Lecture 1 Introduction

Mathematical Modeling

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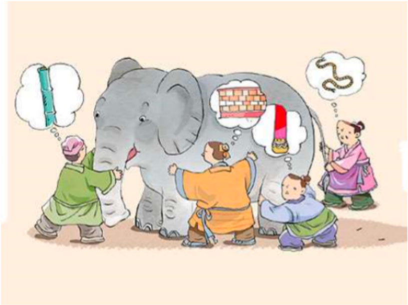
John Godfrey Saxe' s (1816-1887) version of the legend:

- First man (feeling the side): like a wall
- Second (the tusk): like a spear
- Third (the trunk): like a snake
- Fourth (the knee): like a tree
- Fifth (the ear): like a fan
- Sixth (the tail): like a rope



Source: http://history.cultural-china.com/en/38History960.html
Check also: http://www.simoncammilleri.com/the_truth_of_the_elephant/

- **What is a model?**
  - A (partial) view of the reality
  - An abstraction of the reality
  - A representation of the (supposedly) main features of the reality, including the connections among them
  - For a given object of study, many models may be given, possibly focusing on different features of the object
- **What a model is not**
  - A model is not the reality
  - A model is not certain!
- **Many types of models exist!**

"All models are wrong, some are useful"

Box, G.E.P., *Robustness in the strategy of scientific model building, in Robustness in Statistics, R.L. Launer and G.N. Wilkinson, Editors. 1979, Academic Press: New York.*

Huge momentum for modelling right now

- **The Human Brain Project (2013-2023)** is a Future and Emerging Technologies (FET) Flagship project of the European Commission
  - Aim: build and simulate complete model of the human brain to better understand its functions
  - Total budget: 1.2 billion euros
- **The BRAIN Activity Map Project (2013-2023)** is a US initiative
  - Aim: map the activity of every neuron in the human brain
  - Seen as the next high-impact project after the human genome project
  - Total budget: at least 3 billion dollars
- Climate change models
- Economic forecast models
- Weather models

"The sciences do not try to explain, they hardly even try to interpret, they mainly make models.

By a model is meant a mathematical construct which, with the addition of certain verbal interpretations describes observed phenomena.

The justification of such a mathematical construct is solely and precisely that it is expected to work."

John von Neumann (1903-1957)

"Every attempt to employ mathematical methods in the study of chemical questions must be considered profoundly irrational and contrary to the spirit of chemistry.

If mathematical analysis should ever hold a prominent place in chemistry - an aberration which is happily almost impossible - it would occasion a rapid and widespread degeneration of that science."

Auguste Comte (full name: Isidore Marie Auguste Francois Xavier Comte; January 17, 1798 - September 5, 1857): Philosophie Positive, 1830

Models?? Yes, please

"The more progress the physical sciences make, the more they tend to enter the domain of mathematics, which is a kind of center to which they all converge.

We may even judge the degree of perfection to which a science has arrived by the facility to which it may be submitted to calculation."

Adolphe Quetelet (1828)

Models: some examples

Models of a building

- The foundation plan
- The water pipes plan
- The electricity plan
- The ventilation plan
- The room division plan
- Division of people into rooms

Maps

- Geographic map
- Political map
- Road map
- ...

- Models of morality
- Family models
- Role models
- Models of society
- Political models
- Election models
- Political representation models
- Weather models
- Traffic models
- Infection models
- Company development models
- ...

A different type of model

Model of a computer

- **Why do we need one?**
  - Can we not just use a real computer?
  - The model(s) came first!
  - What can you do with a model that you cannot do with a real computer?
  - Reason about computing!
- **Is there only one model? What is the difference?**
  - Turing machine
    - The factory worker model
  - Register machines (e.g., counter machines)
    - The arithmetic model
  - Biocomputing
    - Based on/inspired by molecular biology
  - Quantum computing
    - Based on/inspired by quantum physics
  - ...
- **Classes of results**
  - Computability
  - Complexity (efficiency)
  - Note: independence from the actual implementation of the computation (i.e., independent of the computer)

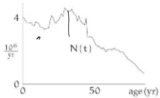
Another type of model: Fermi approximations

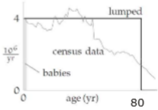
Enrico Fermi: rough quantitative estimates from very little data
Examples:

- **How many piano tuners are in Chicago?**
  - About 5 million people in Chicago
  - In average about 2 persons in each household
  - About one household in 20 has a piano that is tuned regularly
  - Pianos are tuned regularly (about once per year)
  - A piano tuner needs two hours to tune a piano (incl. driving)
  - A piano tuner works 8 hours per day, 5 days a week, 50 weeks per year
- **Result:**
  - $((5,000,000/2)/20) * 1 = 125,000$ piano tunings per year
  - $(50 * 5 * 8)/2 = 1000$ piano tunings per year for one tuner
  - $125,000/1000 = 125$ piano tuners in Chicago

From S. Mahajan: Street-fighting mathematics, MIT Press, 2010
Problem: how many babies (0-2 year olds) are in the US?

- **Exact solution:** look at the plot with the birth dates of every person in the US; numerical integration
- **Drawback:** huge effort; collected every 10 years by the US Census Bureau

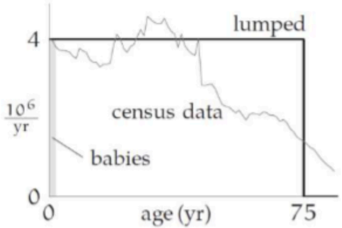




Approximation

- US population: 300 million in 2008
- Assume a life expectancy of 80 (a model where everybody still alive at 80 dies abruptly on their 80th birthday)
- Lump the curve into a rectangle: width of 80, height to be calculated

- Height of the rectangle:
  - Total population of US: 300 million (2008)
  - Easier to divide to 75 than to 80
    - The inaccuracy is not larger than the error made by lumping; might even cancel the lumping error
  - Height: 300,000,000 / 75 = 4,000,000
- Result: calculate the area of a rectangle with height 4,000,000 and width 2
  - Result: 8,000,000 babies 0-2 year of age
  - Compare with the Census Bureau's figure: 7,980,000!!



Models: other examples

- **Computer network architecture**
  - Design chosen so that the server services are available at all times (with large probability)
    - Depending on the probability of failure of a computer, or of the network links, a certain design is chosen
    - Change the reliability model and the architecture will stop being robust
- **Cryptography**
  - A cryptographic system should be provably secure
    - Prove mathematically that an attacker would not be able to break the system in “reasonable” time
    - What computing power should one assume for the attacker?
    - What knowledge of the system should one assume for the attacker?
    - What level of access to the cryptographic system?

- **Casino gambling**
  - The games/machines are designed in such a way that in average the casino has (slightly) better chances of winning
    - Some gamblers will win, others will lose
  - Change the rules and the chances of winning may change
    - Casinos are very carefully monitoring players with a gambling system

We focus in this course on mathematical and computational models

- As we saw, many other types of models exist
- "Model" is indeed a very overloaded word
- In this way, we define a model as a **mathematical** representation of the reality
- Models that mimic the reality by using the language of mathematics

Goal of the course

- An introduction to the process of mathematical modeling
- Give a number of techniques used for:
  - Building a model
  - Analyzing a model
  - Using a model
  - Simulating a model
- Not a course in mathematics, rather in the use of some mathematical techniques for the purpose of modeling
  - How to use various tools
  - Little on the math properties of the tools

- **What is a mathematical model**
  - Possible definition: An abstract, simplified, mathematical construct related to a part of reality and created for a particular purpose.
- **Why mathematical modeling?**
  - It allows for a precise formulation of the chosen aspects of the reality
    - Must formulate ideas precisely; less implicit assumptions
  - It allows for a precise formulation of the current knowledge of the reality
    - Models as knowledge representations
  - It allows for precise reasoning about the reality
    - Based on the concise language, models can be analyzed in a formal way
  - It allows for some types of analysis that would be impossible to perform on the reality
    - Large body of math tools available

Mathematical models

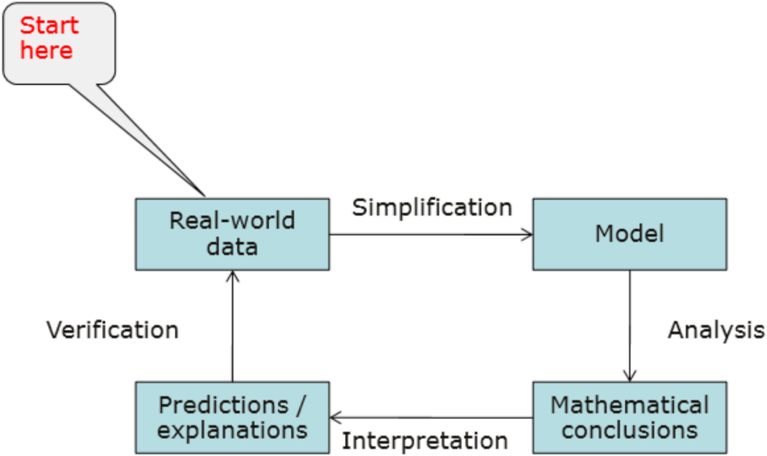
- **Starting point for modeling: divide the world into 3 parts**
  - Things whose effects are neglected
    - Ignore them in the model
  - Things that affect the model but whose behavior the model is not designed to study
    - External variables, considered as parameters, input, or independent variables
  - Things the model is designed to study the behavior of
    - Internal (or dependent) variables of the model
- **Deciding what to model and what not is difficult**
  - Wrong things neglected: the model is no good
  - Too much included: hopelessly complex model
  - Choose the internal variables wrongly: the model will not capture its target
  - How general should the model be: model a table (any table?) or the specific table in front of the modeler

Building a model

- **Formulate the problem**
  - What do you want to know/understand?
- **Outline the model**
  - Divide the world into the three categories in the previous slide
  - Specify the interrelations among the variables
  - A challenging stage: not always clear what is important and what is not useful is the resulting model
- **How useful is the resulting model?**
  - Can you obtain the needed data for the model?
  - Can the available data be used in the model?
  - If not, reformulate the model and perhaps even the problem
  - **Note:** sometimes a model needs no data; all included in the assumptions of the model

- **Test the model**
  - Use the model to make predictions that can be checked against known data or common-sense
  - **Note:** sometimes difficult/expensive/impossible to test a model
- **Use the model**
  - Careful that the situations where it is used were captured in the modeling process
  - Not very different than those where the model was tested
  - Every application of the model is a test for the model

Modeling



Start here
Real-world data
Simplification
Model
Analysis
Mathematical conclusions
Interpretation
Predictions / explanations
Verification

An example

- **Goal: build a model for the long term growth of a population**
  - Choose as independent (external) variables:
    - Birth rate per individual: b
    - Death rate per individual: d
    - Population size at $t = 0$: $P(0)$
  - Write the model: the change in the population per unit of time is the gain through birth minus the loss through death:

$$\frac{dP}{dt} = bP(t) - dP(t)$$

  - Solve the model:

$$P(t) = P(0)e^{(b-d)t}$$

- Test the model:
  - If $b < d$: the population eventually dies out: $\lim_{t\to\infty} P(t) = 0$
  - If $b > d$: the population explodes: $\lim_{t\to\infty} P(t) = \infty$
  - If $b = d$: the population remains constant
  - **Note:** good to test the model in limit cases
- Conclusion:
  - The model is fragile: unreasonable conclusion unless it predicts no change in the population
  - May be good for short-term predictions

An example (continued)

- **Error in the model formulation:**
  - The growth rate in a population is not a constant
  - It should depend on the size of population: e.g., exhaustion of food supply
- **Reformulate the model:**
  - The growth rate $r = b - d$ is now a (unknown) function of P: $r(P)$
  - $\frac{dP}{dt} = r(P)P$

- **Model is less useful (incomplete) because the function $r(P)$ may be hard (impossible) to deduce:**
  - Obtain rough estimates of $r(P)$
  - For example: assume that $r(P)$ gets to 0 as $P(t)$ gets close to a maximum value M
  - For example take $r(P) = s(M - P(t))$, for some constants s, M
- **Model analysis:**
  - The model has M as a steady state
- **Conclusion:**
  - Reaching a steady state was built in the choice of $r(P)$
  - Weak point: the model shows no fluctuations around the steady state

- **Reformulate the model:**
  - Introduce a time delay:
    - The death rate d is the same regardless of the age
    - The birth rate changes in time from 0 to a constant that is reached at age p
  - Reformulate the model for the interval $[0, p]$:

$$-\frac{dP}{dt} = -dP(t) + bP(t-p)$$

- **Other objections to the model:**
  - Refine the death and the birth rate depending on the age and the sex division of the population:
    - Divide the population into sex and age groups
    - Model the interrelations among these groups
  - Model the seasonality of the birth rate:
    - Short term models: consider the details
    - Long terms models: eliminate the problem by averaging over an entire year

- **Model validation**
  - Any model must always be subjected to experimental validation against the reality
  - A model may be invalidated by experimental data
  - No set of experimental data can confirm the “truthfulness” of a model

- **Great traditions of mathematical modeling in science and engineering**
  - Physics
  - Chemistry
  - Chemical engineering
  - Computer science
  - Biology

- **Mathematical models**
  - Continuous vs. discrete mathematics
  - Deterministic vs. stochastic mathematics
- **Only somewhat elementary techniques to be discussed in this course**
  - list on the next slide
  - only elementary math concepts, techniques
  - wide applicability
- **A second course on modeling to be offered in period 2**
  - based on concepts, techniques from computer science
  - more advanced
  - more focus on software tools, computational simulations
  - fresh material, developed in the last 10-15 years
  - more specialized, narrower applicability

Course content

- Modeling change
- Modeling proportionality, geometric similarity
- Model fitting
- Data-driven modeling
- Simulation-based modeling
- Discrete probabilistic modeling
- Discrete optimization modeling
- Dimensional analysis
- Modeling with ODEs
- Continuous optimization modeling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