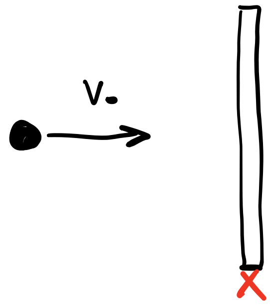
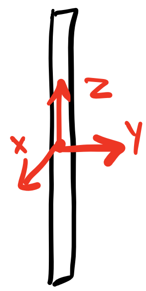
1)



a) <u>Center of Mass</u>



Rod symmetric about x and y

Let $\rho = \frac{dm}{dz}$

$$I_{xx} = I_{yy}$$

$$I_{xx} = \int \cancelto{0}{y^2} + z^2 \, dm$$

$$= \rho \int_{-l/2}^{l/2} z^2 \, dz$$

$$= \rho \left[\frac{1}{3} z^3 \right]_{-l/2}^{l/2}$$

$$= \rho \frac{2}{3} \left(\frac{l}{2} \right)^3$$

$$= \frac{1}{12} m l^2$$

$$I_{zz} = \int \cancelto{0}{x^2} + \cancelto{0}{y^2} \, dm$$

$$= 0$$

Due to symmetry,

$$I_{xy} = I_{xz} = I_{yz} = 0$$

$$\boxed{\overleftrightarrow{I}_{cm} = \begin{bmatrix} \frac{1}{12} m l^2 & 0 & 0 \\ 0 & \frac{1}{12} m l^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

Pivot

Using parallel axis theorem

$$I'_{xx} = I_{xx} + m\left(\frac{l}{2}\right)^2 = \frac{1}{3}ml^2$$

$$I'_{yy} = I_{yy} + m\left(\frac{l}{2}\right)^2 = \frac{1}{3}ml^2$$

$$I'_{zz} = 0$$

$$\boxed{\overset{\Rightarrow}{I}_{pivot} = \begin{bmatrix} \frac{1}{3}ml^2 & 0 & 0 \\ 0 & \frac{1}{3}ml^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

b) Energy: collision is elastic
no external force applied over distance
$\Rightarrow$ conserved

Linear Momentum: external force applied at pivot
$\Rightarrow$ not conserved

Angular Momentum: When taken about pivot, no external forces that cause moments.
$\Rightarrow$ conserved about pivot.

c)

Energy: $\frac{1}{2} m V_o^2 = \frac{1}{2} m V_f^2 + \frac{1}{2}\left(m \frac{\ell^2}{3}\right) \omega_f^2$

Ang. Momentum: $\frac{\ell}{2} m V_o = \frac{\ell}{2} m V_f + \left(m \frac{\ell^2}{3}\right) \omega_f$

$$\Rightarrow \quad V_o^2 = V_f^2 + \frac{\ell^2}{3} \omega_f^2$$

$$V_o = V_f + \frac{2}{3} \ell \omega_f \quad \Rightarrow \quad V_f = V_o - \frac{2}{3} \ell \omega_f$$

$$V_o^2 = \left(V_o - \frac{2}{3} \ell \omega_f\right)^2 + \frac{\ell^2}{3} \omega_f^2$$

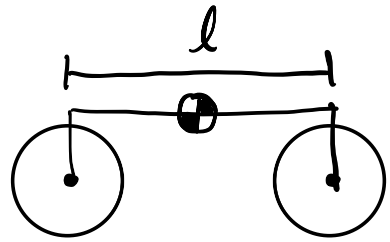
$$V_o^2 = V_o^2 - \frac{4}{3} \ell \omega_f V_o + \frac{4}{9} \ell^2 \omega_f^2 + \frac{\ell^2}{3} \omega_f^2$$

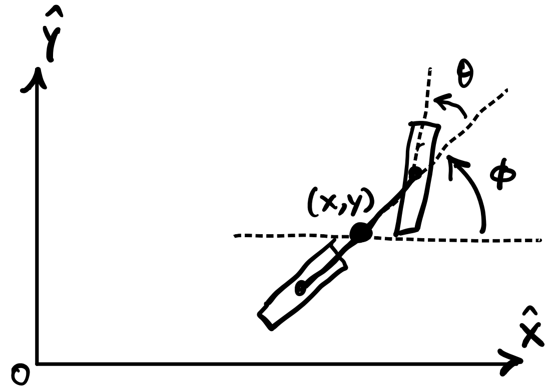
$$\Rightarrow \quad \frac{7}{9} \ell^2 \omega_f^2 - \frac{4}{3} \ell V_o \omega_f = 0$$

$$\omega_f \left(\frac{7}{9} \ell \omega_f - \frac{4}{3} V_o\right) = 0$$

$$\Rightarrow \quad \boxed{\omega_f = \frac{12}{7} \frac{V_o}{\ell}}$$

2)





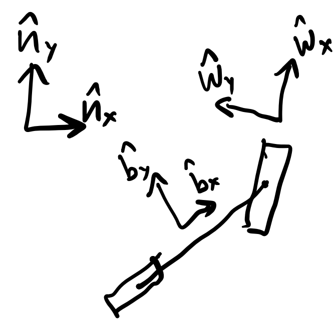
a) Gen Coord: x, y, ϕ, θ

We can ignore the rotations of the wheels.

There are no holonomic constraints

$\Rightarrow \boxed{4 \text{ DOF}}$

b) Define three frames:



Inertial frame N: $\hat{n}_x, \hat{n}_y$

Motorcycle frame B: $\hat{b}_x, \hat{b}_y$

Front wheel frame W: $\hat{w}_x, \hat{w}_y$

$$\vec{r}_{cm} = x\hat{n}_x + y\hat{n}_y$$

$$\vec{r}_{front} = x\hat{n}_x + y\hat{n}_y + \frac{l}{2}\hat{b}_x$$

$$\vec{r}_{rear} = x\hat{n}_x + y\hat{n}_y - \frac{l}{2}\hat{b}_x$$

$\Rightarrow$ Take derivative for velocity.

$$\vec{V}_{front} = \dot{x}\,\hat{n}_x + \dot{y}\,\hat{n}_y + \dot{\phi}\frac{l}{2}\,\hat{b}_y$$

$$\vec{V}_{rear} = \dot{x}\,\hat{n}_x + \dot{y}\,\hat{n}_y - \dot{\phi}\frac{l}{2}\,\hat{b}_y$$

Constraints: $\vec{V}_{front} \cdot \hat{w}_y = 0$

$\vec{V}_{rear} \cdot \hat{b}_y = 0$

$$\boxed{\begin{aligned} f_1 &= -\dot{x}\sin(\phi+\theta) + \dot{y}\cos(\phi+\theta) + \dot{\phi}\frac{l}{2}\cos(\theta) = 0 \\ f_2 &= -\dot{x}\sin(\phi) + \dot{y}\cos(\phi) - \dot{\phi}\frac{l}{2} = 0 \end{aligned}}$$

c) $T = \frac{1}{2} m \vec{V}_{cm} \cdot \vec{V}_{cm} + \frac{1}{2} I \dot{\phi}^2$

$= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} I \dot{\phi}^2$

$V = 0$

$$\boxed{\mathcal{L} = T - V = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} I \dot{\phi}^2}$$

There are no non-conservative forces present $\Rightarrow \sum Q_j = 0$

x) $$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} + \mu_1 \frac{\partial f_1}{\partial \dot{x}} + \mu_2 \frac{\partial f_2}{\partial \dot{x}} = 0}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = m\ddot{x} \qquad \frac{\partial \mathcal{L}}{\partial x} = 0$$

$$\frac{\partial f_1}{\partial \dot{x}} = -\sin(\phi+\theta) \qquad \frac{\partial f_2}{\partial \dot{x}} = -\sin(\phi)$$

$$m\ddot{x} - \mu_1 \sin(\phi+\theta) - \mu_2 \sin(\phi) = 0$$

y) $$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) - \frac{\partial \mathcal{L}}{\partial y} + \mu_1 \frac{\partial f_1}{\partial \dot{y}} + \mu_2 \frac{\partial f_2}{\partial \dot{y}} = 0}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) = m\ddot{y} \qquad \frac{\partial \mathcal{L}}{\partial y} = 0$$

$$\frac{\partial f_1}{\partial \dot{y}} = \cos(\phi+\theta) \qquad \frac{\partial f_2}{\partial \dot{y}} = \cos(\phi)$$

$$m\ddot{y} + \mu_1 \cos(\phi+\theta) + \mu_2 \cos(\phi) = 0$$

ϕ) $$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) - \frac{\partial \mathcal{L}}{\partial \phi} + \mu_1 \frac{\partial f_1}{\partial \dot{\phi}} + \mu_2 \frac{\partial f_2}{\partial \dot{\phi}} = 0}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = I\ddot{\phi} \qquad \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

$$\frac{\partial f_1}{\partial \dot{\phi}} = \frac{l}{2}\cos\theta \qquad \frac{\partial f_2}{\partial \dot{\phi}} = -\frac{l}{2}$$

$$I\ddot{\phi} + \mu_1 \frac{l}{2}\cos\theta - \mu_2 \frac{l}{2} = 0$$

θ)

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} + \mu_1 \frac{\partial f_1}{\partial \dot{\theta}} + \mu_2 \frac{\partial f_2}{\partial \dot{\theta}} = 0}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = 0 \quad \frac{\partial \mathcal{L}}{\partial \theta} = 0 \quad \frac{\partial f_1}{\partial \dot{\theta}} = 0 \quad \frac{\partial f_2}{\partial \dot{\theta}} = 0$$

$\Rightarrow$ Degenerate equation (θ is a completely free input)
due to approximation (nothing to solve)

$$\boxed{\begin{aligned}
&m\ddot{x} - \mu_1 \sin(\phi+\theta) - \mu_2 \sin(\phi) = 0 \\
&m\ddot{y} + \mu_1 \cos(\phi+\theta) + \mu_2 \cos(\phi) = 0 \\
&I\ddot{\phi} + \mu_1 \frac{l}{2}\cos\theta - \mu_2 \frac{l}{2} = 0 \\
&-\dot{x}\sin(\phi+\theta) + \dot{y}\cos(\phi+\theta) + \dot{\phi}\frac{l}{2}\cos(\theta) = 0 \\
&-\dot{x}\sin(\phi) + \dot{y}\cos(\phi) - \dot{\phi}\frac{l}{2} = 0 \\
&\theta \text{ is by choice} \Rightarrow \text{no restrictions (not well-defined)}
\end{aligned}}$$

3)

$$\psi,\ \theta,\ \phi$$

Spacecraft : 3, 2, 1

Ground : 3, 1, 3

$\psi = 45°$

$\theta = 30°$

$\phi = 45°$

a)

Rotation matrix mapping from Orbit frame to S/C frame

$$\Rightarrow \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & 1/2 \\ 0 & -1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \sqrt{2}/2 & \sqrt{6}/4 & \sqrt{2}/4 \\ -\sqrt{2}/2 & \sqrt{6}/4 & \sqrt{2}/4 \\ 0 & -1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1/2 - \sqrt{3}/4 & 1/2 + \sqrt{3}/4 & \sqrt{2}/4 \\ -1/2 - \sqrt{3}/4 & -1/2 + \sqrt{3}/4 & \sqrt{2}/4 \\ \sqrt{2}/4 & -\sqrt{2}/4 & \sqrt{3}/2 \end{bmatrix}$$

(confirm in handout)

Take transpose for body $\rightarrow$ orbit frame

$$\begin{bmatrix} \frac{1}{2}-\frac{\sqrt{3}}{4} & -\frac{1}{2}-\frac{\sqrt{3}}{4} & \frac{\sqrt{2}}{4} \\ \frac{1}{2}+\frac{\sqrt{3}}{4} & -\frac{1}{2}+\frac{\sqrt{3}}{4} & -\frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} \frac{1}{2}-\frac{\sqrt{3}}{4} \\ \frac{1}{2}+\frac{\sqrt{3}}{4} \\ \frac{\sqrt{2}}{4} \end{bmatrix}}$$

b) space $\rightarrow$ body

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right) & \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right) & -\frac{1}{2} \\ -\frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2}\right)+\frac{\sqrt{2}}{2}\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) & \frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2}\right)+\frac{\sqrt{2}}{2}\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) & \frac{\sqrt{3}}{2}\left(\frac{\sqrt{2}}{2}\right) \\ \frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2}\right)+\frac{\sqrt{2}}{2}\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) & -\frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2}\right)+\frac{\sqrt{2}}{2}\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) & \frac{\sqrt{3}}{2}\left(\frac{\sqrt{2}}{2}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{6}}{4} & \frac{\sqrt{6}}{4} & -\frac{1}{2} \\ -\frac{1}{4} & \frac{3}{4} & \frac{\sqrt{6}}{4} \\ \frac{3}{4} & -\frac{1}{4} & \frac{\sqrt{6}}{4} \end{bmatrix}$$

$\Rightarrow$ Take transpose for body $\rightarrow$ space

$$\begin{bmatrix} \frac{\sqrt{6}}{4} & -\frac{1}{4} & \frac{3}{4} \\ \frac{\sqrt{6}}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{2} & \frac{\sqrt{6}}{4} & \frac{\sqrt{6}}{4} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} \frac{\sqrt{6}}{4} \\ \frac{\sqrt{6}}{4} \\ -\frac{1}{2} \end{bmatrix}}$$

c) $\vec{H} = \vec{\vec{I}}\vec{\omega}$, but we need to match the frames.

Let ${}_BR_S$ be the rotation matrix mapping from space to body frame. Then

$$ {}_BR_S \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} {}_BR_S \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} $$

$\vec{\omega}$ in orbit frame coord.

inverse

$$ {}_BR_S \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} \sqrt{6}/4 & \sqrt{6}/4 & -1/2 \\ -1/4 & 3/4 & \sqrt{6}/4 \\ 3/4 & -1/4 & \sqrt{6}/4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} $$

$$ = \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 3\sqrt{6}/4 - 3/2 \\ 5/4 + 3\sqrt{6}/4 \\ 1/4 + 3\sqrt{6}/4 \end{bmatrix} $$

$$ = \begin{bmatrix} \frac{3\sqrt{6}-6}{16} \\ \frac{3\sqrt{6}+5}{12} \\ \frac{3\sqrt{6}+1}{12} \end{bmatrix} \leftarrow $$

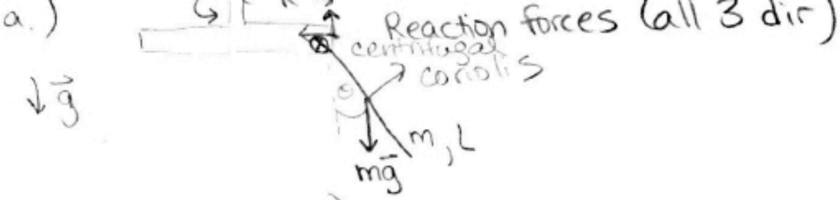
$\vec{\omega}$ in body frame coordinates.

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = {}_B R_S^T \begin{bmatrix} (3\sqrt{6}-6)/16 \\ (3\sqrt{6}+5)/12 \\ (3\sqrt{6}+1)/12 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{6}/4 & -1/4 & 3/4 \\ \sqrt{6}/4 & 3/4 & -1/4 \\ -1/2 & \sqrt{6}/4 & \sqrt{6}/4 \end{bmatrix} \begin{bmatrix} (3\sqrt{6}-6)/16 \\ (3\sqrt{6}+5)/12 \\ (3\sqrt{6}+1)/12 \end{bmatrix}$$

From here on out this becomes painful, so we will accept this as a solution.

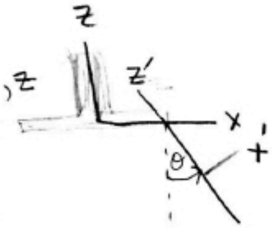
a.)



$F_c = \vec{\omega} \times (\vec{\omega} \times \vec{r})$

$\vec{F}_c = 2(\vec{\omega} \times \vec{v}_r)$

b.) inertial: x, z
body: x', z'



$\boxed{\text{gen. coord: } \theta \text{ \& } \psi}$

$$\vec{\omega} = \dot{\psi}\hat{z} - \dot{\theta}\hat{y} \quad \text{and} \quad \hat{z} = -\sin\theta\,\hat{x}' + \cos\theta\,\hat{z}'$$

$$\vec{\omega} = -\dot{\psi}\sin\theta\,\hat{x}' - \dot{\theta}\hat{y}' + \dot{\psi}\cos\theta\,\hat{z}'$$

pivot: $\vec{v}_p = R\dot{\psi}\hat{y}'$

CM: $\vec{v}_{cm} = R\dot{\psi}\hat{y}' + \left[-\dot{\psi}\sin\theta\,\hat{x}' - \dot{\theta}\hat{y}' + \dot{\psi}\cos\theta\,\hat{z}'\right] \times \left(-\frac{L}{2}\hat{z}'\right)$

$$\vec{v}_{CM} = \frac{L}{2}\dot{\theta}\hat{x}' + \dot{\psi}\left(R - \frac{L}{2}\sin\theta\right)\hat{y}'$$

$$\vec{H}_{CM} = I_{CM}\left(-\dot{\psi}\sin\theta\,\hat{x}' - \dot{\theta}\,\hat{y}'\right)$$

$$T = \frac{1}{2}m\vec{v}_{cm}\cdot\vec{v}_{cm} + \frac{1}{2}\vec{\omega}\cdot\vec{H}_{CM} + \frac{1}{2}I_1\dot{\psi}^2$$

$$T = \frac{1}{2}m\left[\left(\frac{L}{2}\dot{\theta}\right)^2 + \dot{\psi}^2\left(R + \frac{L}{2}\sin\theta\right)^2\right] + \frac{1}{2}I_1\dot{\psi}^2 + \frac{1}{2}I_{CM}\left(\dot{\psi}^2\sin^2\theta + \dot{\theta}^2\right)$$

$$V = -mg\frac{L}{2}\cos\theta \quad \text{and} \quad \mathcal{L} = T - V$$

c.) $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\psi}} - \frac{\partial \mathcal{L}}{\partial \psi} = \Gamma$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} = 0$$

5)

a) Measure $\overleftrightarrow{I}$ for $\hat{x}, \hat{y}, \hat{z}$ "body-fixed" about Center of mass

$$I_{ij} = \sum_k m_k \left(\delta_{ij} \vec{r}_k \cdot \vec{r}_k - (\vec{r}_k \cdot \vec{e}_i)(\vec{r}_k \cdot \vec{e}_j) \right)$$

$$\Rightarrow I_{xx} = m_1(0^2 + b^2) + m_2(0^2 + (-b)^2)$$

$$= 2mb^2$$

$$I_{yy} = 2mb^2$$

$$I_{zz} = 0$$

$$I_{xy} = I_{xz} = I_{yz} = 0$$

$$\boxed{\overleftrightarrow{I} = \begin{bmatrix} 2mb^2 & 0 & 0 \\ 0 & 2mb^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{for } \hat{x}, \hat{y}, \hat{z} \text{ about center of mass}}$$

b) $$\vec{\omega} = \omega \left[\cos\alpha \, \hat{z} + \sin\alpha \, \hat{y} \right]$$

$$= \omega \sin\alpha \, \hat{y} + \omega \cos\alpha \, \hat{z} \Rightarrow \boxed{\begin{aligned} \omega_x &= 0 \\ \omega_y &= \omega \sin\alpha \\ \omega_z &= \omega \cos\alpha \end{aligned}}$$

c)

$$\boxed{\vec{H} = \vec{\vec{I}}\,\vec{\omega} = 2mb^2\omega\sin\alpha\,\hat{y}}$$

$$\boxed{\begin{aligned} H_x &= 0 \\ H_y &= 2mb^2\omega\sin\alpha \\ H_z &= 0 \end{aligned}}$$

$$\vec{M} = \cancel{\vec{\vec{I}}\,\dot{\vec{\omega}}}^{\,0} + \vec{\omega} \times \vec{\vec{I}}\,\vec{\omega} \quad \text{(Euler's EOM)}$$

$$= \begin{bmatrix} 0 \\ \omega\sin\alpha \\ \omega\cos\alpha \end{bmatrix} \times \begin{bmatrix} 2mb^2 & 0 & 0 \\ 0 & 2mb^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \omega\sin\alpha \\ \omega\cos\alpha \end{bmatrix}$$

$$= \det\begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & \omega\sin\alpha & \omega\cos\alpha \\ 0 & 2mb^2\omega\sin\alpha & 0 \end{bmatrix}$$

$$= -2mb^2\omega^2\sin\alpha\cos\alpha\,\hat{x}$$

$$\boxed{\begin{aligned} M_x &= -2mb^2\omega^2\sin\alpha\cos\alpha \\ M_y &= 0 \\ M_z &= 0 \end{aligned}}$$