Lecture 8 Discrete optimization modeling II

Mathematical Modeling

Prof. Dr. Jingzhi Li

Department of Mathematics,
Southern University of Science and Technology

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1. Linear Programming: Simplex Method

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Recall: what kind of problem are we solving

- Focus here on the following type of discrete optimization problems:
  - Optimize (i.e. minimize or maximize) linear function $f(x_1, ..., x_m)$ such that the following linear constraints are satisfied:
    - $g_1(x_1, ..., x_m) \geq b_1, \; ..., \; g_{p_1}(x_1, ..., x_m) \geq b_{p_1}$
    - $g_{p_1+1}(x_1, ..., x_m) = b_{p_1+1}, \; ..., \; g_{p_1+p_2}(x_1, ..., x_m) = b_{p_1+p_2}$
    - $g_{p_1+p_2+1}(x_1, ..., x_m) \leq b_{p_1+p_2+1}, \; ...,$ $g_{p_1+p_2+p_3}(x_1, ..., x_m) \leq b_{p_1+p_2+p_3}$
  - $(x_1, ..., x_m)$ are the variables of the model
  - f is the objective function (to be optimized)
    - multi-objective optimization often reduced to single-objective optimization
    - multiple objectives may not sometimes be optimized simultaneously; often replaced with a single combined objective

Recall from the last lecture

- Without loss of generality, we can assume that the problem to solve is of the form

$$\text{Minimize/maximize } c_1 x_1 + \cdots + c_n x_n$$

$$\text{subject to constraints } a_{i1} x_1 + \cdots + a_{in} x_n = b_i, \quad 1 \leq i \leq p$$

$$\text{where } x_1, \ldots, x_n \geq 0, \quad b_1, \ldots, b_p \geq 0$$

- Already discussed turning inequalities to equations through new non-negative variables (called *slack variables*)
- For variables with a lower bound other than 0, say $x_1 \geq 3$, introduce a new variable $y_1 \geq 0$ and replace all occurrences of x_1 with $y_1 + 3$
- Replace each unrestricted variable, say z, with $z^+ - z^-$, where $z^+, z^- \geq 0$

Standard form

- The **standard form** of a linear program:
  - Maximize $c \cdot x$
  - Subject to $Ax = b$, $x_i \geq 0$

  where
  - $x = (x_1, ..., x_n)$ is the vector of variables
  - $c = (c_1, ..., c_n)$ are the coefficients of the objective function
  - A is a $p \times n$ matrix, and
  - $b = (b_1, ..., b_p)$ are constants with $b_i \geq 0$
- **Note:** The case of minimization problems is similar, requires some minor modifications in the algorithm; skip it here.

Recall from the last lecture: the geometric approach

- Consider again the carpenter example
  - The objective function to maximize was $f(x_1, x_2) = 25x_1 + 30x_2$
  - Note the plots below:
    - The level curves are parallel to each other
    - The maximum level curve touching the feasible region will do so at an extreme point
    - It may also touch it at two extreme points simultaneously (along one of its borders)

Figure 7.7
The level curves of the objective function f are parallel line segments in the first quadrant; the objective function either increases or decreases as we move in a direction perpendicular to the level curves.

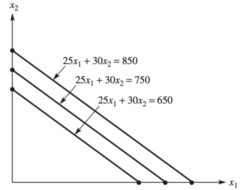
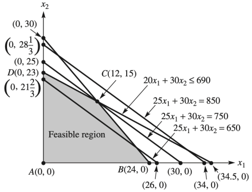


Figure 7.8
The level curve $25x_1 + 30x_2 = 750$ is tangent to the feasible region at extreme point C.

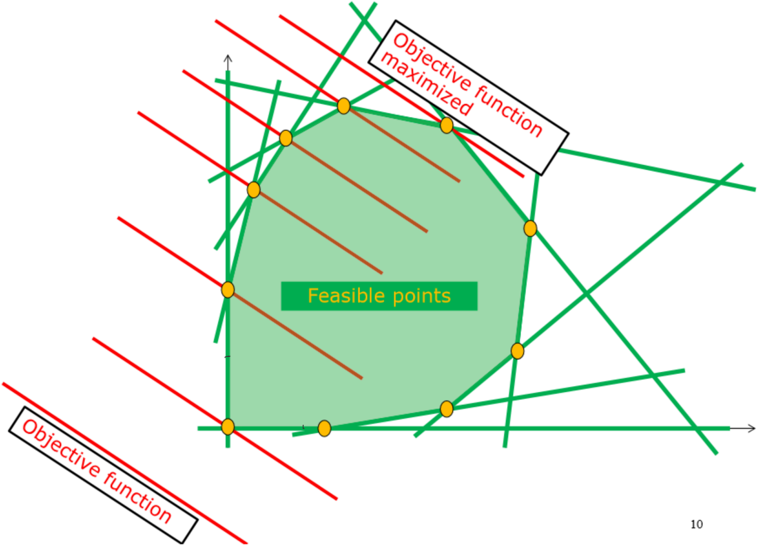


Recall from the last lecture: the algebraic approach

- Enumerating all intersection points for $m > 2$ variables
  - Without loss of generality assume that the constraints are linear inequalities: $a_1 x_1 + \cdots + a_m x_m \leq b$, where $x_i \geq 0$
  - Add a new variable y_1, and transform the inequality into a linear equation $a_1 x_1 + \cdots + a_m x_m + y_1 = b$, where $x_i \geq 0$, $y_1 \geq 0$
    - $y_1 = 0$ is equivalent to a point on the border of the constraint
  - Repeat for all other constraints (new variable for each)
    - We will have now $m + n$ variables and n linear equations
    - Solve them over non-negative integers
  - For all possible sets S of m variables, set the variables in S to 0 and solve the resulting set of n equations with n unknowns to yield all intersection points

The algebraic approach yields intersection points that may be unfeasible. Improvements:

- **Idea 1:** drop out the computation of *unfeasible points*
  - Intersection points simultaneously with the constraints
- **Idea 2:** drop out the computation of intersection points that cannot improve the current optimum
- **Idea 3:** if an extreme point is not optimal, then there is a better extreme point adjacent to it (based on the geometric argument of the sliding parallel lines given by the linear objective function)
  - traversal across edges
- **Idea 4 (greedy approach):** improve the optimum as much as possible in every step of the algorithm
  - Unlike in the general Greedy algorithms, we will not miss the optimum (based on the geometrical argument with the optimization function as a set of parallel lines)


Objective function maximized
Feasible points
Objective function
10

The simplex method

- Solution: the simplex method
  - Proposed by George Dantzig (1947)
  - Earlier developed independently (and secretly) by Leonid Kantorovich (1939)
  - John von Neumann another independent contributor (1947)
  - Historical note: Dantzig’ s solutions to two open problems in statistics while a graduate student at Berkeley (1939)

Simplex algorithm: input

- The linear program is in standard form:
  - Maximize $c \cdot x$
  - Subject to $Ax = b$, $x_i \geq 0$
- Rework the objective function into a constraint-like condition
  - to maximize $c \cdot x$ is to maximize a new variable $z \geq 0$ such that $z = c \cdot x$, or in other words $-c \cdot x + z = 0$, for all x

An example

- $20x_1 + 30x_2 + y_1 = 690$
- $5x_1 + 4x_2 + y_2 = 120$
- $-25x_1 - 30x_2 + z = 0$
- This corresponds to point $(x_1, x_2, y_1, y_2, z) = (0, 0, 690, 120, 0)$
- Will move to an intersection point that increases x_2 because it is what will increase the value of z the most (-30 vs. -25)

- To do this, we will scale down to 1 the coefficient of x_2 in either the first or the second equation –will choose the equation with the smaller coefficient on the right hand side; this corresponds to the geometric intuition that we move along one edge to the next intersection point, not to an intersection point further on that edge
  - $\frac{2}{3}x_1 + x_2 + \frac{1}{30}y_1 = 23$
  - $\frac{5}{4}x_1 + x_2 + \frac{1}{4}y_2 = 30$
  - $-25x_1 - 30x_2 + z = 0$
- So, we scale down the coefficient of x_2 in the first equation, keep the first and the third unchanged
  - -$2/3x_1 + x_2 + 1/30y_1 = 23$
  - $5x_1 + 4x_2 + y_2 = 120$
  - $-25x_1 - 30x_2 + z = 0$

An example (2)

- Eliminate x_2 from the equations 2 and 3: multiply the first equation with the suitable coefficient and add it to eq. 2; the same for eq. 3
    - $2/3x_1 + x_2 + 1/30y_1 = 23$
    - $7/3x_1 - 2/15y_1 + y_2 = 28$
    - $-5x_1 + y_1 + z = 690$
- Same as before: will move to an intersection point that increases the value of x_1 because that will increase the value of z (its coefficient in the last equation is -5)
    - Scale down the coefficient of x_1 to 1 in equations 1 and 2 and choose the one with the smaller right hand side
        - $x_1 + 3/2x_2 + 1/20y_1 = 69/2$
        - $x_1 - 2/35x_2 + 3/7y_2 = 12$
    - Select the second equation because $12 < 69/2$

An example (3)

- $2/3x_1 + x_2 + 1/30y_1 = 23$
- $x_1 - 2/35x_2 + 3/7y_2 = 12$
- $-5x_1 + y_1 + z = 690$
- Eliminate x_1 from the first and the third equations
  - $x_1 + 1/14y_1 - 2/7y_2 = 15$
  - $x_1 - 2/35x_2 + 3/7y_2 = 12$
  - $5/7y_1 + 15/7y_2 + z = 750$
- There is no more variable with a negative coefficient in the last equation –this is the signal that z cannot be increased anymore, i.e. we maximized our objective function
- Solution: $(12, 15, 0, 0, 750)$, i.e., maximum value is 750, for $x_1 = 12$, $x_2 = 15$

Simplex algorithm: tableau format

- The linear program is in standard form:
  - Maximize $c \cdot x$
  - Subject to $Ax = b$, $x_i \geq 0$
- Rework the objective function into a constraint-like condition
  - to maximize $c \cdot x$ is to maximize a new variable $z \geq 0$ such that $z = c \cdot x$, or in other words $-c \cdot x + z = 0$, for all x
- Represent now the linear program as a tableau (matrix):
  - one column for each of the variables, both the original ones and the slack variables
  - one last column for the right hand side of all equations
  - one row for each of the equations, write in each column the coefficient of the corresponding variable in the equation
  - the original variables will form the initial set of **independent** variables, the slack variables will form the initial set of **dependent** variables
    - the variable added to the objective function remains a dependent variable throughout the algorithm

Tableaux format: example

- Example: the carpenter's problem. Maximize $25x_1 + 30x_2$ such that:
  - $20x_1 + 30x_2 \leq 690$; $5x_1 + 4x_2 \leq 120$
  - $x_1 \geq 0$; $x_2 \geq 0$
- Transform the problem into:
  - $20x_1 + 30x_2 + y_1 = 690$; $5x_1 + 4x_2 + y_2 = 120$
  - $x_1 \geq 0$; $x_2 \geq 0$; $y_1 \geq 0$; $y_2 \geq 0$
  - Maximize $z \geq 0$ such that $-25x_1 - 30x_2 + z = 0$
- Tableau:

x_1	x_2	y_1	y_2	z	RHS
20	30	1	0	0	690 $(= y_1)$
5	4	0	1	0	120 $(= y_2)$
−25	−30	0	0	1	0 $(= z)$

Dependent variables: $\{y_1, y_2, z\}$
Independent variables: $x_1 = x_2 = 0$
Extreme point: $(x_1, x_2) = (0, 0)$
Value of objective function: $z = 0$

Outline of the Simplex Method

- **Step 0 [Initialization]** Present a given linear programming problem in standard form and set up the initial tableau.
- **Step 1 [Optimality test]** If all entries in the objective row are nonnegative, then stop: the tableau represents an optimal solution.
- **Step 2 [Find entering variable]** Select the smallest negative entry in the objective row. Mark its column to indicate the entering variable and the pivot column.

- **Step 3 [Find exiting variable]** For each positive entry in the pivot column, calculate the θ-ratio by dividing that row's entry in the rightmost column by its entry in the pivot column. If there are no positive entries in the pivot column, then stop: the problem is unbounded. Find the row with the smallest θ-ratio, mark this row to indicate the exiting variable and the pivot row.
- **Step 4 [Form the next tableau]** Divide all the entries in the pivot row by its entry in the pivot column. Subtract from each of the other rows, including the objective row, the new pivot row multiplied by the entry in the pivot column of the row in question. Replace the label of the pivot row by the variable's name of the pivot column and go back to Step 1.

First step: find one initial extreme point

- The algorithm starts implicitly by choosing the origin as the initial extreme point
  - all independent variables equal to 0

Next step: apply the optimality test

- **Examine the last equation (corresponding to the objective function)**
  - if all its coefficients are nonnegative, then stop: the current extreme point is optimal
  - otherwise, choose the variable with the smallest negative coefficient (i.e. largest module) as the new entering variable (into the set of dependent variables)
    - intuition: this is the one to contribute the most to the maximization of the target variable
  - effectively, we choose a column corresponding to an independent variable

x_1	x_2	y_1	y_2	z	RHS
20	30	1	0	0	$690\ (= y_1)$
5	4	0	1	0	$120\ (= y_2)$
−25	−30	0	0	1	$0\ (= z)$

Next step: apply the feasibility test

- For each row of the tableau, divide its right-hand-side values by the coefficient values in the chosen column
- Choose the exiting variable (from the set of dependent variables) to be the one corresponding to the smallest positive ratio after this division (the only one with a non-zero coefficient on that row)
    - this variable is called the pivot
    - Intuition:
        - We move from one intersection point to another intersection point along the line given by one of the constraints
        - We move by only one segment, not more, so as not to go outside the set of feasible points

Entering variable (x_2)

x_1	x_2	y_1	y_2	z	RHS	Ratio
20	30	1	0	0	690	23 (= 690/30) ← Exiting variable
5	4	0	1	0	120	30 (= 120/4)
−25	−30	0	0	1	0	*

Next step: pivoting

- **Pivoting to solve for the new dependent variable values**
  - derive a new, equivalent system of equations by eliminating the entering variable from all equations that do not contain the exiting variable
  - divide the row containing the exiting variable by the coefficient of the entering variable in that row; the entering variable gets coefficient 1
  - eliminate the entering variable from the other rows
  - assign value 0 to the variables in the new independent set (remove from it the exiting variable, add to it the entering one)
  - calculate the value of the target variable

x_1	x_2	y_1	y_2	z	RHS
20	30	1	0	0	690
5	4	0	1	0	120
-25	-30	0	0	1	0

Entering variable

Exiting variable

x_1	x_2	y_1	y_2	z	RHS
0.66	1	0.033	0	0	23($=x_2$)
2.33	0	-0.133	1	0	28 ($=y_2$)
-5	0	1	0	1	690($=z$)

Dependent variables: $\{x_2, y_2, z\}$
Independent variables: $x_1=y_1=0$
Extreme point: $(x_1,x_2)=(0,23)$
Value of the objective function: $z=690$

Iterate

- A new intersection point has thus been found
- Repeat the whole procedure starting from this new intersection point

Example (continued)

Entering variable

x_1	x_2	y_1	y_2	z	RHS
0.66	1	0.033	0	0	23(=x_2)
2.33	0	-0.133	1	0	28(=y_2)
-5	0	1	0	1	690(=z)

⇨

x_1	x_2	y_1	y_2	z	RHS	ratio
0.66	1	0.033	0	0	23(=x_2)	23/0.66
2.33	0	-0.133	1	0	28(=y_2)	28/2.33
-5	0	1	0	1	690(=z)	*

Dependent variables: $\{x_2,y_2,z\}$
Independent variables: $x_1=y_1=0$
Extreme point: $(x_1,x_2)=(0,23)$
Value of the objective function: z=690

Exiting variable

x_1	x_2	y_1	y_2	z	RHS
0	1	0.071	-0.285	0	15 (=x_2)
1	0	-0.057	0.428	0	12 (=x_1)
0	0	0.714	2.142	1	750 (=z)

Dependent variables: $\{x_1,x_2,z\}$
Independent variables: $y_1=y_2=0$
Extreme point: $(x_1,x_2)=(12,15)$
Value of the objective function: z=750

Solution:
- $(x_1,x_2)=(12,15)$;
- maximum value 750

Another example

$$\begin{aligned} \text{maximize} \quad & z = 3x + 5y + 0u + 0v \\ \text{subject to} \quad & x + y + u = 4 \\ & x + 3y + v = 6 \\ & x \geq 0,\ y \geq 0,\ u \geq 0,\ v \geq 0 \end{aligned}$$

	x	y	u	v		
u	1	1	1	0	4	4
v	1	3	0	1	6	2
	−3	−5	0	0	0	

current feasible sol. (0, 0, 4, 6)

$z = 0$

	x	y	u	v		
u	$\frac{2}{3}$	0	1	$-\frac{1}{3}$	2	3
y	$\frac{1}{3}$	1	0	$\frac{1}{3}$	2	6
	$-\frac{4}{3}$	0	0	$\frac{5}{3}$	10	

current feasible sol. (0, 2, 2, 0)

$z = 10$

	x	y	u	v	
x	1	0	$\frac{3}{2}$	$-\frac{1}{2}$	3
y	0	1	$-\frac{1}{2}$	$\frac{1}{2}$	1
	0	0	2	1	14

current feasible sol. (3, 1, 0, 0)

$z = 14$

Computational complexity of simplex

- Simplex performs well in practice
- It has exponential worst-case complexity
  - there is a class of linear programs for which it runs in exponential time
  - for every variation of simplex that has been proposed, the same had been found
- It has polynomial average-time complexity under various probability distributions for random matrices
- There are methods for solving linear programs in polynomial time
  - the ellipsoid method (1979) –polynomial but impractical: $O(n^6 L)$, where n is the number of variables and L is the number of bits of input to the algorithm
  - Karmarkar's algorithm (1984) –the first reasonably efficient algorithm for linear programming: $O(n^{3.5} \cdot L^2 \cdot \log L \cdot \log\log L)$; performance comparable "in practice" to that of simplex

Learning objectives

- Understand the conceptual difference between solving an optimization problem over real numbers and over integers
- The different tools to be used in these two cases: calculus vs. combinatorics (with support from geometry, algebra, or algorithmics)
- Understand the concept of linear program, integer problem
- Understand the basic concepts in the geometric approach to linear programs: a linear function (or line) splitting the plane in two different regions; intersection points; optimization at the intersection points
- Understand the algorithmic strategy in simplex: greedy algorithm, moving from one intersection point to another along the edges of the feasible set