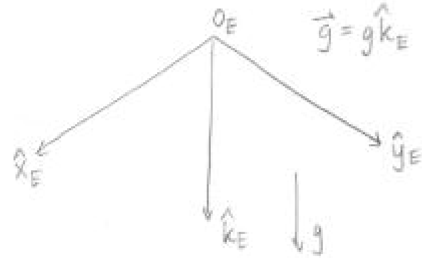
AA 242A Homework 7 Solution

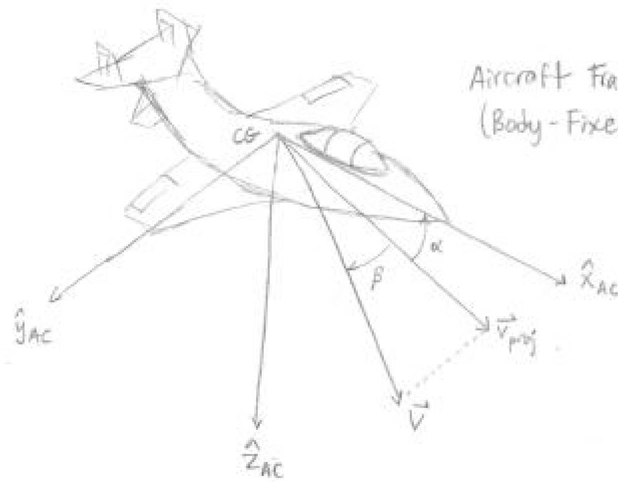
November 20, 2019

1 Problem 1

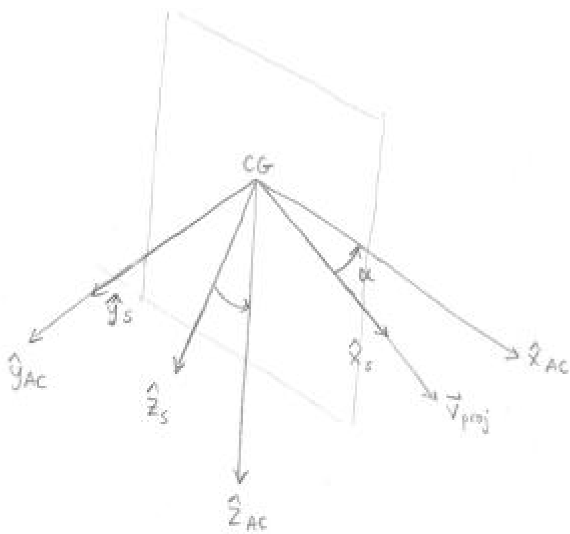
(a)



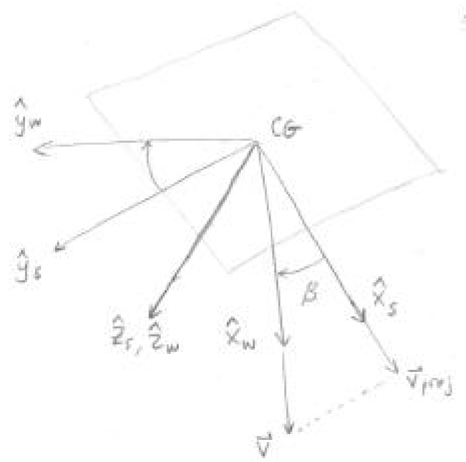
Earth Frame
(Inertial)



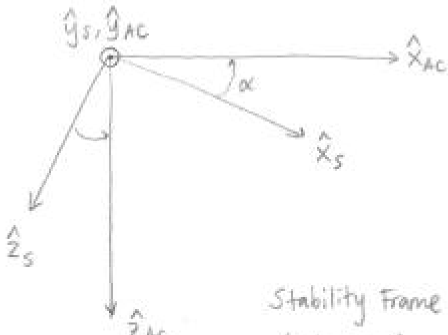
Aircraft Frame
(Body-Fixed)



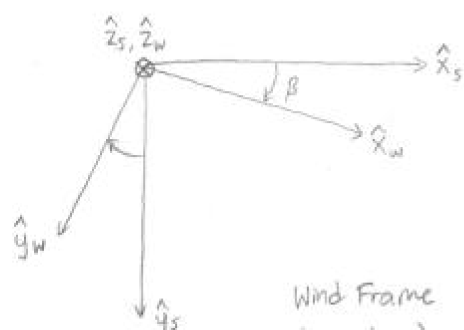
Stability Frame (3D View)



Wind Frame (3D View)



Stability Frame
(Side View)



Wind Frame
(Top View)

a). So we can see from these diagrams that the transformation from stability to aircraft frame is just a rotation of angle α around the local y-axis:

$$\overleftrightarrow{A}_{AC/S} = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

similarly, the transformation from wind to stability frame is a rotation of $-\beta$, so as a result:

$$\begin{aligned} \overleftrightarrow{A}_{AC/W} &= \overleftrightarrow{A}_{AC/S}\overleftrightarrow{A}_{S/W} \\ &= \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos\alpha\cos\beta & -\cos\alpha\sin\beta & -\sin\alpha \\ \sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & -\sin\alpha\sin\beta & \cos\alpha \end{bmatrix} \end{aligned}$$

If the Euler rotation matrices $\overleftrightarrow{A}_\Psi, \overleftrightarrow{A}_\Theta, \overleftrightarrow{A}_\Phi$ are all known, then by definition of these maneuvers we know:

$$\begin{aligned} \begin{bmatrix} \hat{x}_{AC} \\ \hat{y}_{AC} \\ \hat{z}_{AC} \end{bmatrix} &= \overleftrightarrow{A}_\Phi \overleftrightarrow{A}_\Theta \overleftrightarrow{A}_\Psi \begin{bmatrix} \hat{x}_E \\ \hat{y}_E \\ \hat{z}_E \end{bmatrix} \\ \overleftrightarrow{A}_{AC/W} \begin{bmatrix} \hat{x}_W \\ \hat{y}_W \\ \hat{z}_W \end{bmatrix} &= \overleftrightarrow{A}_\Phi \overleftrightarrow{A}_\Theta \overleftrightarrow{A}_\Psi \begin{bmatrix} \hat{x}_E \\ \hat{y}_E \\ \hat{z}_E \end{bmatrix} \\ \Rightarrow \begin{bmatrix} \hat{x}_W \\ \hat{y}_W \\ \hat{z}_W \end{bmatrix} &= \overleftrightarrow{A}^T_{AC/W} \overleftrightarrow{A}_\Phi \overleftrightarrow{A}_\Theta \overleftrightarrow{A}_\Psi \begin{bmatrix} \hat{x}_E \\ \hat{y}_E \\ \hat{z}_E \end{bmatrix} \\ \Rightarrow \begin{bmatrix} \hat{x}_E \\ \hat{y}_E \\ \hat{z}_E \end{bmatrix} &= \overleftrightarrow{A}^T_\Psi \overleftrightarrow{A}^T_\Theta \overleftrightarrow{A}^T_\Phi \overleftrightarrow{A}_{AC/W} \begin{bmatrix} \hat{x}_W \\ \hat{y}_W \\ \hat{z}_W \end{bmatrix} \end{aligned}$$

The last two equations are obtained by observing that for rotation matrix $\overleftrightarrow{A}$, $\overleftrightarrow{A}^{-1} = \overleftrightarrow{A}^T$

b).

$$
\begin{aligned}
\overrightarrow{F}_W &= \overrightarrow{D} + \overrightarrow{S} + \overrightarrow{L} = \begin{bmatrix} -D \\ S \\ -L \end{bmatrix}_W \\
\Rightarrow \overrightarrow{F}_{AC} &= \overleftrightarrow{A}_{AC/W} * \overrightarrow{F}_W \\
&= \begin{bmatrix} \cos\alpha\cos\beta & -\cos\alpha\sin\beta & -\sin\alpha \\ \sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & -\sin\alpha\sin\beta & \cos\alpha \end{bmatrix} \begin{bmatrix} -D \\ S \\ -L \end{bmatrix}_W \\
&= \begin{bmatrix} L\sin\alpha - D\cos\alpha\cos\beta - S\cos\alpha\sin\beta \\ -D\sin\beta + S\cos\beta \\ -L\cos\alpha - D\sin\alpha\cos\beta - S\sin\alpha\sin\beta \end{bmatrix}
\end{aligned}
$$

c). We want a way to relate the angular velocity of the wind frame relative to the aircraft frame using α, β, and other derivatives. To do this, it is simplest to think of the rotation of the wind frame relative to the AC frame as a sum of the two ratations α and β. However, to do this we need to the sum of rotations defined from the AC frame to the wind frame as expressed in AC frame. Note that α is defined as the rotation from stability frame to AC frame around the local y-axis, while β is defined as the rotation from stability frame to wind frame in the local z-axis, so we have:

$$
\overrightarrow{w}_W = -\dot{\alpha}\hat{y}_{AC} + \dot{\beta}\hat{z}_S
$$

recall:

$$
\overleftrightarrow{A}_{AC/S} = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix}
$$

so:

$$
\begin{aligned}
\begin{bmatrix} \hat{x}_S \\ \hat{y}_S \\ \hat{z}_S \end{bmatrix} &= \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{bmatrix} \hat{x}_{AC} \\ \hat{y}_{AC} \\ \hat{z}_{AC} \end{bmatrix} \\
\Rightarrow \hat{z}_S &= -\sin\alpha\hat{x}_{AC} + \cos\alpha\hat{z}_{AC} \\
\Rightarrow \overrightarrow{w}_W &= \begin{bmatrix} -\dot{\beta}\sin\alpha \\ -\dot{\alpha} \\ \dot{\beta}\cos\alpha \end{bmatrix}_{AC}
\end{aligned}
$$

and since everything is in AC frame now, we can just take the derivative to find the accelaration (note that this operation is based on the fact that we are only trying to find the acceleration as seen by AC frame; to find the inertial acceleration we need to take the derivatives of the AC frame unit vectors as well, which will not be zero in inertial frame):

$$
\overrightarrow{\dot{\alpha}}_W = \begin{bmatrix} -\ddot{\beta}\sin\alpha - \dot{\alpha}\dot{\beta}\cos\alpha \\ -\ddot{\alpha} \\ \ddot{\beta}\cos\alpha - \dot{\alpha}\dot{\beta}\sin\alpha \end{bmatrix}_{AC}
$$

2 Problem 2

Recall the equation for an ellipsoid in Cartesian coordinates is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

First we need to determine center of mass of the ellipsoid. But in the case of uniform density, by symmetry we can argue that the COM must be located at the origin in the above frame. (Note that if the object is symmetric but has an asymmetric mass distribution, this argument would not work).

Now let us first compute the products of inertia:

$$I_{12} = \iiint_{body} (xy)\, dm = \rho \iiint_V (xy)\, dV$$

It is always possible to carry out the integral, but it saves a lot of time to notice that this is an integral of an odd function over symmetric regions of integration around the symmetry point, so it must be zero. (only in a uniform density case!) And using symmetry again we conclude:

$$I_{12} = 0 = I_{13} = I_{23}$$

so we see that, since our frame is already at the center of mass and it is aligned with the body in such a way as to produce zero value products of inertia, then our coordinate frame must be aligned with the body's principal frame.

Now to find the moments of inertia:

$$\begin{aligned} I_{11} &= \iiint_{body} \left(x^2 + y^2\right) dm \\ &= \rho \int_{-a}^{a} \int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} \int_{-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} \left(x^2 + y^2\right) dz dy dx \end{aligned}$$

A common technique to handle complicated looking integrals is through change of variables:

$$\begin{aligned} \tilde{x} &= \frac{x}{a} \rightarrow dx = a d\tilde{x} \\ \tilde{y} &= \frac{y}{b} \rightarrow dy = b d\tilde{y} \\ \tilde{z} &= \frac{z}{c} \rightarrow dz = c d\tilde{z} \\ \Rightarrow I_{11} &= \rho abc \iiint_{UnitSphere} \left(b^2 \tilde{y}^2 + c^2 \tilde{z}^2\right) d\tilde{x} d\tilde{y} d\tilde{z} \end{aligned}$$

now switch to spherical coordinates:

$$
\begin{aligned}
\tilde{x} &= r\cos\theta\sin\phi \\
\tilde{y} &= r\sin\theta\sin\phi \\
\tilde{z} &= r\cos\phi \\
d\tilde{x}d\tilde{y}d\tilde{z} &= r^2\sin\phi d\phi d\theta dr \\
\Rightarrow I_{11} &= \rho abc\iiint_{UnitSphere}\left(b^2(r\sin\theta\sin\phi)^2 + c^2(r\cos\phi)^2\right)r^2\sin\phi d\phi d\theta dr
\end{aligned}
$$

Things after this point is pure mathematical manipulation. Through trignometric identities and integration formulae, we have:

$$
\begin{aligned}
I_{11} &= \frac{1}{5}m(b^2+c^2) \\
\text{by symmetry} \Rightarrow I_{22} &= \frac{1}{5}m(a^2+c^2) \\
I_{33} &= \frac{1}{5}m(a^2+b^2) \\
\Rightarrow \boldsymbol{I} &= \frac{1}{5}m\begin{bmatrix} (b^2+c^2) & 0 & 0 \\ 0 & (a^2+c^2) & 0 \\ 0 & 0 & (a^2+b^2) \end{bmatrix}
\end{aligned}
$$

To find the new inertia tensor in the rotated frame, we use the similarity transformation:

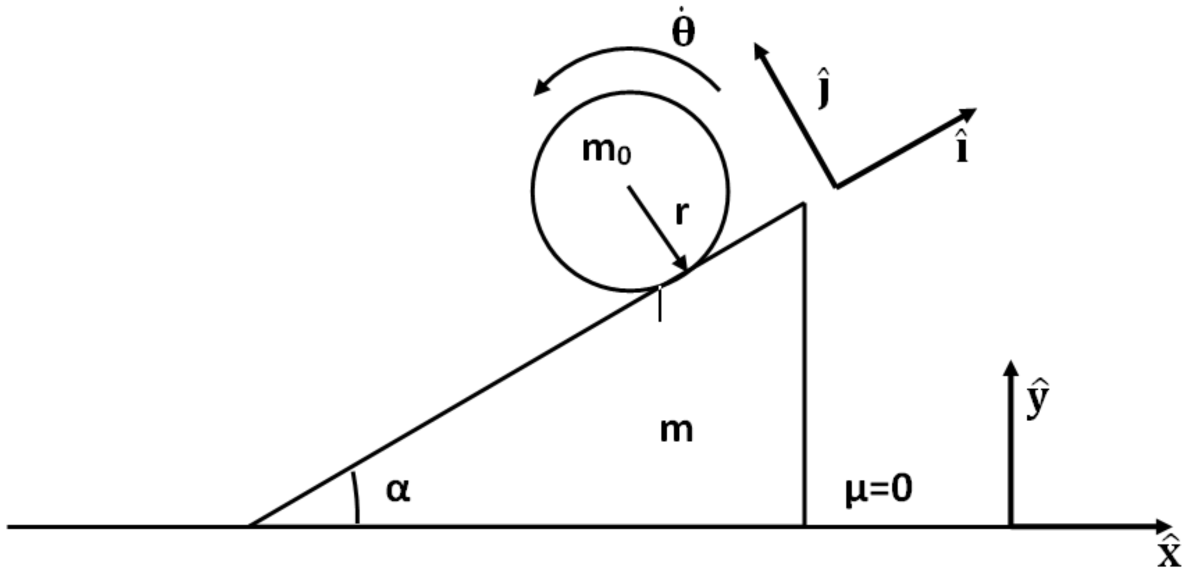
$$\boldsymbol{I'} = \overleftrightarrow{R}\boldsymbol{I'}\overleftrightarrow{R}^{-1}$$

where $\overleftrightarrow{R}$ is the rotation matrix and in this case:

$$
\begin{aligned}
\overleftrightarrow{R} &= \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \\
\Rightarrow \boldsymbol{I'} &= \frac{1}{5}m\begin{bmatrix} b^2 + (c^2\cos\theta^2 + a^2\sin theta^2) & 0 & -\frac{1}{2}(c^2-a^2)\sin 2\theta \\ 0 & a^2+c^2 & 0 \\ \frac{1}{2}(c^2-a^2)\sin 2\theta & 0 & b^2 + (c^2\cos\theta^2 + a^2\sin theta^2) \end{bmatrix}
\end{aligned}
$$

Notice the non-zero products of inertia in the rotated frame, caused by the asymmetric geometry after the rotation. This leads to the conclusion that the rotated frame is not principal; however, at $a = c$ we see that these off diagonal terms disappear! This should be apparent, since with $a = c$, the y-axis is the axis of symmetry so no matter how you rotate around it the frame would always be principal.

3 Problem 3



We will use Lagrange to solve this problem. While force balance is possible, it is more tedious and less straightforward. Do note that friction does exist between the sphere and the block, and that is what enables the sphere to roll.

First let's form the Lagrangian:

$$\begin{aligned}
\overrightarrow{v_m} &= \dot{x}\hat{x} \\
\overrightarrow{v_{m_0}} &= \dot{x}\hat{x} - r\dot{\theta}\hat{i} \\
T &= \frac{1}{2}m(\overrightarrow{v_m}\cdot\overrightarrow{v_m}) + \frac{1}{2}m_0(\overrightarrow{v_{m_0}}\cdot\overrightarrow{v_{m_0}}) + \frac{1}{2}I\omega^2 \\
&= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m_0(\dot{x}^2 + r^2\dot{\theta}^2 - 2r\dot{\theta}\dot{x}\cos\alpha) + \frac{1}{5}m_0r^2\dot{\theta}^2 \\
V &= -m_0gr\theta\sin\alpha \\
L &= T - V \\
&= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m_0(\dot{x}^2 + r^2\dot{\theta}^2 - 2r\dot{\theta}\dot{x}\cos\alpha) + \frac{1}{5}m_0r^2\dot{\theta}^2 + m_0gr\theta\sin\alpha
\end{aligned}$$

Lagragian math:

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) &= \frac{d}{dt}\left(m\dot{x} + m_0\dot{x} - m_0r\dot{\theta}\cos\alpha\right) \\
&= (m+m_0)\ddot{x} - m_0r\ddot{\theta}\cos\alpha
\end{aligned}$$

$$\begin{aligned}
\frac{\partial L}{\partial x} &= 0 \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) &= \frac{d}{dt}\left(m_0 r^2\dot{\theta} - m_0 r\dot{x}\cos\alpha + \frac{2}{5}m_0 r^2\dot{\theta}\right) \\
&= m_0 r^2\ddot{\theta} - m_0 r\ddot{x}\cos\alpha + \frac{2}{5}m_0 r^2\ddot{\theta} \\
&= \frac{7}{5}m_0 r^2\ddot{\theta} - m_0 r\ddot{x}\cos\alpha \\
\frac{\partial L}{\partial \theta} &= m_0 g r\sin\alpha
\end{aligned}$$

We then have the two equations of motion:

$$\begin{aligned}
(m+m_0)\ddot{x} - m_0 r\ddot{\theta}\cos\alpha &= 0 \\
\frac{7}{5}m_0 r^2\ddot{\theta} - m_0 r\ddot{x}\cos\alpha - m_0 g r\sin\alpha &= 0
\end{aligned}$$

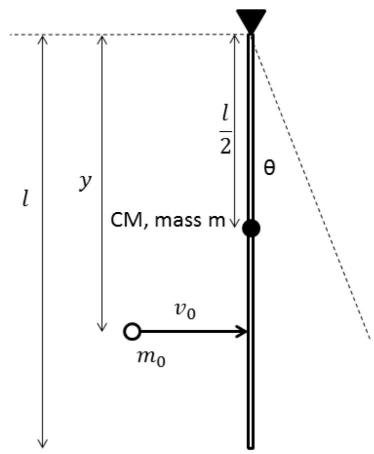
Substituting the first equation into the second gives:

$$\begin{aligned}
\frac{7}{5}m_0 r^2\left(\frac{m+m_0}{m_0 r\cos\alpha}\right)\ddot{x} - m_0 r\ddot{x}\cos\alpha - m_0 g r\sin\alpha &= 0 \\
\frac{7}{5}\left(\frac{m+m_0}{m_0\cos\alpha}\right)\ddot{x} - \ddot{x}\cos\alpha - g\sin\alpha &= 0 \\
\left(\frac{7}{5}(m+m_0) - m_0\cos^2\alpha\right)\ddot{x} &= g m_0\sin\alpha\cos\alpha \\
\ddot{x} &= \frac{5 g m_0\sin\alpha\cos\alpha}{7(m+m_0) - 5m_0\cos^2\alpha}
\end{aligned}$$

Integrating will give (with initial conditions):

$$\dot{x} = \frac{5m_0 g\sin\alpha\cos\alpha}{7(m+m_0) - 5m_0\cos^2\alpha}t$$

4 Problem 4



First notice that whenever a mass particle “sticks” with another mass, we can almost always conclude that energy is not conserved. If there is no impulse at the pivot point at the instant of impact, we can see that no external force/impulse acts on the system defined as the point mass m_0 and the stick, so the total linear momentum in the horizontal direction is conserved; also, at the moment of impact both the center of mass of the rod and the point particle is directly under the pivot point, so gravity exerts no external torque on the system as well: angular momentum is also conserved at the instant of impact:

$$\begin{aligned}\overrightarrow{p}(0^-) &= \overrightarrow{p}(0^+)\\ \overrightarrow{H}(0^-) &= \overrightarrow{H}(0^+)\end{aligned}$$

Assume that the point mass strikes the rod and sticks at distance y from the pivot; furthermore let the angular velocity of the rod-mass system be $\dot{\theta}$ right after impact. Now we can write all quantities using these two variables and the known constants:

$$
\begin{aligned}
\overrightarrow{p}(0^-) &= m_0 v_0 \\
\overrightarrow{H}(0^-) &= m_0 v_0 y \\
\overrightarrow{p}(0^+) &= m_0 y\dot{\theta} + m\frac{l}{2}\dot{\theta} = \left(\frac{ml}{2} + m_0 y\right)\dot{\theta} \\
\overrightarrow{H}(0^+) &= m_0 y\dot{\theta} y + \frac{ml^2}{3}\dot{\theta} = \left(\frac{ml^2}{3} + m_0 y^2\right)\dot{\theta} \\
\text{2 equations, 2 variables} \Rightarrow y &= \frac{2}{3}l \\
\dot{\theta} &= \frac{m_0 v_0}{\left(\frac{1}{2}m + \frac{2}{3}m_0\right)l} = \frac{6 m_0 v_0}{(3m + 4m_0)l}
\end{aligned}
$$