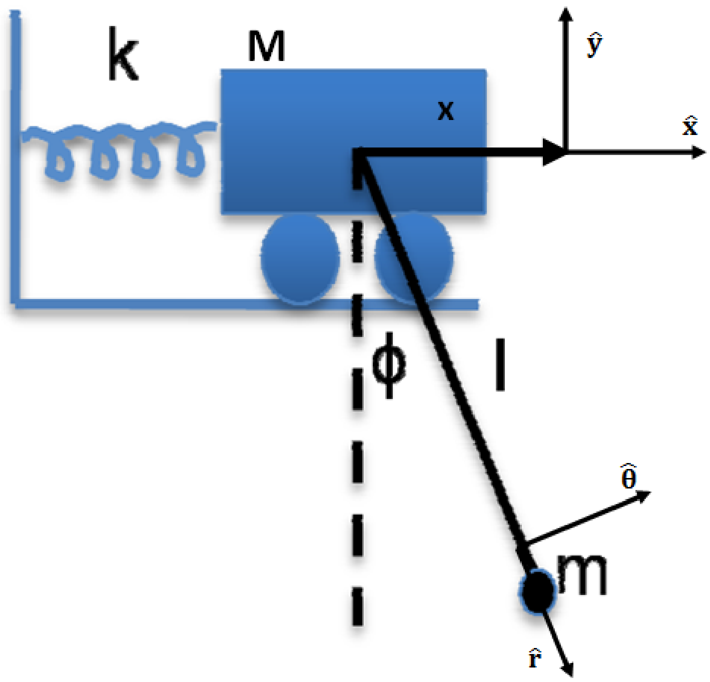
AA 242A Homework 5 Solution

2020.10.23

1 Problem 1



a)Form velocities:

$$\overrightarrow{v}^M = \dot{x}\hat{x}$$
$$\overrightarrow{v}^m = \dot{x}\hat{x} + l\dot{\phi}\hat{\theta}$$

Form K.E.:

$$\begin{aligned} T &= \frac{1}{2}M\overrightarrow{v}^M \cdot \overrightarrow{v}^M + \frac{1}{2}m\overrightarrow{v}^m \cdot \overrightarrow{v}^m \\ &= \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}m(l^2\dot{\phi}^2 + 2l\dot{x}\dot{\phi}\cos\phi) \end{aligned}$$

Form P.E.:

$$V = -mgl\cos\phi + \frac{1}{2}kx^2$$

Form Lagrangian:

$$
\begin{aligned}
L &= T - V \\
&= \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}m(l^2\dot{\phi}^2 + 2l\dot{x}\dot{\phi}\cos\phi) + mgl\cos\phi - \frac{1}{2}kx^2
\end{aligned}
$$

There is no external forces acting on the system except for gravity and the spring, which are both conservative. Hence $Q_x = Q_p = 0$.

Lagrangian math:

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) &= \frac{d}{dt}\Big((M+m)\dot{x} + ml\cos\phi\dot{\phi}\Big) \\
&= (M+m)\ddot{x} + ml\ddot{\phi}\cos\phi - ml\dot{\phi}^2\sin\phi \\
\frac{\partial L}{\partial x} &= \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -kx \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) &= \frac{d}{dt}\Big(ml^2\dot{\phi} + ml\cos\phi\dot{x}\Big) \\
&= ml^2\ddot{\phi} + ml\ddot{x}\cos\phi - ml\dot{x}\dot{\phi}\sin\phi \\
\frac{\partial L}{\partial \phi} &= -ml\dot{x}\dot{\phi}\sin\phi - mgl\sin\phi
\end{aligned}
$$

Final equations:

$$
\begin{aligned}
(M+m)\ddot{x} + ml\ddot{\phi}\cos\phi - ml\dot{\phi}^2\sin\phi + kx &= 0 \\
l\ddot{\phi} + \ddot{x}\cos\phi + g\sin\phi &= 0
\end{aligned}
$$

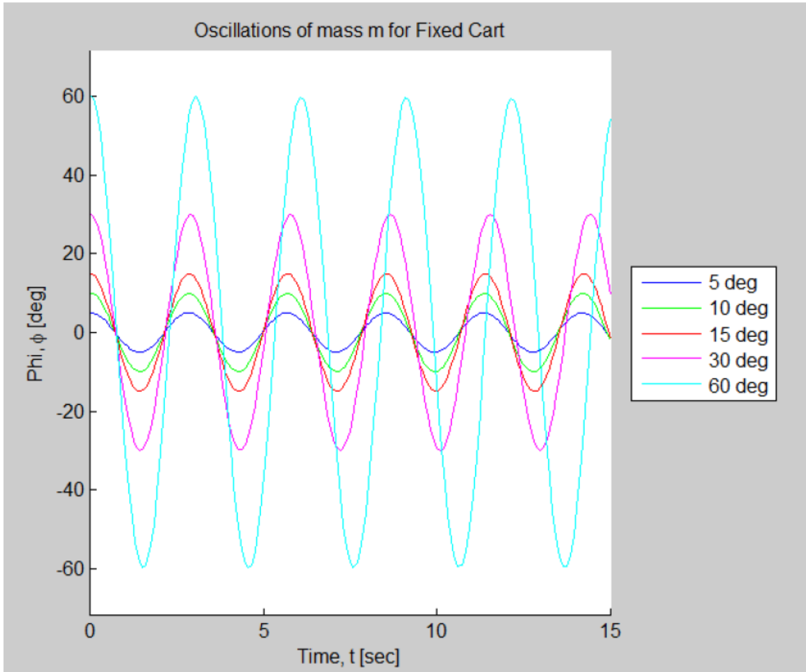
b) Cart fixed at $\dot{x} = \ddot{x} = 0$. Small angles so $\sin\phi \approx \phi$, $\cos\phi \approx 1$
The 2nd EOM then becomes:

$$
\ddot{\phi} + \frac{g}{l}\phi = 0
$$

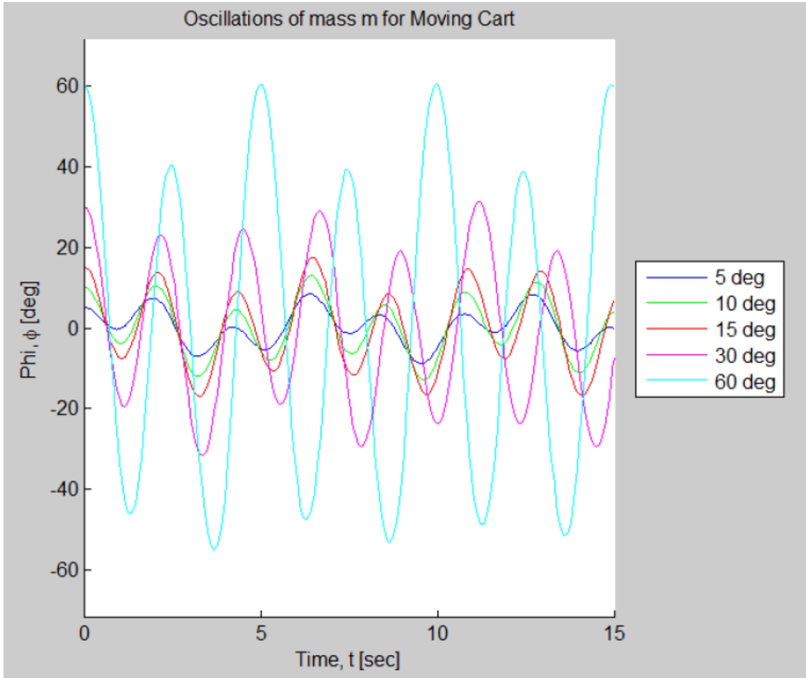
which gives

$$
\omega_n = \sqrt{\frac{g}{l}}
$$

c) As expected, the small angle assumption result holds well enough for 5, 10 degrees and even up to 15 degrees, but starts diverging for 30 degrees and is completely inaccurate for 60 degrees.



d) We no longer have pure oscillation; each oscillation of the pendulum has a different shape and a changing amplitude from one peak to the next. The response is much harder to predict, and would not have been predicted well using a small angle assumption


Oscillations of mass m for Moving Cart
60
40
20
0
-20
-40
-60
Phi, ϕ [deg]
0
5
10
15
Time, t [sec]
5 deg
10 deg
15 deg
30 deg
60 deg

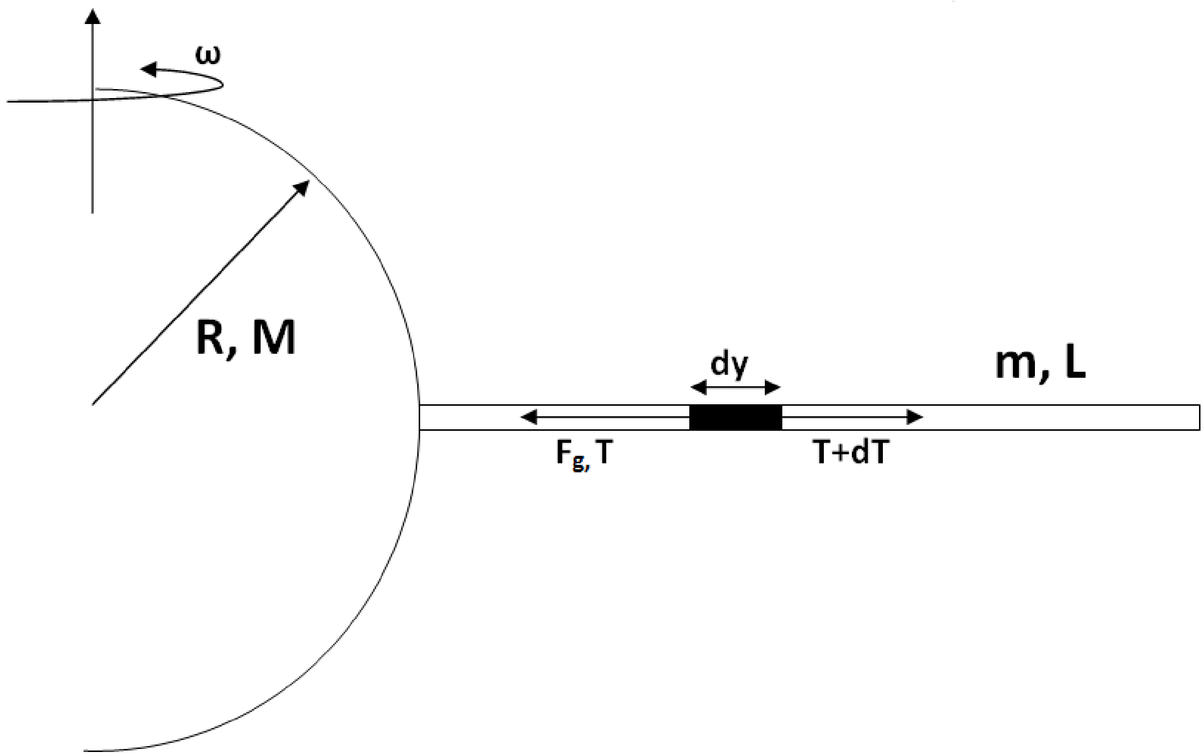
Matlab code:

```
function AA242A_HW4_Prob1
clc; clear all; close all;

t0 = 0; % Start time
tf = 15; % End time
deg2rad = pi/180; % Degrees-to-radians conversion factor

% Homework 4, Problem 1c
phi0 = deg2rad.*[5, 10, 15, 30, 60]; % Initial conditions for phi
phi_dot0 = 0; % Initial conditions for phi_dot
plot_format = ['b', 'g', 'r', 'm', 'c'];
figure; hold on; figure; hold on;

for k = 1:length(phi0)
    y0 = [phi0(k); phi_dot0];
    [t,y] = ode45(@fixed_cart_EOMs, [t0, tf], y0);
    phi = y(:,1);
    figure(1);
    plot(t, phi, plot_format(k));
    figure(2);
    plot(t, phi/phi0(k), plot_format(k));
end

figure(1);
legend('5 deg', '10 deg', '15 deg', '30 deg', '60 deg', 'Location', ...
    'EastOutside');
title('Oscillations of mass m for Fixed Cart');
xlabel('Time, t [sec]');
ylabel('Phi, \phi [deg]');
ylim((180/pi).*[-1.25, 1.25]);
figure(2);
legend('5 deg', '10 deg', '15 deg', '30 deg', '60 deg', 'Location', ...
    'EastOutside');
title('Oscillations of mass m for Fixed Cart [Normalized by \phi(0)]');
xlabel('Time, t [sec]');
ylabel('\phi/\phi_0');
ylim([-1.25, 1.25]);

% Homework 4, Problem 1d
x0 = 0; % Initial conditions for x
x_dot0 = 0.5; % Initial conditions for x_dot
figure; hold on;

for k = 1:length(phi0)
    y0 = [x0; x_dot0; phi0(k); phi_dot0];
    [t,y] = ode45(@moving_cart_EOMs, [t0, tf], y0);
    phi = y(:,3);
    plot(t, phi, plot_format(k));
```

```matlab
end

legend('5 deg', '10 deg', '15 deg', '30 deg', '60 deg', 'Location', ...
    'EastOutside');
title('Oscillations of mass m for Moving Cart');
xlabel('Time, t [sec]');
ylabel('Phi, \phi [deg]');
ylim((180/pi).*[-1.25, 1.25]);
end

function ydot = fixed_cart_EOMs(t,y)
% Define constant parameters
g = 9.81;
l = 2;

% Extract variables from state vector, y
phi = y(1);
phi_dot = y(2);
ydot = zeros(size(y));

% Define state derivatives and store in ydot
ydot(1) = phi_dot;
ydot(2) = -(g/l)*sin(phi);
end

function ydot = moving_cart_EOMs(t,y)
% Define constant parameters
g = 9.81;
m = 1;
l = 2;
k = 5;
M = 2;

% Extract variables from state vector, y
x = y(1);
x_dot = y(2);
phi = y(3);
phi_dot = y(4);
ydot = zeros(size(y));

% Define state derivatives and store in ydot
ydot(1) = x_dot;
ydot(2) = (m*g*cos(phi)*sin(phi) + m*l*sin(phi) - k*x) / (M + ...
    m*(sin(phi)^2));
ydot(3) = phi_dot;
ydot(4) = (-m*sin(phi)*cos(phi)*phi_dot^2 + (cos(phi)/l)*k*x - ...
    (M+m)*(g/l)*sin(phi)) / (M + m*(sin(phi)^2));
end
```

2 Problem 2

a) The diagram below shows the rope over right, in particular an infinitesimal piece of rope labeled **dy**. The forces on it are gravity, F_g, tension towards the Earth, T, and tension away from the Earth, $T + dT$. The reason for the differential tension is due to the changing gravitational field along the rope, so every section of the rope must be balanced.



First let $\mu = \frac{m}{L}$, the linear mass density of the rope. We can then do force balance on the segment of the rope dy:

$$\begin{aligned}
(T + dT) - T - \frac{GMdm}{(R + y)^2} &= a_r dm \\
dT - \frac{GM\mu dy}{(R + y)^2} &= (-(R + y)\omega^2)(\mu dy) \\
dT &= \left(\frac{GM}{(R + y)^2} - (R + y)\omega^2 \right) \mu dy \\
T &= -\frac{GM\mu}{R + y} - \left(Ry + \frac{y^2}{2} \right) \omega^2 \mu + C
\end{aligned}$$

At the free end of the rope, $y = L$, $T = 0$ gives the necessary boundary condition to solve for C:

$$0 = -\frac{GM\mu}{R+L} - \left(RL + \frac{L^2}{2}\right)\omega^2\mu + CC = \frac{GM\mu}{R+L} + \left(RL + \frac{L^2}{2}\right)\omega^2\mu$$

Thus the entire expression is:

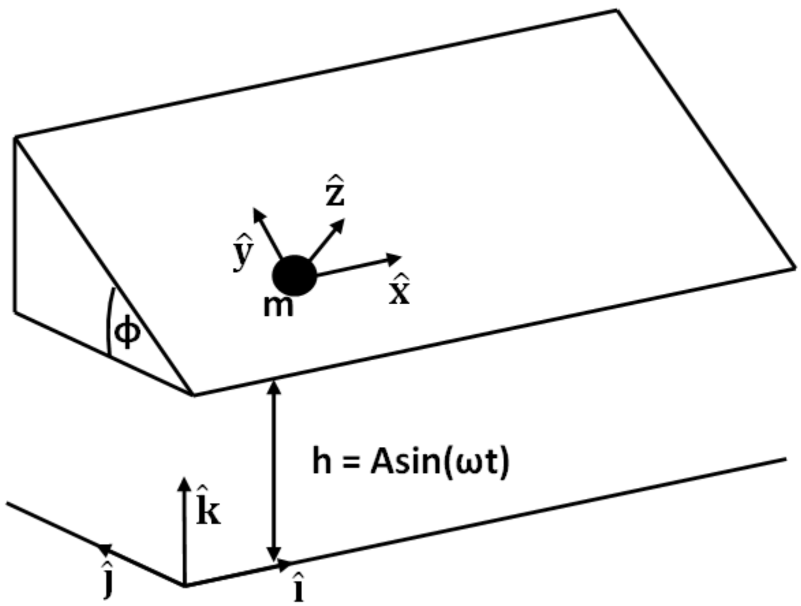
$$T = GM\mu\left(\frac{1}{R+L} - \frac{1}{R+y}\right) + \frac{L-y}{2}\left(2R+L+y\right)\omega^2\mu$$

b) If the rope were to hang freely without an attachment, then $T(0) = 0$.

$$0 = GM\mu\left(\frac{1}{R+L} - \frac{1}{R}\right) + \mu\omega^2\left(RL + \frac{1}{2}L^2\right)$$

$$L = -\frac{3}{2}R + \frac{1}{2}\sqrt{9R^2 - \frac{4}{R\omega^2}(2R^3\omega^2 - 2GM)}$$

3 Problem 3



a) Force balance method

$$\begin{aligned}
\overrightarrow{r} &= h\hat{k} + x\hat{x} + y\hat{y} \\
&= h\hat{k} + x\hat{i} + y(\cos\phi\hat{j} + \sin\phi\hat{k}) \\
&= x\hat{i} + y\cos\phi\hat{j} + (h + y\sin\phi)\hat{k} \\
\overrightarrow{v} &= \dot{x}\hat{i} + \dot{y}\cos\phi\hat{j} + (\dot{h} + \dot{y}\sin\phi)\hat{k} \\
\overrightarrow{a} &= \ddot{x}\hat{i} + \ddot{y}\cos\phi\hat{j} + (\ddot{h} + \ddot{y}\sin\phi)\hat{k}
\end{aligned}$$

We can also identify the forces on the mass:

$$\begin{aligned}
\overrightarrow{F}_{total} &= \overrightarrow{F_g} + \overrightarrow{F_N} \\
&= -mg\hat{k} + N\hat{z} \\
&= -mg\hat{k} + N(-\sin\phi\hat{j} + \cos\phi\hat{k})
\end{aligned}$$

So we balance $F = ma$. First in $\hat{i}$:

$$\begin{aligned}
m\ddot{x} &= 0 \\
\dot{x} &= v_0 \\
x &= x_0 + v_0 t
\end{aligned}$$

Look at $\hat{j}$ direction:

$$
\begin{aligned}
m\ddot{y}\cos\phi &= -N\sin\phi \\
N &= -m\ddot{y}\frac{\cos\phi}{\sin\phi}
\end{aligned}
$$

Look at $\hat{k}$ direction:

$$
\begin{aligned}
m(\ddot{h}+\ddot{y}\sin\phi) &= N\cos\phi - mg \\
m(\ddot{h}+\ddot{y}\sin\phi) &= -m\ddot{y}\frac{\cos^2\phi}{\sin\phi} - mg \\
\ddot{y}\frac{\sin^2\phi+\cos^2\phi}{\sin\phi} + \ddot{h} + g &= 0
\end{aligned}
$$

$$
\begin{aligned}
\ddot{y} &= A\omega^2\sin(\omega t)\sin\phi - g\sin\phi \\
\dot{y} &= -A\omega\cos(\omega t)\sin\phi - gt\sin\phi + C \\
y &= -A\sin(\omega t)\sin\phi - \frac{1}{2}gt^2\sin\phi + Ct + D
\end{aligned}
$$

We can solve for C and D by noting that $\dot{y}(0) = 0$ and $y(0) = y_0$ to get:

$$
\begin{aligned}
&y = -A\sin(\omega t)\sin\phi - \frac{1}{2}gt^2\sin\phi + A\omega t\sin\phi + y_0 \\
&x = x_0 + v_0 t
\end{aligned}
$$

b) Lagrange multipliers

$$
\begin{aligned}
\overrightarrow{r} &= h\hat{k} + x\hat{x} + y\hat{y} + z\hat{z} \\
&= A\sin(\omega t)\hat{k} + x\hat{x} + y\hat{y} + z\hat{z} \\
\overrightarrow{v} &= A\omega\cos(\omega t)\hat{k} + \dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z}
\end{aligned}
$$

We then note that the particle is constrained to move only in the local xy plane, which means:

$$z = 0$$

This is our constraint equation. Going on to form the Lagrangian:

$$
\begin{aligned}
T &= \frac{1}{2}m\overrightarrow{v}\cdot\overrightarrow{v} \\
&= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 + \dot{h}^2 + 2\dot{h}\dot{y}\sin\phi + 2\dot{h}\dot{z}\cos\phi)
\end{aligned}
$$

$$
\begin{aligned}
T &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 + A^2\omega^2\cos^2(\omega t) + 2A\omega\sin\phi\cos(\omega t)\dot{y} + 2A\omega\cos\phi\cos(\omega t)\dot{z}) \\
V &= mg(h + y\sin\phi + z\cos\phi) = mg(A\sin(\omega t) + y\sin\phi + z\cos\phi) \\
L &= T - V
\end{aligned}
$$

Lagrangian math:

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) &= \frac{d}{dt}(m\dot{x}) = m\ddot{x} \\
\frac{\partial L}{\partial x} &= 0 \\
\frac{\partial f}{\partial x} &= 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) &= \frac{d}{dt}(m\dot{y} + mA\omega\sin\phi\cos(\omega t)) = m\ddot{y} - mA\omega^2\sin\phi\sin(\omega t) \\
\frac{\partial L}{\partial y} &= -mg\sin\phi \\
\frac{\partial f}{\partial y} &= 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) &= \frac{d}{dt}(m\dot{z} + mA\omega\cos\phi\cos(\omega t)) = m\ddot{z} - mA\omega^2\cos\phi\sin(\omega t) \\
\frac{\partial L}{\partial z} &= -mg\cos\phi \\
\frac{\partial f}{\partial z} &= 1
\end{aligned}
$$

We have thus the 4 equations:

$$
\begin{aligned}
m\ddot{x} &= 0 && (1) \\
m\ddot{y} - mA\omega^2\sin\phi\sin(\omega t) + mg\sin\phi &= 0 && (2) \\
m\ddot{z} - mA\omega^2\cos\phi\sin(\omega t) + mg\cos\phi + \lambda &= 0 && (3) \\
z &= 0 && (4)
\end{aligned}
$$

We are only concerned with solving for x and y, so we can focus only on (1) and (2). Note that combining (3) and (4) will yield $\lambda = mA\omega^2\cos\phi\cos(\omega t) - mg\cos\phi$, the (negative) normal force! But going on:

$$
\begin{aligned}
m\ddot{x} &= 0 \\
\dot{x} &= v_0 \\
x &= x_0 + v_0 t
\end{aligned}
$$

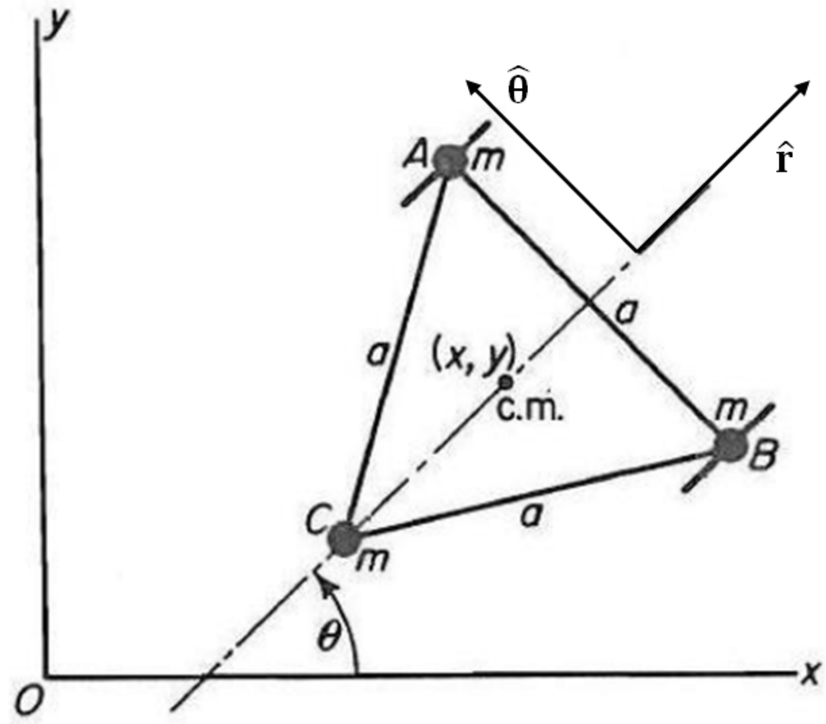
$$
\begin{aligned}
\ddot{y} &= A\omega^2 \sin(\omega t)\sin\phi - g\sin\phi \\
\dot{y} &= -A\omega\cos(\omega t)\sin\phi - gt\sin\phi + C \\
y &= -A\sin(\omega t)\sin\phi - \frac{1}{2}gt^2\sin\phi + Ct + D
\end{aligned}
$$

Again, we can solve for C and D by noting that $\dot{y}(0) = 0$ and $y(0) = y_0$ to get:

$$
\begin{aligned}
&y = -A\sin(\omega t)\sin\phi - \frac{1}{2}gt^2\sin\phi + A\omega t\sin\phi + y_0 \\
&x = x_0 + v_0 t
\end{aligned}
$$

Force balance and Lagrange yield the same answer! This is good.

4 Problem 4



a) First form velocity of the CoM:

$$\overrightarrow{v}^{CoM} = \dot{x}\hat{x} + \dot{y}\hat{y}$$

We can then easily form the kinetic energy, consisting of the translating motion and the rotational motion of the cart:

$$T_{cm} = \frac{1}{2}(3m)(\dot{x}^2 + \dot{y}^2)$$
$$T_{rot} = \frac{1}{2}(3m)(\frac{a}{\sqrt{3}}\dot{\theta})^2$$
$$T = \frac{3}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}ma^2\dot{\theta}^2$$

Since there is only horizontal motion, $V = 0$, and hence our Lagrangian is simply $L = T$. To form the constraint equations, we need to find the velocity of the wheels. In particular let's find the velocity of wheel A:

$$
\begin{aligned}
\overrightarrow{r}^{A/O} &= x\hat{x} + y\hat{y} + (\frac{\sqrt{3}}{6}a\hat{r} + \frac{1}{2}a\hat{\theta}) \\
\overrightarrow{v} &= \dot{x}\hat{x} + \dot{y}\hat{y} + \dot{\theta}\hat{z} \times (\frac{\sqrt{3}}{6}a\hat{r} + \frac{1}{2}a\hat{\theta}) \\
&= \dot{x}\hat{x} + \dot{y}\hat{y} + \frac{\sqrt{3}}{6}a\dot{\theta}\hat{\theta} - \frac{1}{2}a\dot{\theta}\hat{r}
\end{aligned}
$$

Dot product the velocity of A with the $\hat{\theta}$ direction and set to 0, we will get the constraint equation:

$$
\begin{aligned}
f &= \overrightarrow{v}^{A} \cdot \hat{\theta} = 0 \\
&= -\dot{x}\sin\theta + \dot{y}\cos\theta + \frac{\sqrt{3}}{6}a\dot{\theta} = 0 \\
&= \dot{x}\sin\theta - \dot{y}\cos\theta - \frac{\sqrt{3}}{6}a\dot{\theta}
\end{aligned}
$$

Following some Lagrangian math with 3 generalized coordinates: x, y, and θ and taking $\frac{\partial f}{\partial \dot{q}}$, we get:

$$
\begin{aligned}
x &: 3m\ddot{x} + \mu\sin\theta = 0 \\
y &: 3m\ddot{y} - \mu\cos\theta = 0 \\
\theta &: ma^2\ddot{\theta} - \mu\frac{\sqrt{3}}{6}a = 0
\end{aligned}
$$

b) First let us work with the 3 equations of motion we got from the Lagrangian such that we substitute out μ:

$$
\begin{aligned}
\frac{\sqrt{3}}{6}a\mu &= ma^2\ddot{\theta} \\
\mu &= 2\sqrt{3}ma\ddot{\theta} \\
3m\ddot{x} &= -\mu\sin\theta \\
\ddot{x} &= -\frac{2\sqrt{3}}{3}a\ddot{\theta}\sin\theta \\
3m\ddot{y} &= \mu\cos\theta \\
\ddot{y} &= \frac{2\sqrt{3}}{3}a\ddot{\theta}\cos\theta
\end{aligned}
$$

To find the equation for $\ddot{\theta}$, we differentiate the constraint equation twice:

$$(\dot{x}\cos\theta + \dot{y}\sin\theta)\dot{\theta} + \ddot{x}\sin\theta - \ddot{y}\cos\theta - \frac{\sqrt{3}}{6}a\ddot{\theta} = 0$$

Plugging in the $\ddot{x}$ and $\ddot{y}$ EOMs and rearranging yields:

$$\ddot{\theta} = \frac{2\sqrt{3}}{5a}(\dot{x}\cos\theta + \dot{y}\sin\theta)\dot{\theta}$$