Lecture 12 Continuous Optimization Modeling

Mathematical Modeling

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Focus

- Linear programming models studied earlier:
  Optimize $f(X)$
  - subject to inequality constraints $g_i(X) \leq b_i$, $h_j(X) \geq c_j$ where f, g, h are linear functions
  - If X is a vector of integers, then we deal with integer programming
- In this lecture: optimize $f(X)$, where f is a continuous, potentially non-linear function
  - might have constraints of the form $g_i(X) = b_i$, where functions g_i may also be non-linear
  - in particular, focus on the case where f and g_i are differentiable functions; apply calculus-based techniques

1. Unconstrained optimization
2. Constrained optimization
3. Summary of the course

Unconstrained optimization

- **Problem**: optimize $f(X)$, where f is a continuous function
  - we only focus here on the case where f is differentiable
- The single-variable case
  - the optimum is among the solutions of the equation $df/dx = 0$
  - to check whether a certain solution is a minimum (or a maximum), consider the second derivative test or numerical approaches
- The multi-variable case: $X = (x_1, \ldots, x_n)$
  - consider the system of partial derivative equations $\partial f/\partial x_i = 0$, for all $i = 1, \ldots, n$
  - to check whether a certain solution is a minimum (or a maximum), consider the second derivative test or numerical approaches

Example 1: inventory

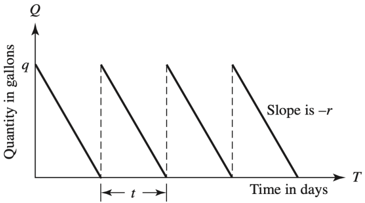
- **Problem**: Minimize the daily cost of delivery and storage
  - Chain of gas stations; determine how often and how much gasoline to deliver to the various stations
- Costs
  - Delivery cost d (in addition to the price of the gasoline delivered)
  - Storage costs: assume a constant storage cost per gallon per day
  - Demand rate: assume a constant demand rate r per day
- Variables
  - Quantity of gasoline to be delivered periodically
  - Time in-between two deliveries
- Objective
  - Minimize the daily storage and delivery costs
  - Do not run out of gasoline

Example 1: inventory (continued)

- Plan
  - Initial delivery;
  - wait until it runs out, then have a second (instantaneous) delivery;
  - again wait until it runs out, etc
  - linear depletion of the stock, with the slope given by the demand rate r

Figure 13.3

An inventory cycle consists of an order quantity q consumed in t days.



Example 1: inventory (continued)

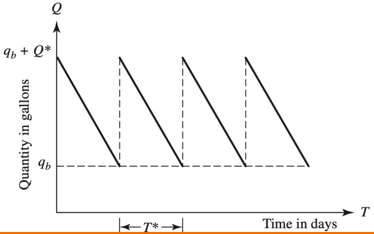
- Given
  - Delivery cost d
  - Demand rate r
- To calculate
  - Daily storage costs
- Calculate storage costs in-between two deliveries
  - constant storage cost per unit
  - calculate the total amount of gasoline stored in an interval of length t
  - in other words: the area under the line curve for the interval $[0, t]$: $qt/2$
  - Answer: the cost is $sqt/2$
- Slope of the curve: $r = q/t$, i.e., $q = rt$
- Cost per delivery cycle: $d + sqt/2 = d + srt^2/2$
- Daily cost: $d/t + srt/2$

Example 1: inventory (continued)

- Our model will be based on equal-interval deliveries (because of constant demand rate), triggered by reaching a minimal value q – this does not change the daily cost
- Problem: find the value $t = T^*$ minimizing the daily cost: $f(t) = d/t + srt/2$
- Approach: solve $df/dt = 0$

$$-\frac{d}{t^2} + \frac{sr}{2} = 0 \qquad T^* = \sqrt{\frac{2d}{sr}}$$

Figure 13.5
A buffer stock q_b helps prevent stock-outs.



Example 2: producer of competing products

- **Problem**: maximize the profit of a company producing two new, competing products (say two versions of a computer)
  - There are some fixed costs to prepare the launch of the two products
  - There is a cost per unit of product for each type of the product
  - Revenue per unit of product for each type
  - Decline in the value of both products with each sold unit of either products
- **Variables**
  - x_i: the number of products i that the producer will make, $i = 1, 2$

- **Costs, prices**
  - F = fixed cost for preparing the launch of the two products
  - C_i = manufacturing cost of one unit of product i
  - P_i = initial selling price of one unit of product i
  - a_i = decline in the price of x_i with each sold unit of product 1
  - b_i = decline in the price of x_i with each sold unit of product 2
- **Objective**: how much to manufacture to maximize the profit

Example 2 (continued)

- **Costs, prices**
  - F = fixed cost for preparing the launch of the two products
  - C_i = manufacturing cost of one unit of product i
  - P_i = initial selling price of one unit of product i
  - a_i = decline in the price of x_i with each sold unit of product 1
  - b_i = decline in the price of x_i with each sold unit of product 2
- **Model building**
  - Price $P_1(x_1, x_2)$ for product 1 after already selling x_1 units of product 1, x_2 units of product 2: $P_1(x_1, x_2) = P_1 - a_1x_1 - b_1x_2$
  - Price $P_2(x_1, x_2)$ for product 2 after already selling x_1 units of product 1, x_2 units of product 2: $P_2(x_1, x_2) = P_2 - a_2x_1 - b_2x_2$
  - Total revenue: $R(x_1, x_2) = x_1P_1(x_1, x_2) + x_2P_2(x_1, x_2)$
  - Total cost: $C(x_1, x_2) = F + C_1x_1 + C_2x_2$
- **Objective**: maximize function $f(x_1, x_2) = R(x_1, x_2) - C(x_1, x_2)$

Example 2 (continued)

- **Objective**: maximize function
  $f(x_1, x_2) = R(x_1, x_2) - C(x_1, x_2)$

  $f(x_1, x_2) = x_1(P_1 - a_1x_1 - b_1x_2) + x_2(P_2 - a_2x_1 - b_2x_2) - F - C_1x_1 - C_2x_2$
- **Note**: our model is discrete
  - However, maximize the function as if it were continuous!
- Calculate the "critical" point – equilibrium
  - $\partial f/\partial x_1 = P_1 - 2a_1x_1 - b_1x_2 - a_2x_2 - C_1 = -2a_1x_1 - (b_1 + a_2)x_2 + P_1 - C_1 = 0$
  - $\partial f/\partial x_2 = -b_1x_1 + P_2 - a_2x_1 - 2b_2x_2 - C_2 = -(b_1 + a_2)x_1 - 2b_2x_2 + P_2 - C_2 = 0$
- Check whether the critical point is a maximum point
  - Check convexity/concavity

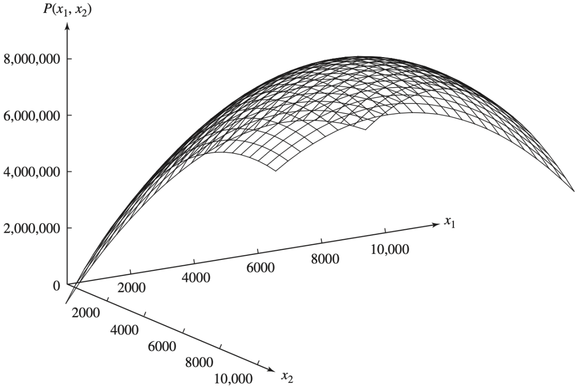


Figure 13.9

The total profit surface
$R(x_1, x_2) = 1440x_1 - 0.1x_1^2 + 1740x_2 - 0.1x_2^2 - 0.07x_1x_2 - 400{,}000$

Constrained optimization

- **Problem**: optimize $f(X)$, subject to some constraints $g_i(X) = b_i$, $\mathrm{i} = 1, 2, \ldots, \mathrm{n}$, where f and $\mathrm{g_i}$ are continuous functions
  - we only focus here on the case where they are differentiable

Constrained optimization

- Unconstrained optimization was in some sense "simple"
  - Only has one goal: optimize the objective function
- Constrained optimization is more difficult
  - It has a potential conflict of requirements: the optimization of the objective function and the feasibility of the optimum
- Many approaches exist; we only focus here on one: **merit functions**

Merit functions for constrained optimization

- Consider here problems of minimization: minimize $f(X)$, subject to the constraint $g(X) = b$,
- **Approach**: replace the constrained problem with an unconstrained problem, whose solution is an *approximation* of the solution of the original problem
- That a point x_0 is an approximation has two meanings here:
  - x_0 is an approximation of a minimum point of f
  - x_0 is an approximation of a solution of equation $g(X) = b$
- Then solve the unconstrained problem as discussed earlier

Merit functions for constrained optimization (continued)

- There is no unique way in which to introduce a merit function
- Examples for the problem minimize $f(X)$, subject to $g(X) = b$
  - **Quadratic penalty function:** $\Phi(x, \mu) = f(x) + \frac{1}{2\mu} |g(x) - b|^2$
  - **Lagrange multiplier merit function:** $\Theta(x, \lambda) = f(x) + \lambda |g(x) - b|$
- Intuition when choosing a merit function
  - Estimate where lies the major effort in solving the original problem: in finding the optimum, or in satisfying the constraint
  - Introduce a smaller (larger, resp.) term based on the constraint
  - No guarantee that a solution to the merit function will be a good approximation of the solution to the original problem; may need to iterate

Example

- Minimize $f(x_1, x_2) = \frac{27}{x_1} + 0.25x_1 + \frac{20}{x_2} + 0.1x_2$, such that $2x_1 + 4x_2 = 24$
- Approach: Lagrange multipliers
  - Introduce the merit function $L(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda(2x_1 + 4x_2 - 24)$
  - $L(x_1, x_2, \lambda) = \left(\frac{27}{x_1} + 0.25x_1 + \frac{20}{x_2} + 0.1x_2\right) + \lambda(2x_1 + 4x_2 - 24)$

Learning objectives

- Understand the concept of constrained/unconstrained optimization
- Be able to solve a simple unconstrained optimization problem through the differential-based method
- Understand the approach for solving constrained optimization based on merit functions
- Be able to define a merit function for a simple constrained optimization problem

- Minimize $L(x_1, x_2, \lambda)$

$$\frac{\partial L}{\partial x_1} = -\frac{27}{x_1^2} + 0.25 + 2\lambda = 0$$

$$\frac{\partial L}{\partial x_2} = -\frac{20}{x_2^2} + 0.1 + 4\lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = 2x_1 + 4x_2 - 24 = 0$$

- **Solution:**
  - $x_1 = 5.0968$; $x_2 = 3.4516$; $\lambda = 0.3497$
  - $f(x_1, x_2) = 12.71$
  - $2x_1 + 4x_2 = 24$
  - Varying x_1, x_2 while obeying the constraint shows that this is a minimum point

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3. Summary of the course

Goal of the course

- Give an introduction to mathematical modeling
  - Formulate the problem
  - Identify the type of problem: data-driven, hypothesis-driven, optimization
  - Identify the main variables to be included
  - Identify the dependencies
  - Formulate the model
  - Solve the problem
- Only concerned with elementary methods and approaches for mathematical modeling
  - Modeling focused on change, proportionality, geometric similarity
  - Data-driven modeling
  - Discrete probabilistic modeling
  - Modeling with difference and with differential equations
  - Discrete and continuous optimization
  - Dimensional analysis

Course content

- Only included elementary methods, only gave an introduction to each of them
- Content kept generic, independent of a specific application field
- Focused only on model formulation and simple mathematical analysis of the models
    - Left out algorithmic issues, complexity aspects, implementation of simulations, computer-based modeling environments, numerical methods, etc.
    - Left out modeling methods stemming from computer science rather than mathematics

Course content

- Focused on learning about the modeling process through lectures and exercises
  - Did not include a student project component into the course
- Some of the omitted topics will be included in the second modeling course
  - Computer science-based approaches
  - Computer-based modeling environments
  - Partly a project-based course

Course content

Modeling paradigms

- Modeling change
- Modeling using proportionality
- Modeling using geometric similarity
- Data-driven modeling
- Hypothesis-driven modeling
- Dimensional analysis
- Modeling of optimization

Modeling methods

- Differential equations
- Discrete-time Markov chains

Data and modeling

- Model fitting
- Model interpolation

Optimization

- Discrete, especially linear programming
- Algebraic and geometric solutions to linear programming
- Simplex
- Continuous optimization, constrained and unconstrained