AA 242A Homework 2 Solution

October 6, 2020

Notation

Before we begin these problems, let us clarify the notation we will encounter:

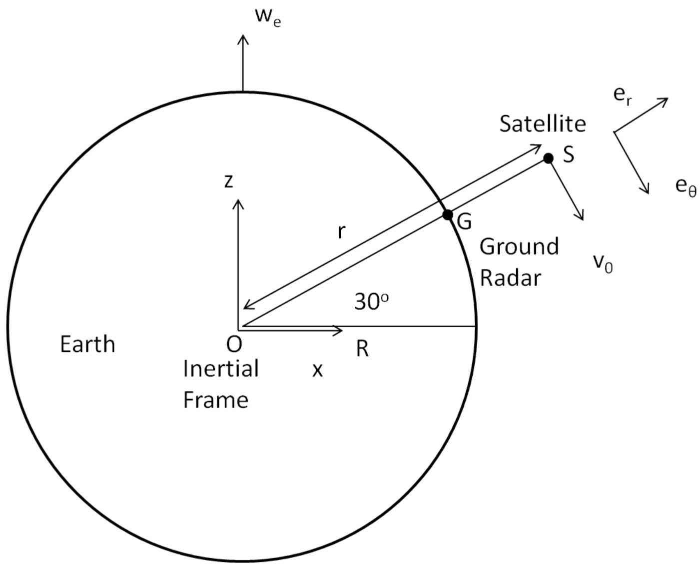
1. $\vec{r}_{A/B}$ is the position of A with respect to B.

2. $\vec{v}_{A/B}\Big|_F$ or ${}^F\vec{v}_{A/B}$ is the velocity of point A with respect to point B. Since the velocity is a time derivative, we also mention the frame in which this velocity is measured, which is F. This may be any frame (inertial or otherwise).

3. $\frac{d\vec{A}}{dt}\Big|_F$, $\dot{\vec{A}}\Big|_F$ or ${}^F\frac{d\vec{A}}{dt}$ denote the time derivative of any vector $\vec{A}$ in frame F. The first option will be used throughout the solutions.

4. $\vec{\omega}_{A/B}$ is the angular velocity of A in frame B. A could be a frame, point, object, anything. This is also sometimes written as ${}^B\vec{\omega}^A$.

1 Problem 1



It is convenient to locate a rotating frame at the radar station, with $\hat{e_r}$ pointing radially outward, $\hat{e_\theta}$ pointing due south and $\hat{e_\phi}$ points into the plane of the paper. The inertial frame is fixed in space at the center of the earth with z pointing in the same direction as the Earth's rotation and x and y in the equatorial plane. What we need to find is the velocity and the acceleration of the satellite as measured from the ground station. If we call the inertial frame the I frame and the station attached to the ground station G, then we want:

$$\vec{v}_{S/G}\Big|_G = \frac{d\vec{r_{S/G}}}{dt}\Bigg|_G$$

and

$$\vec{a}_{S/G}\Big|_G = \frac{d^2\vec{r_{S/G}}}{dt^2}\Bigg|_G$$

$$\vec{r}_{G/O} = R\hat{r}$$

$$\left.\frac{d\vec{r}_{S/G}}{dt}\right|_I = \left.\frac{d\vec{r}_{S/G}}{dt}\right|_G + \vec{\omega}_{G/I} \times \vec{r}_{S/G}$$

$$\Rightarrow \vec{v}_{S/G}\big|_G = \vec{v}_{S/G}\big|_I - \vec{\omega}_{G/I} \times \vec{r}_{S/G}$$

$$\begin{aligned}
\vec{v}_{S/G}\big|_I &= \vec{v}_{S/O}\big|_I - \vec{v}_{G/O}\big|_I \\
&= v_0\hat{e}_\theta - \left(\cancel{\vec{v}_{G/O}\big|_G} + \vec{\omega}_{G/I} \times \vec{r}_{G/O}\right)
\end{aligned}$$

$$\vec{\omega}_{G/I} = \omega_e\hat{z} = \omega_e\left(\cos\theta\hat{e}_r - \sin\theta\hat{e}_\theta\right)$$

$$\begin{aligned}
\Rightarrow \vec{v}_{S/G}\big|_I &= v_0\hat{e}_\theta - \omega_e\left(\cos\theta\hat{e}_r - \sin\theta\hat{e}_\theta\right) \times R\hat{e}_r \\
&= v_0\hat{e}_\theta - R\omega_e\sin\theta\hat{e}_\phi
\end{aligned}$$

$$\vec{\omega}_{G/I} \times \vec{r}_{S/G} = \vec{\omega}_{G/I} \times (r-R)\hat{e}_r = (r-R)\omega_e\sin\theta\hat{e}_\phi$$

Note that the last equation is only true at the very instant that the spacecraft passes over the ground station. Putting everything together, we get:

$$\begin{aligned}
\vec{v}_{S/G}\big|_G &= v_0\hat{e}_\theta - r\omega_e\sin\theta \\
&= v_0\hat{e}_\theta - \frac{\sqrt{3}}{2}r\omega_e
\end{aligned}$$

To get the acceleration, we repeat this process again.

$$\left.\frac{d}{dt}\right|_G \vec{v}_{S/G}\big|_I = \left.\frac{d}{dt}\right|_I \vec{v}_{S/G}\big|_I - \vec{\omega}_{G/I} \times \vec{v}_{S/G}\big|_I$$

$$\left.\frac{d}{dt}\right|_G \left(\vec{v}_{S/G}\big|_G + \vec{\omega}_{G/I} \times \vec{r}_{S/G}\right) = \vec{a}_{S/G}\big|_I - \vec{\omega}_{G/I} \times \vec{v}_{S/G}\big|_I$$

$$\vec{a}_{S/G}\big|_G = \vec{a}_{S/G}\big|_I - \vec{\omega}_{G/I} \times \vec{v}_{S/G}\big|_I - \vec{\omega}_{G/I} \times \vec{v}_{S/G}\big|_G$$

Now we can start analyzing term by term:

$$\vec{a}_{S/G}\big|_I = \left.\frac{d}{dt}\right|_I \left(\vec{v}_{S/G}\big|_I\right) = \left.\frac{d}{dt}\right|_I \left(\vec{v}_{S/O}\big|_I - \vec{v}_{G/O}\big|_I\right) = \vec{a}_{S/O}\big|_I - \vec{a}_{G/O}\big|_I$$

The first term on the right hand side is simply the centripetal acceleration of the satellite in the inertial frame given by

$$\vec{a}_{S/O}\big|_I = -\frac{v_0^2}{r}\hat{e}_r$$

while for the second term, we get:

$$\begin{aligned}\vec{a}_{G/O}\big|_I &= \vec{\omega}_{G/I} \times \vec{v}_{G/O}\big|_I \\ &= \omega_e \left(\cos\theta \hat{e}_r - \sin\theta \hat{e}_\theta\right) \times R\omega_e \sin\theta \hat{e}_\phi \\ &= -R\omega_e^2 \sin\theta \left(\cos\theta \hat{e}_\theta + \sin\theta \hat{e}_r\right)\end{aligned}$$

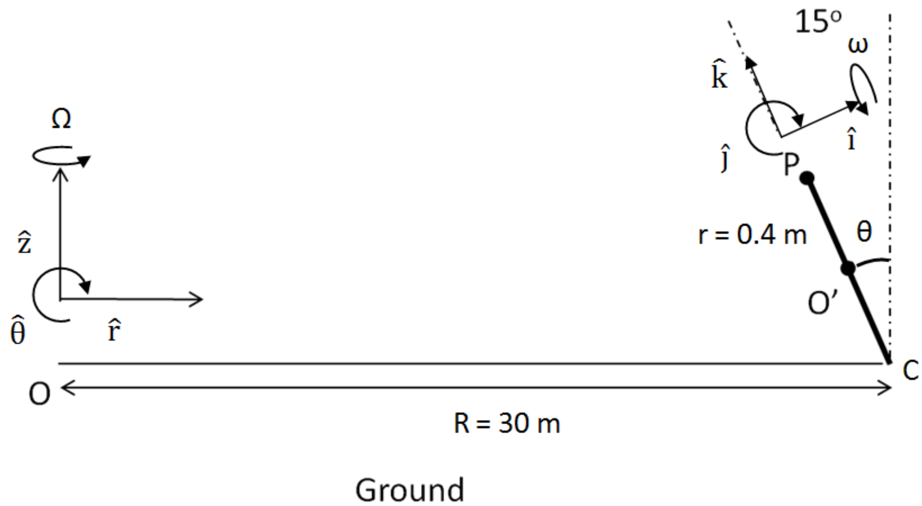
Substituting, we get:

$$\vec{a}_{G/O}\big|_I = -\frac{\sqrt{3}}{4}\omega_e^2 R\hat{e}_\theta - \frac{3}{4}\omega_e^2 R\hat{e}_r$$

We know the quantities $\vec{\omega}_{G/I}$, $\vec{v}_{S/G}\big|_G$ and $\vec{v}_{S/G}\big|_I$.
And finally:

$$\vec{a}_{S/G}\big|_G = -\left(\frac{v_0^2}{r} + \frac{3}{4}r\omega_e^2\right)\hat{e}_r - \frac{\sqrt{3}}{4}r\omega_e^2\hat{e}_\theta - \omega_e v_0 \hat{e}_\phi$$

2 Problem 2



A few definitions regarding the frames:

1. Unit vectors $\hat{r}$, $\hat{z}$ and $\hat{\theta}$ rotate such that $\hat{r}$ always points to the point of contact.

2. Unit vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$ rotate with the wheel as it goes around O.

3. By this definition, the first frame rotates with angular velocity $\Omega\hat{z}$ and the second frame rotates with angular velocity $\Omega\hat{z} - \omega\hat{i}$.

4. The contact point moves around at a velocity of 10 m/s, but the velocity of the point on the wheel in contact with the ground is actually 0.

Let us first find the value of Ω. It is simply:

$$
\begin{aligned}
R\Omega &= v_C \\
\Omega &= \frac{v_C}{R} \\
\Omega &= \frac{10\,\mathrm{m/s}}{30\,\mathrm{m}} = \frac{1}{3}\mathrm{rad/s}
\end{aligned}
$$

To find the value of ω, we need to use the no slip condition. Let us define the position vector from O to the contact point on the bicycle:

$$\vec{r}_{C/O} = R\hat{r} - r\sin\theta\hat{r} + r\cos\theta\hat{z} - r\hat{k}$$

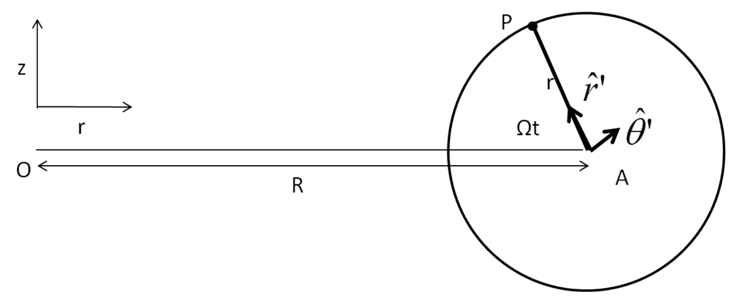
Note the use of mixed basis: We are trying to define a position vector true for ALL time. Differentiate the above to get v and find ω (Note that $\hat{j}$ and $\hat{\theta}$ are in the same direction in this instant in time, but this is not true for ALL time). All time derivatives that follow are with respect to the inertial frame - a " $\left.\frac{d}{dt}\right|_I = \frac{d}{dt}$ " is therefore implied over all velocities/accelerations.

$$\begin{aligned}
\vec{v}_{C/O} &= (R - r\sin\theta)\dot{\hat{r}} + r\cos\theta\dot{\hat{z}} - r\dot{\hat{k}} \\
0 &= (R - r\sin\theta)\Omega\hat{z}\times\hat{r} + 0 - r(\Omega\hat{z} - \omega\hat{i})\times\hat{k} \\
0 &= (R - r\sin\theta)\Omega\hat{\theta} - r\Omega(\sin\theta\hat{i} + \cos\theta\hat{k})\times\hat{k} - r\omega\hat{j} \\
0 &= (R - r\sin\theta)\Omega\hat{\theta} + r\Omega\sin\theta\hat{j} - r\omega\hat{j} \\
0 &= (R\Omega - r\omega)\hat{\theta} \\
\omega &= \frac{R}{r}\Omega = 25\,\text{rad/s}
\end{aligned}$$

We now take the second derivative of $\vec{r}_{P/O}$ to get acceleration. Be careful about differentiation here, and note that $\dot{\hat{z}} = 0$:

$$\begin{aligned}
\vec{r}_{P/O} &= R\hat{r} - r\sin\theta\hat{r} + r\cos\theta\hat{z} + r\hat{k} \\
\vec{v}_{P/O} &= (R - r\sin\theta)\dot{\hat{r}} + r\cos\theta\dot{\hat{z}} + r\dot{\hat{k}} \\
&= (R - r\sin\theta)\Omega\hat{z}\times\hat{r} + r(\Omega\hat{z} - \omega\hat{i})\times\hat{k} \\
&= (R - r\sin\theta)\Omega\hat{z}\times\hat{r} + r\Omega\hat{z}\times\hat{k} + r\omega\hat{j} \\
\vec{a}_{P/O} &= (R - r\sin\theta)\Omega\hat{z}\times\dot{\hat{r}} + r\Omega\hat{z}\times\dot{\hat{k}} + r\omega\dot{\hat{j}} \\
&= (R - r\sin\theta)\Omega\hat{z}\times\Omega\hat{\theta} + r\Omega\hat{z}\times[(\Omega\hat{z} - \omega\hat{i})\times\hat{k}] + r\omega(\Omega\hat{z} - \omega\hat{i})\times\hat{j} \\
&= -(R - r\sin\theta)\Omega^2\hat{r} + r\Omega\hat{z}\times(-\Omega\sin\theta\hat{\theta} + \omega\hat{\theta}) - r\omega\Omega\hat{r} - r\omega^2\hat{k} \\
&= [-(R - r\sin\theta)\Omega^2 + r\Omega^2\sin\theta - 2r\omega\Omega + r\omega^2\sin\theta]\hat{r} - r\omega^2\cos\theta\hat{z} \\
&\approx 54.7\,\text{m/s}^2\hat{r} - 241.5\,\text{m/s}^2\hat{z}
\end{aligned}$$

3 Problem 3



Let's set up our coordinate system first. There is a frame with basis vectors $[\hat{r}, \hat{\theta}, \hat{z}]$ at the origin O where the radial vector always points at the center of the propeller (call this point A). Note that this is a non-inertial reference frame. There is a second reference frame, let's call it F - this reference frame has unit vectors $[\hat{r}', \hat{\theta}', \hat{z}']$. Here, $\hat{r}'$ **always** points from point A to point P. Note that $\hat{r}', \hat{\theta}'$ are always in the same plane as $\hat{r}, \hat{z}$ and $\hat{z}'$ always points in the direction of $\hat{\theta}$.

What we want, is the inertial acceleration of point P about O. That is:

$$\left.\frac{d^2}{dt^2}\right|_I \vec{r}_{P/O} = \left.\frac{d^2}{dt^2}\right|_I \vec{r}_{P/A} + \left.\frac{d^2}{dt^2}\right|_I \vec{r}_{A/O}$$

Now, let's go about solving this problem.

$$\begin{aligned}
\left.\frac{d\vec{r}_{P/A}}{dt}\right|_I &= \left.\frac{d\vec{r}_{P/A}}{dt}\right|_F + \vec{\omega}_{F/I} \times \vec{r}_{P/A} \\
&= \vec{0} + \vec{\omega}_{F/I} \times \vec{r}_{P/A} \\
\vec{\omega}_{F/I} &= \Omega\hat{\theta} + \frac{v}{R}\hat{z} \\
\vec{r}_{P/A} &= -r\cos(\Omega t)\hat{r} + r\sin(\Omega t)\hat{z} \\
\Rightarrow \left.\frac{d\vec{r}_{P/A}}{dt}\right|_I &= \Omega r\sin(\Omega t)\hat{r} - v\frac{r}{R}\cos(\Omega t)\hat{\theta} + \Omega r\cos(\Omega t)\hat{z}
\end{aligned}$$

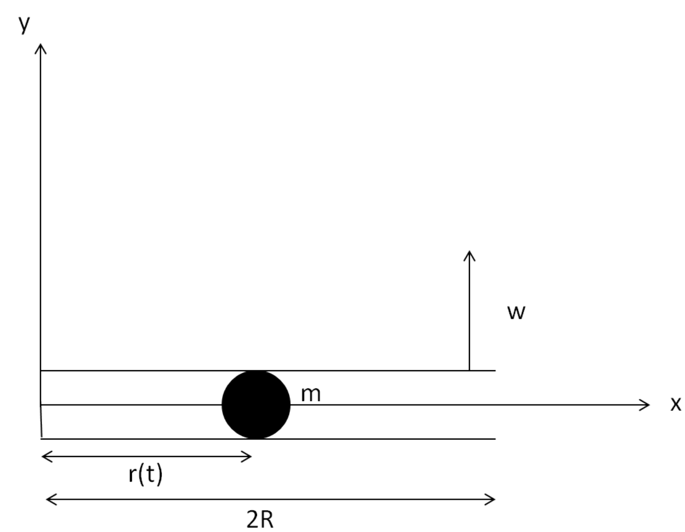
Once we have this, all we need to do is to take the inertial time derivative of the equation above. Here, Ω, R, r and v are all constant, but the unit vectors change with time. So,

$$\left.\frac{d^2\vec{r}_{P/A}}{dt^2}\right|_I = \Omega r \left.\frac{d}{dt}\right|_I [\sin(\Omega t)\hat{r}] - v\frac{r}{R}\left.\frac{d}{dt}\right|_I \left[\cos(\Omega t)\hat{\theta}\right] + \Omega r \left.\frac{d}{dt}\right|_I [\cos(\Omega t)\hat{z}]$$
$$= r\left(\frac{v^2}{R^2} + \Omega^2\right)\cos(\Omega t)\hat{r} + \frac{2r}{R}v\Omega\sin(\Omega t)\hat{\theta} - r\Omega^2\sin(\Omega t)\hat{z}$$

The inertial acceleration of point A is simply the centripetal acceleration towards O. Add that to the above equation and we get our final answer:

$$\left.\vec{a}_{P/O}\right|_I = \left[-\frac{v^2}{R} + r\left(\frac{v^2}{R^2} + \Omega^2\right)\cos(\Omega t)\right]\hat{r} + \frac{2r}{R}v\Omega\sin(\Omega t)\hat{\theta} - r\Omega^2\sin(\Omega t)\hat{z}$$

4 Problem 4



We choose two coordinate frames here. One is the inertial frame where x always points along the horizontal and y always points along the vertical direction. The r, θ frame rotates with the tube. Therefore the position vector of the bead with respect to the origin can be written as:

$$\vec{r}_{P/O} = r\hat{r}$$

Because the tube is frictionless, we know that there is no radial forces acting on the mass, therefore:

$$\frac{F_r}{m} = \ddot{r} - \omega^2 r = 0 \tag{1}$$

Solving the second order ODE gives $r(t) = A\cosh\omega t + B\sinh\omega t = Ce^{\omega t} + De^{-\omega t}$. Plugging in the initial conditions $r(0) = R$ and $\dot{r}(t) = 0$ will give the values for the two unknowns:

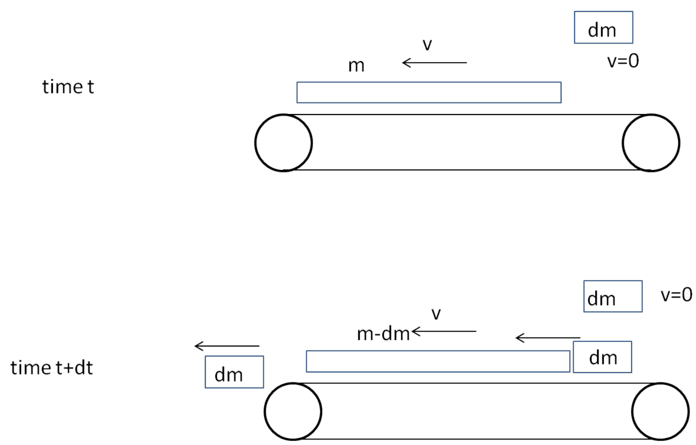
$$r(t) = R\cosh\omega t = \frac{R}{2}(e^{\omega t} + e^{-\omega t}) \tag{2}$$

Given that $\theta(0) = 0$, if we plug in $r(t_0) = 2R$ (the moment the bead leaves the tube), solve for the values of $\cosh\omega t_0$ and $\sinh\omega t_0$ (or $e^{\pm\omega t_0}$) and substitute into the expression $\dot{r}(t)$:

$$\begin{aligned}
v_r &= \sqrt{3}R\omega = 1.73R\omega \\
v_t &= 2R\omega \\
|v| &= \sqrt{7}R\omega = 2.646R\omega \\
\theta &= \cosh^{-1}\left(\frac{r(t_0)}{R}\right) = \cosh^{-1}(2) = 75.5^\circ \\
\theta_{r\theta} &= \tan^{-1}\left(\frac{v_t}{v_r}\right) = 47.1^\circ \\
\theta_{xy} &= \theta + \theta_{r\theta} = 122.5^\circ \\
\vec{v} &= (-1.5\hat{x} + 2.2\hat{y})R\omega
\end{aligned}$$

Here, $\theta_{r\theta}$ is the angle relative to the tube and θ is the angle the tube makes with the positive x-axis.

5 Problem 5



Ignoring direction:

$$\begin{aligned}
p(t) &= mv \\
p(t+dt) &= (m-dm)v + vdm + vdm = v(m+dm) \\
F &= \frac{p(t+dt)-p(t)}{dt} = \frac{vdm}{dt} = v\beta \\
\frac{2M}{R} &= F \\
\Rightarrow M &= \frac{vR\beta}{2}
\end{aligned}$$

One mistake often made by students is attempt to (incorrectly) use the energy approach:

$$\begin{aligned}
Fv &= \frac{dE}{dt} = \frac{\beta v^2}{2} \\
F &= \frac{v\beta}{2} \\
\Rightarrow M &= \frac{vR\beta}{4}
\end{aligned}$$

This approach ignored the fact that the differential mass dm has been accelerated from $v = 0$ to v in dt time, which implies that the average velocity is $\frac{v}{2}$ instead of v.