Midterm 2022

1a) Ang. mom. conserved (no external torque or change in mass)

Lin. mom. conserved (no external force or change in mass)

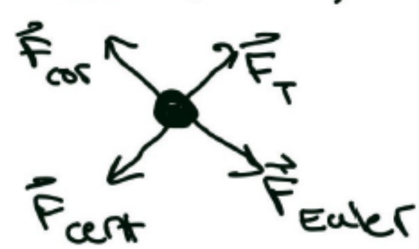
Energy not conserved (reeling in $\Rightarrow$ $dE/dt \neq 0$)

b.) Body fixed frame is rotating

Euler : $-m\frac{d\vec{\omega}}{dt}\times\vec{r} \Rightarrow -(\hat{k}\times\hat{r}) = -\hat{\theta}$

Coriolis : $-2m\vec{\omega}\times\dot{\vec{r}} \Rightarrow -(\hat{k}\times-(\hat{r})) = \hat{\theta}$

centrifugal : $-m\vec{\omega}\times(\vec{\omega}\times\vec{r}) \Rightarrow -\hat{k}\times(\hat{k}\times\hat{r}) = \hat{r}$



c) 

$$\vec{R} = \frac{1}{M}\sum_i m_i \vec{r}_i = \frac{3l_0}{4}$$

$$\vec{H} = \vec{r} \times \vec{p} = mr^2\omega$$

$$H_i = \left[m\left(\frac{3}{4}l_0\right)^2 + 3m\left(\frac{1}{4}l_0\right)^2\right]\omega_0 = \frac{3}{4}ml_0^2\omega_0$$

After reeling $l = l_0/3$

$$H_f = \frac{3m}{4}\left(\frac{l_0}{3}\right)^2\omega = \frac{1}{12}ml_0^2\omega$$

$$H_i = H_f \rightarrow \frac{3}{4}ml_0^2\omega_0 = \frac{1}{12}ml_0^2\omega$$

$$\boxed{\omega = 9\omega_0}$$

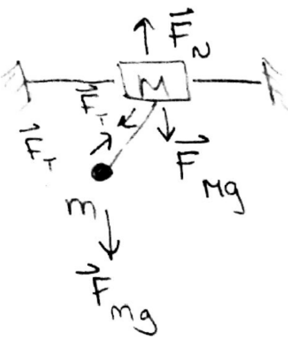
d.) String tension is $F = mr\omega_f^2$

$$\vec{R} = \frac{1}{4m}3ml = \frac{3}{4}l = \frac{l_0}{4}$$

$$F = m\left(\frac{l_0}{4}\right)\omega^2 = m\left(\frac{l_0}{4}\right)(9\omega_0)^2$$

$$\boxed{F = \frac{81}{4}ml_0\omega_0^2}$$

a)



b.) $x, \theta \rightarrow 2$ DOF

c.) $y = 0$; $l = \text{const}$ $\Big\}$ holonomic

d.) $\mathcal{L} = T - V$

$V = -mgl\cos\theta$ (no V for M)

$T = \frac{1}{2}Mv_M^2 + \frac{1}{2}mv_m^2$

we can use Cartesian coordinates since M moves only in x direction

$\vec{r}_M = x\hat{x}$ so $v_M^2 = \dot{x}^2$

$\vec{r}_m = (x + l\sin\theta)\hat{x} + (-l\cos\theta)\hat{y}$

$\vec{v} = (\dot{x} + l\dot{\theta}\cos\theta)\hat{x} - l\dot{\theta}\sin\theta\,\hat{y}$

$v_m^2 = \dot{x}^2 + l^2\dot{\theta}^2\cos^2\theta + 2l\dot{x}\dot{\theta}\cos\theta + l^2\dot{\theta}^2\sin^2\theta$

$= \dot{x}^2 + l^2\dot{\theta}^2 + 2l\dot{x}\dot{\theta}\cos\theta$

$$\boxed{\mathcal{L} = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + l^2\dot{\theta}^2 + 2l\dot{x}\dot{\theta}\cos\theta) + mgl\cos\theta}$$

e.) $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} = 0$

$$\frac{d}{dt}\left(M\dot{x} + m\dot{x} + ml\dot{\theta}\cos\theta\right) = 0$$

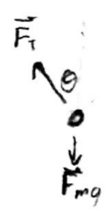
$$\boxed{M\ddot{x} + m\left(\ddot{x} + l\ddot{\theta}\cos\theta - l\dot{\theta}^2\sin\theta\right) = 0}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} = 0$$

$$\frac{d}{dt}\left(ml^2\dot{\theta} + ml\dot{x}\cos\theta\right) - \left(ml\dot{x}\dot{\theta}(-\sin\theta) - mgl\sin\theta\right) = 0$$

$$ml^2\ddot{\theta} + ml\ddot{x}\cos\theta - ml\dot{x}\dot{\theta}\sin\theta + ml\dot{x}\dot{\theta}\sin\theta + mgl\sin\theta =$$

$$\boxed{l\ddot{\theta} + \ddot{x}\cos\theta + g\sin\theta = 0}$$



$$F_r = mg\cos\theta - F_T$$

$$F_\theta = -mg\sin\theta$$

$$\vec{r} = l\hat{r}$$

$$\dot{\vec{r}} = l\dot{\hat{r}} = l\dot{\theta}\hat{\theta}$$

$$\ddot{\vec{r}} = l\ddot{\theta}\hat{\theta} + l\dot{\theta}\dot{\hat{\theta}} = l\ddot{\theta}\hat{\theta} - l\dot{\theta}^2\hat{r}$$

$$-ml\dot{\theta}^2 = mg\cos\theta - F_T$$

$$ml\ddot{\theta} = -mg\sin\theta \Rightarrow l\ddot{\theta} + g\sin\theta = 0$$

(if linearized, same as M>>m case)