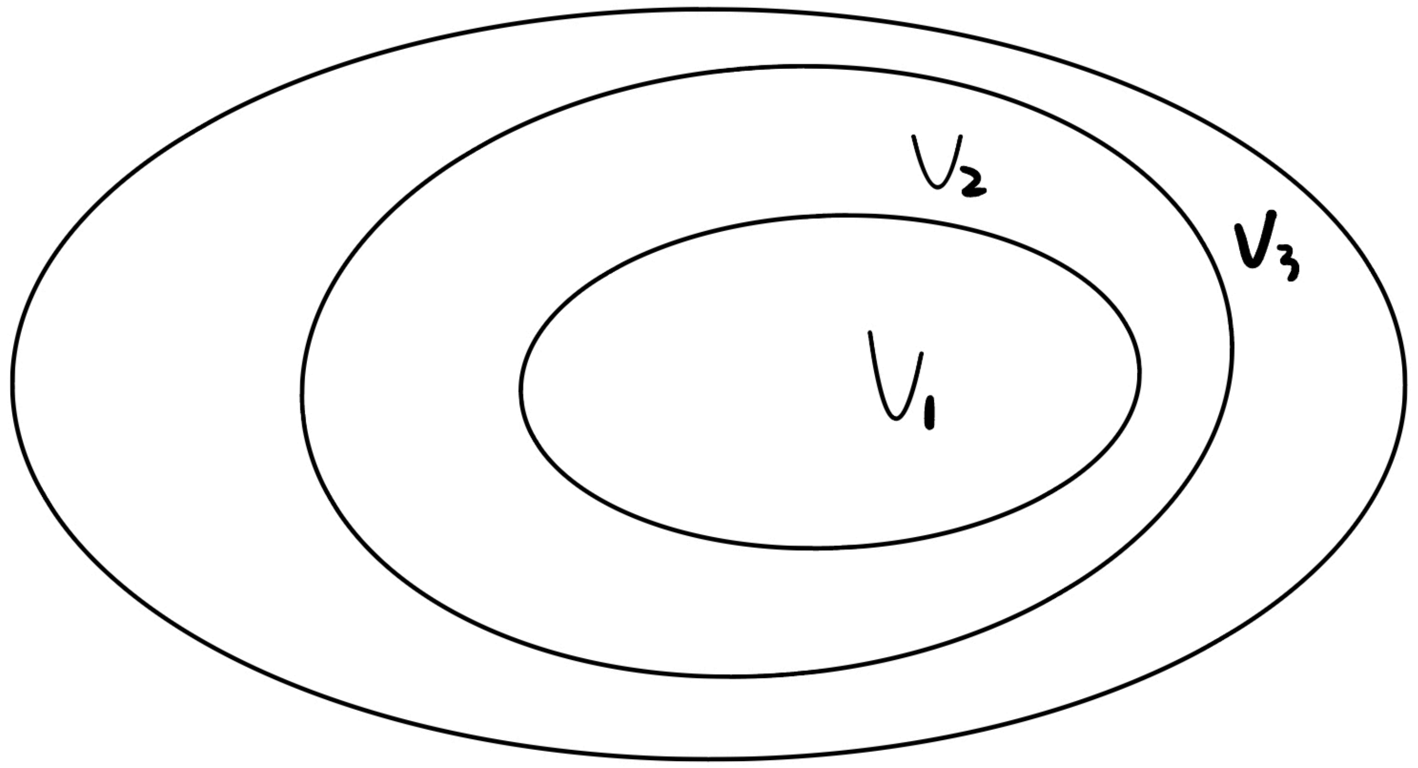
Discrete Functions

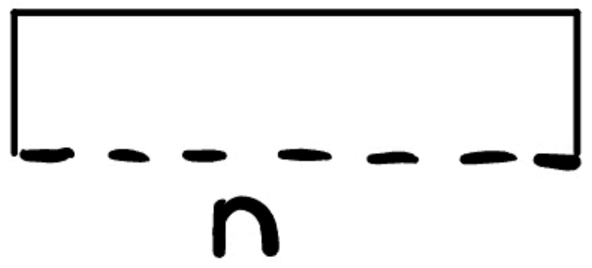
- Idea: In real life, mostly, we can only sample discrete datasets
- So we need discrete wavelet transform
- Use Haar Wavelet as an example

What is Wavelet

Discrete Functions

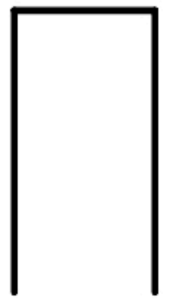
- Haar Wavelet





$$\psi(t)$$

$$\psi(t) = \frac{1}{\sqrt{2}}(\sqrt{2}\psi(2t) + \sqrt{2}\psi(2t-n))$$



$$\psi(t) = \frac{1}{2}(2\psi(4t) + 2\psi(4t-n) + 2\psi(4t-2n) + 2\psi(4t-3n))$$

$$\psi_{j,k}(x) = 2^{\frac{j}{2}}\psi(2^j x - k)$$

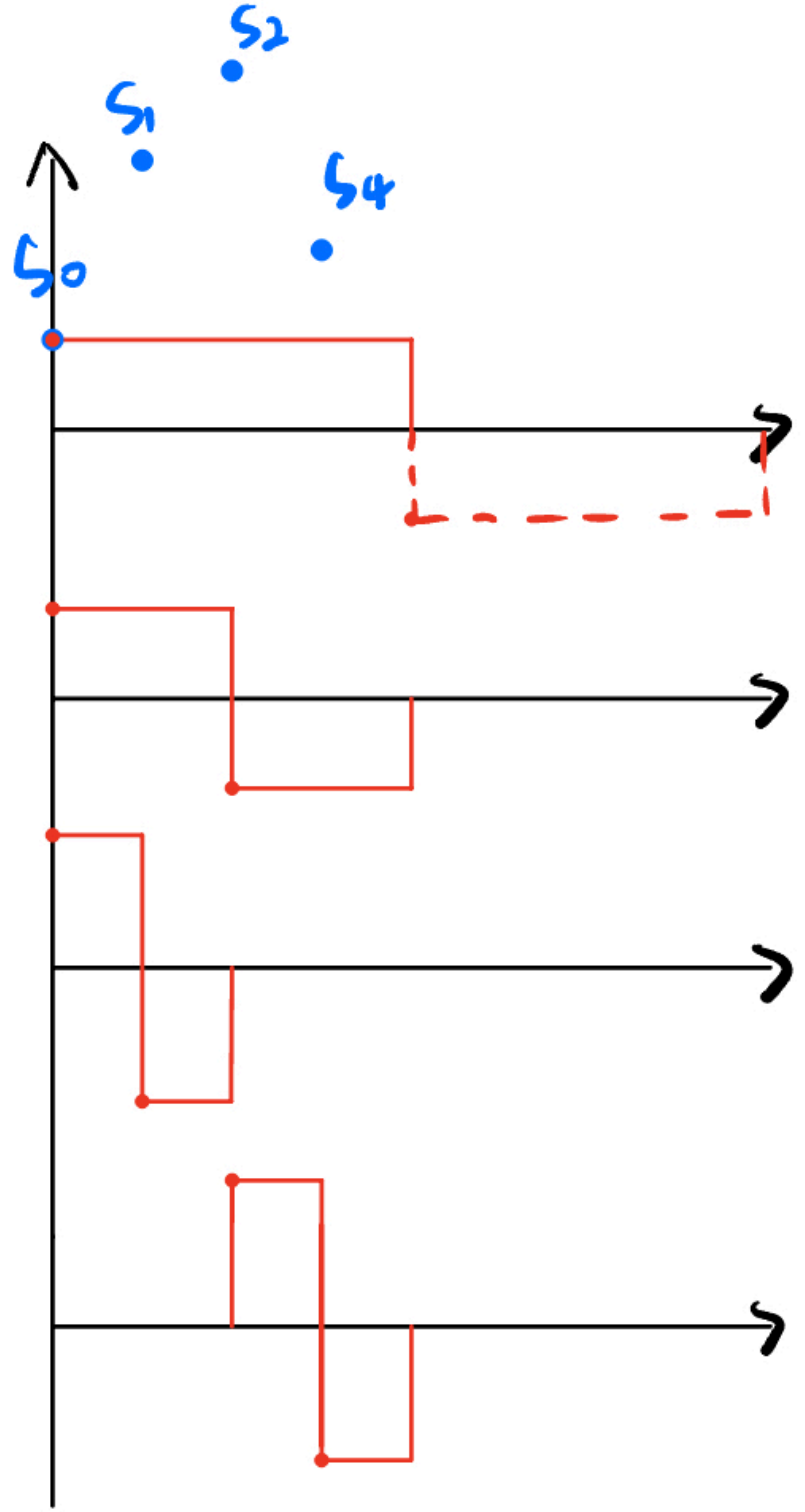
Wavelet Transform

Discrete Functions

- For example, [s0, s1, s2, s3]

$$W = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} f(x)\psi(x)$$

$$\begin{pmatrix} \frac{1}{2}\sum_{x=0}^{3} \quad \varphi_0(x) \\ \frac{1}{2}\sum_{x=0}^{3} f(x)\,\varphi_1(x) \\ \frac{1}{2}\sum_{x=0}^{3} f(x)\,\varphi_2(x) \\ \frac{1}{2}\sum_{x=0}^{3} f(x)\,\varphi_3(x) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(s_0 + s_1 + s_2 + s_3) \\ \frac{1}{2}(s_0 + s_1 - s_2 - s_3) \\ \frac{1}{2}(\sqrt{2}s_0 - \sqrt{2}s_1 + 0 + 0) \\ \frac{1}{2}(0 + 0 + \sqrt{2}s_3 - \sqrt{2}s_4) \end{pmatrix}$$



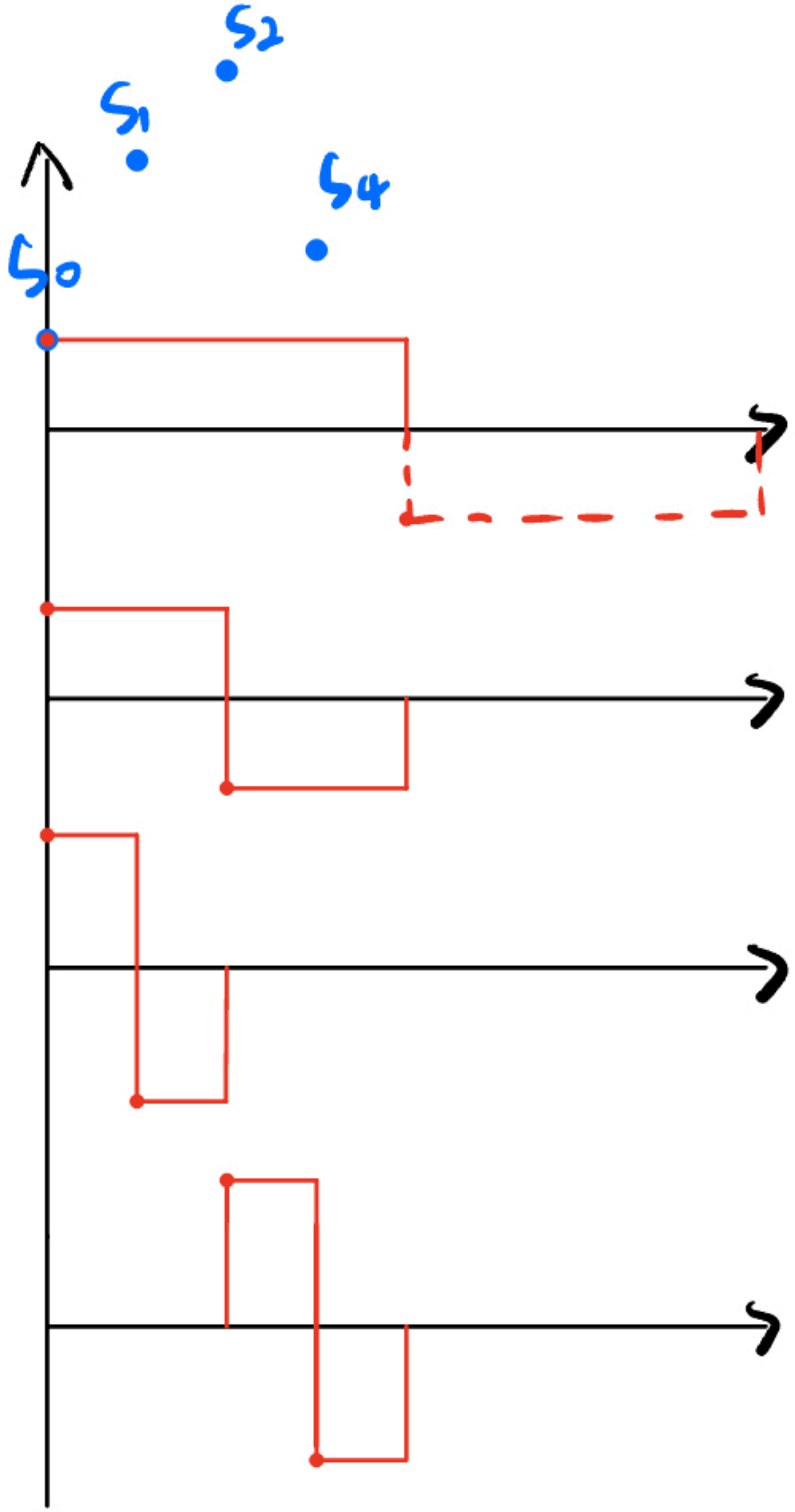
Wavelet Transform

Discrete Functions

- For example, [s0, s1, s2, s3]

$$W = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} f(x)\psi(x)$$

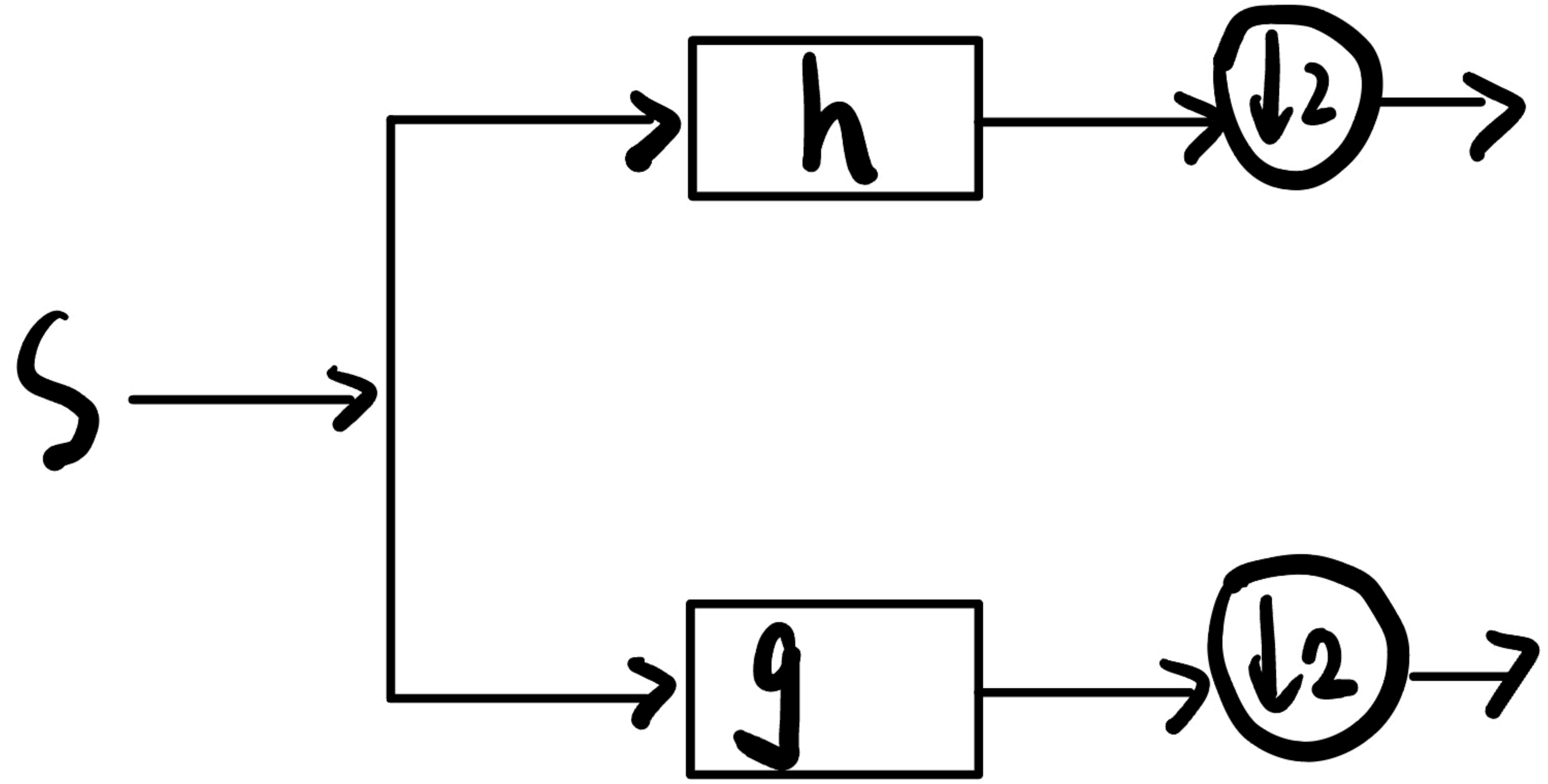
$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix}$$



Wavelet Transform

Discrete Functions: 1-D Fast wavelet transform

- Idea: can be consider as two filers
- For example, [s0, s1], haar wavelet

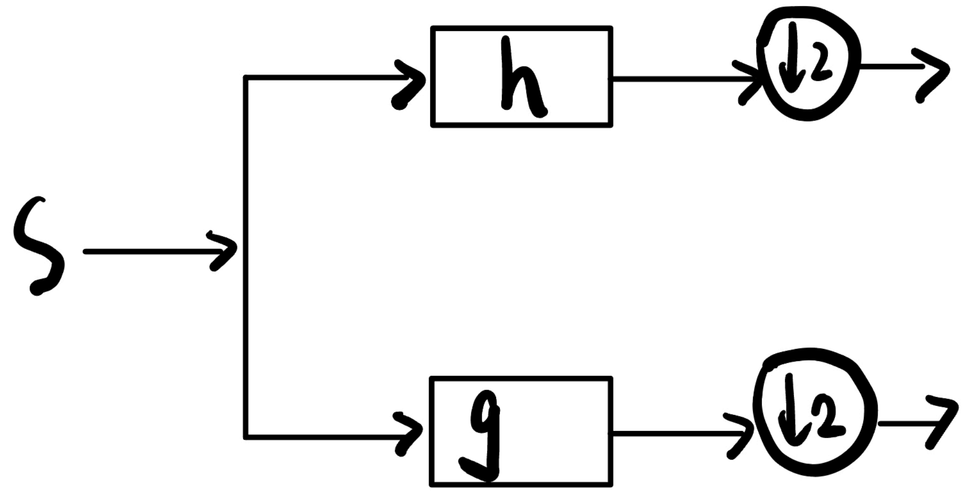


$h_0 = \frac{1}{\sqrt{2}}$, $h_1 = \frac{1}{\sqrt{2}}$

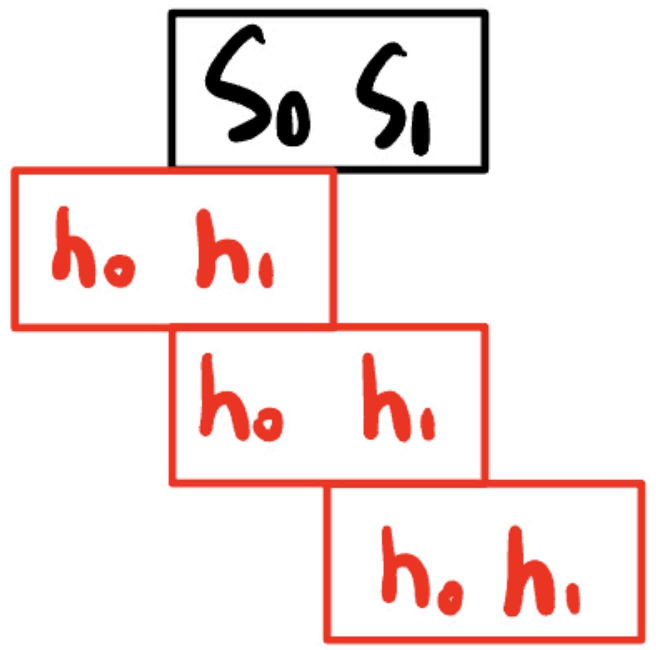
$g_0 = \frac{1}{\sqrt{2}}$, $g_1 = -\frac{1}{\sqrt{2}}$

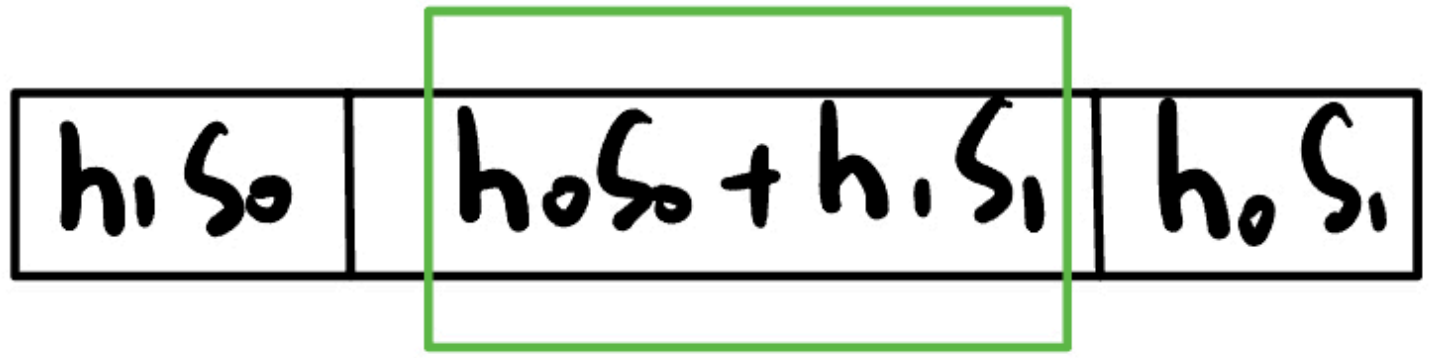
Wavelet Transform

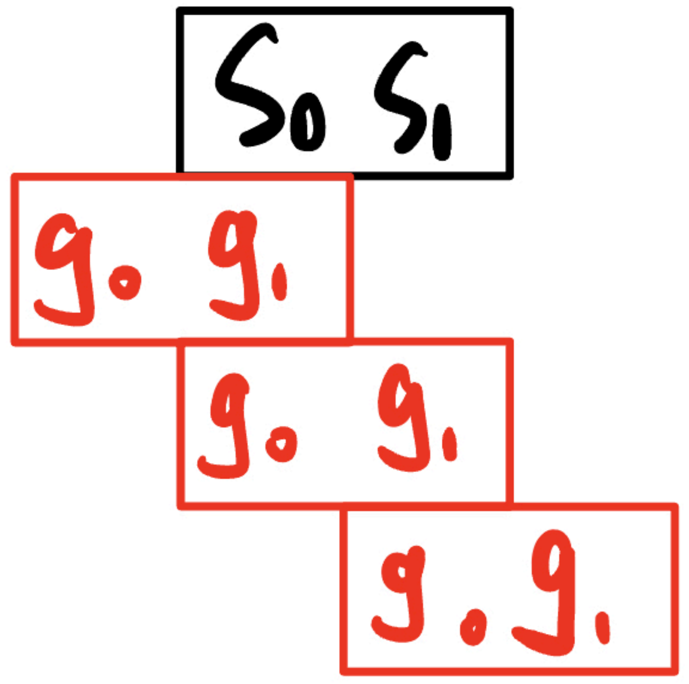
Discrete Functions: 1-D Fast wavelet transform

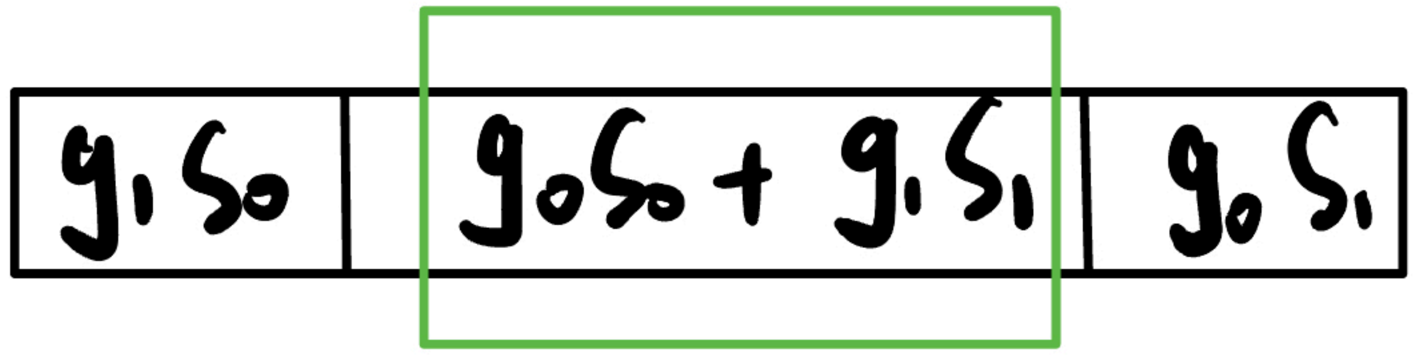


- For example, [s0, s1], haar wavelet









Wavelet Transform

Discrete Functions: 1-D Fast wavelet transform

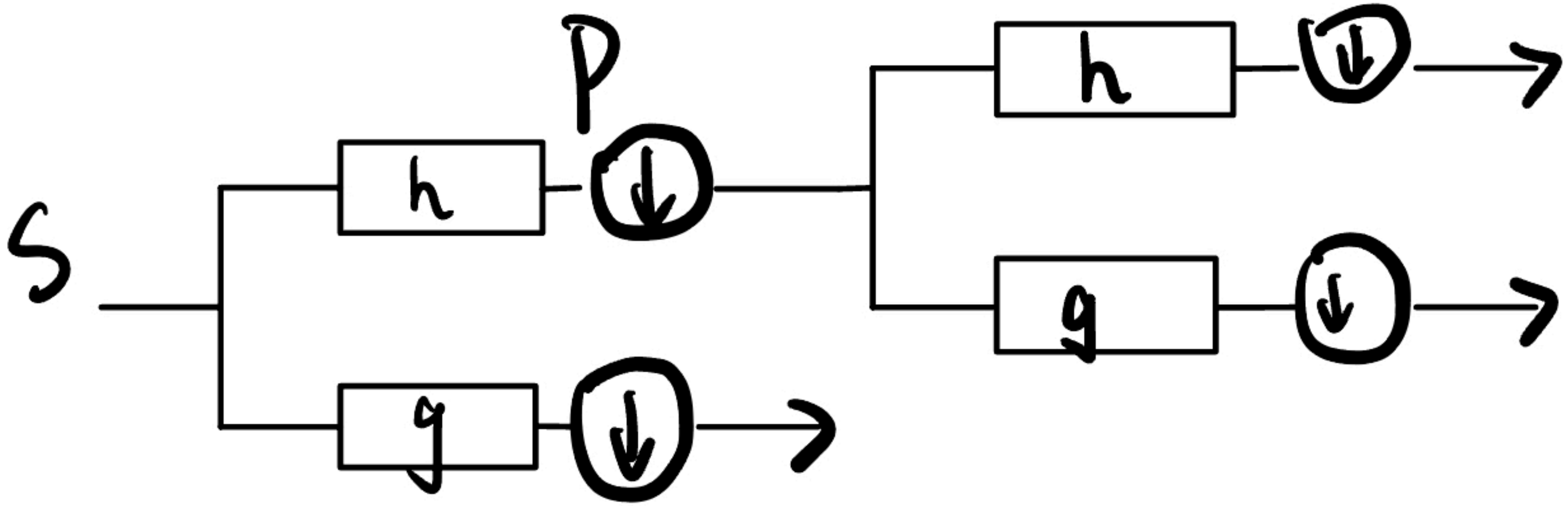
- For example, [s0, s1], haar wavelet
- 2-d Haar matrix

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} s_0 \\ s_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} s_0 + \frac{1}{\sqrt{2}} s_1 \\ \frac{1}{\sqrt{2}} s_0 - \frac{1}{\sqrt{2}} s_1 \end{bmatrix}$$

Wavelet Transform

Discrete Functions: 1-D Fast wavelet transform

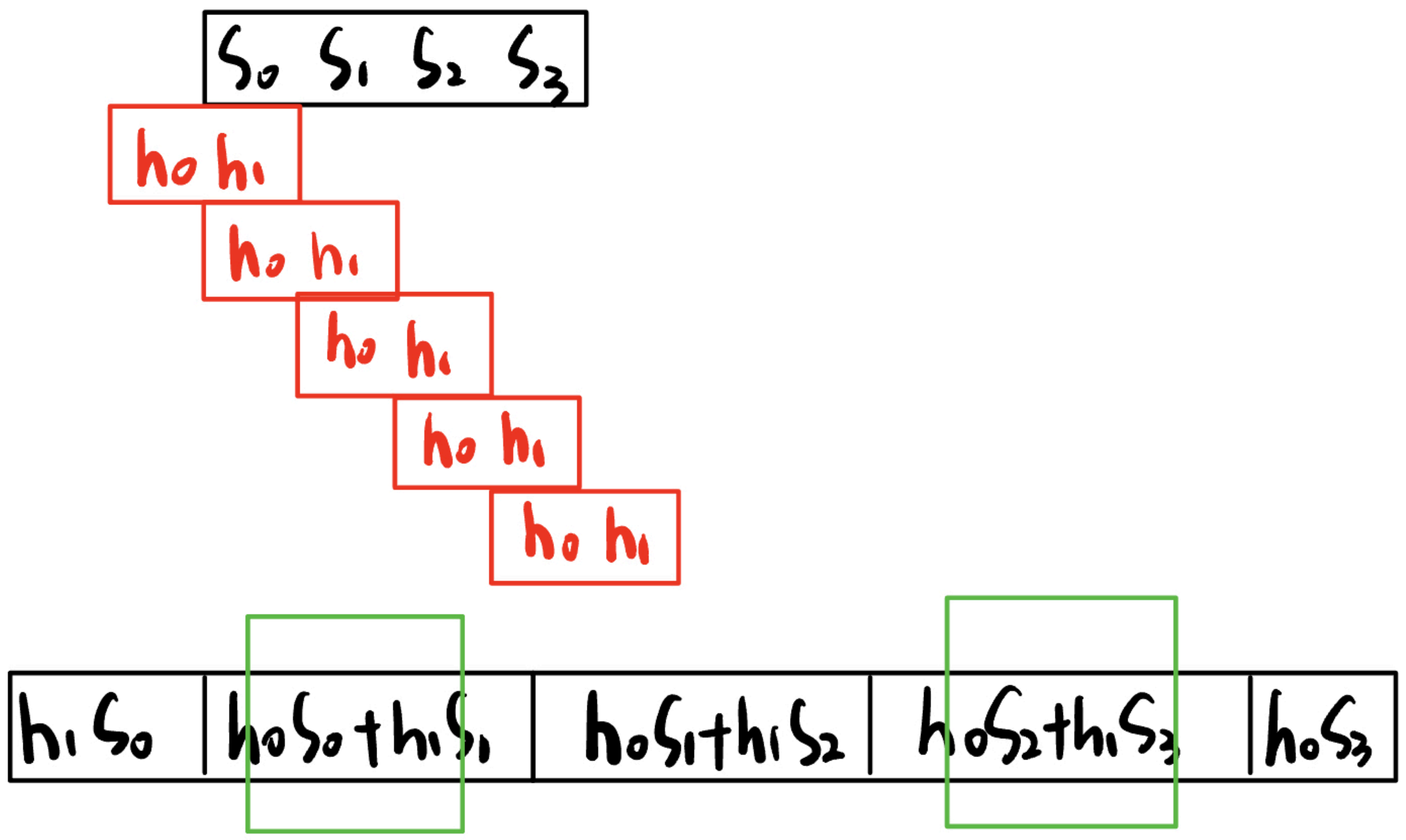
- For example, [s0, s1, s2, s3], haar wavelet



Wavelet Transform

Discrete Functions: 1-D Fast wavelet transform

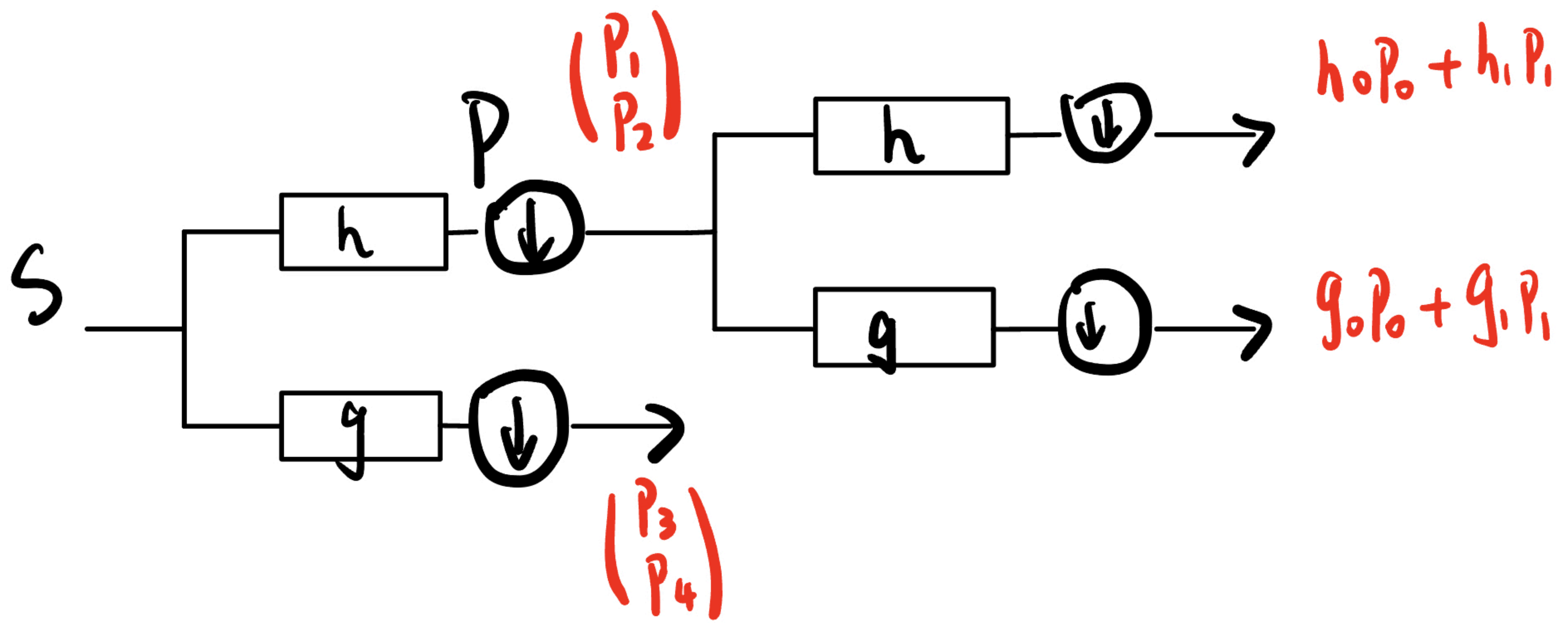
- For example, [s0, s1, s2, s3]

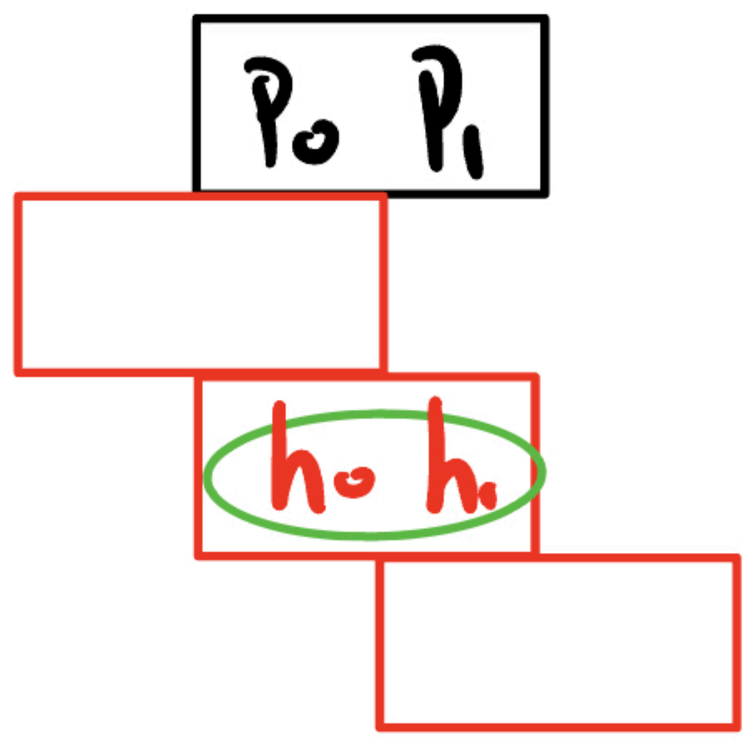


$$\begin{pmatrix} h_0 & h_1 & 0 & 0 \\ 0 & 0 & h_0 & h_1 \\ g_0 & g_1 & 0 & 0 \\ 0 & 0 & g_0 & g_1 \end{pmatrix} \begin{pmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} h_0 s_0 + h_1 s_1 \\ h_0 s_2 + h_1 s_3 \\ g_0 s_0 + g_1 s_1 \\ g_0 s_2 + g_1 s_2 \end{pmatrix} = \begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix}$$

Wavelet Transform

Discrete Functions: 1-D Fast wavelet transform





$$\begin{pmatrix} h_0 & h_1 \\ g_0 & g_1 \end{pmatrix} \begin{pmatrix} P_0 \\ P_1 \end{pmatrix} = \begin{pmatrix} h_0P_0 + h_1P_1 \\ g_0P_0 + g_1P_1 \end{pmatrix}$$

Wavelet Transform

Discrete Functions: 2-D Fast wavelet transform

- picture (1024 x 1024 x 3)

Wavelet Transform

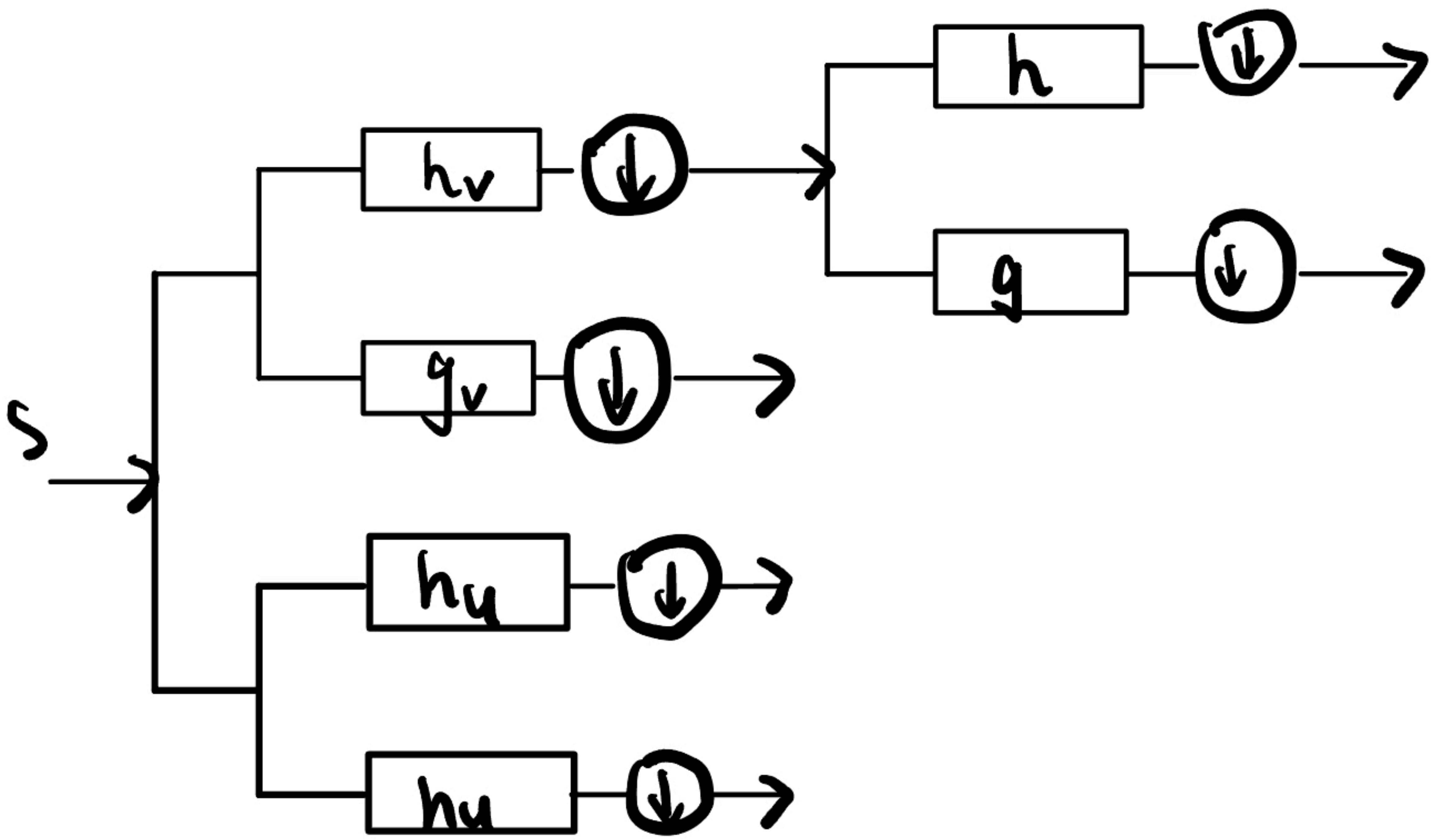
Discrete Functions: 2-D Fast wavelet transform

- Transform from RGB to Gray.

Wavelet Transform

Discrete Functions: 2-D Fast wavelet transform

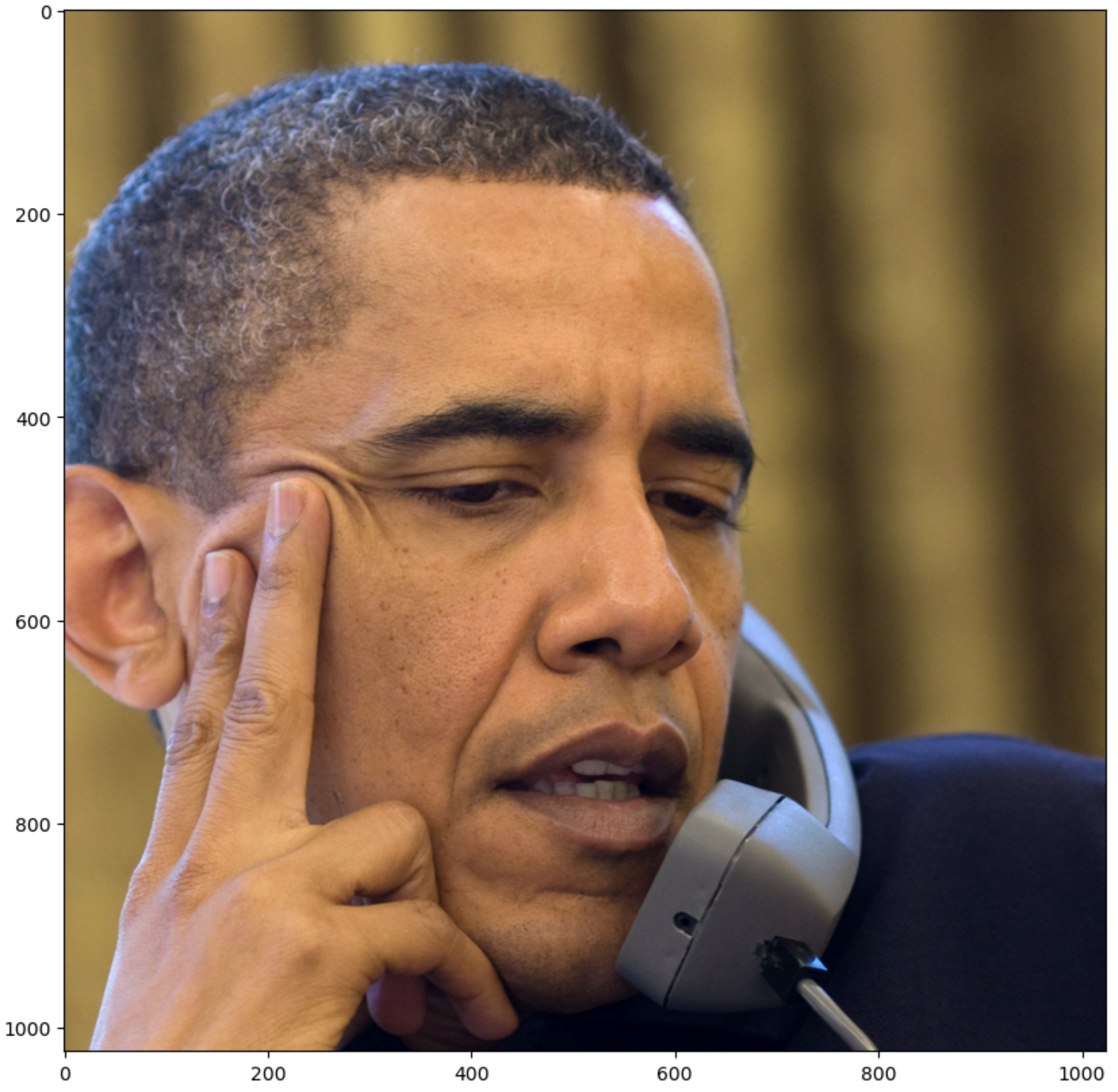
- Fast Wavelet Transform



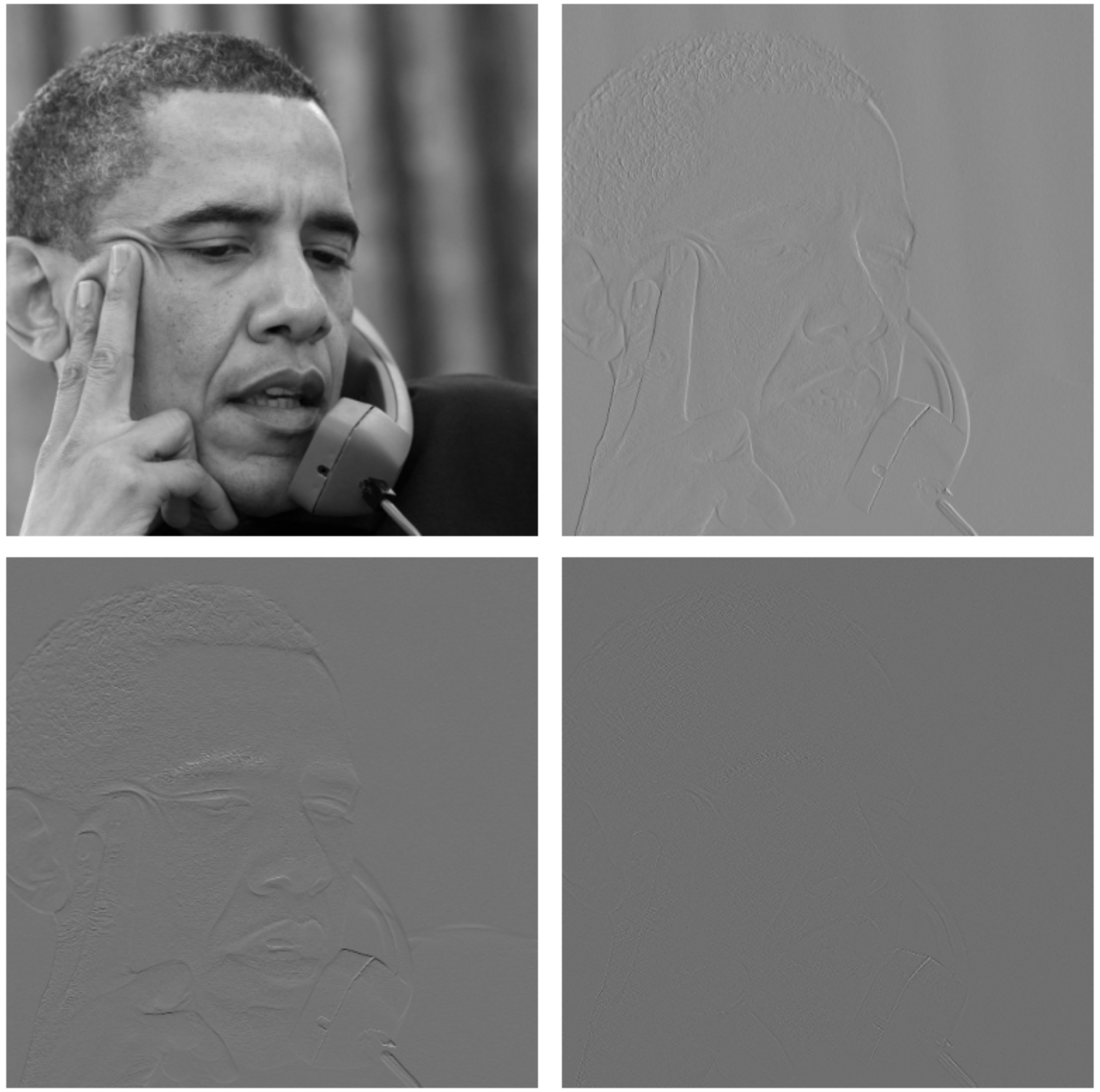
Wavelet Transform

Discrete Functions: 2-D Fast wavelet transform

image



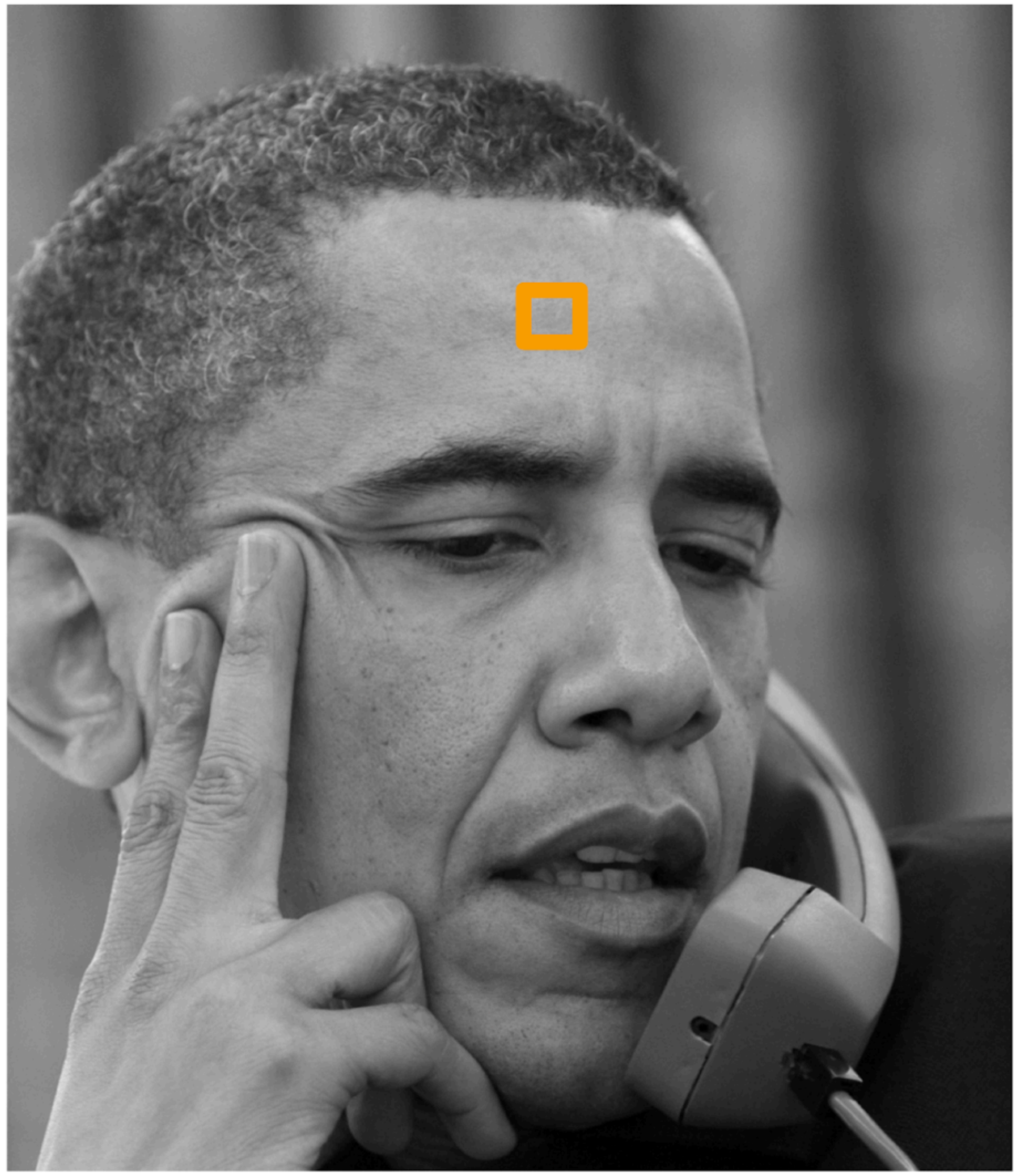
WT



Wavelet Transform

Discrete Functions Example: JPEG 2000

- Divides the original image into 8-by-8 blocks and do DWT



Wavelet Transform

Discrete Functions Example: JPEG 2000

- Each block is a 8x8 matrix.

	1	2	3	4	5	6	7	8
1	137	134	132	133	133	134	133	130
2	134	134	137	135	134	134	131	128
3	137	134	135	137	135	136	132	129
4	138	134	132	135	135	136	136	132
5	133	135	137	136	132	134	136	130
6	133	135	136	135	133	132	130	129
7	135	135	134	135	133	133	133	132
8	135	134	131	134	131	129	134	132

Wavelet Transform

Discrete Functions Example: JPEG 2000

- Wavelet transform with Haar wavelet (Haar Matrix)
- $L = AH$ is row Haar transform. $S = H^T L$ is column Haar transform.

	1	2	3	4	5	6	7	8
1	133.6250	1.0938	0.0938	0.8438	0.4375	-0.3750	-0.1250	1.4375
2	0.3125	-0.1563	0.2813	0.7813	0.8125	-0.1250	-0.2500	0.1875
3	-0.6250	0.2500	-0.1250	0	-0.5000	0.7500	0.1250	-0.1250
4	0.1875	0.2500	-0.8125	0.6875	-0.6250	0.7500	-0.3750	0.5000
5	-0.0625	-0.4375	1.2500	-0.6250	0.7500	-0.7500	-0.2500	0
6	-0.1875	0.6875	-0.7500	0.8750	-0.2500	0.2500	0	-0.2500
7	0.6250	-0.3750	-0.2500	-0.7500	0	0	-0.7500	1.2500
8	0.6250	0	-0.3750	0.8750	-0.2500	0.5000	-0.5000	-0.2500

Wavelet Transform

Discrete Functions Example: JPEG 2000

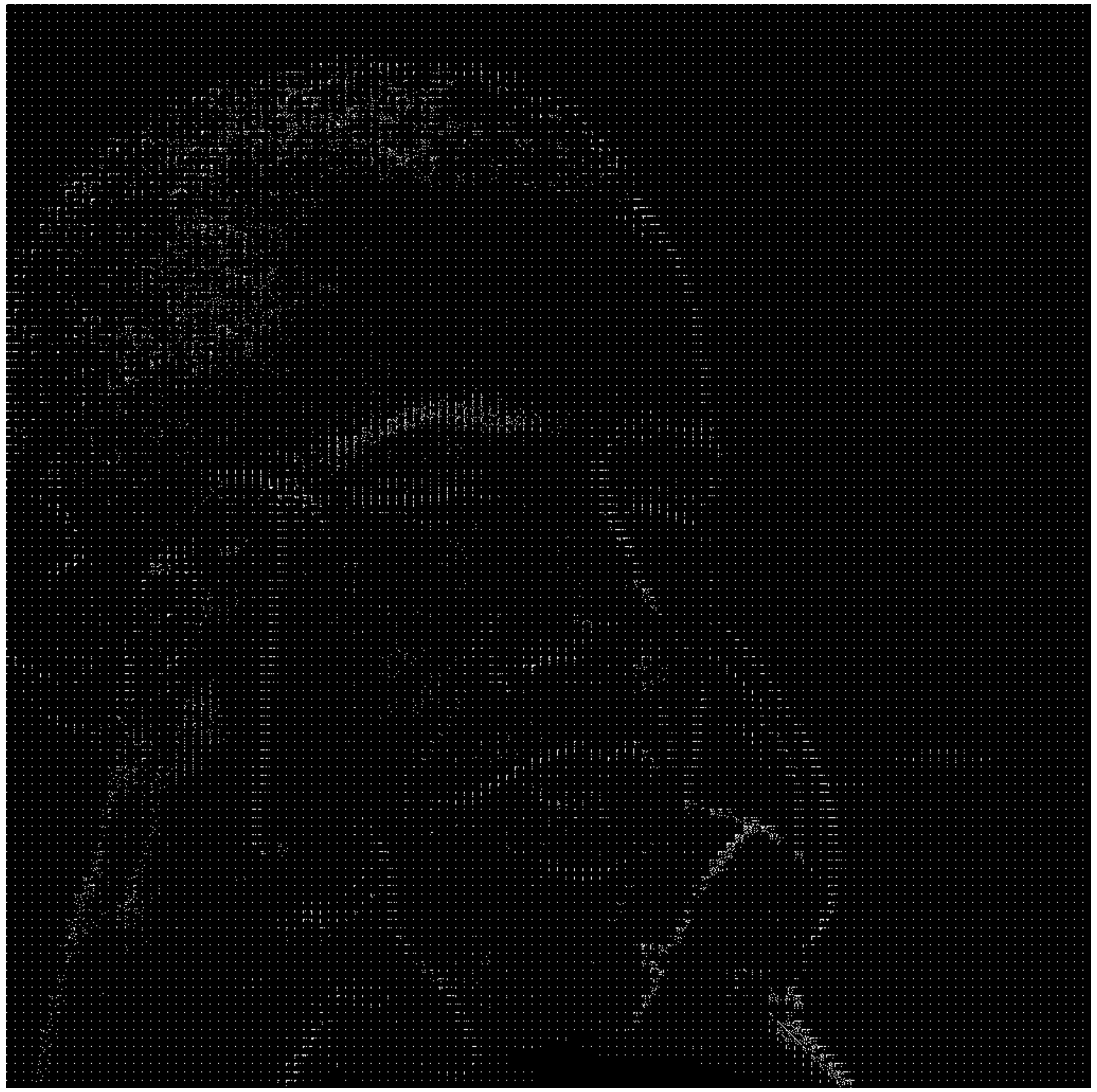
- Choose a threshold. If the absolute value of elements is lower than the threshold, set them to 0.

	1	2	3	4	5	6	7	8
1	133.6250	1.0938	0	0	0	0	0	1.4375
2	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0
5	0	0	1.2500	0	0	0	0	0
6	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	1.2500
8	0	0	0	0	0	0	0	0

Wavelet Transform

Discrete Functions Example: JPEG 2000

- Transform to image.



Wavelet Transform

Discrete Functions Example: JPEG 2000

- Code the sparse matrix.

	1	2	3	4	5	6	7	8
1	133.6250	1.0938	0	0	0	0	0	1.4375
2	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0
5	0	0	1.2500	0	0	0	0	0
6	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	1.2500
8	0	0	0	0	0	0	0	0

- Position coding: $(133.6250, 1, 1)$

Wavelet Transform

Discrete Functions Example: JPEG 2000

- Decompression

Thanks !

https://github.com/QianboZang