Lecture 7 Discrete optimization modeling

Mathematical Modeling

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Model fitting as an optimization problem

- Discussed in Lecture 4 about model fitting
  - gave several formulations for model fitting
  - they all are optimization problems
  - discussed in Lecture 4 a calculus-based approach to solving the least squares formulation of model fitting
- Discuss here about a discrete approach to optimization, in particular to model fitting
  - focus on linear programming
  - Chebyshev criterion (minimize the greatest deviation) leads to a linear program in the case of linear models

Generalities

- Discrete optimization problems: the variables are assumed to take only discrete values
  - large field of research, many different branches
- Focus here on the following type of discrete optimization problems:
  - Optimize (i.e. minimize or maximize) functions $f_1(x_1,\ldots,x_m),\ f_2(x_1,\ldots,x_m),\ \ldots,\ f_n(x_1,\ldots,x_m)$ such that the following constraints are satisfied:

$$
\begin{aligned}
&g_1(x_1,\ldots,x_m) \geq b_1,\ \ldots,\ g_{p_1}(x_1,\ldots,x_m) \geq b_{p_1},\\
&g_{p_1+1}(x_1,\ldots,x_m) = b_{p_1+1},\ \ldots,\ g_{p_1+p_2}(x_1,\ldots,x_m) = b_{p_1+p_2},\\
&g_{p_1+p_2+1}(x_1,\ldots,x_m) \geq b_{p_1+p_2+1},\ \ldots,\\
&g_{p_1+p_2+p_3}(x_1,\ldots,x_m) \leq b_{p_1+p_2+p_3}.
\end{aligned}
$$

- $(x_1, \ldots, x_m)$ are the variables of the model
- $f_1, f_2, \ldots, f_n$ are the objective functions (to be optimized)
    - multi-objective optimization often reduced to single-objective optimization
    - multiple objectives may not be optimized simultaneously; often replaced with a single combined objective

Example: production schedule

- Carpenter making tables and bookcases. Question: how many of each should he make each week so that he maximizes his profit?
  - net profit for selling a table: $25, a bookcase: $30
  - amount of lumber available weekly: 600 board feet
  - amount of labor per week: 40 hours
  - lumber needed for a table: 20 board feet; for a bookcase: 30 board feet
  - work needed for a table: 5 hours; for a bookcase: 4 hours
  - he has firm contracts for 4 tables and 2 bookcases every week
- Problem formulation: maximize $25x_1 + 30x_2$ such that:
  - $20x_1 + 30x_2 \leq 600$ (lumber)
  - $5x_1 + 4x_2 \leq 40$ (labor)
  - $x_1 \geq 4$ (contract)
  - $x_2 \geq 2$ (contract)

Linear programs

- A discrete optimization problem is called a linear program if:
  - there is a unique objective function
  - the objective functions and the constraint functions are linear
  - the objective functions and the constraint functions are non-parametric
  - the variables take value over rational numbers
- If there are no constraint functions: unconstrained problem
  - Example: model fitting through the sum of absolute deviations:

$$\text{minimize} \sum_i |y_i - y(x_i)|$$

- A linear program is called an integer program if at least one variable is restricted to integer values only

Example of an integer problem

- The knapsack problem
  - Given a set of items, each with a weight and a value
  - Determine the number of each item to include in a collection so that the total weight is less than or equal to a given limit and the total value is as large as possible.
- Formulation:
  - we have m items with weights and values as follows:

$$(w_1, v_1), \dots, (w_m, v_m)$$

  - we have a rucksack of capacity at most C
  - maximize $n_1 v_1 + \cdots + n_m v_m$ over the non-negative integer variables $(n_1, \dots, n_m)$ such that

$$n_1 w_1 + \cdots + n_m w_m \leq C$$

- Computational complexity
  - NP-hard in the formulation above
  - NP-complete as a decision problem: can a value of at least V be achieved without exceeding total weight W?

Example of a multiobjective program

- Investment problem: an investor has \$40.000 to invest in
  - savings with return 7%
  - municipal bonds at 9%
  - stocks that have consistently averaged at 14%
  - various option have different degrees of risk
- Goals for the investment:
  1. yearly return of at least \$5000
  2. invest at least \$10.000 in stocks
  3. investment in stocks should not exceed the combined total in bonds and savings
  4. have a liquid savings account between \$5000 and \$15.000

- A first attempt to a solution: invest as much as possible in the highest-return option available, as little as possible in the lowest-return option available
  - savings have the smallest return: only invest \$5000 in them (constraint 4)
  - the rest into stocks and bonds with as much as possible into stocks: \$20.000 in stocks, \$15.000 in bonds (see 3)
  - return: $0.07 \times 5000 + 0.09 \times 15000 + 0.14 \times 20000 = 4500$, violating constraint 1

Example (continued)

- Denote by x,y,z the investment in savings, bonds, stocks, resp.
- Formulate the constraints:
  1. $0.07x + 0.09y + 0.14z \geq 5000$
  2. $z \geq 10000$
  3. $z \leq x + y$
  4. $5000 \leq x \leq 15000$
  5. $x + y + z \leq 40000$
- Clear from our first attempt at a solution (Greedy) that not all constraints can be satisfied

- Reformulate the problem: we can fail goals 3 and 4, but we should minimize the total amount with which we fail the two goals
  - Minimize $G_3 + G_4$ such that:
    1. $0.07x + 0.09y + 0.14z \geq 5000$
    2. $z \geq 10000$
    3. $z - G_3 \leq x + y$
    4. $5000 - G_4 \leq x \leq 15000$
    5. $x + y + z \leq 40000$
  - where x,y,z are non-negative
  - A solution for the reformulated problem can be found, e.g., through the Simplex method

Dynamic programming

- A type of optimization where not all decision have to be taken at the same time – they can be taken in stages
- Example:
  - a rancher with an initial herd of k cattle
  - he wants to retire after N years, when he will sell all remaining cattle
  - each year he decides how many cattle to sell: profit p_i in year i
  - the number of cattle kept in year i will double in year i+1
  - characteristic of the problem: each year he takes a (local) decision of taking a larger profit vs building the future; the optimization goal is a global one, concerned with the profit after N years.

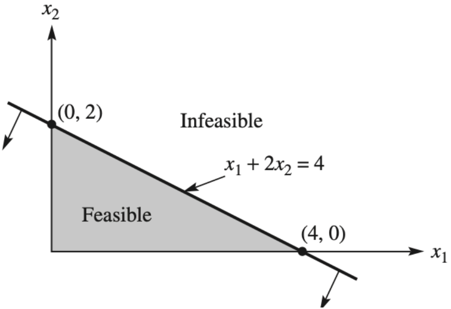
- Focus in the remaining of the lecture on solving linear programs
  - geometric solutions
  - algebraic solutions
  - the Simplex method

Linear programs: geometric solutions

- Basic insight:
  - a linear equation $a_1 x_1 + a_2 x_2 = b$ can be seen as a line in the bidimensional plane (x_1, x_2)
  - the line will separate the bidimensional plane (x_1, x_2) into two regions (half-planes), one where $a_1 x_1 + a_2 x_2 < b$, and the other where $a_1 x_1 + a_2 x_2 \geq b$
  - identify which region of the plane satisfies the constraint
    - choose any point (x_1, x_2) in the region, calculate $a_1 x_1 + a_2 x_2$ and compare it to b
  - repeat the same for all constraints
  - take the intersection of all the identified regions
  - Note: the same is true for more than 2 variables; a linear equation $a_1 x_1 + ... + a_n x_n = b$ can be seen as a hyperplane (dimension $n - 1$) in the n-dimensional plane

Figure 7.2

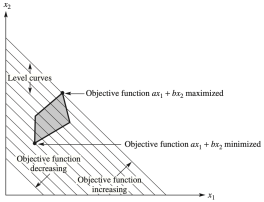
The feasible region for the constraints $x_1 + 2x_2 \le 4$, $x_1, x_2 \ge 0$



Geometric solutions to linear programs (2)

- **Theorem.** The points satisfying the constraint of a linear program (the feasible region) form a convex set.
  - If the feasible region is bounded, then the objective function attains both its maximum and its minimum value at extreme points of the feasible region.
  - If the feasible region is unbounded, the objective function may not have a finite optimum. However, if the optimum exists, then it will be attained at an extreme point.

Figure 7.10
A linear function assumes its maximum and minimum values on a nonempty and bounded convex set at an extreme point.

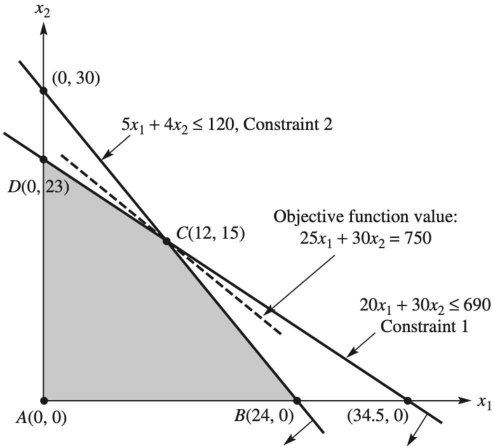


Example

- Recall the carpenter's problem: maximize $25x_1 + 30x_2$ such that:
  - $20x_1 + 30x_2 \leq 600$ (lumber)
  - $5x_1 + 4x_2 \leq 40$ (labor)
  - $x_1 \geq 0$ (contract)
  - $x_2 \geq 0$ (contract)
- Find the solution at the extreme points
  - A(0,0): $0
  - B(24,0): $600
  - C(12,15): $750
  - D(0,23): $690
- Answer: make 12 tables and 15 bookshelves for a profit of $750

■ **Figure 7.4**

The set of points satisfying the constraints of the carpenter's problem form a convex set.



Another example

- A model fitting problem
  - The model is $y = cx$
  - The data is $(1, 2), (2, 5), (3, 8)$
  - Fit the model to minimize the largest absolute deviation
  - Model formulation: let the largest absolute deviation be r
  - $|2 - c| \leq r; \quad |5 - 2c| \leq r; \quad |8 - 3c| \leq r$
  - Equivalently:
    - $r - (2 - c) \geq 0; \quad r + (2 - c) \geq 0$
    - $r - (5 - 2c) \geq 0; \quad r + (5 - 2c) \geq 0$
    - $r - (8 - 3c) \geq 0; \quad r + (8 - 3c) \geq 0$
  - Geometrical approach: plot the graphs of the linear functions obtained by transforming each inequality above into an equation

Example (continued)

- The intersection forms a convex set in the (c, r) plane with corners B-C
  - check the value of the objective function $f(r) = r$ in points B and C
  - minimum reached in B
  - Solution: $c = 5/2; r = 1/2$
  - Check the solution: plot $y = 5/2\,x$ against the data

Figure 7.5
The feasible region for fitting $y = cx$ to a collection of data

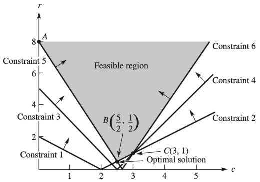
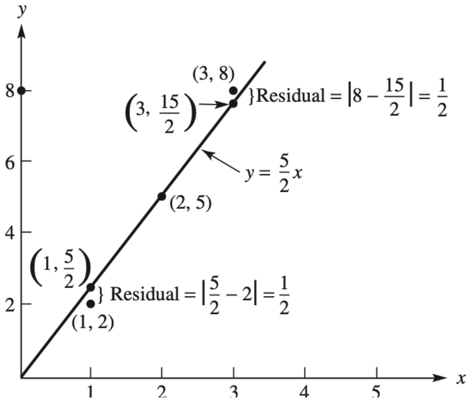


Figure 7.6

The line $y = (5/2)x$ results in a largest absolute deviation $r_{\max} = \frac{1}{2}$, the smallest possible $r_{\max}$.



Empty and unbounded feasible regions

- **Empty feasible regions**
  - might get empty feasible regions because of inconsistent constraints
  - Example: $x < 2$ and $x > 4$
  - in this case there is obviously no solution to the optimization problem
- **Unbounded feasible regions**
  - we had such a case in the last example
  - depending on the optimization problem to solve, there might not be a solution
  - example: what if in the previous example we had to maximize function $f(r) = 2r + 3$

Level curves of the objective function

- Consider again the carpenter example
  - The objective function to maximize was $f(x_1, x_2) = 25x_1 + 30x_2$
  - Note the plots below
    - The level curves are parallel to each other
    - The maximum level curve touching the feasible region will do so at an extreme point
    - It may also touch it at two extreme points simultaneously (along one of its borders)

Figure 7.7
The level curves of the objective function f are parallel line segments in the first quadrant; the objective function either increases or decreases as we move in a direction perpendicular to the level curves.

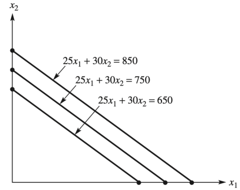
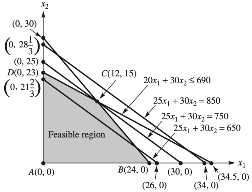


Figure 7.8
The level curve $25x_1 + 30x_2 = 750$ is tangent to the feasible region at extreme point C.



Algebraic approach

- **Approach:**
  - Find all intersection points of the constraints
    - Consider all possible pairs of constraints
  - Determine which intersection points are feasible; this gives the extreme points
    - Check the intersection points against all constraints
  - Evaluate the objective function at the extreme points
  - Choose the extreme point giving the optimal value for the objective function

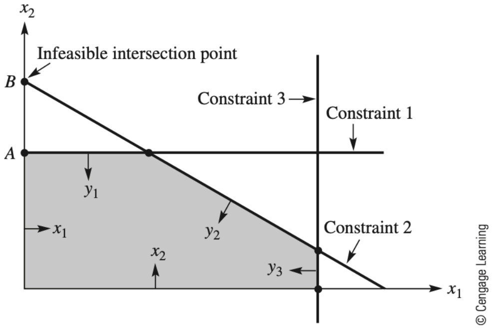


Figure 7.11

The variables x_1, x_2, y_1, y_2, and y_3 measure the satisfaction of each of the constraints; intersection point A is characterized by $y_1 = x_1 = 0$; intersection point B is not feasible because y_1 is negative; the intersection points surrounding the shaded region are all feasible because none of the five variables is negative there.

Algebraic approach (2)

- Enumerating all intersection points for 2-dimensional problems
  - Without loss of generality assume that the constraints are linear inequalities: $a_1x_1 + a_2x_2 \leq b$, where $x_i \geq 0$
  - Add a new variable y_1 and transform the inequality above into a linear equation $a_1x_1 + a_2x_2 + y_1 = b$, where $x_i \geq 0,\ y_1 \geq 0$
    - $y_1 = 0$ is equivalent to a point on the border of the constraint
  - Repeat for all other constraints (new variable for each)
    - We will have now $2 + n$ variables and n linear equations
    - Solve them over non-negative integers

- Finding the intersection point comes to solving a system of two linear equations in non-negative variables
  - Some intersection points will be between two border lines, i.e., with the corresponding y variables set to 0
  - Other intersection points will be between a border line and an axis –an y variable and an x variable set to 0
  - Another intersection point will be the origin: $x_1 = x_2 = 0$
  - In other words: for any pair of variables, set them to 0 and solve the remaining system of n equations with n variables

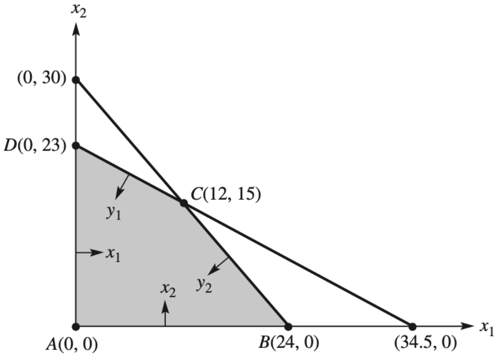
Example: the carpenter's problem

- Recall the carpenter's problem: maximize $25x_1 + 30x_2$ such that:
  - $20x_1 + 30x_2 \leq 690$; $5x_1 + 4x_2 \leq 120$
  - $x_1 \geq 0$; $x_2 \geq 0$ (assume here for simplicity no pre-existing contract)
- Convert each of the inequalities into equations:
  - $20x_1 + 30x_2 + y_1 = 690$; $5x_1 + 4x_2 + y_2 = 120$
  - $x_1 \geq 0$; $x_2 \geq 0$; $y_1 \geq 0$; $y_2 \geq 0$

- Consider the set of variables $\{x_1, x_2, y_1, y_2\}$
  - For all pairs of variables, set the chosen pair to 0 and solve the resulting system of 2 equations with 2 unknowns: 6 possibilities
    1. $x_1 = x_2 = 0$; in this case we obtain $y_1 = 690$, $y_2 = 120$
    2. $x_1 = y_1 = 0$; in this case we obtain $x_2 = 23$, $y_2 = 28$
    3. $x_1 = y_2 = 0$; in this case we obtain $x_2 = 30$, $y_1 = -210$; *unfeasible point*
    4. $x_2 = y_1 = 0$; in this case we obtain $x_1 = 34.5$, $y_2 = -52.5$; *unfeasible point*
    5. $x_2 = y_2 = 0$; in this case we obtain $x_1 = 24$, $y_1 = 210$
    6. $y_1 = y_2 = 0$; in this case we obtain $x_1 = 12$, $x_2 = 15$

Figure 7.12

The variables $\{x_1, x_2, y_1, y_2\}$ measure the satisfaction of each constraint; an intersection point is characterized by setting two of the variables to zero.



Algebraic approach (3)

- **Enumerating all intersection points for $m > 2$ variables**
  - Without loss of generality assume that the constraints are linear inequalities: $a_1x_1 + \cdots + a_mx_m \leq b$, where $x_i \geq 0$
  - Add a new variable y_1 and transform the inequality above into a linear equation $a_1x_1 + \cdots + a_mx_m + y_1 = b$, where $x_i \geq 0$, $y_1 \geq 0$
    - $y_1 = 0$ is equivalent to a point on the border of the constraint
  - Repeat for all other constraints (new variable for each)
    - We will have now $m + n$ variables and n linear equations
    - Solve them over non-negative integers
  - For all possible sets S of m variables, set the variables in S to 0 and solve the resulting set of n equations with n unknowns to yield all intersection points

- **Computational complexity**
  - Identify $\frac{(m+n)!}{(m!n!)}$ intersection points
  - Solve a linear system of n equations, n unknowns for each
- **Note:** some (potentially many/most) intersection points might not be feasible