Lecture 9 Dimensional Analysis

Mathematical Modeling

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1. Focus in this lecture on modeling of some physical processes
2. Dimensional Analysis
3. Using similitude and prototyping - an example

1. Focus in this lecture on modeling of some physical processes
2. Dimensional Analysis
3. Using similitude and prototyping - an example

Background

- Explored so far mostly hypothesis-driven and data-driven modeling
  - In the former case, we know the main interactions and test how well they fit the data
    - Focus on how much the hypothesis can explain the data
    - Mostly a problem of **model fitting**
  - In the later case, we only have data and are willing to accept any (reasonable) model confirming them
    - Interested in describing the trends of the data
    - Often a problem of **interpolation**

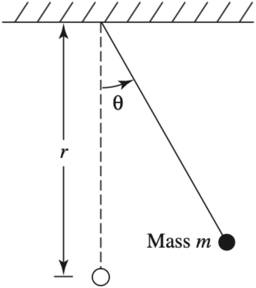
- Discuss here a different approach to modeling, based on **dimensional analysis**
    - In this case we know the main variables important for the model
        - We know their units of measure (or dimension)
    - We do not know their interactions
    - We do not have the data at the moment when we build the model
    - We have in this case a problem of **model identification**

Dimensional Analysis –Basics

- It is a method for determining how the selected variables are related
- Important also in reducing significantly the amount of experimental data needed later on for fitting the model
- Main ideas in dimensional analysis:
  - Physical quantities have dimensions
  - Physical laws are not altered by changing the units measuring dimensions
  - Any equation must be expressed in the same dimension on both sides of the equality (equations must be dimensionally correct)

Example: a simple pendulum

- **Problem:** for a given pendulum, determine its period
  - the time required for the pendulum bob to swing through one complete cycle and return to its initial position
- **Assumptions**
  - Identify the factors influencing the period:
    - the length r
    - the acceleration due to gravity g
    - the mass m
    - the initial angle θ
    - frictional forces
    - the drag on the pendulum
  - For simplicity, assume that the hinge is frictionless, the drag force is negligible, and the mass of the pendulum is concentrated at one end of the pendulum



Example: a simple pendulum

- **Problem formulation:** determine/approximate the function $t = f(r, m, \theta, g)$
- **Data-driven approach**
  - Gravity is constant
  - t only depends on r, m, θ
  - Collect enough data on the value of t depending on various values of r, m, θ
  - **Question:** how many experiments are needed?
    - Assume that N points are to be considered for each variable
    - Then we need 3^N experiments
  - Clear need to reduce the number of independent variables as much as possible
    - In fact, minimize the number of independent variables in function f

About dimensions

- **Some elementary concepts in physics:**
  - mass, length, time, velocity, acceleration, force, energy, work, pressure
  - For each such concept there is a unit of measurement
  - For a physical law, e.g., $F = ma$, to make sense, the units of measurement on both sides of the equality have to be consistent
    - If the mass is measured in kilograms and the acceleration in meters per second squared, then the force must be measured in newtons: the MKS (meter-kilogram-second) system
    - It would be inconsistent with the equation $F = ma$ to measure mass in slugs, acceleration in feet per second squared, and force in newtons
    - It would become consistent if force were expressed in pounds: the American Engineering System of measurement

About dimensions

- The three main physical quantities we consider in this lecture are **mass**, **length**, and **time**
  - Associate with these quantities the (generic) dimensions M, L, and T, respectively
    - We do not specify which unit of measurement to use for each
  - The dimension of other quantities with respect to M, L, T follow from basic laws of physics
  - Numerical (dimensionless) constants may be involved in defining some more complex quantities; they are ignored in finding the dimension of that concept
  - There are also dimensional variables that in our application may be considered a constant, such as the gravity - for them we need to consider their dimensions

About dimensions

- More complex entities, such as momentum or kinetic energy, may be defined as products of simpler entities, which in turn are products of M, L, T)
- Each product will get a dimension associated to it, in terms of a product $M^n L^p T^q$
  - Dimensionless quantities: $n = p = q = 0$

Table 14.1 Dimensions of physical entities in the *MLT* system

Mass	M	Momentum	MLT^{-1}
Length	L	Work	ML^2T^{-2}
Time	T	Density	ML^{-3}
Velocity	LT^{-1}	Viscosity	$ML^{-1}T^{-1}$
Acceleration	LT^{-2}	Pressure	$ML^{-1}T^{-2}$
Specific weight	$ML^{-2}T^{-2}$	Surface tension	MT^{-2}
Force	MLT^{-2}	Power	ML^2T^{-3}
Frequency	T^{-1}	Rotational inertia	ML^2
Angular velocity	T^{-1}	Torque	ML^2T^{-2}
Angular acceleration	T^{-2}	Entropy	ML^2T^{-2}
Angular momentum	ML^2T^{-1}	Heat	ML^2T^{-2}
Energy	ML^2T^{-2}		

Dimensions

- **Dimensionally compatible sums**
  - All terms in a sum must have compatible dimensions
  - Example: $F = mv + v^2$ must be incorrect: mv has dimension MLT^{-1}, whereas v^2 has dimension L^2T^{-2}
- **Dimensional homogeneity**
  - Equations that are true regardless of the system of units for its variables are called dimensionally homogeneous
  - Example: the time a body falls a distance s under gravity g, neglecting air resistance:

$$t = \sqrt{\frac{2s}{g}} \quad \text{is dimensionally homogeneous}$$

$$t = \sqrt{\frac{s}{16.1}} \quad \text{is not dimensionally homogeneous}$$

Applying dimensional analysis

- **Main assumption**: the model is given by a dimensionally homogenous equation in terms of the appropriate variables
- **Task**: determine the form of that equation
- **Approach**: find an appropriate **dimensionless** equation and solve it for the dependent variable of interest
    - Choose which are the relevant variables to consider
    - Determine/describe all possible dimensionless products of those variables
    - Use some of these products to construct dimensionally homogeneous equations

Another example

Wind force on a van: you are driving a van on a highway with gusty winds

- **Problem**: how does the speed of the van affect the wind force you are experiencing?
- **Assumptions**: the force F of the wind is affected by the speed v of the van and the surface area A of the van directly exposed to the wind's direction
- **Hypothesis**: the force is proportional to some power of the speed times some power of the surface area:

$$F = kv^a A^b$$

Variable	F	k	v	A
Dimension	MLT^{-2}	$M^0L^0T^0$	LT^{-1}	L^2

Example(continued)

- In terms of dimensions:

$$MLT^{-2} = (M^0 L^0 T^0)(LT^{-1})^a (L^2)^b$$

- But this cannot be correct: M has a zero exponent on the right-hand side
- **Conclusion**: error in the model formulation
- **Rework the hypothesis**: the strength of the wind is also affected by its density ρ

$$F = kv^a A^b \rho^c$$

Dimensionally: $MLT^{-2} = (M^0 L^0 T^0)(LT^{-1})^a (L^2)^b (ML^{-3})^c$

Example (continued)

- Dimensionally:

$$MLT^{-2} = (M^0 L^0 T^0)(LT^{-1})^a (L^2)^b (ML^{-3})^c$$

- Equivalent with:

$$c = 1, \quad a + 2b - 3c = 1, \quad -a = -2$$

- In other words: $a = 2, b = 1, c = 1$
- **Final model:**

$$F = kv^2 A\rho$$

- **Note**: for a particular situation, the density ρ is a constant
  - One might reformulate the model to say that F is proportional to $v^2 A$
  - The reason this model seemed to fail in the beginning of our reasoning is because we assumed the proportionality constant to be dimensionless

Example: simple pendulum revisited

- **The main variables of interest:**

Variable	m	g	t	r	θ
Dimension	M	LT^{-2}	T	L	$M^0L^0T^0$

- Find all of their dimensionless products: $m^a g^b t^c r^d \theta^e$
  - Its dimension is $(M)^a(LT^{-2})^b(T)^c(L)^d(M^0L^0T^0)^e$
  - In other words: $M^a L^{b+d} T^{-2b+c}$
  - Dimensionless condition: $a = 0$, $b + d = 0$, $-2b + c = 0$
  - In other words: $a = 0$, $c = 2b$, $d = -b$, where b, e are arbitrary
  - Infinitely many dimensionless products
- Consider here only two of them, for $b = 0, e = 1$ and for $b = 1, e = 0$

Example (continued)

- Two dimensionless products:
  - $b = 0$, $e = 1$: $\Pi_1 = m^0 g^0 t^0 r^0 \theta^1 = \theta$
  - $b = 1$, $e = 0$: $\Pi_2 = m^0 g^1 t^2 r^{-1} \theta^0 = \frac{gt^2}{r}$
- Any other solution of the linear system on the previous slide is a linear combination of the solutions for $(b, e) = (0, 1)$ and for $(b, e) = (1, 0)$
- In other words, any other dimensionless product is of the form $\Pi_1^a \Pi_2^b$
- **Recall our main assumption:** the solution is given by a dimensionally homogeneous equation in terms of the appropriate variables
  - By rewriting that equation, we get a dimensionless equation
  - But a dimensionless equation can only involve an expression in terms of Π_1, Π_2
- In other words, a solution to the problem must be of the form $f(\Pi_1, \Pi_2) = 0$, for some (as yet unknown) function f

Example (continued)

- $f(\Pi_1, \Pi_2) = 0$, i.e., $f(\theta, gt^2/r) = 0$
  - If we managed to solve this equation in terms of t (invert function f), then the expression we got would be $gt^2/r = \varphi(\theta)$ for some function φ
- **Final solution:** $t = \sqrt{r/g} \cdot h(\theta)$, for some function $h(\theta) = (\varphi(\theta))^{1/2}$
  - Test the solution: keep g and θ constant, vary r; check the dependency between t and $r^{1/2}$
  - Determine the unknown function h: collect data on t for fixed r and various values of θ, do a data-driven modeling approach

Figure 14.4
Testing the assumptions of the simple pendulum model by plotting the period t versus the square root of the length r for constant displacement θ

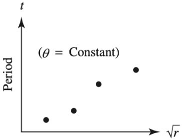
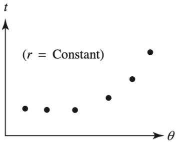


Figure 14.5
Determining the unknown function h



The process of dimensional analysis

Recall:

- **Main assumption:** the solution is given by a dimensionally homogenous equation in terms of the appropriate variables
- **Task:** determine the form of that equation
- **Approach:** find an appropriate dimensionless equation and solve it for the dependent variable of interest

- Focus on determining the form of all dimensionless products
  - Similarly as in our previous example, this leads to a linear system of equations, where each equation stands for the exponent of one of the basic dimensions you consider (in our examples, M, L, and T)
  - Elementary result in linear algebra: the solution of any linear system of equations is a linear space;
  - **Idea:** determine its linear basis: a complete, independent set of solutions
  - Each of these "basic" solutions gives a "basic" dimensionless product
  - Any solution of the system of equations is a linear combination of the basic solutions
  - Any dimensionless product is a product of some powers of the basic dimensionless products

The process of dimensional analysis (2)

- **Main tool:** Buckingham's theorem

Intuition:

- We have a physically meaningful (dimensionally homogenous) equation involving n physical variables, and these variables are expressible in terms of k independent fundamental physical quantities
- Then the original equation is equivalent to an equation involving a set of $p = n - k$ dimensionless parameters constructed from the original variables
- This provides a method for computing sets of dimensionless parameters from the given variables, even if the form of the equation is still unknown
- **Note:** the choice of dimensionless parameters is not unique; up to the modeler to choose the most "physically meaningful" such parameters

The process of dimensional analysis (3)

Buckingham's theorem / the pi-theorem (1914):
An equation

$$f(q_1, \ldots, q_n) = 0$$

is *dimensionally homogeneous*, where q_i are physical variables expressed in terms of k independent physical units if and only if the equation can be restated as

$$F(\pi_1, \ldots, \pi_p) = 0$$

where π_j are *dimensionless parameters* constructed from the q_i by $n - k$ equations of the form

$$\pi_j = q_1^{a_1} q_2^{a_2} \ldots q_n^{a_n}$$

where the exponents a_i are integers

Example

Problem: determine the terminal velocity v of a raindrop falling from a motionless cloud

Variables to consider:

- size of the raindrop (say its radius r),
- density ρ of the air,
- viscosity μ of the air (resistance to motion; sort of internal molecular friction)
- gravity g

Variable	Dimension
v	LT^{-1}
r	L
g	LT^{-2}
ρ	ML^{-3}
μ	$ML^{-1}T^{-1}$

- By Buckingham's theorem, we are interested in all dimensionless products of powers of these variables: $v^a r^b g^c \rho^d \mu^e$
- Dimensionally: $(LT^{-1})^a (L)^b (LT^{-2})^c (ML^{-3})^d (ML^{-1}T^{-1})^e$ is dimensionless
- Equivalently: $d + e = 0$; $a + b + c - 3d = 0$; $-a - 2c - e = 0$
- Solution: $b = 1.5d - 0.5a$; $c = 0.5d - 0.5a$; $e = -d$; a, d arbitrary
- Consider the independent solutions obtained for $(a, d) = (1, 0)$, $(a, d) = (0, 1)$

Example (continued)

- Basic solutions: $\Pi_1 = vr^{-1/2}g^{-1/2}$, $\Pi_2 = r^3 g^{1/2}\rho\mu^{-1}$
- Buckingham's theorem: there is a function f such that

$$f(vr^{-1/2}g^{-1/2}, r^3 g^{1/2}\rho\mu^{-1}) = 0$$

- Assuming we could invert this function, we obtain $vr^{-1/2}g^{-1/2}$ as a function of $r^3 g^{1/2}\rho\mu^{-1}$
  - Equivalently:

$$v = \sqrt{rg} \cdot h\left(\frac{r^{3/2}g^{1/2}\rho}{\mu}\right)$$

  where h is some (unknown) function

Notes

- The $n - k$ dimensionless products stated in Buckingham's theorem are not unique
  - For the purpose of modeling in the way we showed in the example, it is necessary to choose them in such a way that the dependent variable of interest only occurs in one of the products
  - In that way we can continue our reasoning with the inversion of the (unknown) function in Buckingham' s theorem
- We did not discuss here when the function in Buckingham's theorem is invertible
  - Deeper discussion related to the implicit function theorem in advanced calculus

Usefulness of Buckingham's theorem for modeling

- Get information about how the variables are related to each other; in some cases, lead to complete model identification
- Identify incompleteness of a model
- Help in experiment design: if $\pi_1 = h(\pi_2, \pi_3, \ldots, \pi_p)$, then design the experiments so that $\pi_3, \ldots, \pi_p$ are constant to isolate the effect of π_2 on π_1
- Reduce the number of experiments needed to fully identify the model: no need to sample n variables $x_1, \ldots, x_n$, only sample the products $\pi_1, \ldots, \pi_p$ where $p < n$
- Helps in presenting the model in a standard form: choose the dimensionless products to be widely used coefficients in that particular field of study (Reynolds number, Froude number, Mach number, etc)

Similitude

- **Using prototypes**
  - Sometimes it is impossible to study the actual object of interest
    - Example: effects of wave action on a large ship at sea, heat loss of a submarine, wind effects on an aircraft wing
  - **Solution**: construct a scaled-down model (prototype) and simulate the environment
  - **Problem**: how do we scale the experiments to ensure that the effects observed on the model are the same as those that will take place on the final product?
  - **Approach here**: use dimensional analysis

Example: drag force on a submarine

- **Assumptions: the variables affecting the drag D are**
  - Fluid velocity v
  - Length of the submarine: r
  - Fluid density ρ
  - Fluid viscosity μ
  - Velocity of sound in the fluid c

Describing the shape factors related to the submarine

- Describing the shape factors related to the submarine
  - **Assumption:** assume the submarine having an ellipsoidal shape; describe it through two ellipses –two cross-sections, one length-wise, the other width-wise
  - **Shape factors**
    - Length of the submarine (an axis in the length-wise cross-section): r
    - Ratio of the lengths of the two axis in the first cross-section: r_1
    - Ratio of the lengths of the two axis in the second cross-section: r_2
- **In general:** collect enough shape factors to be able to describe in enough details the 3D shape of the object of interest

Example (continued)

Dimensional analysis

- We are highly interested in the drag D –the pressure (force per unit area) times the area
  - For geometrically similar objects, the area is proportional to r^2
  - So D is proportional to pr^2
- We have six fluid mechanics variables: D, v, r, ρ, μ, c
- We have two other dimensionless (shape) factors: r_1, r_2
- The MLT-based dimensional analysis will give an equation in $6 - 3 = 3$ dimensional variables and in the 2 dimensionless variables
  - We can choose three independent variables that are widely used in physics
  - For example: Reynolds number $R = \frac{vr\rho}{\mu}$, Mach number $M = \frac{v}{c}$, pressure coefficient $P = \frac{p}{\rho v^2}$

Example (continued)

- Buckingham's theorem gives an equation $h(P, M, R, r_1, r_2) = 0$
- Assuming we could solve it for P, we get $P = H(M, R, r_1, r_2)$
- Substitute $P = \dfrac{p}{\rho v^2}$ and we get $p = \rho v^2 H(M, R, r_1, r_2)$
- Drag force is proportional to pr^2, i.e. $D = k\rho v^2 r^2 H(M, R, r_1, r_2)$

Example (continued)

- Drag force is proportional to pr^2, i.e. $D = k\rho v^2 r^2 H(M, R, r_1, r_2)$
- A condition of the same form holds both for the model/prototype and for the final object:

$$D_m = k\rho_m v_m^2 r_m^2 H(M_m, R_m, r_{1m}, r_{2m})$$

Conclusion: design conditions for the model

1. $R_m = R$, i.e. $v_m r_m \rho_m / \mu_m = vr\rho/\mu$
2. $M_m = M$, i.e. $v_m / c_m = v/c$
3. $r_{1m} = r_1$
4. $r_{2m} = r_2$

- In these conditions, $\dfrac{D_m}{D} = \dfrac{\rho_m v_m^2 r_m^2}{\rho v^2 r^2}$
  - Compute D_m and then calculate D

Discuss the design conditions

- Conditions c and d imply geometric similarity
- If the velocities are small compared to the speed of sound in a fluid, then v/c can be considered constant, as required by condition b
- If the same fluid is used both for the model and the final object, then condition a is satisfied iff $v_m r_m = vr$, i.e. $v_m/v = r/r_m$
    - Setting the velocity of the model in this way yields $D = D_m$

Learning objectives

- Understand the principles of dimensional analysis: physical quantities have dimensions, models should be dimensionally coherent
- Be able to perform dimensional analysis for simple models
- Be able to apply Buckingham's theorem to simple models
- Understand the application of dimensional analysis in the form of similitude analysis