Games with Incomplete Information: Bayesian Nash Equilibrium in the Volunteer's Dilemma game

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1 Introduction

Game theory has been applied to solve practical problems in many fields such as pedestrian modelling, evolutionary biology, or even network routing. In particular, we studied and utilized games with perfect information, where the structure of a game being played (i.e. players, payoffs, previous actions) is known among all the agents. Examples of games with perfect information includes the classic Prisoner's Dilemma, or the game of chess, where both the rules of the games and the actions are communicated clearly to the players. One relaxation of the perfect information game assumptions is the complete but imperfect information games. This category of games includes games where the rule (payoff table) is known among the players, but previous or current actions are not available to all players. Assuming imperfect information accounts for the variance caused by players being forgetful or unaware of previous events.

However, the assumption of imperfect information still does not capture the complexity of real world problems. In fact, almost all economic or social environments of interest include some form of incompleteness of information, where there is no common knowledge of the structure of the game being played [8]. For instance, a company's pricing strategies are developed without complete information of its competitors' costs; countries may negotiate climate change agreements having different beliefs about the costs and benefits of the global climate change. The prior of different beliefs is not reflected in a static payoffs table.

Therefore, this report aims to provide an overview on Bayesian games which allow for the modelling of games with incomplete information. Particularly, the report will focus on an analysis of the Bayesian version of the Volunteer's Dilemma (VDI) [9]. The problem is attainable yet it captures the essence of a lot of public goods problems which will be covered in section 2. Then, with a gentle summary of Bayesian games and the concept of type spaces, we derive the Bayesian Nash Equilibrium in the Volunteer's Dilemma game. Finally, the report will present simulation results of VDI with varying priors, decision strategies, group sizes, and number of volunteers required to produce the public good. Section 2 and 3 of this report (and its notation) are loosely based on section 2 and 3 of [9].

2 Problem Definitions

2.1 The Volunteer's Dilemma (VOD)

2.1.1 Definition

The Volunteer's Dilemma (VOD) is an N-player game that models the production of public goods [4]. Each player can either *cooperate* ("volunteer") with a cost of C or *defect* ("abstain") with a cost of 0. If the number of players choosing to volunteer is at least k, the public good is produced.

	0	...	$k-2$	$k-1$	k	...	$N-1$
C	0	...	0	B-C	B-C	...	B-C
D	0	...	0	0	B	...	B

Table 1: Payoff table of a player w.r.t. the number of people volunteered for threshold k

	0	...	$k-2$	$k-1$	k	...	$N-1$
C	0	...	0	$1-\mu$	$1-\mu$	...	$1-\mu$
D	0	...	0	0	1	...	1

Table 2: Payoff table of a player **(normalized)** w.r.t. the number of people volunteered for threshold k

Everyone (including those who abstained) gets a payoff B minus their cost (see Table. 1). Otherwise, everyone receives payoff 0. After normalizing this is equivalent to the payoff table presented in Table. 2, where $\mu = C/B$, $\mu \in [0,1]$.

One can verify easily that the N cases where one player cooperates and all the others defect are asymmetric equilibria. Furthermore, since there is no intrinsic differences between each player and this is a finite game, there must also exist a symmetric equilibrium where each player employs the same strategy. Everyone all cooperating or all defecting is not an equilibrium. Therefore, the symmetric equilibrium must entail a mixed strategy. This equilibrium is shown to be each player defects w.p. $\alpha = \mu^{1/(n-1)}$ [3].

2.1.2 Asymmetric VOD

Also introduced by Diekmann [3], asymmetric VOD is a game where one of the players has a lower cost to benefit ratio than the rest of the players, making it a "strong" player. It can be shown that the evolutionary stationary point (ESS) of the Asymmetric VOD includes 1) the strong player always cooperates and the week players free-ride 2) the strong player always defects and the weak players employ a mixed strategy [6]. Later in 2.3, we will see why the Bayesian Volunteer's Dilemma game is a generalization of Asymmetric VODs.

2.1.3 Application

While VOD's application can be seen in many areas such as sociology, economics, evolutionary biology, and computer science, the author of this report find the recent implementation of VOD games in the publishing industry fascinating. In 2020, Subscribe to Open (S2O) as a business model was introduced for publishers to support open access sustainably. The S2O model sets a threshold for subscription revenue every fiscal year, and once that threshold is reached, all articles published next year can be openly accessed [2]. The publishers specifically target existing large subscribers such as universities and libraries for renewal, while the less privileged can benefit from the knowledge being open access. This is a typical instantiating of Asymmetric VOD's first type of equilibrium. By the time of writing, multiple publishing associations has adopted or trailing S2O, including the European Mathematical Society [1].

2.2 Bayesian Game

Intuitively, Bayesian games model games where players each keep some private information that this only known to that specific player but not to other players or the observer. Bayesian games can be interpreted as having a distribution over payoff tables instead of always having a fixed payoff table. To formally define Bayesian games, we adopt the following definition from [] with slight modifications.

Definition A game with incomplete information $G = (\Theta, \Pi, F, u)$ consists of:

1. A set $\Theta = \Theta_1 \times ... \times \Theta_N$, where Θ_i is the (finite) set of possible types for player i.
2. A set $\Pi = \Pi_1 \times ... \times \Pi_N$, where Π_i is the set of possible strategies for player i.
3. A joint probability distribution $F(\Theta_1, ..., \Theta_N)$ over types. For finite type space, assume that $p(\mu_i) > 0$ for all $\mu_i \in \Theta_i$.
4. Payoff functions $u_i : \Pi \times \Theta \to \mathbb{R}$

To analyze Bayesian games, we introduce *Nature* as a player who always adopts a fixed mixed strategy that assign types privately to each player based on the joint probability distribution F (the *prior*). This converts the game with incomplete information into a game with complete but imperfect information. A Bayesian Nash Equilibrium is simply a Nash Equilibrium of the game where Nature moves first, chooses $\mu \in \Theta$ from F with probability $p(\mu)$ and reveals μ_i to player i [5].

2.3 The Volunteer's Dilemma with Incomplete Information (VDI)

We have seen that in Asymmetric VODs, one or more of the players can have a lower cost-benefit ratio than the other players, and this information is available to all the players in the game. We can generalize it to the case where each player has a cost-benefit ratio μ_i which is privately known to that player. This gives us the Volunteer's Dilemma Game with Incomplete Information (VDI) [9]. Namely, VDI is a two stage game where in the first stage, Nature assigns μ_is to each player based on F; in the second stage, a standard VOD is played.

Definition VDI is a Bayesian game with $\Theta_i = [0, 1]$, $\Pi_i = \{C(\textit{cooperate}), D(\textit{defect})\}$. Types of each player μ_i is independently drawn from a distribution $F(\Theta)$ defined over $[0, 1]$. u_i w.r.t. μ_i is the same as that of VOD (Table 3).

	0	...	$k-2$	$k-1$	k	...	$N-1$
C	0	...	0	$1-\mu_i$	$1-\mu_i$	...	$1-\mu_i$
D	0	...	0	0	1	...	1

Table 3: Payoff table of player i in VDI

2.4 Bayesian Nash Equilibrium of VDIs

Here we derive the Bayesian Nash Equilibrium for VDI with $k = 1$ by considering the expected payoff of player i [9]. The expected payoff of cooperating is always $u_i(C_i; \mu_i, \pi_i, F) = 1 - \mu_i$. The expected payoff of defecting is 0 when everyone defects, and 1 otherwise. Let $\pi(\mu) = 1$ denotes the player cooperates,

$$u_i(D_i; \mu_i, \pi_i, F) = \mathbb{E}_F \left[1 - \prod_{j \neq i} \pi_j(\mu_j) \right] = 1 - \prod_{j \neq i} \mathbb{E}_F[\pi_j(\mu_j)] = 1 - \prod_{j \neq i} p(D_j)$$

where $p(D_j)$ is the probability that player j defects. It immediately follows that a symmetric equilibrium strategy is to cooperate when $1 - \mu_i > 1 - \prod_{j \neq i} p(D_j)$ and defect otherwise. One implication of this equilibrium strategy is that similar to the VOD with a strong player, in VDIs, the equilibrium strategy also results in the strongest player surely contributing and everyone else surely defecting. Indeed, next section will explore the this strategy show that it is an equilibrium in VDIs with different parameters.

3 Simulation

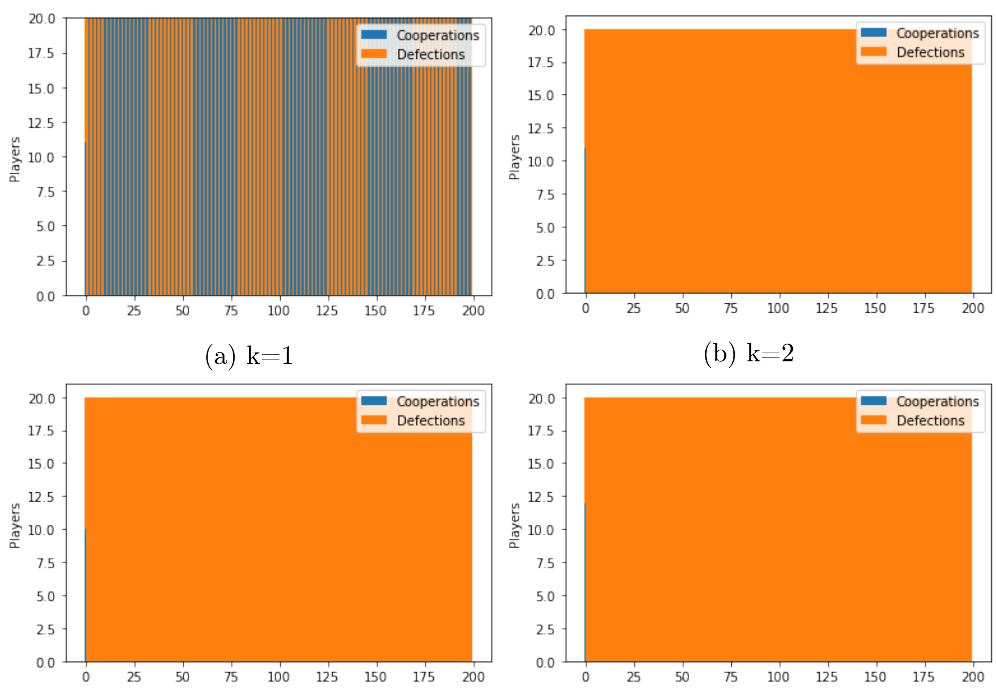
A VDI game is set up as described in section 2.3. Three different strategies are evaluated as we will describe later. The simulation also evaluates two choices of priors:

1. Bernoulli(p): $\mu_i = 0.2$ w.p. p, and $\mu_i = 0.8$ w.p. $1 - p$
2. Beta(α, β): $\mu_i \in [0, 1]$, $\mu_i \sim \text{Beta}(\alpha, \beta)$)

The code is available at https://github.com/QianqianF/Volunteers-dilemma

3.1 Simple Pure Strategy

We first explore a simple pure strategy, where each player chooses the best response to the outcome of the previous round of the game. We initialize each session with all players taking an action $\sim \text{Ber}(\mu_i)$, and run the game for $n = 200$ rounds. Not surprisingly, when $k = 1$, all players oscillates between cooperating and defecting (Figure 3a). When $k > 1$, since switching from D to C for each individual player doesn't produce an immediate increase in payoff, the public good is never produced except for the first round (Figure 3b3c3d).

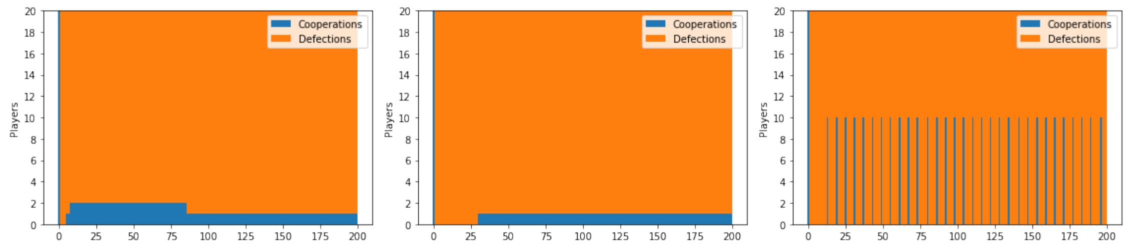


(a) k=1
(b) k=2
(c) k=8
(d) k=16

Figure 1: Number of cooperations and defections throughout 200 rounds

3.2 Equilibrium Strategy

In section 2.4, we derived the Bayesian Nash Equilibrium of VDIs when $k = 1$. Instead of deriving $p(D_j)$ from F, we aggregate all the past actions to get an estimate of the probability that each player defects. Implementing the equilibrium strategy when $F = \mathcal{U}(0, 1)$ indeed yields the equilibrium where the strongest player contributes every round. When F is the beta distribution with different configurations of α and β (other than $\alpha = \beta = 1$), the game behaves similarly.



(a) $F = \mathcal{U}(0, 1)$ (b) $F = \text{Beta}(5, 1)$ (c) $F = \text{Ber}(0.5)$

Figure 2: Number of cooperation and defections throughout 200 rounds with equilibrium strategy

Interestingly, when $F = \text{Ber}(0.5)$, in this particular case that exactly 10 players are strong players out of the 20 players in total, we get the oscillating effect again where these 5 strong players either all volunteer or all defect. This is due to the symmetric and fixed equilibrium strategy used so the 10 strong players can't decide who volunteers.

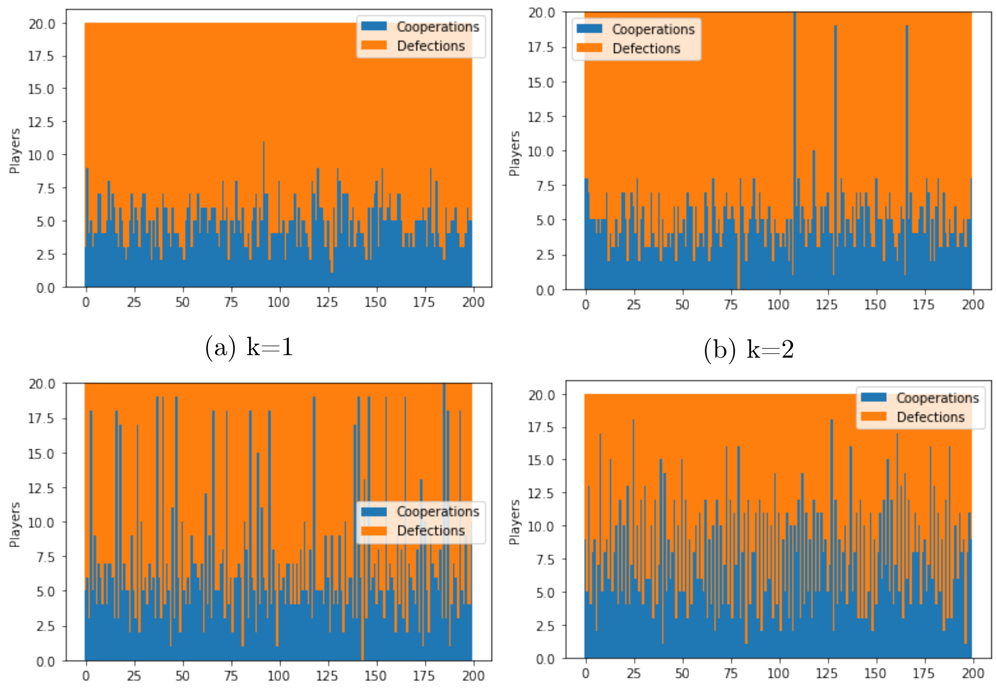
3.3 Mixed Strategy

We are also interested in how mixed strategies would behave in a game with incomplete information, so the following strategy is also evaluated.

- Initialize each session with all players taking an action $\sim \text{Ber}(\mu_i)$
- Each player evaluate the potential payoff of both actions based on their types μ_i, assuming that other players will take the same action as last round
- Normalize the payoffs for each player so that they form a Bernoulli distribution over actions Π_i
- Draw an action from the new Bernoulli distribution

It also allows us to gain intuition into how the number of volunteers required to produced the common good (k) affects the game without having to analytically calculate $p(D_j)$ based on different priors. In this simulation, F is set to be $\mathcal{U}(0, 1)$. Intuitively, this strategy encourages cooperation when the objective is not being met, and encourages defection when the number of players contributing is more than necessary. Moreover, strong players also have a larger probability to contribute than the weaker players.

As shown in Figure. 3, the contribution rate is relatively low when $k = 1$ and the stochasticity introduced resulted in the objectives being obtained all the time. When the number of players contributing is more than necessary, it quickly drops to around k. As k increases, we get a larger variance of players' contribution. When k further increases, the incentive to contribute reduces because switching from defecting to cooperating requires more existing contributions. Furthermore, the current strategy prevents everyone from all contributing because there is always a non-zero probability to defect. However, it is still interesting to observe that under this mixed strategy, the number of times that the objective is reached decrease almost linearly as k increases (Figure 4).



(a) k=1 (b) k=2

(c) k=4 (d) k=8

Figure 3: Number of cooperations and defections throughout 200 rounds

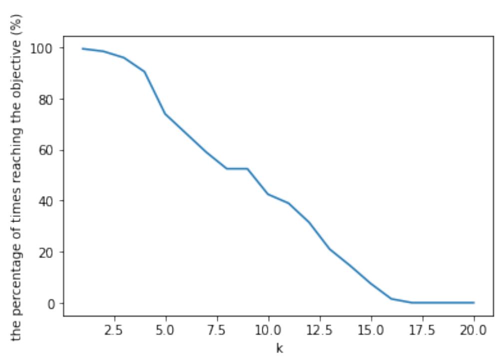


Figure 4: Percentage of times that the objective is reached out of the 200 rounds

4 Discussion

There are a lot of aspects of VDIs that can be further explored. For instance, in a repeated game setting, the payoff is discounted with time as the game progress and it is a more accurate representation of real world challenges such as climate change and other public goods production. It is also an interesting topic to derive the Bayesian Nash Equilibrium for VDIs with $k > 1$. Furthermore, there are other Bayesian decision based models that takes the conformity into account [7]. In addition to incomplete information, it also models the fact that individual players want to play actions similar to a the group. This is shown to closely model human participants in a VDI game [7].

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