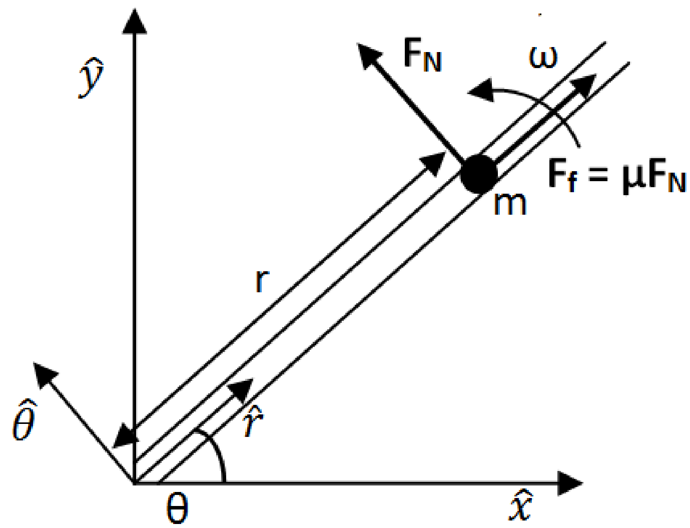
# AA 242A Homework 3 Solution

# 1 Problem 1

Diagram of the mass in the tube, spinning with angular velocity $\omega$. The normal force, $F_n$ and frictional force, $F_f$ are shown as well.




Let us first form the acceleration of the particle $m$:

$$\begin{aligned}
\vec{p} &= r\hat{r} \\
\dot{\vec{p}} &= \vec{v} \\
&= \dot{r}\hat{r} + r\omega\hat{\theta} \\
\ddot{\vec{p}} &= \vec{a} \\
&= (\ddot{r} - r\omega^2)\hat{r} + (2\dot{r}\omega + r\dot{\omega})\hat{\theta} \\
&= (\ddot{r} - r\omega^2)\hat{r} + 2\dot{r}\omega\hat{\theta}
\end{aligned}$$

We can then equate the forces with the acceleration. Let's first look at forces in the $\hat{\theta}$ direction:

$$F_n = 2m\dot{r}\omega$$

We recognize that since $F_n$ is always positive, then so $\dot{r}$ must also always be positive. We can then go on to look at forces in the $\hat{r}$ direction (note that friction always acts opposite to the direction of velocity, hence the $sgn(\dot{r})$ in the equation):

$$\begin{aligned} F_f &= (\ddot{r} - r\omega^2)m \\ -\mu F_n sgn(\dot{r}) &= (\ddot{r} - r\omega^2)m \\ -2m\mu\dot{r}\omega &= (\ddot{r} - r\omega^2)m \\ \ddot{r} + 2\mu\omega\dot{r} - \omega^2 r = 0 & \end{aligned}$$

The above second order, linear, homogeneous, ODE has solutions of the form $e^{\lambda t}$. The characteristic equation is:

$$\lambda^2 + 2\mu\omega\lambda - \omega^2 = 0$$

with roots $\lambda_{1,2} = (-\mu \pm \sqrt{1+\mu^2})\omega$

Hence, we have the general solution:

$$r = Ae^{(-\mu+\sqrt{1+\mu^2})\omega t} + Be^{(-\mu-\sqrt{1+\mu^2})\omega t}$$

However, we are given initial conditions:

$$\begin{aligned} r(0) &= R \\ \dot{r}(0) &= 0 \end{aligned}$$

which we can use to solve for $A$ and $B$:

$$\begin{aligned} A &= \frac{1}{2}R\left(1 + \frac{\mu}{\sqrt{1+\mu^2}}\right) \\ B &= \frac{1}{2}R\left(1 - \frac{\mu}{\sqrt{1-\mu^2}}\right) \end{aligned}$$

And so we have:

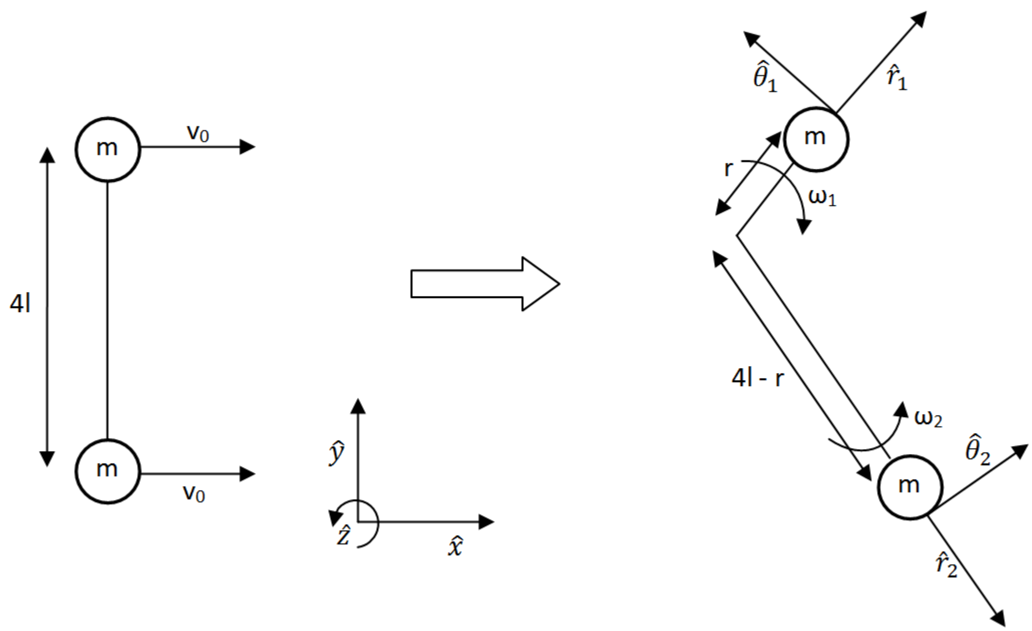
$$r(t) = \frac{1}{2}Re^{-\mu\omega t}\left[\left(1 + \frac{\mu}{\sqrt{1+\mu^2}}\right)e^{\sqrt{1+\mu^2}\omega t} + \left(1 - \frac{\mu}{\sqrt{1+\mu^2}}\right)e^{-\sqrt{1+\mu^2}\omega t})\right]$$

Alternate form is:

$$r(t) = Re^{-\mu\omega t}\left(\cosh(\sqrt{1+\mu^2}\omega t) + \frac{\mu}{\sqrt{1+\mu^2}}\sinh(\sqrt{1+\mu^2}\omega t)\right)$$

## 2 Problem 2

The figure shows the initial state and then the whirling state of the two masses. Reference frames in the second case are attached to the rope so that they rotate with their respective masses.




Let us first form the acceleration of the masses.

$$
\begin{aligned}
\vec{p_1} &= r\hat{r}_1 \\
\dot{\vec{p_1}} &= \dot{r}\hat{r}_1 - r\omega_1\hat{\theta}_1 \\
\ddot{\vec{p_1}} &= (\ddot{r} - r\omega_1^2)\hat{r}_1 - (2\dot{r}\omega_1 + r\dot{\omega}_1)\hat{\theta}_1 \\
\vec{p_2} &= (4l - r)\hat{r}_2 \\
\dot{\vec{p_2}} &= -\dot{r}\hat{r}_2 + (4l - r)\omega_2\hat{\theta}_2 \\
\ddot{\vec{p_2}} &= (-\ddot{r} - (4l - r)\omega_2^2)\hat{r}_2 + (-2\dot{r}\omega_2 + (4l - r)\dot{\omega}_2)\hat{\theta}_2
\end{aligned}
$$

**a)** Conservation of linear momentum does not apply to this question since the pin is exerting a force on the entire system, and so it is treated as an external force. We can, however, use the conservation of angular momentum. Particularly the angular momentum of each mass about the pin is conserved:

$$\vec{H}_1 = -lmv_0\hat{z} = r\hat{r}_1 \times m(-r\omega_1)\hat{\theta}_1$$
$$\omega_1 = \frac{lv_0}{r^2}$$
$$\vec{H}_2 = 3lmv_0\hat{z} = (4l-r)\hat{r}_2 \times m((4l-r)\omega_2)\hat{\theta}_2$$
$$\omega_2 = \frac{3lv_0}{(4l-r)^2}$$

When mass 1 is the farthest away from the pin, $\dot{r} = 0$. However, be careful as $\ddot{r} \neq 0$. Continuing on, kinetic energy is conserved for the entire system:

$$2\left(\frac{1}{2}mv_0^2\right) = \frac{1}{2}m(r\omega_1)^2 + \frac{1}{2}m((4l-r)\omega_2)^2$$
$$mv_0^2 = \frac{1}{2}mr^2\left(\frac{lv_0}{r^2}\right)^2 + \frac{1}{2}m(4l-r)^2\left(\frac{3lv_0}{(4l-r)^2}\right)^2$$
$$r^2(4l-r)^2 = \frac{1}{2}l^2(4l-r)^2 + \frac{9}{2}l^2r^2$$
$$2r^4 - 16lr^3 + 32r^2l^2 = l^2r^2 - 8l^3r + 16l^4 + 9l^2r^2$$
$$0 = r^4 - 8lr^3 + 11r^2l^2 + 4l^3r - 8l^4$$

If we solve the above 4th order, the only physically meaningful root is:

$$r = 1.653l$$

**b)** *Note that this is vestigial from a previous iteration of this homework. Its inclusion is for instructive purposes only.* To solve for the minimum tension, we need to go back to our acceleration formulas and use force balance:

$$-T = m(\ddot{r} - r\omega_1^2)$$
$$-T = m(-\ddot{r} - (4l-r)\omega_2^2)$$

Using the conservation equations with the above equations, we can substitute out $\ddot{r}$ and eventually get:

$$-T = -m\left(-\frac{T}{m} + r\omega_1^2\right) - (4l-r)m\omega_2^2$$
$$-T = T - mr\left(\frac{lv_0}{r^2}\right)^2 - (4l-r)m\left(\frac{3lv_0}{(4l-r)^2}\right)^2$$

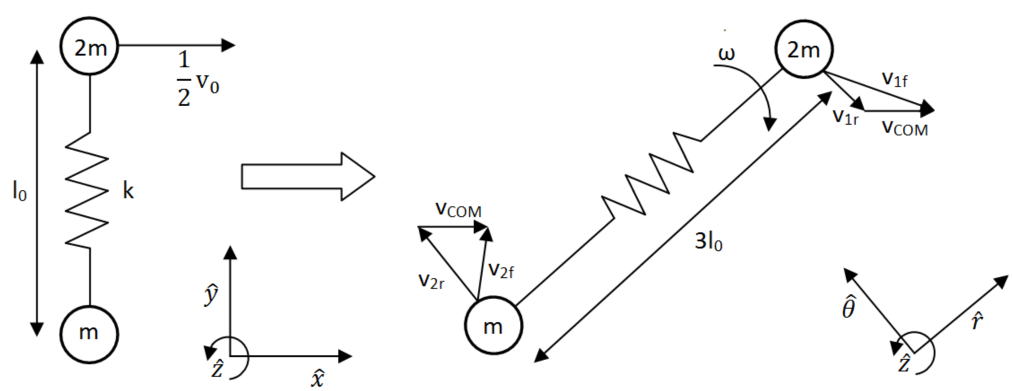
$$T = \frac{ml^2v_0^2}{2}\left(\frac{1}{r^3} + \frac{9}{(4l-r)^3}\right)$$

To solve the above equation, you can either graph it, or take the derivative and find when it is equal to 0. This occurs when $r = 1.464l$. So it follows that the final result is:

$$T_{min} = 0.4353\frac{mv_0^2}{l}$$

## 3 Problem 3

The diagram below shows the two points in time: the first being right after the impact and the second when the spring is stretched to its maximum length. Note that $v_{1f}$ and $v_{2f}$ are both composed of two vectors, $v_{1r}$ and $v_{COM}$ in the first case, and $v_{2r}$ and $v_{COM}$ in the second. This is due to the fact that the velocity of masses with respect to a stationary observer is the combination of the velocity of the center of mass with the rotational velocity of each respective mass. Also note that there is no velocity component along the direction of the spring, since at maximum length $\dot{l}$ is 0.




An important realization that must be made is that the center of the mass of the whole system is travelling at a constant $\frac{1}{3}v_0\hat{x}$. This can be calculated as follows:

$$\begin{aligned} v_{COM} &= \frac{mv_0 + m(0) + m(0)}{m + m + m} \\ &= \frac{1}{3}v_0 \end{aligned}$$

Another important realization is that the COM does not vary in the $\hat{y}$ direction. This means that the COM must also be located $l_0$ from $2m$ and $2l_0$ from $m$ in the case where spring is streched to a length of $3l_0$.

First of all, to find the initial velocity of $2m$ right after the collision, use conservation of linear momentum on the top two particles:

$$\begin{aligned} mv_0 + m(0) &= 2mv \\ v &= \frac{1}{2}v_0 \end{aligned}$$

Let's now look at the conservation of angular momentum. To make calculations simpler, let's take the angular momentum about the **moving** COM for the first instance:

$$\frac{1}{3}l_0\hat{y} \times 2m\left(\frac{1}{2}v_0 - \frac{1}{3}v_0\right)\hat{x} - \frac{2}{3}l_0\hat{y} \times m\left(0 - \frac{1}{3}v_0\right)\hat{x} = -\frac{1}{3}ml_0v_0\hat{z}$$

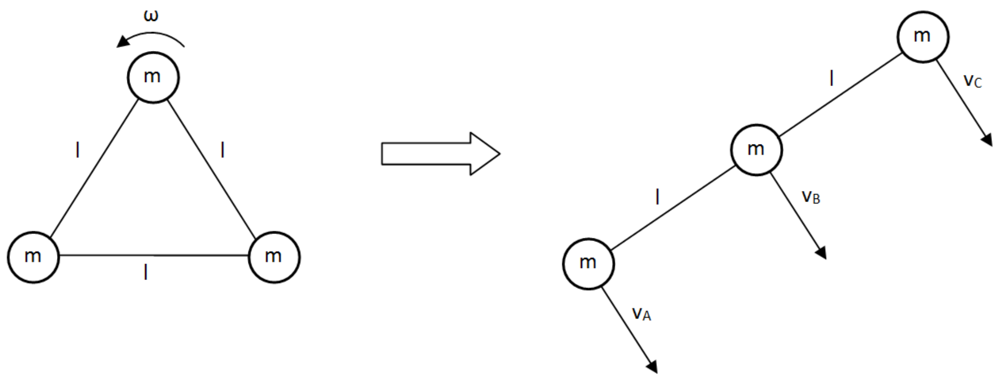
Relate this to the second instance:

$$\begin{aligned} l_0\hat{r} \times 2m(\vec{v_{1f}} - \frac{1}{3}v_0\hat{x}) - 2l_0\hat{r} \times m(\vec{v_{2f}} - \frac{1}{3}v_0\hat{x}) &= -\frac{1}{3}ml_0v_0\hat{z} \\ l_0\hat{r} \times 2m(-v_{1r}\hat{\theta}) - 2l_0\hat{r} \times m(v_{2r}\hat{\theta}) &= -\frac{1}{3}ml_0v_0\hat{z} \\ l_0\hat{r} \times 2m(-l_0\omega\hat{\theta}) - 2l_0\hat{r} \times m(2l_0\omega\hat{\theta}) &= -\frac{1}{3}ml_0v_0\hat{z} \\ -6ml_0^2\omega\hat{z} &= -\frac{1}{3}ml_0v_0\hat{z} \\ \omega = \frac{v_0}{18l_0} \end{aligned}$$

Finally let's use the conservation of energy to relate the two instances. Let's take the energy relative to the the fixed reference frame:

$$\begin{aligned} \frac{1}{2}(2m)\left(\frac{1}{2}v_0\right)^2 &= \frac{1}{2}(2m)v_{1f}^2 + \frac{1}{2}mv_{2f}^2 + \frac{1}{2}k(3l_0 - l_0)^2 \\ \frac{1}{4}mv_0^2 &= m\left(\frac{1}{3}v_0\hat{x} - l_0\omega\hat{\theta}\right)^2 + \frac{1}{2}m\left(\frac{1}{3}v_0\hat{x} + 2l_0\omega\hat{\theta}\right)^2 + 2kl_0^2 \\ \frac{1}{4}mv_0^2 &= \frac{1}{9}mv_0^2 - \frac{2}{3}mv_0l_0\omega\hat{x}\cdot\hat{\theta} + ml_0^2\omega^2 + \frac{1}{18}mv_0^2 + \frac{2}{3}mv_0l_0\omega\hat{x}\cdot\hat{\theta} + 2ml_0^2\omega^2 + 2kl_0^2 \\ -2kl_0^2 &= \left(\frac{1}{9} + \frac{1}{18} - \frac{1}{4}\right)mv_0^2 + 3ml_0^2\left(\frac{v_0}{18l_0}\right)^2 \\ -2kl_0^2 &= -\frac{8}{108}mv_0^2 \\ v_0 &= 3l_0\sqrt{3\frac{k}{m}} \end{aligned}$$

# 4 Problem 4




**a)** We realize that at the initial point in time, the linear momentum of the entire system is 0. That is, the entire system rotates about the COM, which does not move. Using this fact, we can use the conservation of angular momentum:

$$\begin{aligned} mv_A + mv_B + mv_C &= 0 \\ v_A + v_B + v_C &= 0 \end{aligned}$$

We can also use the conservation of angular momentum. Using a little trig, we can arrive at the conclusion that the COM is located at the center of the 3 masses, which is exactly $\frac{\sqrt{3}}{3}l$ from each mass. Thus the angular momentum about the COM:

$$\begin{aligned} 3\left(\frac{\sqrt{3}}{3}lm\left(\frac{\sqrt{3}}{3}l\omega\right)\right) &= lmv_A + (0)mv_B - lmv_C \\ l\omega &= v_A - v_C \end{aligned}$$

Finally we have the conservation of energy:

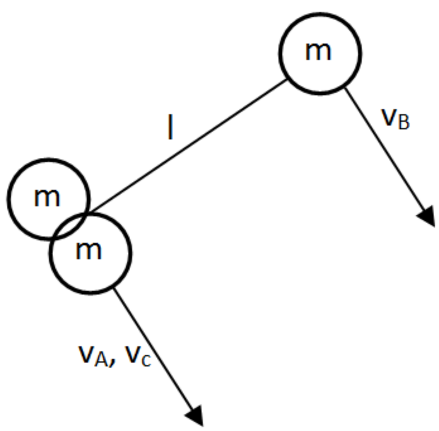
$$\begin{aligned} 3\left(\frac{1}{2}m\left(\frac{\sqrt{3}}{3}l\omega\right)^2\right) &= \frac{1}{2}m(v_A^2 + v_B^2 + v_C^2) \\ l^2\omega^2 &= v_A^2 + v_B^2 + v_C^2 \end{aligned}$$

Combining the three equations, we can solve for $v_A$, $v_B$, and $v_C$:

$$
\begin{aligned}
v_A &= \frac{1}{2}\left(1 - \frac{1}{\sqrt{3}}\right) l\omega \\
v_B &= \frac{1}{\sqrt{3}} l\omega \\
v_C &= \frac{1}{2}\left(-1 - \frac{1}{\sqrt{3}}\right) l\omega
\end{aligned}
$$

There is also a solution where $v_C$ and $v_A$ are switched. Also depending on how you defined which way your velocities are pointing, you will get slightly different answers (although just sign differences).

**b)**




To prove that the masses A and C never collide, we can use the same formulation as part a. Do note that the COM is now located $\frac{1}{3}l$ from masses A and C and $\frac{2}{3}l$ from mass B. First form linear momentum same as before:

$$
\begin{aligned}
mv_A + mv_B + mv_C &= 0 \\
v_A + v_B + v_C &= 0
\end{aligned}
$$

Form angular momentum:

$$
\begin{aligned}
3\left(\frac{\sqrt{3}}{3}lm\left(\frac{\sqrt{3}}{3}l\omega\right)\right) &= \frac{1}{3}lmv_A - \frac{2}{3}lmv_B + \frac{1}{3}lmv_C \\
v_A - 2v_B + v_C &= 3l\omega
\end{aligned}
$$

Form energy (same as before):

$$
\begin{aligned}
3\left(\frac{1}{2}m\left(\frac{\sqrt{3}}{3}l\omega\right)^2\right) &= \frac{1}{2}m(v_A^2 + v_B^2 + v_C^2) \\
l^2\omega^2 &= v_A^2 + v_B^2 + v_C^2
\end{aligned}
$$

If we try to solve these three equations, we get:

$$
\begin{aligned}
v_A &= \frac{1}{2}\left(1 + i\right)l\omega \\
v_B &= -l\omega \\
v_C &= \frac{1}{2}\left(1 - i\right)l\omega
\end{aligned}
$$

This result is nonphysical. Thus masses A and C never collide.

An alternative approach assumes that the masses collide inelastically. This is equivalent to setting $v_A = v_C = v_{AC}$ in the previous equations, rather than using conservation of energy which is violated in these types of collisions. This results in the system of equations

$$
\begin{aligned}
2v_{AC} + v_B &= 0 \\
2v_{AC} - 2v_B &= 3l\omega
\end{aligned}
$$

Combining these yields $v_B = -2v_{AC} = l\omega$. There doesn't seem to be a contradiction yet, until we check the energy of this state.

$$
T_f = \frac{1}{2}mv_B^2 + \frac{1}{2}(2m)v_{AC}^2 = \frac{3}{4}m\omega^2 l^2 > \frac{1}{2}m\omega^2 l^2 = T_i
$$

Since the inelastic collision results in a higher kinetic energy, this configuration is impossible. Note that it is not enough here to simply state that the energies are different since this is true of all inelastic collisions.