Lecture 4
Modeling with Proportionality and Geometric Similarity

Mathematical Modeling

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1. Modeling using proportionality and geometric similarity

1 Modeling using proportionality and geometric similarity

Modeling using proportionality

- y is proportional to x, denoted $y \sim x$, if $y = kx$, for some $k > 0$

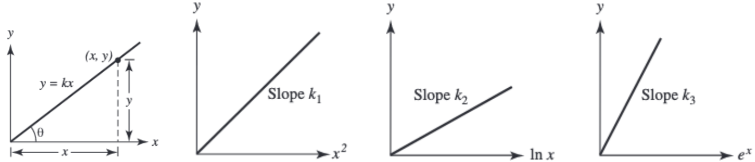


图 1: Giordano et al. A first course in mathematical modeling. (5th edition) Fig. 2.8, 2.9, page 71

- $y \sim x^2$ if $y = k_1 x^2$, for some constant $k_1 > 0$
  - In this case, $x \sim y^{1/2}$
- $y \sim \ln(x)$ if $y = k_2 \ln(x)$, for some constant $k_2 > 0$
- $y \sim e^x$ if $y = k_3 e^x$, for some constant $k_3 > 0$
- Note: If $z \sim y$ and $y \sim x$, then $z \sim x$

Modeling using geometric similarity

- Two objects are said to be geometrically similar if there is a one-to-one correspondence between the points of the objects such that the ratio of distances between corresponding points is the same for all pairs of points.

- **Example: two boxes**
  - $l/l' = w'/w = h/h' = k$, with $k > 0$
  - Ratio of their volumes: $\frac{V'}{V} = \left(\frac{lwh}{l'w'h'}\right) = k^3$
  - Ratio of their surface areas:

$$\frac{S'}{S} = \frac{2lh + 2wh + 2wl}{2l'h' + 2w'h' + 2w'l'} = k^2$$

  - In other words:

$$-\frac{S}{S'} = \frac{l^2}{|l'^2|}, \quad \frac{V}{V'} = \frac{l^3}{l'^3}$$

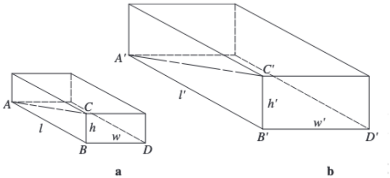
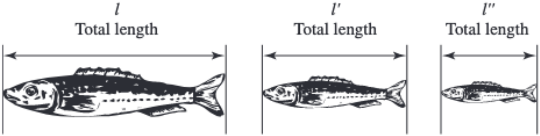
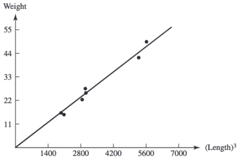


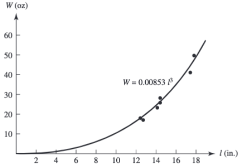
图 2: Giordano et al. A first course in mathematical modeling. (5th edition) Fig. 2.18,, page 81

Example: Bass fishing derby

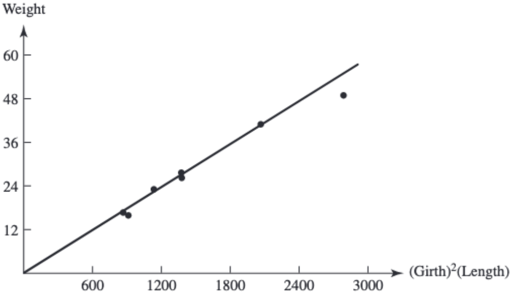
- **Problem:** predict the weight of a fish in terms of some easily measurable dimensions
- **Solution:** assume geometric similarity
  - $V \sim l^3$; $W \sim V$
  - Conclusion: $W \sim l^3$
  - Calculate the proportionality constant based on just one fish







- **The rule above does not reward catching a fat fish.**
- **Changes in the model:**
  - Assume that the majority of the weight comes from the main body, without head and tail
  - Length of the main body (effective length): l_{eff}
  - The main body has a varying cross-sectional area. Consider the average cross-sectional area: A_{avg}
  - **Model:** $W \sim V \sim A_{\text{avg}} l_{\text{eff}}$



- How do we measure/estimate the effective length and the average cross-sectional area?
  - Answer: Model it through geometric similarity!
  - Assume $l_{\text{eff}} \sim l$
  - Measure the circumference of the fish at its widest point: the girth g
  - Assume $A_{\text{avg}} \sim g^2$
  - Final model: $W \sim l^2 g^2$
  - Test the model: $W = 0.0187 l^2 g^2$

Learning objectives for this lecture

- Understand the concept of modeling the change in a discrete dynamical systems.
- Able to write a linear and an affine model with difference equations for a simple real-life phenomenon.
- Understand the diverse behavior that a linear dynamical system model can have.
- Understand the notion of equilibrium point.
- Understand the different types of stability of equilibrium points.
- Ability to write multi-variable models as systems of difference equations
- Understand the basic paradigm of modeling proportionality.
- Understand the basic paradigm of modeling geometric similarity.