Lecture 2

Mathematical Modeling

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1. Modeling Change
2. Linear Dynamical Systems: $a_{n+1} = ra_n$
3. Affine dynamical Systems: $a_{n+1} = ra_n + b$
4. Equilibrium Points

Modeling change

Basic paradigm:

- Future value = present value + change
- Change = future value - present value

Discrete time $\rightarrow$ difference equation

- change takes place in discrete time intervals (e.g., the depositing of interest in an account)
- in this lecture

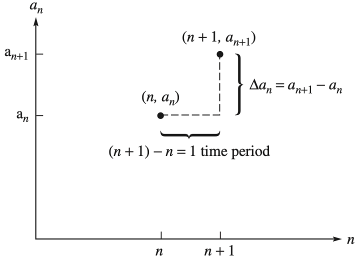
Continuous time $\rightarrow$ differential equation

- change takes place continuously (e.g., the position of a moving car)
- in a later lecture in this course

Modeling change with difference equations

For a sequence of numbers $a_1, a_2, \ldots, a_n$, their differences are:

- $\Delta a_0 = a_1 - a_0$
- $\Delta a_1 = a_2 - a_1$
- $\Delta a_2 = a_3 - a_2$
- $\ldots$
- $\Delta a_n = a_{n+1} - a_n$



1. Modeling Change
   - A few examples
2. Linear Dynamical Systems: $a_{n+1} = ra_n$
3. Affine dynamical Systems: $a_{n+1} = ra_n + b$
4. Equilibrium Points

Example: savings deposit

A savings deposit

- A savings deposit initially worth 1000 eur
- Interest rate of 1% per month
- Calculate the value growth of the certificate

Denote by a_n the value of the deposit n months after the first deposit:

- $a_0 = 1000$

The value at time $n + 1$ as a function of the value at time n:

- $\Delta a_n = a_{n+1} - a_n = 0.01a_n$
- $a_{n+1} = 1.01a_n, \quad$ for all $n \geq 0$

Solution: $a_n = a_0 1.01^n$

Note

- We have an infinite set of algebraic equations
- This is called a (discrete) dynamical system
- The change may depend on several previous terms and/or external terms

How about if you withdraw 50 eur each month?

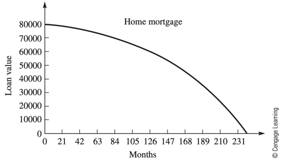
- $a_{n+1} - a_n = 0.01a_n - 50$

Example: mortgage

Mortgaging a home

- Get a mortgage of 80,000 eur with an interest rate of 1% per month
- Monthly payment: 880.87 eur
- Question 1: after n payments, how much is there still to pay?
- Question 2: how long does it take to pay the whole mortgage?
    - Denote by b_n the value of the loan after n payments:
    - $b_0 = 80,000$
    - $b_{n+1} - b_n = 0.01b_n - 880.87$; in other words,
      $b_{n+1} = 1.01b_n - 880.87$
    - What is the smallest n such that $b_n \leq 0$?

Months n	Amount owed b_n
0	80000.00
1	79919.13
2	79837.45
3	79754.96
4	79671.64
5	79587.48
6	79502.49
7	79416.64
8	79329.94
9	79242.37
10	79153.92
11	79064.59
12	78974.37

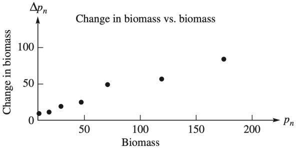


Example: growth of a yeast culture

We are given data on the growth of a yeast culture

- Measurements on the size of the culture at various time points
- Problem: propose a model for the growth
  - Idea: look at the change with respect to the population size
  - Observe that the change can be *approximated* as a straight line
  - Measure the slope of the line: 0.5
- Proposed model: $\Delta p_n = p_{n+1} - p_n = 0.5 p_n$
  - Solution: $p_{n+1} = 1.5 p_n$
  - Good model for the little data we have
  - The model predicts infinite growth; unlikely to hold against more data

Time in hours n	Observed yeast biomass p_n	Change in biomass $p_{n+1} - p_n$
0	9.6	8.7
1	18.3	10.7
2	29.0	18.2
3	47.2	23.9
4	71.1	48.0
5	119.1	55.5
6	174.6	82.7
7	257.3	



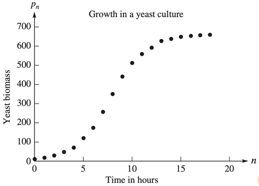
Example (continued)

Get more data on the yeast culture; plot the change per hour against the population size at the time

Note

- The culture levels out at about 665 units
- Growth rate slows down to almost 0 as the population approaches the max value
- **Old model:** $p_{n+1} - p_n = kp_n$
- **New model:** replace the constant k with a simple function that approaches 0 as p_n approaches the max value
- **New model:** $p_{n+1} - p_n = r(665 - p_n)p_n$

Time in hours n	Yeast biomass p_n	Change/hour $p_{n+1} - p_n$
0	9.6	8.7
1	18.3	10.7
2	29.0	18.2
3	47.2	23.9
4	71.1	48.0
5	119.1	55.5
6	174.6	82.7
7	257.3	93.4
8	350.7	90.3
9	441.0	72.3
10	513.3	46.4
11	559.7	35.1
12	594.8	34.6
13	629.4	11.4
14	640.8	10.3
15	651.1	4.8
16	655.9	3.7



Example (continued)

New model:

- $p_{n+1} - p_n = r(665 - p_n)p_n$

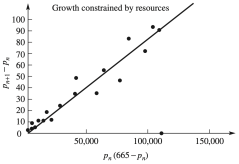
How do we test if the model makes sense?

- Compare the differences $p_{n+1} - p_n$ and $(665 - p_n)p_n$
- Check if there is a reasonable proportionality
- Answer: YES!
- $r = 0.00082$

Final model:

- $p_{n+1} = p_n + 0.00082(665 - p_n)p_n$

$p_{n+1} - p_n$	$p_n(665 - p_n)$
8.7	6291.84
10.7	11,834.61
18.2	18,444.00
23.9	29,160.16
48.0	42,226.29
55.5	65,016.69
82.7	85,623.84
93.4	104,901.21
90.3	110,225.01
72.3	98,784.00
46.4	77,867.61
35.1	58,936.41
34.6	41,754.96
11.4	22,406.64
10.3	15,507.36
4.8	9050.29
3.7	5968.69
2.2	3561.84



Example (continued)

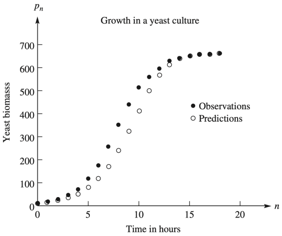
Final model:

- $p_{n+1} = p_n + 0.00082(665 - p_n)p_n$

Test the model

- **Result:** very good fit
- Comment: discuss later how to measure the "goodness" of a fit; for now only through visual inspection

Time in hours	Observation	Prediction
0	9.6	9.6
1	18.3	14.8
2	29.0	22.6
3	47.2	34.5
4	71.1	52.4
5	119.1	78.7
6	174.6	116.6
7	257.3	169.0
8	350.7	237.8
9	441.0	321.1
10	513.3	411.6
11	559.7	497.1
12	594.8	565.6
13	629.4	611.7
14	640.8	638.4
15	651.1	652.3
16	655.9	659.1
17	659.6	662.3
18	661.8	663.8



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Solutions to linear dynamical systems

Linear dynamical systems: $a_{n+1} = ra_n$, for some constant r

- By induction: the solution is $a_n = r^n a_0$
- $r = \frac{a_1}{a_0}$

Linear dynamical systems: $a_n = r^n a_0$, $r = 0, 1, -1$

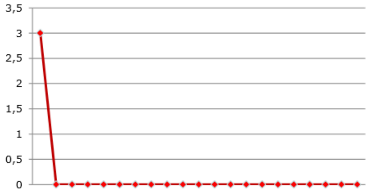

图 1: r=0

图 2: r=1

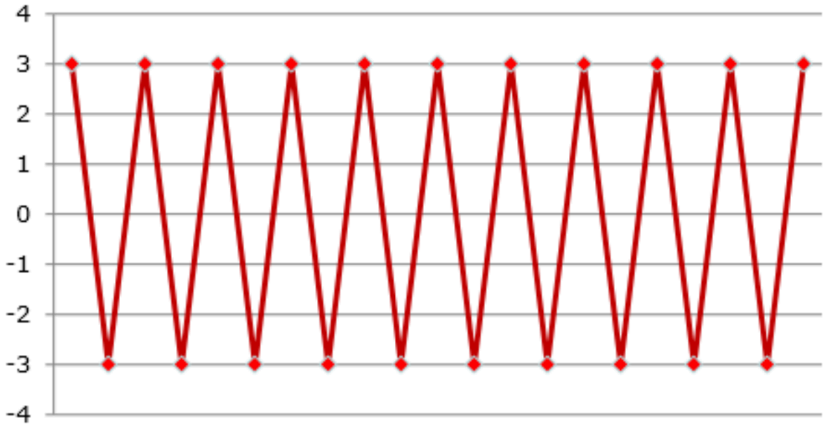

图 3: r=-1

Linear dynamical systems: $a_n = r^n a_0$, $r > 1$, $r < -1$

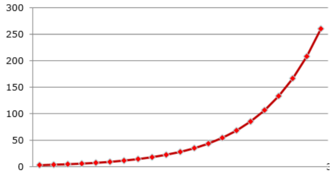

图 4: r=1.25

图 5: r=-1.25

Linear dynamical systems: $a_n = r^n a_0$, $-1 < r < 1$

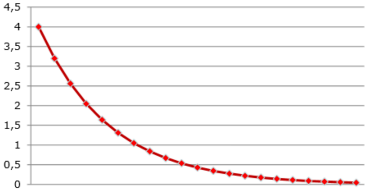

图 6: r=0.8

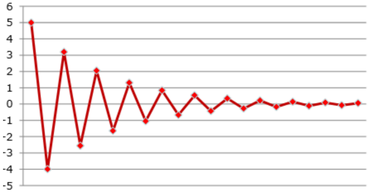

图 7: r=-0.8

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Example: prescription for digoxin (treatment of some heart conditions)

Problem: Prescribe an amount that keeps the concentration of digoxin in the bloodstream above an effective level, without exceeding a safe level (variation here among patients)

Several questions to settle:

- what is the decay of a single dose
- at what intervals to give the consecutive doses (not considered in this example)
- what doses to give

Example (continued)

First question: what is the decay of a single dose?

- Give a single dose and measure the amount of digoxin remaining in the bloodstream after n days

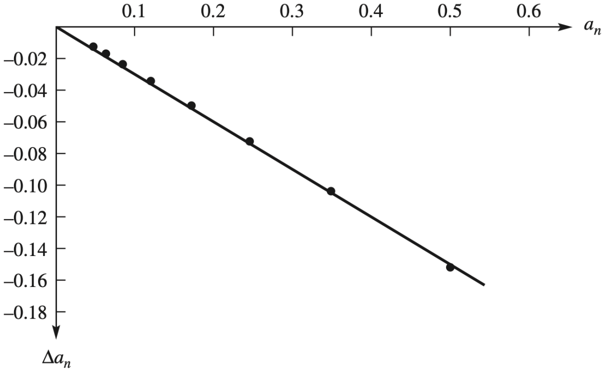
n	a_n	Δa_n
0	0.500	
1	0.345	-0.155
2	0.238	-0.107
3	0.164	-0.074
4	0.113	-0.051
5	0.078	-0.035
6	0.054	-0.024
7	0.037	-0.017
8	0.026	-0.011

Example (continued)

Plot Δa_n against a_n

Conclusion:

- $a_{n+1} - a_n = -0.5a_n$
- $a_{n+1} = 0.5a_n$



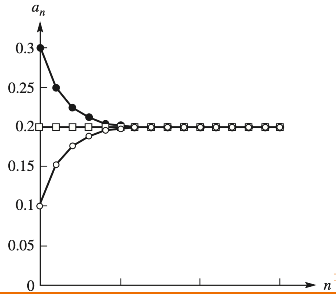
Example (continued)

Second part: additional doses

- We add daily a dosage of 0.1
- Model: $a_{n+1} = 0.5a_n + 0.1$

Initial value a_0: might be different than the subsequent doses

- $a_0 = 0.1$ (series A)
- $a_0 = 0.2$ (series B)
- $a_0 = 0.3$ (series C)



Analytic solutions of affine dynamical systems

Affine dynamical systems:

- $a_{n+1} = ra_n + b$, for some constants r, b with $r \neq 1$

Analytic solution:

$$a_n = r^n c + \frac{b}{1-r}, \quad \text{where } c = \frac{a_0 - b}{1-r}$$

- $a_n = ra_{n-1} + b$
- $a_{n+1} - a_n = r(a_n - a_{n-1})$
- Let $x_n = a_{n+1} - a_n$, $x_0 = a_1 - a_0 = ra_0 + b - a_0$
- Then $x_n = rx_{n-1}$, i.e., $x_n = r^n x_0$
- So $a_{k+1} - a_k = r^k x_0$.Sum this relation from $k = 0$ to $n-1$:
- $a_n - a_0 = x_0(1 - r^n)/(1-r) = -r^n x_0/(1-r) + x_0/(1-r)$
  - Replace $x_0/(1-r) = -a_0 + b/(1-r)$
- $a_n - a_0 = r^n c - a_0 + b/(1-r)$, where $c = a_0 - b/(1-r)$
- $a_n = r^n c + b/(1-r)$

Equilibrium

Consider a dynamical system: $a_{n+1} = f(a_n)$

- A number a is called an equilibrium point (or a fixed point) of the dynamical system if $a = f(a)$
- In other words, if we start with initial value $a_0 = a$, then $a_n = a$, for all $n \geq 0$
- Important to identify the equilibrium points of a dynamical system (and their properties) to know about its asymptotic behavior

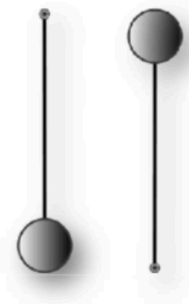
Example. Linear dynamical systems: $a_{n+1} = ra_n$, $r \neq 0$

- Look for equilibrium points: solve the equation $x = rx$, i.e., $(1 - r)x = 0$
- If $r \neq 1$, then 0 is the only equilibrium point
- If $r = 1$, then all numbers are equilibrium points

Types of equilibrium

Types of equilibrium points (informal definitions)

- **Stable:** starting from a nearby initial point will give an orbit that remains nearby the original orbit
  - **Asymptotically stable (attractor):** starting from a nearby initial point will give an orbit that converges towards the original orbit
  - Example: a pendulum in the lowest position
- **Unstable:** starting from a nearby initial point *may* give an orbit that goes away from the original orbit
  - Example: a pendulum in the highest position



Stable-unstable equilibrium
Source for picture: Wikipedia

Types of equilibrium points

Affine dynamical systems: $a_{n+1} = ra_n + b, \quad r \neq 0$

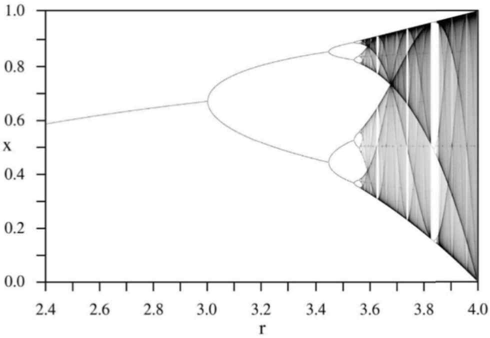
- Look for equilibrium points: solve the equation $x = rx + b$, i.e., $(1-r)x = b$
  - If $r \neq 1$, then $\frac{b}{1-r}$ is the only equilibrium point
  - If $r = 1$ and $b = 0$, then all numbers are equilibrium points
  - If $r = 1$ and $b \neq 0$, then the dynamical system has no equilibrium point
- Assume $r \neq 1$. What kind of equilibrium point is $\frac{b}{1-r}$?
  - (asymptotically stable, unstable?)
- Recall that $a_n = r^n c + \frac{b}{1-r}$, where $c = a_0 - \frac{b}{1-r}$
  - $|r| < 1$: $\frac{b}{1-r}$ is asymptotically stable
  - $|r| > 1$: $\frac{b}{1-r}$ is unstable
  - $r = -1$: $\frac{b}{2}$ is stable
    - Two constant subsequences (the odd and the even terms) on either side of the equilibrium
    - Depending on the value of a_0, the two subsequences can be close to $\frac{b}{2}$

Nonlinear systems

- $a_{n+1} = f(a_n)$, where f is a non-linear function.
- **Example**: $a_{n+1} = r(1 - a_n)a_n$, that came up earlier in this lecture
  - The system can have very different behavior depending on r
  - Also called the "logistic map"
  - Typical example for how chaotic behavior can rise from very simple (non-linear) dynamics

Bifurcation diagram for $a_{n+1} = r(1 - a_n)a_n$

- For each r, the diagram shows the period p of the dynamical system and the attractors of its p convergent subsequences a_{np+i} with $i = 0, 1, \ldots, p - 1$
- Period doubles as r increases
- Eventually it leads to chaos



Learning objectives

- Understand the concept of modeling the change in a discrete dynamical system
- Able to write a linear and an affine model with difference equations for a simple real-life phenomenon
- Understand the diverse behavior that a linear dynamical system model can have
- Understand the notion of equilibrium point
- Understand the different types of stability of equilibrium points