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SCHOOL OF ECONOMICS AND MANAGEMENT

Time-varying Commodity Portfolio Optimization

by

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Abstract

Commodity future is basically a way of hedging against physical products, interest rates, and the stock market. However, commodity portfolios alone can be an investment choice based on their return and risk characteristics. In order to analyze the interconnectedness among commodities and build commodity portfolios, we apply multivariate GARCH models to get the time-varying conditional statistics of the metal, energy, and agricultural commodities from 1991 to 2023. By doing portfolio optimization and then obtaining the optimal time-varying commodity weights, we can analyze the performance of different portfolio strategies over time. Finally, investors can reduce the risk of commodity portfolios by considering the hedge ratios of different commodity pairs.

Keywords: commodity, portfolio optimization, time-varying volatility

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1. Introduction

Commodities are raw materials that can be classified as metal, energy, and agricultural products (see Forbes Advisor). Their prices are easily influenced by various factors. For example, energy commodities are influenced by heating while agricultural commodities are influenced by weather conditions and seasonal demand. In order to avoid the associated risk of these factors, the first commodity futures trading exchange was established in Japan for trading rice futures in 1730 (Dojima Rice Exchange). The original commodity trading of futures serves as a risk management tool for producers, allowing them to avoid the impact of price changes by transferring risks to speculators. In modern times, a third group of transactors has emerged and become increasingly important (Gilbert, 2008). These kinds of investors view commodity futures as one kind of asset based on the derivatives' return and risk nature. The majority of these kinds of investors use commodity futures as a way of hedging the stock market (Alshammari & Obeid, 2023), real estate risk (Raza et al., 2018), or Bitcoin (Joo & Park, 2024). However, instead of using the commodity portfolios as a way of hedging, commodity futures alone can be the investment portfolios due to their volatile price.

Our study focuses on commodity futures, despite other commodity trading ways such as physical commodity purchases, commodities stocks, commodities ETFs, and Mutual Funds (see Forbes Advisor). This is because commodity futures are now the most common way of trading commodities. Also, commodity futures have the advantage of their liquidity and standardized contracts, along with minimal counterparty risk (Ruano & Barros, 2022). While there are commodity future indexes that can provide us with broader information about commodity sectors, we use the future prices of individual commodities. These can help us capture the potential relationship between specific commodity groups (Daigler et al., 2017). By using the commodity future price as the representation of commodity price, we can analyze the dependence and interconnectedness among commodities. Apart from analyzing individual commodities, we can build portfolios such as equally weighted portfolios, maximum Sharpe ratio portfolios, and minimum CVaR portfolios. Based on investment objectives and subjective preferences, investors can make commodity portfolio investment decisions. Additionally, by analyzing the time-varying portfolio performance, investors can build on their portfolios differently based on market conditions.

In order to build commodity portfolios, we begin by analyzing the statistical characteristics of the selected commodities, which are classified into three categories: metal, energy, and agricultural groups. This is done by plotting the returns and calculating the descriptive statistics. Also, by applying the statistical tests, we can capture the commodity return data features, which is crucial for the subsequent analysis. The empirical part of our thesis starts by using the multivariate GARCH model, which can help us get the time-varying correlation and covariance matrix during the period from 1991 to 2023. Next, we build seven portfolios of equal weight, mean-variance, minimum variance, minimum correlation, maximum Sharpe ratio, maximum Sortino ratio, and mean-CVaR, which focus on different risk objectives and provide a comprehensive framework of investment strategies. By considering short-sell and no short-sell opportunities and thus using the optimization process, we get the time-varying weight for different portfolios. Based on the performance metrics of the seven portfolios under different times, we can make investment decisions and adjust accordingly based on market times.

2. Literature Review

2.1 Interconnectedness among Commodities

Commodity portfolios can be used for hedging purposes, as different commodities have varying levels of risk and returns. Therefore, analyzing the interconnections between commodity markets and implementing portfolio optimization strategies can help investors optimize their portfolio allocations and mitigate risks. We focus on three commodity portfolios: metal, energy, and agricultural commodities. Through our analysis, we aim to explore their interrelationships, including metal-energy, metal-agricultural, and energy-agricultural connections.

2.1.1 Metal and Energy Commodities

Gold has been recognized as a safe haven asset in hedging against risk and serving as a reliable source of maintaining value during volatile times. Investors often use precious metals such as gold as a hedge to protect against currency depreciation during the slowing down economy which is indicated by the increasing oil price (Bedoui et al., 2023). Hammoudeh and Yuan (2008) find that gold and silver may be partial substitutes for hedgers under volatile times. Instead, copper is not suitable for hedging as the transitory volatility vanishes much more rapidly for copper commodities. Also, the comovement of metal and energy commodities becomes negative during financial booms, which brings out the possibility of portfolio diversification (Albulescu et al., 2020). In contrast, Reboredo (2013) finds that gold and oil show a strong positive dependence, suggesting that gold might not effectively hedge against oil price movements. However, they also find tail independence, which suggests that the hedge is effective under bad market times. Sari et al. (2010) also find that there are minor but positive effects between precious metals and oil. Specifically, if the commodity price of silver increases, it may signal that the price of oil will also increase because silver is highly volatile like oil. Similarly, Rehman and Vo (2021) study the cointegration of precious metals and oil, and they find that precious metals and oil prices tend to move together in the long run under both bearish and normal market conditions. As a result, they suggest that investors should avoid building portfolios focusing solely on energy and precious metal commodities due to their high interdependence, especially during extreme market conditions.

2.1.2 Metal and Agricultural Commodities

A large number of researchers study the relationship between metal and agricultural commodities. Hanif et al. (2023) study the time-varying relationship among commodity markets and find that the metal commodities portfolio is less risky compared to the agricultural commodities portfolio. According to their portfolio optimization result, investors in metal markets should allocate the majority of the commodities to gold, followed by aluminum, platinum, and lead. While for the agricultural market, the greatest amount should be allocated to timber, followed by wheat, cocoa, and soybeans. For energy and agricultural commodities, the prices of their futures are highly correlated and the increase in energy commodity prices will drive that of agricultural commodity prices (Koirala et al., 2015). Naeem et al. (2022) also find that before the global financial crisis, oil is the safe-haven asset for the majority of agricultural products, except sugar. But after the volatile times, it loses this property. Apart from that, researchers also suggest contagion effects among the two commodity portfolios, which indicates that the two commodities might not be a suitable choice for portfolio diversification during financial volatile times (Khalfaoui et al., 2023). Hernandez et al. (2019) also find that wheat and rice prices are slightly affected only in extreme downside oil price scenarios. However, corn stands out as the only commodity that can diversify and safeguard against extreme oil market risk.

2.1.3 Energy and Agricultural Commodities

In recent years, there has been significant cross-market linkage between energy and agricultural markets. This is due to the substitute of bio-fuel for the traditional fossil fuel. Studies show that energy-agricultural commodities connectedness is time-varying and thus there are times that exhibit limited causal effects, such as post-1980s (Shahzad et al., 2021) or during the COVID-19 pandemic times (Furuoka et al., 2023). By analyzing the systematic risk across markets, Kumar et al. (2021) recommend that investors should consider choosing other commodities when the energy and agricultural commodity market collapse due to regime-switching risk spillovers. Moreover, when both oil and agricultural commodity markets are going up or decline, it's more likely to experience losses than gains simultaneously. As for the energy and agricultural commodity portfolios, researchers suggest that investors should allocate a lower proportion of energy commodities in order to achieve higher profits based on the performance of dynamic optimal portfolio weights for energy-agricultural commodities portfolios (Furuoka et al., 2023). Shiferaw (2019) studies the relationship between energy prices and agricultural commodity prices focusing on South Africa and finds that the cost of agricultural products is directly impacted by the volatile global oil prices. As a result, the changing oil price causes fluctuations in agricultural commodity prices. Recently, Miljkovic and Vatsa (2023) have shown that crop prices typically exhibit lags to oil prices, although there are times that the reverse is observed. Also, they group similar commodity price movements, which helps them identify two major clusters: one is oil and agricultural commodity prices, and the other is coal and natural gas prices.

2.2 Models for Studying the Commodities' Interconnectedness

Our analysis focuses on exploring the interconnections among energy, metal, and agricultural commodity markets. Existing articles generally have four empirical methods: GARCH, copula, granger causality, and quantile methods.

2.2.1 GARCH Models

A wide range of researchers use different GARCH models such as DCC (Shiferaw, 2019), Asymmetric DCC (Raza et al., 2018; Trabelsi et al., 2022), Beta-Skew-t-EGARCH (Gaete & Herrera, 2023), GARCH-MIDAS-X Framework (Yaya et al., 2022) to study the correlation and spillover among commodity markets or their interactions with other markets. The core of these kinds of models is to capture the volatility patterns and study how the volatility of one commodity impacts the volatility of others, but they might not be helpful in analyzing the relationship between different commodities. In order to analyze the dependence structure between agricultural and energy commodities, Shiferaw (2019) chooses eight commodities and applies the Bayesian multivariate GARCH model with skewness and heavy tails. The researcher finds that the DCC model with the error skewed-mvt distribution assumption performed better than other competitive methods, such as the CCC model with various distributions based on its AIC, BIC, and DIC values. Additionally, Gaete and Herrera (2023) use the Beta-Skew-t-EGARCH model to capture the volatility between equities and commodities. By modeling the volatility and the standardized residuals, they can get the score-driven dynamics for solving the mean-variance optimization problem. Similarly, the DCC-MIDAS framework applied by Yaya et al. (2022) helps investigate the conditional correlations and volatility between oil and precious metal prices. The DCC-MIDAS means the dynamic conditional correlation with mixed data sampling. This framework incorporates information from different frequencies of data, which is efficient when one variable is observed at a higher frequency than the others.

2.2.2 Copula Models

Copula models can characterize the tail dependence structure between returns or risks of different commodities (Albulescu et al., 2020; Kumar et al., 2021; Koirala et al., 2015; Adhikari & Putnam, 2020; Fousekis & Grigoriadis, 2017; Hanif et al., 2023). The copula models have flexibility in modeling different types of dependencies and are particularly effective for handling tail risk which is helpful for understanding how extreme movements in one commodity affect other commodities (see Albulescu et al., 2020). To model the asymmetric dependencies in the tails, Albulescu et al. (2020) use the Gumbel copula for upper-tail and the Clayton copula for lower-tail dependence. Also, the rotated versions of both Gumbel and Clayton copulas are also considered to capture different dependency structures. Similar to them, Koirala et al. (2015) also use a mixture of Clayton and Gumbel copulas. For the dependence parameters estimation, the researchers use the maximum likelihood method. The parameter is then translated into Kendall's tau to provide a measure of correlation. There exist various copula models. Kumar et al.(2021) apply a regime-switching copula approach to model the dependence between oil and agricultural commodity returns. The model considers two regimes of positive and negative correlation. By calculating the transition probabilities between positive and negative correlation regimes, the model helps capture the dynamic and asymmetric nature of market relationships, particularly under extreme conditions.

2.2.3 Granger Causality Test

The time-varying Granger causality test is another way to analyze how commodities impact each other's returns and volatility over different time periods. To identify the time-varying causal relationships, Mohamad and Fromentin (2023) apply the time-varying Granger causality methods to energy commodities and ethical investment indices. Additionally, in order to find the instability in causal relationships, forward expanding (FE), rolling (RO), and recursive evolving (RE) are also employed to incorporate the recursive estimation of Wald statistics from the Vector Autoregressive model, which helps in detecting changes in the intensity and direction of causality over time. Similarly, Shahzad et al. (2021) observe significant fluctuations in causal relationships as time changes and those variations are not always in line with stress time periods. Based on the basic Granger causality model, Meta-Granger addresses the heterogeneity by considering commodity type, sample period, and control variables (Wimmer et al., 2021). Focusing on the relationship between future and spot prices, Joseph et al. (2014) study the bi-directional causalities between futures and spot prices of gold, silver, and crude oil commodities. They use the frequency domain analysis to decompose the causality across different frequencies to capture the dynamics of causality over short, medium, and long-term periods.

2.2.4 Quantile Methods

Various quantile methods are available for assessing interconnectedness. In order to examine the directional influence between crude oil and precious metal prices across different quantiles, Shafiullah et al. (2021) combine the Granger causality with the quantile method. The researchers first test for unit roots using the quantile unit root test as introduced by Galvao (2009) and then use the Kuriyama (2016) test to analyze the distributional aspects of quantile cointegration. Finally, they apply the Troster (2018) method to examine Granger causality in quantiles. These series of quintiles are helpful for enhancing robustness against outliers, nonlinearities, and distributional dynamics. Also, a widely used quantile method is the quantile-on-quantile regression, which is useful for estimating correlation in divergent return quantiles (Duan et al., 2023; Naeem et al., 2022). Duan et al. (2023) investigate the linkage between Shanghai crude oil futures prices and WTI crude oil futures prices. The researchers use nonparametric estimation to examine how the quantiles of independent variables affect the conditional quantiles of dependent variables. This quantile-on-quantile regression approach incorporates cross-validation to find a suitable bandwidth, which helps balance the residuals and variance during the estimation and thus increases the robustness to capture dynamic patterns of information transmission between markets. Similar to them, Naeem et al. (2022) employ a bivariate quantile-on-quantile regression model to analyze the relationship between oil and gold prices on industrial metals and agricultural commodities across different time periods. The model captures the relationship during bearish, normal, and bullish market phases, representing lower, median, and higher quantiles respectively.

3. Methodology

In this section, we present our study's methodologies. First, we consider using the normal and student-t distribution to take into account the error features. Second, we construct multivariate GARCH models of diagonal VECH, diagonal BEKK, and CCC, which can help us model the time-varying volatility and correlation. Finally, we find the optimal investment weight based on seven portfolios: equal weight, mean-variance, minimum variance, minimum correlation, maximum Sharpe ratio, maximum Sortino ratio, and mean-CVaR. By analyzing their performance, we can find the fittest portfolio over time.

3.1 Data

This study uses the daily commodity closing price groups of metal, energy, and agricultural from 1991 to 2023 as shown in Table 1. There are 14 individual commodities, including gold, silver, palladium, platinum, copper, WTI crude oil, Brent crude oil, natural gas, corn, cocoa, cotton, coffee, lean hogs, and soybeans. The price data is obtained from Refinitiv Eikon. The commodity products are traded on different commodity markets and each market has its own unique focus. The New York Mercantile Exchange serves as the primary trading market for Palladium, Platinum, WTI Crude Oil, and Natural Gas, focusing on energy and precious metal sectors. Meanwhile, the Intercontinental Exchange Europe mainly exchanges Brent Crude Oil, Cocoa, Cotton, and Coffee, which is related to the global energy and agricultural markets. Corn and Soybeans are the center products of the Chicago Board of Trade, which focuses on food and fuel industries. Additionally, the Chicago Mercantile Exchange concentrates on the Lean Hogs trading.

Table 1: Commodity Data

Group	**Commodity**	**Market**	**Full Name**
Metal	Gold	COMEX	COMEX Gold Composite Commodity Future Continuation 1
	Silver	COMEX	COMEX Silver Composite Commodity Future
	Palladium	NYMEX	NYMEX Palladium Electronic Commodity Future Continuation 1
	Platinum	ICE Europe	ICE Europe Brent Crude Electronic Energy Future
	Copper	CBoT	CBoT Corn Composite Commodity Future
Energy	WTI Crude Oil	NYMEX	NYMEX Light Sweet Crude Oil (WTI) Electronic Energy Future Continuation 1
	Brent Crude Oil	CBoT	CBoT Soybeans Composite Commodity Future Continuation 1
	Natural Gas	COMEX	COMEX Copper Composite Commodity Future Continuation 1
Agriculture	Corn	ICE-US	ICE-US Cocoa Futures Electronic Commodity Future Continuation 1
	Cocoa	ICE-US	ICE-US Coffee C Futures Electronic Commodity Future Continuation 1

Cotton	ICE-US	ICE-US Cotton No. 2 Futures Electronic Commodity Future Continuation 1
Coffee	NYMEX	NYMEX Platinum Electronic Commodity Future Continuation 1
Lean Hogs	NYMEX	NYMEX Henry Hub Natural Gas Electronic Energy Future Continuation 1
Soybeans	CME	CME Lean Hogs Electronic Commodity Future Continuation 1

Note: The table includes 14 commodity futures grouped into metal, energy, and agricultural. Market denotes the place where the commodities are traded. COMEX is the Commodities Exchange, which is now part of the New York Mercantile Exchange. NYMEX is the New York Mercantile Exchange. ICE Europe is the Intercontinental Exchange Europe. CBoT is the Chicago Board of Trade, which is now part of the Chicago Mercantile Exchange Group. ICE-US is the Intercontinental Exchange US. CME is the Chicago Mercantile Exchange. Continuation 1 means that the selected futures contract is the first continuation contract.

There are high correlations between commodity futures and spot prices (Gulley & Tilton, 2014), and particularly a strong unidirectional relationship from futures to spots for both metal and agricultural commodities (Joseph et al., 2014). However, the choice of commodity price is often the commodity futures rather than the spot price considering the absence of arbitrage (Rad et al., 2022). After acquiring the daily prices of the commodities of the chosen period, we get the continuously compounded daily returns by taking the difference in the log of two consecutive prices. The log returns can provide us with a normalized measure of returns and can help reduce the problem of non-stationarity or unit roots during the estimation. The daily return of the commodities is defined as:

$$R_t = ln\frac{P_t}{P_{t-1}} = lnP_t - lnP_{t-1}$$

where P_t is the closing price at time t. By taking the average of the individual metal, energy, and agricultural products, we can get the 3 groups of return data for different commodity markets. Through grouping, we can better understand different market dynamics by studying the interconnectedness among these sectors using the multivariate GARCH models. This approach helps us identify common dependencies and risk factors across related commodities, which is useful for building optimal portfolios.

3.2 Descriptive Statistic

We have three groups of metal, energy, and agricultural commodities. Figure 1 shows the daily returns of the metal, energy, and agriculture commodities, in a total of 14 commodities. Metal commodities experienced a significant decline in 2008, which might be caused by the global financial crisis. Also, the primary factor driving the silver price in 2005 is likely to be the investment demand of the successful launch of Barclays' Global Investors iShares Silver Trust Exchange Traded Fund (Silver Institute). For the energy commodities, the variations at the beginning of 1991 and 2020 might be due to the early 1990s recession and the COVID-19 pandemic, respectively. Additionally, the energy commodities show significant variations around

2022. These are most likely attributed to the economic impact of the Russia-Ukraine War. The war's influence on global energy supplies, especially oil and natural gas, caused volatility in the energy markets. The agricultural commodities show a pattern of seasonal volatility. These variations are caused by seasonal factors such as weather conditions, which can have a significant impact on agricultural yields.

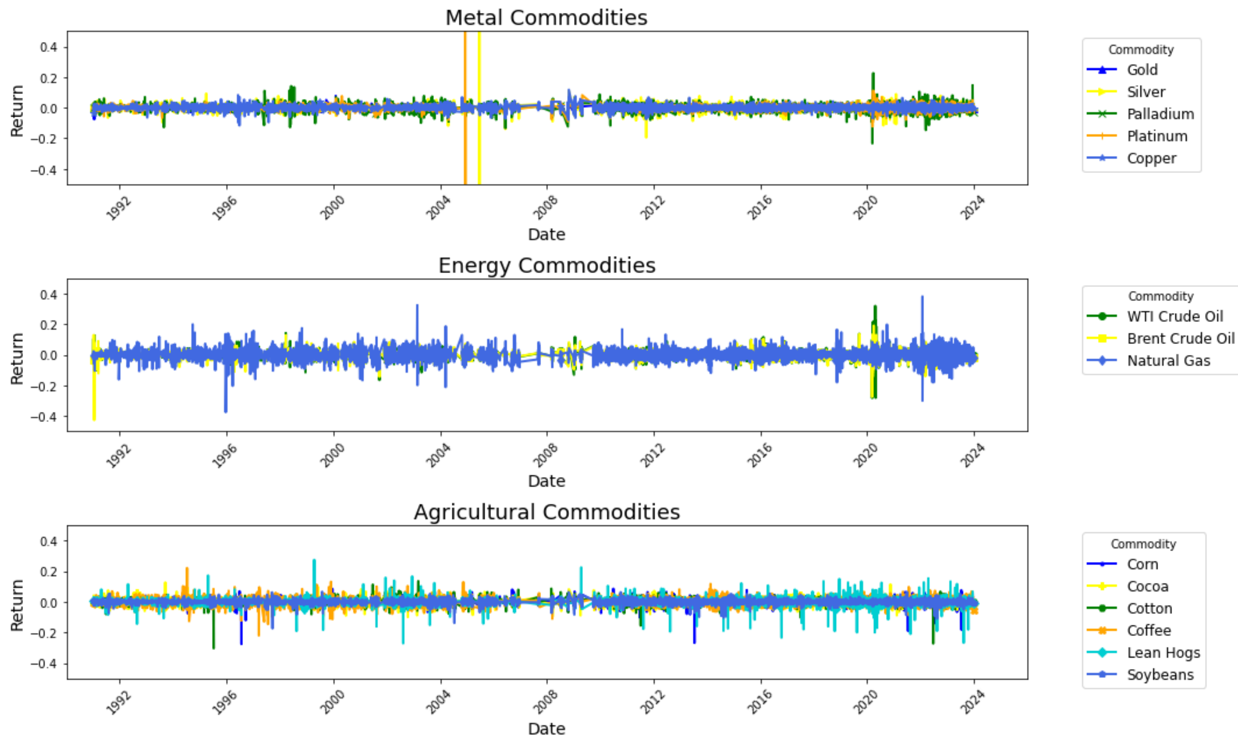

Figure 1: Commodity Return from 1991 to 2023

Table 2 presents descriptive statistics for the returns of the 3 commodity future return groups. For the commodity groups, metal and energy commodities exhibit higher returns, with metal having the highest at 0.02%, followed by metal at 0.01%, and agricultural with the lowest at 0.00%. Additionally, energy commodity shows the highest standard deviation of 2.07%, which corresponds to its highest mean return. Conversely, agricultural commodity displays the lowest volatility at 0.94%. All the commodities groups exhibit slightly negative skewness, which indicates that our data is left-skewed and has asymmetry features. Also, the metal commodity shows an extremely high kurtosis of 44.61, which suggests the presence of heavy tails and implies a higher risk of extreme values. Energy also has a high kurtosis of 11.27. Different from metal and energy commodities, agricultural commodities have a kurtosis of 2.63, which indicates that it approximates a normal distribution.

Table 2: Descriptive Statistics

	Mean	**Std.**	**Min.**	**Max.**	**Skewness**	**Kurtosis**
Metal	0.01%	1.26%	-0.22	0.23	-0.36	44.61
Energy	0.02%	2.07%	-0.30	0.14	-0.53	11.27
Agricultural	0.00%	0.94%	-0.06	0.04	-0.41	2.63

Note: This is the descriptive statistics of metal, energy, and agricultural commodity future returns. A skewness value around 0 indicates symmetry, with a negative skewness indicating left-skewed and a positive skewness indicating right-skewed. A kurtosis value around 3 indicates normal distribution, with a lower value indicating lighter tails and a higher value indicating heavy tails.

We conduct some statistical tests for the three commodity groups of return data. We do Jarque-Bera tests, and the extremely high JB test statistics for all the commodities indicate that there is no normality for our dataset. Furthermore, we applied the Augmented Dickey-Fuller test to test for stationarity. The significant negative values of the ADF statistics suggest that our data is stationary, reflecting that it does not have significant trends over time. Similarly, the supplementary KPSS tests with a p-value of 0.1 can reject the null hypothesis, indicating that the data series is stationary. We use the Ljung-Box test to diagnose if there is autocorrelation for the commodities. The LBQ results for lag 15 show that metal and energy commodities have autocorrelation at a 10% significance level, while agricultural commodity has autocorrelation at a 1% significance level, indicating the presence of dependence over time for these commodities. To test if the GARCH model is fit for our dataset, we conduct Engle's Lagrange Multiplier test (Engle, 1982) to see if our data has ARCH effects. The p-value of the LM test strongly rejects the null hypotheses, which show the conditional heteroscedasticity features of commodity returns. In summary, our data generally shows no normality, stationary, heteroscedasticity, and autocorrelation characteristics.

Table 3: Statistical Tests

	JB	**ADF**	**PP**	**KPSS**	**LBQ**	**LM**
Metal	524814.92***	-83.47***	-84.54***	0.02	22.74*	1952.31***
Energy	33814.35***	-80.36***	-80.35***	0.07	22.65*	407.00***
Agricultural	2011.31***	-28.13***	-75.08***	0.07	39.67***	93.77***

Note: This table shows the statistical test results for metal, energy, and agricultural commodity future returns. JB denotes the Jarque-Bera normality test, with H0: data is normally distributed. ADF denotes the Augmented Dickey-Fuller unit root test, with H0: data has a unit root, indicating non-stationary. PP denotes the Phillips-Perron unit root test, with H0: data has a unit root, indicating non-stationary. KPSS denotes the Kwiatkowski-Phillips-Schmidt-Shin unit root test, with H0: data is stationary around a deterministic trend. LBQ denotes the Ljung-Box test for autocorrelation using lag 15, with H0: data has no autocorrelation. LM denotes the White's Lagrange Multiplier test for Heteroskedasticity, with H0: data has no heteroskedasticity. The asterisk ***, **, and * indicate the significance at 1%, 5%, and 10% level.

3.3 Multivaraite GARCH Model

We use the normal distribution and student t distribution to model the error distribution of returns. In order to model the time-varying volatility and correlations, we use the multivariate GARCH model to account for heteroscedasticity and assist in portfolio management. The multivariate GARCH models for studying the time-varying dynamics among commodities include the diagonal VECH, diagonal BEKK, and CCC. For setting up the models, we define the total number of assets to be N, with each individual commodity denoted as i or j. The total length of the sample period is T, with each day represented as t. The order of the GARCH models is shown as p and q, which means the autocorrelation and moving average terms, respectively.

For the general framework of the multivariate GARCH model, the selected models of VECH, BEKK, and CCC all have the same form of the mean equation and the standardized residual equation.

$$R_{it} = \mu_{it} + \epsilon_{it}$$

$$\epsilon_{it} = \sqrt{h_{it}} \times z_{it}$$

In the mean equation, R_{it} is the actual commodity return, μ_{it} is the expected conditional mean, and ϵ_{it} is the residuals. z_{it} is a random variable that follows a normal or student t distribution. In the standardized residual equation, $h_{ij,t}$ is the conditional variance-covariance of the residuals. The combination of $h_{ij,t}$ is the $N \times N \times T$ conditional covariance matrix H_t .

For the estimation part of these GARCH models, we use the maximizing the log-likelihood function, which is defined as follows:

$$l(\theta) = -\frac{TN}{2} log2\pi - \frac{1}{2}\sum_{t=1}^{T} \left(log\left|H_t\right| + \epsilon_t\ H_t^{-1}\epsilon_t'\right)$$

where θ is the parameters to be estimated. In order to maximize θ, we need to maximize the log-likelihood function $l(\theta)$. By using the numerical optimization method BFGS, which is the Broyden–Fletcher–Goldfarb–Shanno algorithm of the quasi-Newton method, we can do non-linear optimizations by doing the approximation of the inverse Hessian matrix (Liu & Nocedal, 1989). Apart from that, the Marquardt Steps which is the Levenberg–Marquardt Algorithm is also used for adding the robustness of the optimization (Lourakis, 2005). The approach modifies the Gauss-Newton direction by adding a dampening parameter, which can provide a balance between the stability of gradient descent and the efficiency of the Gauss-Newton method.

3.3.1 Diagonal VECH

The first multivariate GARCH model is the VEC model, which is useful for modeling variances and covariances among multiple time series data (Bollerslev et al., 1988). However, the VECH model has limits on its extremely large numbers of estimated parameters even under small dimensions, and the condition of positive-definite conditional covariance matrices needs to be fulfilled (see de Almeida et al., 2018). As a result, the restricted VECH model of the diagonal VECH is more fitted in practice as it can deal with those limits (see Bollerslev et al., 1988; de Almeida et al., 2018). The equation of the diagonal VECH model is as follows:

$$VECH(H_t) = \omega + \sum_{k=1}^{p} A_k VECH(\epsilon_{t-k}\epsilon'_{t-k}) + \sum_{l=1}^{q} B_k VECH(H_{t-l})$$

where H_t is the conditional variance-covariance matrix of the residuals. In this equation, $VECH$ is the operator that stretches the lower triangular part of a matrix including the diagonal into a vector form. ω is the vector with the dimension of $\frac{N\times(N+1)}{2}$, while A_k and B_k are the parameters of matrices to be estimated. For the diagonal restriction, A_k and B_k become diagonal matrices with the dimension of $\frac{N\times(N+1)}{2}$ and $\frac{N\times(N+1)}{2}$. This means that one individual commodity's conditional variance will be determined only by its own autocorrelation and moving average, not by other commodities. The diagonal VECH model will estimate the total number of $(p+q+1)\times\frac{N\times(N+1)}{2}$ parameters. For example, the estimated parameters of the diagonal VECH(1,1) with 3 commodities is $(1+1+1)\times\frac{3\times(3+1)}{2} = 18$.

3.3.2 Diagonal BEKK

The BEKK model is named Baba, Engle, Kraft, and Kroner GARCH model (Engle & Kroner, 1995). This model is a more restricted form of the VECH model. The variance equation is shown as follows:

$$H_t = \omega\omega' + \sum_{k=1}^{p} \sum_{i=1}^{N} A'_{ik}\epsilon_{t-k}\epsilon'_{t-k}A_{ik} + \sum_{l=1}^{q} \sum_{i=1}^{N} B'_{ik}H_{t-l}B_{ik}$$

where ω, A_{ik}, and B_{ik} are the $N\times N$ parameter matrices. The constant matrix ω is the lower triangular matrix, which contains parameters with the number of $\frac{N\times(N+1)}{2}$. Similar to the diagonal VECH model, the diagonal BEKK model also has the diagonal form of A_{ik} and B_{ik}, which contains an equal number of estimated parameters of N. We use the scalar BEKK which is the most restricted version of the diagonal BEKK with $A = aI$ and $B = bI$, where a and b are scalars (Bunnag, 2016). The total number of estimated parameters is $(p+q)\times N+\frac{N\times(N+1)}{2}$. For example, the number of estimated parameters of 3 commodities using BEKK (1,1) is $(1+1)\times 3+\frac{3\times(3+1)}{2} = 12$.

3.3.3 CCC

The CCC model is the constant conditional correlation model where the conditional correlation is assumed to be constant while the conditional variances are time-varying (Bollerslev, 1990). The CCC model uses the univariate GARCH to get the conditional variance h_{it}, the variance equation is as follows:

$$h_{i,t} = \omega_i + \sum_{k=1}^{p} \alpha_{ik}\epsilon_{it-k}^2 + \sum_{l=1}^{q} \beta_{ik}h_{it-l}$$

where h_{it} is the conditional variance of the commodity return of i at time t, which is the expected volatility based on information available at previous times. h_{it-l}^2 is the lagged conditional variance at time $t-l$. In this variance equation, ω_i ,α_{ik} and β_{ik} are the parameters to be estimated, where ω_i measures the long-term average variance,α_{ik} captures the effect of lagged past shocks, and β_{ik} reflects the persistence of past conditional variances. ϵ_{it-k}^2 are the squared innovation at time $t-k$.

$$D_t = diag\left\{\sqrt{h_{i,t}}\right\}$$

$$H_t = D_t \ R \ D_t$$

where H_t is the conditional variance-covariance matrix of residuals, D_t is the conditional standard deviations. R is the constant correlation matrix.

3.4 Time-varying Portfolio Optimization

By considering different risk and return objectives. We construct seven portfolios for optimization: equal weight, mean-variance, minimum variance, minimum correlation, maximum Sharpe ratio, maximum Sortino ratio, and mean-CVaR.

3.4.1 Equal Weight Portfolio

We build the equally weighted portfolio by setting $w_{it} = \frac{1}{N}$, where w_{it} is the weight for commodity i, and N is the number of commodity types in the portfolio. This portfolio is not rebalanced through time, and it serves as a benchmark for all other portfolios.

3.4.2 Mean-Variance Portfolio

The basic portfolio optimization method is the mean-variance method developed by Markowitz (1952), in which investors seek to find the maximum returns and the minimum volatility of the portfolio. The return and variance of the portfolio based on the multivariate GARCH models are as follows:

$$E(R_p) = \sum_{i=1}^{n} \mu_{it} w_{it}$$

$$Var(R_p) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} w_{it} w_{jt} = w^T Q_t \ w$$

By assuming no short-selling opportunities, we can build a quadratic optimization process by using the time-varying covariance matrix and the expected mean return (see for Ghaemi Asl et al., 2024):

$$Max \left\{ w^T \mu - \frac{\gamma}{2} w^T Q_t \; w \right\}, \qquad \text{subject to } \sum_{i=1}^{n} w_{it} = 1 \text{ and } w_{it} > 0$$

where γ is the risk aversion coefficient, Q_t is the covariance matrix with the dimension of $i \times i \times T$, μ is the expected mean return calculated in the GARCH model, and w is the weight of each commodity.

3.4.3 Minimum Variance Portfolio

We can create a portfolio that reduces the investor's risk by minimizing the portfolio's variance. To define the minimum variance portfolio, we use a function that incorporates the variances and covariances of the returns of the assets within the portfolio. The objective is to find the set of weights for the assets that result in the lowest possible variance for the portfolio's returns. The function is defined as follows:

$$Min \left\{ w^T Q_t \; w \right\}, \qquad \text{subject to } \sum_{i=1}^{n} w_{it} = 1 \text{ and } w_{it} > 0$$

where Q_t is the conditional covariance matrix.

3.4.4 Minimum Correlation Portfolio

We create the minimum correlation portfolio based on the time-varying conditional correlation matrix. The correlation between assets has a significant impact on the risk and return of the portfolio. If assets are highly positively correlated, when one asset suffers losses, others may also be affected, thereby increasing the overall risk of the portfolio. Conversely, if assets exhibit a negative correlation or low correlation, it may provide some degree of hedging when some assets perform poorly, thereby reducing the overall risk of the portfolio. By constructing a minimum correlation portfolio, we can further reduce the overall risk of the portfolio and seek more effective asset allocation.

$$Min \left\{ w^T R_t \; w \right\}, \qquad \text{subject to } \sum_{i=1}^{n} w_{it} = 1 \text{ and } w_{it} > 0$$

where R_t is the time-varying conditional correlation matrix.

3.4.5 Maximum Sharpe Ratio Portfolio

The maximum Sharpe ratio aims to optimize the ratio of the portfolio's excess return to its standard deviation, thus maximizing risk-adjusted return. The function for the maximized Sharpe ratio with a fixed risk-free rate is as follows:

$$Max \left\{ \frac{\mu - R_f}{\sqrt{w^T Q_t \; w}} \right\}, \qquad \text{subject to } \sum_{i=1}^{n} w_{it} = 1 \text{ and } w_{it} > 0$$

where R_f is the risk-free rate.

3.4.6 Maximum Sortino Ratio Portfolio

The maximum Sortino ratio is similar to the maximum Sharpe ratio, but it focuses on downside risk rather than total standard deviation of returns, making it more suited for portfolios that aim to prevent losses rather than maximize risk-adjusted returns.

$$Max \left\{ \frac{\mu - R_f}{\sqrt{w^T D_t \; w}} \right\}, \qquad \text{subject to } \sum_{i=1}^{n} \; w_{it} \; = 1 \text{ and } w_{it} \; > 0$$

where D_t is the downside risk, which considers only the variances-covariances of returns falling below the risk-free rate.

3.4.7 Mean-CVaR Portfolio

The mean-CVaR (conditional value-at-risk) focuses on the extreme losses during the sample period. During the optimization process, VaR is calculated using the historical simulation as no assumptions of distribution are required for this non-parametric method. The CVaR is calculated as below the mean return and the optimization is the balance of return and extreme losses.

$$VaR_\alpha = -sorted\ returns_{(1-\alpha)N}$$

$$CVaR_\alpha = E(R|R \leq VaR_\alpha)$$

$$Min \left\{ \frac{\gamma}{2} CVaR_\alpha \;\; (R) - (1 - \frac{\gamma}{2}) E(R) \right\} \qquad \text{subject to } \sum_{i=1}^{n} \; w_{it} \; = 1 \text{ and } w_{it} \; > 0$$

where α represents the confidence level of VaR, N is the sample size used for VaR calculation, and γ is a parameter used to balance the trade-off between conditional value-at-risk and expected return. By adjusting γ, investors can tune the optimization based on their preferences between expected returns and extreme losses.

3.5 Hedge Ratios

The commodity market is highly volatile, which can be caused by a range of factors. For metal commodities, industrial metal prices can fluctuate due to changes in industry demand, mining production levels, and geopolitical events. Energy prices, such as oil and gas, are influenced by wars and heating. Agricultural commodities are particularly sensitive to seasonal changes and weather conditions, which will affect crop yields and livestock production. One effective way to reduce the above risk is hedging. By focusing on commodity futures, we can analyze inter-commodity hedging. The primary benefit of inter-commodity hedging is diversification. By spreading risk across different but related commodities, investors can reduce the impact of price volatility in the commodity future market. We use conditional volatility from the multivariate GARCH model to construct the hedge ratios (Kroner & Sultan, 1993).

$$\beta_{ij,t} = \frac{h_{ij,t}}{h_{jj,t}}$$

where $\beta_{ij,t}$ is the hedge ratio over time between asset i and asset j. $h_{ij,t}$ are the conditional covariance between asset i and asset j, while $h_{jj,t}$ is the conditional variance of the returns of asset j at time t. This hedge ratio normalizes the conditional covariance, which is useful for determining the size of the hedge position. When the conditional covariance between the two commodities is high and the variance of the hedging commodity is low, the hedge ratio will be higher. This means that we need a larger position in the hedging commodity to effectively hedge the risk associated with the primary commodity.

4. Results

This section presents the results by using the returns of the three commodity groups of metal, energy, and agricultural. We get the estimated parameters and goodness of fit from the multivariate GARCH model: diagonal VECH, diagonal BEKK, and CCC. By using the time-varying covariance and correlation of these multivariate models, we can do portfolio optimization, and thus get the time-varying weights for the individual commodities groups. By calculating the portfolio performance of average return, variance, Sharpe ratio, and CVaR, we can compare the portfolios under different market conditions. Additionally, in order to hedge against the risk associated with the commodities, we calculate the hedge ratios for commodity pairs. In conclusion, the presented result will be the GARCH result, average optimal weights, portfolio performance, and hedge ratios.

4.1 GARCH Model

As our data displays asymmetry and fat-tailedness features, we choose different distributions of normal and student-t distribution to fit the errors. The multivariate GARCH models that are used for comparison are diagonal VECH, diagonal BEKK, and CCC.

4.1.1 GARCH Model Results

As shown in Table 4, the criteria used for comparison are the log-likelihood, AIC, and BIC. Under the normal distribution assumption, the diagonal VECH model estimates a total of 24 parameters and has the highest log-likelihood of 56886.42. It also has the lowest AIC and BIC values, indicating its best fit within the normal distribution assumption. The diagonal BEKK and CCC model has lower log-likelihood, indicating less goodness of fit. When using the student t distribution, we will get one more estimated parameter compared to the normal distribution. This additional estimated parameter is the degree of freedom. Under this assumption, the diagonal VECH also achieves the highest log-likelihood of 57459.74 among other student-t distribution models. Similarly, the diagonal BEKK and CCC model still has lower log-likelihood values, along with higher AIC and BIC values. Finally, by comparing the results of normal and student-t distributions, we can see that the models assuming student-t distributions generally show better goodness of fit compared to models assuming normal distributions. These results align with the data features of the high kurtosis observed in metal and energy commodities because the student t distribution is better at handling heavy tails and extreme values compared to the normal distribution.

Table 4: Multivariate GARCH (1, 1) models' goodness of fit

		Num_para	**Log-likelihood**	**AIC**	**BIC**
Norm	VECH	24	56886.42	-17.97	-17.95
	BEKK	18	56837.02	-17.96	-17.94
	CCC	18	56798.86	-17.95	-17.93
T	VECH	25	57459.74	-18.15	-18.13
	BEKK	19	57415.96	-18.14	-18.12
	CCC	19	57406.78	-18.14	-18.12

Note: Norm is the normal distribution, and t is the student t distribution. Num_para is the number of parameters to be estimated during the estimation process. AIC is Akaike Information Criterion and BIC is Bayesian Information Criterion. Higher values of log-likelihood, along with lower values of AIC and BIC, indicate better goodness of fit.

4.1.2 Diagnostic Tests

Table 5 presents the results of diagnostic tests for standardized residuals using the Ljung-Box test with 15 lags. The tests evaluate the correlation and covariance matrices of residuals under normal and student-t distributions for the multivariate GARCH models. Across all models, the P-values are relatively high, with none below 0.10, indicating that there is no significant autocorrelation in the standardized residuals. Models under the student-t distribution generally show slightly higher P-values, implying a marginally better fit in handling autocorrelation compared to those under the normal distribution. This supports the effectiveness of the models in capturing the autocorrelation present in the original data, validating their performance.

Table 5: Diagnostic Tests for Standardized Residuals

		Cor		**Cov**	
		Q-Stat	P-value	Q-Stat	P-value
Norm	VECH	151.80	0.15	150.42	0.17
	BEKK	154.36	0.12	153.38	0.13
	CCC	147.96	0.21	147.90	0.21
T	VECH	145.87	0.25	144.86	0.27
	BEKK	151.85	0.15	151.13	0.16
	CCC	141.67	0.33	141.71	0.33

Note: The table is the result of the Ljung-Box test with lags of 15. Norm is the normal distribution, and t is the student t distribution. Cor and Cov denote the square root of the correlation and covariance matrix of residuals. By using the standardized residuals of the square root of correlation and covariance, we can adjust for autocorrelation in the residuals.

4.2 Optimal Portfolio Weights

After using the multivariate GARCH model, we get the time-varying conditional covariance matrix Q_t and correlation matrix R_t with the dimension of $N \times N \times T$. Based on the loglikelihood, AIC, and BIC, we choose the diagonal VECH model as the most suitable model for modeling commodity volatility and dependence. By applying the seven portfolio methods: equal weight, mean-variance, minimum variance, minimum correlation, maximum Sharpe ratio, maximum Sortino ratio, and CVaR, we can obtain the time-varying weight for the 3 commodities to form portfolios. The average optimal portfolio weights of the three commodity groups are shown in Table 6.

Different portfolios are efficient for different investment objectives. The equal weight portfolio serves as the benchmark for all the portfolio strategies. The mean-variance portfolio optimized the weight by maximizing the return and minimizing the risk. When we do not consider short-selling opportunities, the result of the optimal weight is mainly focusing on allocating weights to metal and energy commodities, and this portfolio strategy has the best performance compared to others under the sample period from 1991 to 2023. With a focus on reducing the risk, the minimum variance portfolio allocates more weight to agricultural products. Also, considering the diversified portfolio, the min correlation portfolio approximately allocated equal weight to the three commodity groups. The performance of this portfolio strategy is similar to the min variance portfolio strategy. In contrast, other portfolio strategies of maximum Sharpe ratio and maximum Sortino ratio consider the risk-adjusted factor with sortino ratio focusing on the downside risk. Their results show that investors should allocate all the weight to energy commodities. However, these two portfolio strategies have higher variance and lower CVaR. In order to manage the risk under extreme scenarios, the mean-CVaR focuses on tail risks. This portfolio allocate the majority of the weight to energy commodities, which suggests that energy commodities might have relatively lower tail risk compared to metal and agricultural commodities.

When the investment objective is achieving higher returns while bearing higher risk, we consider the short-selling opportunity. As shown in Table 6, the mean-variance, Sharpe ratio, Sortino ratio, and mean-CVaR portfolios exit short-selling weights. Also, these strategies are all short-sell agricultural commodities. This suggests that agricultural commodities are underperformed compared to metals and energy commodities. In such cases, short-selling agricultural commodities allows the investor to benefit from the anticipated decline. Additionally, short-sell agricultural commodities can also be attributed to the seasonally predictable volatilities for the agricultural products. Regarding the portfolio performance of the short selling conditions, portfolio strategies of mean-variance and maximum Sharpe ratio archive higher return and volatility compared to no short selling opportunity conditions. However, the performance of short-selling maximum sortino ratio and mean-CVaR ratio is generally worse than conditions without short-selling. This is because short selling can increase volatility and the chance of significant losses, which makes the portfolios more risky.

Table 6: Average Weights of Portfolios

	Equal Weight	Mean-Variance	Min Variance	Min Correlation	Sharpe Ratio	Sortino Ratio	Mean-CVaR
				No short Selling			
Weights							
Metal	33%	58%	37%	32%	0%	0%	17%
Energy	33%	40%	10%	35%	100%	100%	71%
Agricultural	33%	2%	53%	33%	0%	0%	11%
Performance							
Return	0.01%	0.02%	0.01%	0.01%	0.01%	-0.32%	-0.06%
Variance	0.01%	0.01%	0.01%	0.01%	0.04%	0.04%	0.04%
Sharpe Ratio	-0.01	0.00	-0.01	-0.01	0.00	-0.17	-0.04
CVaR	-0.02	-0.02	-0.02	-0.02	-0.05	-0.05	-0.05
				Short Selling			
Weights							
Metal	33%	96%	37%	32%	100%	100%	30%
Energy	33%	51%	10%	35%	100%	100%	85%
Agricultural	33%	-46%	53%	33%	-100%	-100%	-15%
Performance							
Return	0.01%	0.03%	0.01%	0.01%	0.03%	-0.33%	-0.10%
Variance	0.01%	0.02%	0.01%	0.01%	0.07%	0.06%	0.06%
Sharpe Ratio	-0.01	0.01	-0.01	-0.01	0.00	-0.14	-0.05
CVaR	-0.02	-0.03	-0.02	-0.02	-0.06	-0.06	-0.06

Note: The optimization process uses the conditional covariance and correlation of the diagonal VECH model with the student t distribution. The significance level for calculating CVaR is 5%.

4.3 Portfolio Performance

In order to analyze how different portfolio strategies perform under different market times, we choose the rolling window of 250 to get the portfolio performance over time. Since the performance of the seven portfolios under short sell and no short sell conditions generally behaves in similar patterns, we draw the plot of the performance without short-selling opportunity as the representation to see how portfolios behave under different market conditions. As shown in Figure

2, during the 1990s and the 2000s, there were significant fluctuations in the returns, which might be due to the early 1990s recessions and the Dot-com bubble at that time. Around 2008, there is a significant decline across all the portfolio strategies, which was caused by the Global Financial Crisis. Additionally, we can see that there are spikes in the return, variance, and Sharpe ratio around the 2020s. This is because, during the COVID-19 volatile times, investors have increased interest in the commodity market to hedge against inflation and economic instability. As a result, the higher returns in the commodity market are also associated with higher variance and extreme risks. Regarding the portfolio strategies, the mean-variance and min-variance portfolios generally show a stable behavior compared to other portfolio strategies. In contrast, the maximum Sharpe ratio, maximum Sortino ratio, and mean-CVaR portfolios exhibit higher sensitivity to extreme market conditions.

Figure 2: Portfolio Performance without Short-selling

4.3 Hedge against Risk

We use conditional volatility from the multivariate GARCH model of the three commodity groups to calculate hedge ratios. The hedge ratios are a measure of how much hedging asset is required to offset the risk of the primary asset, where the hedging asset holds a short position and the primary asset holds a long position. Figure 2 shows the hedge ratios for different commodity pairs from 1991 to 2023, with metal, energy, and agricultural commodities as the primary assets in each subplot. With metal commodities as the primary commodities, the hedge ratios show occasional fluctuations, which might be due to big events such as the early 1990s recession, the global financial crisis, and COVID-19. Also, the high hedge ratios imply a high correlation and similar volatility between the two commodities. Hedging ratios for energy commodities display the most significant fluctuations compared with the other two commodity groups. This implies that the energy market has higher risk and uncertainty. For hedging agricultural commodities, the hedge ratios remain relatively stable during the entire sample period. This indicates that the relationships between agricultural commodities with metal and energy are less volatile, indicating a lower correlation. Overall, the plot shows that the hedge ratios for metal and agricultural commodities are more stable compared to energy commodities. Stable hedge ratios indicate more predictable interconnectedness, which makes it easier to hedge risks. In contrast, volatile hedge ratios of energy commodities suggest that strategies for hedging should be more flexible and should be changing according to specific market situations.

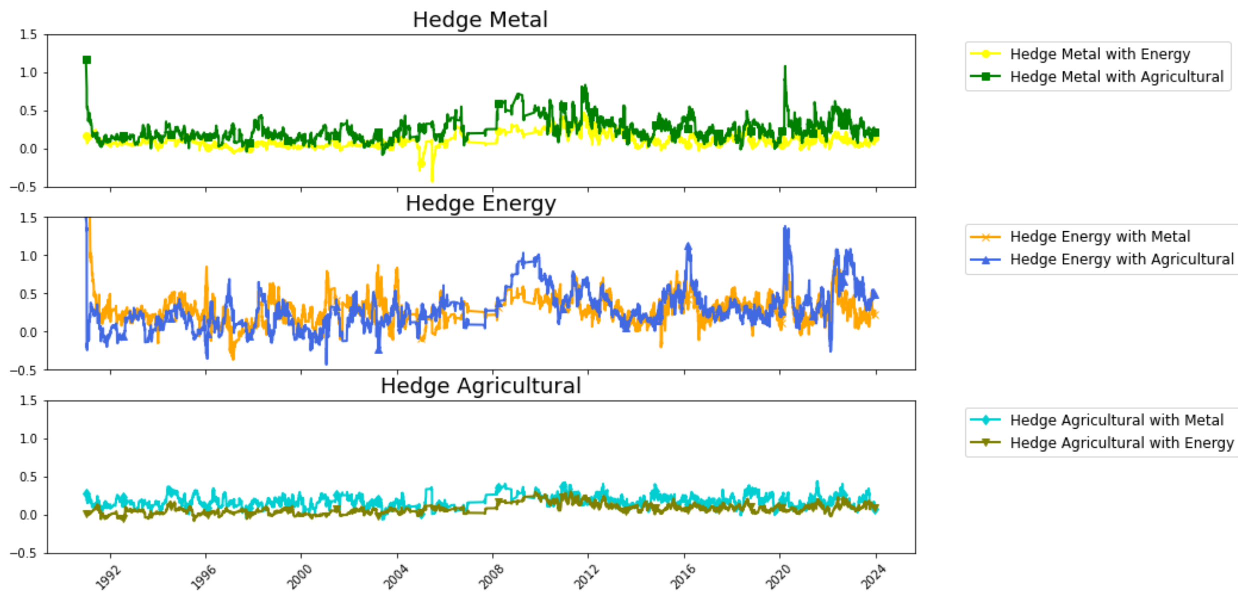

Figure 3: Hedge Ratios

5. Conclusion

Focusing on the commodity market, this thesis analyzes the dynamics of metal, energy, and agricultural commodity groups. We begin by exploring the statistical characteristics of the three commodity groups. All three commodity groups exhibit slightly negative skewness, indicating asymmetry. Moreover, metals and energy display high kurtosis, suggesting heavy-tailed distributions, while agricultural commodities demonstrate lighter tails and approximate normal distribution. Additionally, the statistical tests reveal that the commodity return data is stationary with autocorrelation and heteroscedasticity. Secondly, We use multivariate GARCH models of diagonal VECH, diagonal BEKK, and CCC to model the volatility of the three commodity groups. The empirical results reveal that the student t-distribution assumption of multivariate GARCH models outperforms the normal distribution assumption. Also, the diagonal VECH model with t-distribution has the highest goodness of fit.

By using the conditional variance and conditional correlation from the multivariate GARCH model, we do portfolio optimization of seven portfolios, which are the mean-variance, minimum variance, minimum correlation, maximum Sharpe ratio, maximum Sortinoz ratio, and mean CVaR portfolio strategies. We consider two conditions, which are the short selling condition and no short selling condition. If we do not consider short-selling opportunities, the mean-variance portfolio has the highest average return during the sample period and the lowest volatility, indicating its efficiency as an investment strategy compared to other portfolio strategies under the sample period. This portfolio approach allocates the majority of the proportion to metals and only a relatively smaller amount of proportion to energy commodities. In contrast, the mean-CVaR portfolio has the lowest average return, Sharpe ratio, and CVaR during the sample period, indicating inefficiencies during the sample period. When considering short-selling opportunities, the portfolio strategies of mean-variance, maximum Sharpe ratio, maximum Sortino ratio, and mean-CVaR all include short positions, particularly in agricultural commodities. Also, the average return performance of these short-selling strategies, notably mean-variance portfolio and maximum Sharpe ratio, outperform those of methods without short-selling opportunities. Conversely, the maximum Sortino ratio and mean CVaR strategies are less efficient compared to conditions without short-selling opportunities. Overall, short-selling increases variance across all the portfolio strategies.

The commodity market is inherently volatile due to the fluctuating nature of physical commodity products over time. Therefore, implementing hedging strategies to reduce risks is an efficient approach for conservative investors. We calculate hedge ratios using conditional volatility from the multivariate GARCH model. The hedge ratio results indicate that the energy commodity group exhibits higher and more volatile hedge ratios, reflecting its strong correlation and volatility with metal and agricultural commodities. In contrast, the agricultural commodity group has stable hedge ratios, which shows that it has lower volatility and more predictable relationships with other asset classes. This stability suggests that agricultural commodities are less affected by extreme market events compared to energy commodities, making it easier to hedge effectively.

Appendix

Table 7: Digonal VECH

	Norm			**T**		
	Coef	z-Stat	P-value	Coef	z-Stat	P-value
Mean Equation						
C(1)	0.00	0.86	0.39	0.00	2.44	0.01
C(2)	0.00	0.03	0.97	0.00	-0.02	0.98
C(3)	0.00	1.45	0.15	0.00	2.55	0.01
C(4)	-0.02	-2.02	0.04	-0.02	-1.31	0.19
C(5)	0.00	0.13	0.90	0.00	1.60	0.11
C(6)	0.04	3.22	0.00	0.03	2.47	0.01
Variance Equation						
M(1,1)	0.00	7.65	0.00	0.00	7.49	0.00
M(1,2)	0.00	3.63	0.00	0.00	3.41	0.00
M(1,3)	0.00	3.25	0.00	0.00	3.24	0.00
M(2,2)	0.00	8.31	0.00	0.00	5.77	0.00
M(2,3)	0.00	2.42	0.02	0.00	2.34	0.02
M(3,3)	0.00	7.63	0.00	0.00	4.91	0.00
A1(1,1)	0.03	33.10	0.00	0.06	12.17	0.00
A1(1,2)	0.02	8.80	0.00	0.02	5.44	0.00
A1(1,3)	0.02	6.75	0.00	0.02	4.91	0.00
A1(2,2)	0.07	21.98	0.00	0.07	12.10	0.00
A1(2,3)	0.02	5.54	0.00	0.02	4.50	0.00
A1(3,3)	0.04	11.74	0.00	0.03	7.12	0.00
B1(1,1)	0.97	1721.36	0.00	0.92	154.10	0.00
B1(1,2)	0.96	326.64	0.00	0.95	129.10	0.00
B1(1,3)	0.97	262.50	0.00	0.95	100.32	0.00
B1(2,2)	0.92	283.73	0.00	0.92	154.66	0.00
B1(2,3)	0.97	232.89	0.00	0.97	152.00	0.00
B1(3,3)	0.94	188.95	0.00	0.94	110.27	0.00

Table 8: Digonal BEKK

	Norm			T		
	Coef	z-Stat	P-value	Coef	z-Stat	P-value
Mean Equation						
C(1)	0.00	0.81	0.42	0.00	2.46	0.01
C(2)	0.00	-0.06	0.95	-0.01	-0.50	0.62
C(3)	0.00	1.38	0.17	0.00	2.45	0.01
C(4)	-0.02	-1.91	0.06	-0.02	-1.38	0.17
C(5)	0.00	-0.04	0.97	0.00	1.60	0.11
C(6)	0.04	3.27	0.00	0.03	2.45	0.01
Variance Equation						
M(1,1)	0.00	8.61	0.00	0.00	7.30	0.00
M(1,2)	0.00	3.47	0.00	0.00	2.78	0.01
M(1,3)	0.00	4.28	0.00	0.00	3.93	0.00
M(2,2)	0.00	8.52	0.00	0.00	6.44	0.00
M(2,3)	0.00	3.46	0.00	0.00	3.45	0.00
M(3,3)	0.00	7.55	0.00	0.00	4.91	0.00
A1(1,1)	0.16	67.16	0.00	0.17	25.41	0.00
A1(2,2)	0.25	45.60	0.00	0.25	24.67	0.00
A1(3,3)	0.16	22.67	0.00	0.15	14.33	0.00
B1(1,1)	0.99	3671.72	0.00	0.98	642.66	0.00
B1(2,2)	0.96	596.12	0.00	0.96	345.57	0.00
B1(3,3)	0.98	426.52	0.00	0.98	260.19	0.00

Table 9: CCC

	Norm			**T**		
	Coef	z-Stat	P-value	Coef	z-Stat	P-value
Mean Equation						
C(1)	0.00	0.75	0.46	0.00	2.41	0.02
C(2)	0.00	0.02	0.98	0.00	0.19	0.85
C(3)	0.00	1.57	0.12	0.00	2.64	0.01
C(4)	-0.02	-1.36	0.17	-0.01	-1.01	0.31
C(5)	0.00	-0.03	0.98	0.00	1.49	0.14
C(6)	0.04	3.40	0.00	0.03	2.53	0.01
Variance Equation						
M(1)	0.00	7.56	0.00	0.00	7.52	0.00
A1(1)	0.03	32.83	0.00	0.07	11.79	0.00
B1(1)	0.97	1690.00	0.00	0.90	119.71	0.00
M(2)	0.00	8.34	0.00	0.00	5.64	0.00
A1(2)	0.08	20.51	0.00	0.07	11.65	0.00
B1(2)	0.91	212.78	0.00	0.91	134.97	0.00
M(3)	0.00	7.14	0.00	0.00	4.66	0.00
A1(3)	0.04	11.50	0.00	0.04	6.84	0.00
B1(3)	0.94	161.04	0.00	0.94	97.47	0.00
R(1,2)	0.18	16.11	0.00	0.18	13.31	0.00
R(1,3)	0.22	19.53	0.00	0.22	16.19	0.00
R(2,3)	0.16	13.65	0.00	0.16	11.66	0.00

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