Mathematical Notes on Robotics

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Appendix

Notations

- $x \in \mathbb{R}^n$ is a vector.
- $A \in \mathbb{R}^{m \times n}$ is a maxtrix.
- $p \in \mathbb{R}^3$ is the vector of position, $p = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$.
- $R \in SO(3)$ is the rotation matrix.
- $g \in SE(3)$ is the rigid body transformation, $g = (R, p) \in SO(3) \times \mathbb{R}^3$. Alternatively, g can be represented in homogeneous coordinates $g = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$, with $g^{-1} = \begin{bmatrix} R^T & R^T p \\ 0 & 1 \end{bmatrix}$.
- ϕ, θ, ψ are the ZYX Euler angles, ϕ- roll, θ-pitch, ψ-yaw.
- $q \in Q$ is the quaternion, $q = \begin{bmatrix} q_s \\ q_x \\ q_y \\ q_z \end{bmatrix}$.
- $\omega \in \mathbb{R}^3$ is the unit axis of rotaiton, $\|\omega\| = 1$, $\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$.
- $\hat{\omega} \in so(3)$ is the skew-symmetric matrix, $\hat{\omega}^T = -\hat{\omega}$, $\hat{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$.

Linear Algebra

Vectors

Inner Product

Given $a, b \in \mathbb{R}^n$, $a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$, the *inner product* (also called dot product) of a and b is

$$a \cdot b = a^T b = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = \sum_{i=1}^{n} a_i b_i,$$

where a^T denotes the transpose. The inner product $a \cdot b$ can also be written as $< a, b >$ or (a, b).

The properties of inner product are as follows:

- $a \cdot b = b \cdot a$;
- $(ka) \cdot b = a \cdot (kb) = k(a \cdot b)$;
- $(a + b) \cdot c = a \cdot c + b \cdot c$;
- $a \cdot a \geq 0, \forall a \neq 0$, and $a \cdot a = 0 \iff a = 0$,

where $a, b, c \in \mathbb{R}^n$, $k \in \mathbb{R}$.

Outer Product

Given $a, b \in \mathbb{R}^3$, $a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, the *outer product* (also called cross product) of a and b is

$$a \times b = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}.$$

Define

$$\hat{a} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix},$$

then $a \times b = \hat{a} b$.

The properties of outer product are as follows:

- $a \times b = -b \times a$;
- $(ka) \times b = a \times (kb) = k(a \times b)$;
- $(a + b) \times c = a \times c + b \times c$;
- $a \times (b \times c) + b \times (c \times a) + c \times (a \times b) = 0$;
- $(a \times b) \times c = b(a \cdot c) - a(b \cdot c)$, $a \times (b \times c) = b(a \cdot c) - c(a \cdot b)$,

where $a, b, c \in \mathbb{R}^3$, $k \in \mathbb{R}$.

Vector Norm

Given $a \in \mathbb{R}^n$, the *norm* of a is

$$\|a\| = \sqrt{< a, a >} = \sqrt{a^T a}.$$

If $\|a\| = 1$, then a is called an unit vector. $\forall a \neq 0$, $\frac{a}{\|a\|}$ is an unit vector.

The properties of norm are as follows:

- $\|a\| \geq 0, \forall a \neq 0$, and $\|a\| = 0 \iff a = 0$;
- $\|ka\| = |k|\|a\|$;
- $\|a + b\| \leq \|a\| + \|b\|$, and $\|a + b\| = \|a\| + \|b\| \iff a$ and b are linearly dependent,

where $a, b \in \mathbb{R}^3$, $k \in \mathbb{R}$.

In addition,

$$\begin{aligned} a \cdot b &= \|a\|\|b\| \cos\theta, \\ \|a \times b\| &= \|a\|\|b\| \sin\theta, \end{aligned}$$

where θ is the angle between the a and b.

Orthogonal Vectors

Given $a, b \in \mathbb{R}^n$, if $a \cdot b = 0$, then a and b are called orthogonal, and denoted as $a \perp b$.

Matrix Norms

Matrix Norms Induced by Vector Norms

Given a matrix $A \in \mathbb{R}^{m \times n}$, the induced norm or operator norm is defined as

$$\begin{aligned} \|A\| &= \sup\{\|Ax\| : x \in \mathbb{R}^n, \|x\| = 1\} \\ &= \sup\{\frac{\|Ax\|}{\|x\|} : x \in \mathbb{R}^n, x \neq 0\}. \end{aligned}$$

Particularly,

$$\begin{aligned} \|A\|_1 &= \max_{1 \leq j \leq n} \sum_{i=1}^{m} |a_{ij}|, \\ \|A\|_\infty &= \max_{1 \leq i \leq m} \sum_{j=1}^{n} |a_{ij}|, \\ \|A\|_2 &= \sigma_{\max}(A), \end{aligned}$$

where $\sigma_{\max}(A)$ represents the largest singular value of matrix A.

Entry-wise Matrix Norms

Given a matrix $A \in \mathbb{R}^{m \times n}$, the entry-wise matrix norms treat the $m \times n$ matrix as a vector of size $m \cdot n$ and define the norms for $p \geq 1$ as

$$\|A\|_p = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^p \right)^{\frac{1}{p}}.$$

Particularly, the special case $p = 2$ yields the *Frobenius norm* or the *Hilbert-Schmidt norm*,

$$\|A\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2} = \sqrt{trace(A^T A)} = \sqrt{\sum_{i=1}^{\min\{m,n\}} \sigma_i^2(A)},$$

where $\sigma_i(A)$ are the singular values of A, the $trace(\cdot)$ function returns the sum of diagonal entries of a square matrix.

The special case $p = \infty$ yields the max norm

$$\|A\|_{\max} = \max_{ij} |a_{ij}|.$$

Norm Equivalence

Any two matrix norms are equivalence. That is, $\forall A \in \mathbb{R}^{m \times n}$, $\exists r, s \in \mathbb{R}_{>0}$, s.t.,

$$r\|A\|_\alpha \leq \|A\|_\beta \leq s\|A\|_\alpha,$$

where $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$, $\alpha, \beta \in \mathbb{N}$ are two matrix norms.

For a matrix $A \in \mathbb{R}^{m \times n}$ of rank r,

$$\begin{aligned}
&\|A\|_2 \leq \|A\|_F \leq \sqrt{r}\|A\|_2, \\
&\|A\|_{\max} \leq \|A\|_2 \leq \sqrt{mn}\|A\|_{\max}, \\
&\frac{1}{\sqrt{n}}\|A\|_\infty \leq \|A\|_2 \leq \sqrt{m}\|A\|_\infty, \\
&\frac{1}{\sqrt{m}}\|A\|_1 \leq \|A\|_2 \leq \sqrt{n}\|A\|_1, \\
&\|A\|_2 \leq \sqrt{\|A\|_1\|A\|_\infty}.
\end{aligned}$$

Proof of Propositions

Proposition 0.0.1 (Induced 2 Norm - Singular Value)**.** *Given a matrix $A \in \mathbb{R}^{m \times n}$,*

$$\|A\|_2 = \sigma_{\max}(A),$$

where $\sigma_{\max}(A)$ represents the largest singular value of matrix A.

Proof. [?] Let $B = A^T A$. Since the matrix fulfills $B^* = B$, where where B^* denotes the conjugate transpose, then B is a Hermitian matrix.

B is symmetric and non-negative definite, that is $B^T = B$, $x^T B x \geq 0$ for all $x \neq 0$. All the eigenvalues of B are real and non-negative. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of B and $\{e_1, \ldots e_n\}$ be the corresponding eigenvectors. The eigenvectors are orthogonal and forms an orthogonal basis of the Euclidean space. Denote $\lambda_{\max} = \max\{\lambda_1, \dots, \lambda_n\}$, and e_{max} is the eigen vector corresponding to the eigen value $\lambda_{\max}$.

Let $x = \alpha_1 e_1 + \cdots + \alpha_n e_n$, $\alpha_i \in \mathbb{R}$, for $i = 1, 2, \dots, n$. Then $\|x\|_2 = \sqrt{\sum_{i=1}^n \alpha_i^2}$, and $Bx = B(\sum_{i=1}^n \alpha_i e_i) = \sum_{i=1}^n \alpha_i B e_i = \sum_{i=1}^n \lambda_i \alpha_i e_i$. Therefore,

$$\begin{aligned}\|Ax\|_2^2 &=< Ax, Ax >=< x, A^T Ax >=< x, Bx > \\ &=< \sum_{i=1}^n \alpha_i e_i, \sum_{i=1}^n \lambda_i \alpha_i e_i >= \sum_{i=1}^n \lambda_i \alpha_i^2 \leq \lambda_{\max} \sum_{i=1}^n \alpha_i^2 \leq \lambda_{\max} \|x\|_2^2.\end{aligned}$$

Since $\|A\|_2 = \sup\{\frac{\|Ax\|_2}{\|x\|_2} : x \neq 0\}$,

$$\|A\|_2^2 \leq \lambda_{\max}.$$

Take $x = e_{\max}$, then

$$\|A\|_2^2 \geq \|Ae_{\max}\|_2^2 =< e_{\max}, Be_{\max} >=< e_{\max}, \lambda_{\max} e_{\max} >= \lambda_{\max}$$

Therefore, $\|A\|_2^2 = \lambda_{\max}$ and $\|A\|_2 = \sqrt{\lambda_{\max}} = \sigma_{\max}(A)$. □

Trace – F Norm

Given two matrics $A, B \in \mathbb{R}^{n \times n}$,

$$trace(AB) = trace(BA). \tag{1}$$

Proof. Let $A = (a_{ij})$, $B = (b_{ij})$, then

$$(AB)_{ii} = \sum_{j=1}^n a_{ij} b_{ji}, \quad (BA)_{jj} = \sum_{i=1}^n b_{ji} a_{ij}.$$

$$trace(AB) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} b_{ji} = \sum_{j=1}^n \sum_{i=1}^n a_{ij} b_{ji} = \sum_{j=1}^n \sum_{i=1}^n b_{ji} a_{ij} = trace(BA).$$

□

$$\|A\|_F = \sqrt{\sum_{i=1}^n \sigma_i^2(A)},$$

where $\sigma_i(A)$ are the singular values of A.

Proof. [**?**]

$\sum_{i=1}^n \sigma_i^2(A) = trace(\Sigma^T \Sigma)$, where $A = U \Sigma V^T$.

$$\begin{aligned}\|A\|_F^2 &= trace(A^T A) = trace(V \Sigma^T U^T U \Sigma V^T) = trace(V \Sigma^T \Sigma V^T) \\ &= trace(\Sigma^T \Sigma V^T V) = trace(\Sigma^T \Sigma) = \sum_{i=1}^n \sigma_i^2(A).\end{aligned}$$

Therefore,

$$\|A\|_F = \sqrt{\sum_{i=1}^n \sigma_i^2(A)},$$

□

Eigen Value Decomposition

Singluar Value Decomposition

Matrix Exponential

Given a matrix $A \in \mathbb{R}^{n \times n}$, the exponential of A is

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots . \tag{2}$$

Given a matrix $\hat{\omega} \in so(3)$ with $\|\omega\| = 1$, and a real number $\theta \in \mathbb{R}$, the exponential of $\hat{\omega}\theta$ is

$$e^{\hat{\omega}\theta} = I + \theta\hat{\omega} + \frac{\theta^2}{2!}\hat{\omega}^2 + \frac{\theta^3}{3!}\hat{\omega}^3 + \cdots . \tag{3}$$

Furthermore,

$$e^{\hat{\omega}\theta} = I + (\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots)\hat{\omega} + (\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \cdots)\hat{\omega}^2. \tag{4}$$

Hence, there is the *Rodrigues' formula [**?**]*

$$e^{\hat{\omega}\theta} = I + \hat{\omega}\sin\theta + \hat{\omega}^2(1 - \cos\theta). \tag{5}$$