Lecture 3 Modeling Changes with Systems of Difference Equations

Mathematical Modeling

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1 Modeling change with systems of difference equations

1. Modeling change with systems of difference equations

Modeling with systems of difference equations

- **Example: car rental company; one office in Tampa, the other in Orlando**
  - Travelers may rent a car from either office and drop it at either one
    - Data shows that 40% of those who rent in Orlando, return it in Tampa, 30% of those who rent in Tampa, return it in Orlando
    - The question is whether the car distribution will eventually become unbalanced and cars will have to be moved (empty) from one place to the other –extra cost
- **Problem: build a model for how the number of cars in the two offices varies**
  - O_n = number of cars in Orlando after n time units
  - T_n = number of cars in Tampa after n time units
  - $O_{n+1} = 0.60O_n + 0.3T_n$
  - $T_{n+1} = 0.40O_n + 0.7T_n$

- **Equilibrium values (O, T):**
  - $O = 0.60O + 0.3T$
  - $T = 0.40O + 0.7T$
  - Solution: $O = 0.75T$
  - Note that $O_n + T_n = O_{n+1} + T_{n+1}$, for all n
  - So, $O + T = O_0 + T_0$
  - Final solution: $O/T = 3/7(O_0 + T_0), 4/7(O_0 + T_0)$

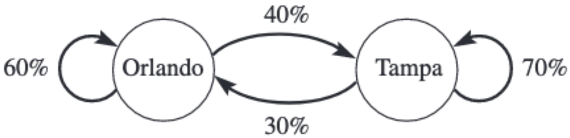


图 1: Giordano et al. A first course in mathematical modeling. (5th edition) Fig. 1.22, page 38

```
N = 7000;
O = zeros(N,1);
T = zeros(N,1);
t = (1:N)';

% initial value
O(1) = 7000;
T(1) = N-N(1);

% parameters
kOO= 0.6;
kOT = 0.3;
kTO = 0.4;
kTT = 0.7;

for n = 1:(N-1)
    O(n+1) = kOO*O(n) + kOT*T(n);
    T(n+1) = kTO*O(n) + kTT*T(n);
end

figure();
plot(t,T,'rs',t,O,'bs');
hold on;
legend('Orlando','Tampa');
axis([0,15,0,8000]);
grid on;
```

图 2: Code

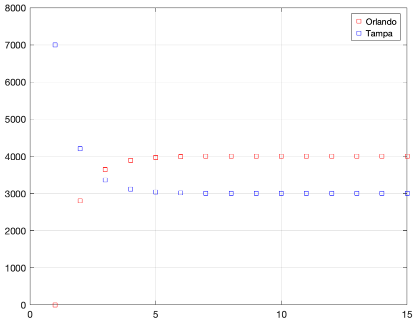
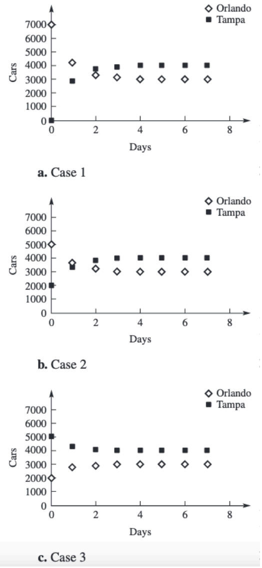


图 3: Figure

Example

- $O_{n+1} = 0.60 O_n + 0.3 T_n$
- $T_{n+1} = 0.40 O_n + 0.7 T_n$
- Total number of cars: 7000
- $(3000, 4000)$ equilibrium
- Start with various initial distributions of the cars between Orlando and Tampa Test the numerical evolution
- Conclusion: after about 7 time units, we approach the equilibrium value $(3000, 4000)$
  - **It suggests** an asymptotically stable equilibrium.



Example: battle of Trafalgar

- Trafalgar, 1805: French and Spanish naval forces under Napoleon against British naval forces under Admiral Nelson
  - French and Spanish: 33 ships
  - British: 27 ships
- Model for battle outcome: during an encounter, each side has a loss equal to 5% of the number of ships of the enemy
- Scenario I: Straight-on series of confrontations
  - $B_{n+1} = B_n - 0.05F_n$
  - $F_{n+1} = F_n - 0.05B_n$

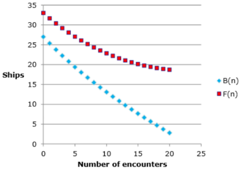


图 5: Scenario I: straight-on battle

- **Recall the model:**
  - French and Spanish: 33 ships
  - British: 27 ships
  - During an encounter, each side has a loss equal to 5% of the number of ships of the enemy
- Scenario II: divide-and-conquer
  - Napoleon's ships were arranged in three groups
  - Strategy:
    - Engage force A with 13 British ships
    - Engage force B with all available ships
    - Engage force C with all available ships

图 6: Giordano et al. A first course in mathematical modeling. (5th edition) Fig. 1.25, page 41

- **Battle A: engage French force A (3 ships) with 13 British ships**
- Outcome of battle A:
  - French force A reduced from 3 to 0.4385
  - French force B, C left intact: 17, 13, resp.
    - did not engage
  - British force:
    - From 13 ships engaged in the battle, 12.5935 remaining
    - 14 did not engage

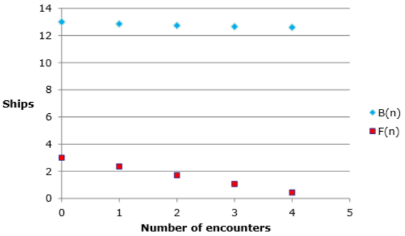


图 7: Scenario II: divide and conquer battle A

- **Battle B: engage French force B (17 ships) with all available British ships (26.5935)**
- Outcome of battle B:
    - French force A: 0.4385
        - did not engage
    - French force B reduced from 17 to 0.1260
    - French force C intact: 13
        - did not engage
    - British force:
        - From 26.5935, reduced to 20.0704

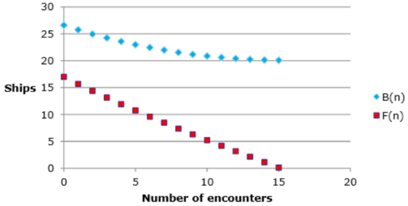


图 8: Scenario II: divide and conquer battle B

- **Battle C: engage French force B (13 ships) with all available British ships (20.0704)**
- Outcome of battle C:
  - French force A: 0.4385
    - did not engage
  - French force B: 0.126
    - did not engage
  - French force C: from 13, reduced to 0.3111
  - British force:
    - From 20.0704, reduced to 15.0101

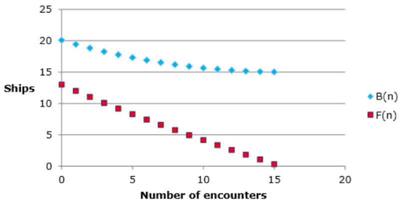


图 9: Scenario II: divide and conquer battle C

Modeling exercise

Another strategy for defeating a superior force is through better technology

- Formulate a model where the British forces have better weaponry and as a result, each encounter results in:
  - A loss for the French-Spanish fleet of 10% of the number of British ships
  - A loss for the British ships of 5% of the number of French-Spanish ships
  - Assume that French-Spanish start with 33 ships, British start with 27 ships

Modeling exercise

Another strategy for defeating a superior force is through better technology

- Formulate a model where the British forces have better weaponry and as a result, each encounter results in:
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  - Assume that French-Spanish start with 33 ships, British start with 27 ships

Result

- $B_{n+1} = B_n - 0.05F_n$
- $F_{n+1} = F_n - 0.15B_n$
- $B_0 = 27, F_0 = 33$

Example: competitive hunter model

- Two species, say owls and hawks co-exist in a habitat: O, H
- In the absence of the other species, each species exhibits unconstrained growth, proportional to its current level:

$$O_{n+1} = k_1 O_n, \quad H_{n+1} = k_2 H_n, \quad \text{where } k_1, k_2 > 0$$

- The effect of each species is to diminish the growth of the other
  - Model it here as proportional to the number of possible encounters between the two species:

$$O_{n+1} = k_1 O_n - k_3 O_n H_n, \quad H_{n+1} = k_2 H_n - k_4 O_n H_n, \quad k_1, k_2, k_3, k_4 > 0$$

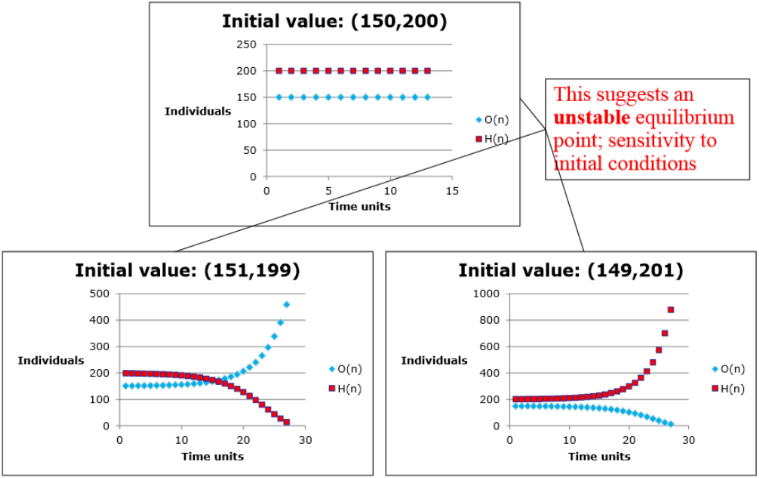
- **Model:**

$$O_{n+1} = (1 + k_1) O_n - k_3 O_n H_n, \quad H_{n+1} = (1 + k_2) H_n - k_4 O_n H_n$$

- **Model:**

$$O_{n+1} = (1+k_1)O_n - k_3 O_n H_n, \quad H_{n+1} = (1+k_2)H_n - k_4 O_n H_n$$

- Question: what are the equilibrium points (O, H)?
  - $O = (1+k_1)O - k_3 OH$, $H = (1+k_2)H - k_4 OH$
  - i.e. $O(k_1 - k_3 H) = 0$; $H(k_2 - k_4 O) = 0$
  - i.e., $(O, H) = (0, 0)$ or $(O, H) = (k_2/k_4, k_1/k_3)$
- **Numerical simulation for** $k_1 = 0.2$**,** $k_2 = 0.3$**,** $k_3 = 0.001$**,** $k_4 = 0.002$
  - Equilibrium points: $(0, 0)$ and $(150, 200)$


Initial value: (150,200)
Individuals
Time units
O(n)
H(n)
This suggests an **unstable** equilibrium point; sensitivity to initial conditions
Initial value: (151,199)
Individuals
Time units
O(n)
H(n)
Initial value: (149,201)
Individuals
Time units
O(n)
H(n)

```
N = 30;
O = zeros(N,1);
H = zeros(N,1);
T = (1:N)';

% initial value
O(1) = 151;
H(1) = 200;

% parameters
k1 = 0.2;
k2 = 0.3;
k3 = 0.001;
k4 = 0.002;

for n = 1:(N-1)
    O(n+1) = (1+k1)*O(n) - k3*O(n)*H(n);
    H(n+1) = (1+k2)*H(n) - k4*O(n)*H(n);
end

figure();
plot(T,O,'*');
hold on;
plot(T,H,'s');
legend('Owl','Hawk');
axis([0,N,0,1000]);
```

图 10: Code

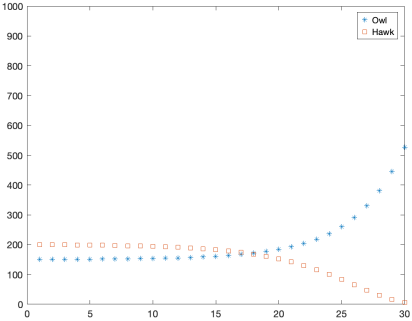


图 11: Figure

Modeling exercise

- Spotted owl's primary food source is mice
- Here is an ecological model for the populations of owls and mice:
  - $M_{n+1} = 1.2M_n - 0.001O_nM_n$
  - $O_{n+1} = 0.7O_n + 0.002O_nM_n$
- Explain the significance of every term and constant in the model

```
N = 30;
O = zeros(N,1);
M = zeros(N,1);
T = (1:N)';

% initial value
O(1) = 150;
M(1) = 200;

% parameters
k1 = 0.2;
k2 = -0.3;
k3 = 0.001;
k4 = 0.002;

for n = 1:(N-1)
    M(n+1) = (1+k1)*M(n) - k3*O(n)*M(n);
    O(n+1) = (1+k2)*O(n) + k4*O(n)*M(n);
end

figure();
plot(T,O,'*');
hold on;
plot(T,M,'s');
legend('Owl','Mice');
axis([0,N,0,1000]);
```

图 12: Code

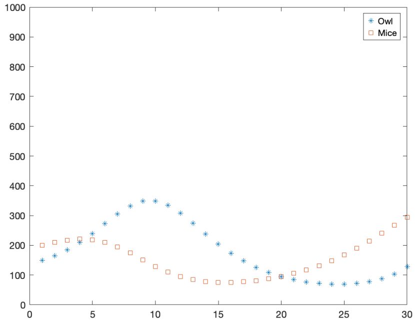
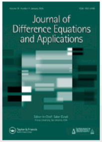


图 13: Figure



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On construction of almost periodic sequences and applications to some discrete population models

Ali Hamidoğlu & Mustafa H. Taghiyev

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Abstract

In this work, we develop a novel approximation strategy for building almost periodic sequences in the theory of almost periodic functions. Here, we create a different perspective for the argument of Dirichlet in the theory of numbers and design an integer approximation strategy in this regard. The idea behind the strategy comes from Kronecker's theorem and it is proven that for given an almost periodic function, it is possible to design its corresponding almost periodic sequence. Moreover, we provide two population models in both continuous and discrete cases where almost periodic sequence solutions are designed under suitable

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Lizhi Fei, Xingwu Chen & Bensan Han

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Abstract

In this paper, a discrete-time predator–prey model with six parameters is investigated. After splitting the parameter space with respect to the number of fixed points, we obtain both transcritical bifurcation surfaces and a Neimark–Sacker bifurcation surface in the six-dimensional parameter space by the normal form method. Then we apply a hybrid control strategy to control the Neimark–Sacker bifurcation so that the positive fixed point of the controlled system is locally asymptotically stable. Numerical simulations are 访达 rmed to illustrate our theoretical results.