AA 242A Homework 1 Solution

Autumn 2021

1 Problem 1

Consider the vector $\boldsymbol{A} = A_x\boldsymbol{i} + A_y\boldsymbol{j} + A_z\boldsymbol{k}$. Find the components A_1, A_2, A_3 of the vector $\boldsymbol{A}$ in a skewed coordinate system whose axes have directions specified by the following unit vector triad:

$$\boldsymbol{e}_1 = \boldsymbol{i}; \boldsymbol{e}_2 = \tfrac{\boldsymbol{i}+\boldsymbol{j}}{\sqrt{2}}; \boldsymbol{e}_3 = \tfrac{\boldsymbol{i}+\boldsymbol{k}}{\sqrt{2}}$$

Solution

In order to find the answer, we must express the components of one frame in terms of the other. We can write the following equality:

$$\boldsymbol{A} = A_x\boldsymbol{i} + A_y\boldsymbol{j} + A_z\boldsymbol{k} = A_1\boldsymbol{e}_1 + A_2\boldsymbol{e}_2 + A_3\boldsymbol{e}_3$$

and then substitute for the basis vectors and rearrange:

$$\begin{aligned} A_x\boldsymbol{i} + A_y\boldsymbol{j} + A_z\boldsymbol{k} &= A_1(\boldsymbol{i}) + A_2\frac{\boldsymbol{i}+\boldsymbol{j}}{\sqrt{2}} + A_3\frac{\boldsymbol{i}+\boldsymbol{k}}{\sqrt{2}} \\ &= (A_1 + \frac{A_2}{\sqrt{2}} + \frac{A_3}{\sqrt{2}})\boldsymbol{i} + (\frac{A_2}{\sqrt{2}})\boldsymbol{i} + (\frac{A_3}{\sqrt{2}})\boldsymbol{k} \end{aligned}$$

Now we can equate the coefficients and solve for A_1, A_3, and A_3:

$$\begin{aligned} A_z = \frac{A_3}{\sqrt{2}} &\quad\Rightarrow\quad A_3 = \sqrt{2}A_z \\ A_y = \frac{A_2}{\sqrt{2}} &\quad\Rightarrow\quad A_2 = \sqrt{2}A_y \\ A_x = A_1 + \frac{A_2}{\sqrt{2}} + \frac{A_3}{\sqrt{2}} &\quad\Rightarrow\quad A_1 = A_x - \frac{A_2}{\sqrt{2}} - \frac{A_3}{\sqrt{2}} = A_x - A_y - A_z \end{aligned}$$

We could also have written these equations in matrix form:

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}$$

$$
\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}^{-1} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}
$$

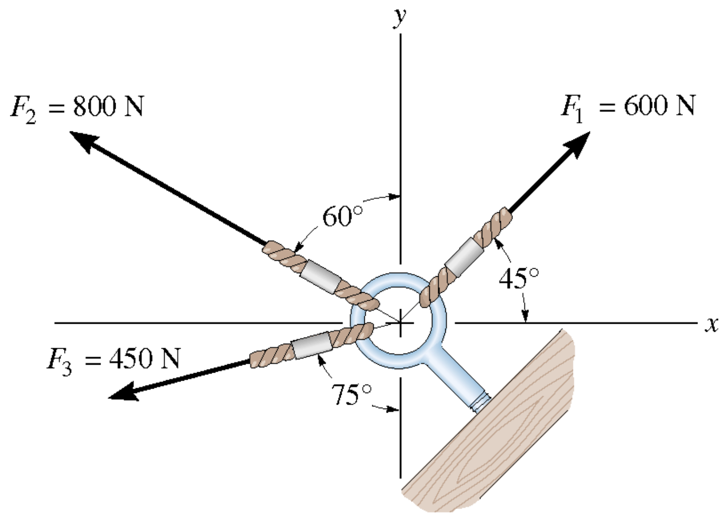
A common mistake is to write:

$$
\boldsymbol{A} = (\boldsymbol{A} \cdot \boldsymbol{e}_1)\boldsymbol{e}_1 + (\boldsymbol{A} \cdot \boldsymbol{e}_2)\boldsymbol{e}_2 + (\boldsymbol{A} \cdot \boldsymbol{e}_3)\boldsymbol{e}_3
$$

This works only if the basis vectors are orthonormal. If the basis vectors are not of unit length, then the dot product does not provide the right scaling. (Try it with a basis of $2\boldsymbol{i}$, $2\boldsymbol{j}$, $2\boldsymbol{k}$ if you are not convinced.) If they are not orthogonal, then the components of each basis are not necessarily the same as the projections since the contribution from one basis can project onto another basis.

2 Problem 2

Determine the magnitude of the resultant force and its direction, measured counterclockwise from the positive x axis.



Solution

The resultant force **F** is equal to the vector sum of $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, and $\boldsymbol{F}_3$. Resolving each vector into the x-y coordinate system shown gives:

$$\begin{aligned}
\boldsymbol{F}_1 &= 600\cos 45^o \hat{\boldsymbol{x}} + 600\sin 45^o \hat{\boldsymbol{y}} \\
\boldsymbol{F}_2 &= 800\sin 60^o(-\hat{\boldsymbol{x}}) + 800\cos 60^o \hat{\boldsymbol{y}} \\
\boldsymbol{F}_3 &= 450\sin 75^o(-\hat{\boldsymbol{x}}) + 450\cos 75^o(-\hat{\boldsymbol{y}})
\end{aligned}$$

Adding these together:

$$\begin{aligned}
\boldsymbol{F} &= \boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 \\
&= (600\cos 45^o - 800\sin 60^o - 450\sin 75^o)\hat{\boldsymbol{x}} + (600\sin 45^o + 800\cos 60^o - 450\cos 75^o)\hat{\boldsymbol{y}} \\
&= -703.2229\hat{\boldsymbol{x}} + 707.7955\hat{\boldsymbol{y}}
\end{aligned}$$

The magnitude of the resultant force F is thus:

$$||\boldsymbol{F}|| = \sqrt{(-703.2229)^2 + (707.7955)^2} = 997.75\text{N}$$

The direction of the resultant force $\boldsymbol{F}$ from the positive x-axis can be found from the dot product of $\boldsymbol{F}$ with $\hat{x}$, which gives:

$$\theta = \arccos \frac{\boldsymbol{F} \cdot \hat{\boldsymbol{x}}}{||\boldsymbol{F}||} = \arccos \frac{-703.2229}{997.75} = 134.8^o$$

From the signs of the components of $\boldsymbol{F}$, we know that it lies in the second quadrant. We could also have used various other inverse trig functions to find the angle and to verify the quadrant ambiguity.

3 Problem 3

An airplane of mass m starts at a speed v_0 in level flight (lift = weight so no vertical motion). Its engine exerts a constant thrust T. The drag experienced by the aircraft is $F_D = bv^2$.
a. Find the speed of the airplane as a function of time.
b. Find the eventual steady-state speed of the airplane.
c. If this were a rocket plane (like the Bell X-1) and you had to consider the mass loss of fuel, how would you change your formulation of the problem? (You don't have to actually solve this problem unless you really want to)

Solution

a. Find the speed of the airplane as a function of time. Using Newton's 2nd law for objects with constant mass:

$$\begin{aligned} m\frac{dv}{dt} &= T - F_D = T - bv^2 \\ \frac{dv}{dt} &= \frac{F}{m} = \frac{T - bv^2}{m} \\ \int_{v_0}^{v} \frac{m}{T - bv^2} dv &= \int_0^t dt \\ \frac{m}{T}\int_{v_0}^{v} \frac{dv}{1 - \frac{b}{T}v^2} &= t \end{aligned}$$

To take the integral, we let $v = \sqrt{\frac{T}{b}}v'$ and $dv = \sqrt{\frac{T}{b}}dv'$, such that

$$\frac{m}{T}\sqrt{\frac{T}{b}}\int_{\sqrt{\frac{b}{T}}v_0}^{\sqrt{\frac{b}{T}}v} \frac{dv'}{1 - v'^2} = t$$

The following hint is given in problem session 1:

$$\int \frac{1}{1 - x^2} dx = \tanh^{-1} x + C, \quad |x| < 1.$$

Since we assume $T > bv^2$, $|v'| < 1$, and we can evaluate the integral,

$$\frac{m}{T}\sqrt{\frac{T}{b}}\int_{\sqrt{\frac{b}{T}}v_0}^{\sqrt{\frac{b}{T}}v} \frac{dv'}{1 - v'^2} = \frac{m}{T}\sqrt{\frac{T}{b}}\left(\tanh^{-1}\left(\sqrt{\frac{b}{T}}v\right) - \tanh^{-1}\left(\sqrt{\frac{b}{T}}v_0\right)\right) = t$$

Solving for v gives,

$$\boxed{v(t) = \sqrt{\frac{T}{b}} \tanh\left(\frac{\sqrt{bT}}{m}(t+C)\right), \quad C = \frac{m}{\sqrt{Tb}} \tanh^{-1}\left(\sqrt{\frac{b}{T}} v_0\right)}$$

Alternative solution: The integral can be solved by hand using partial fractions decomposition, to get the following result:

$$\int \frac{1}{1-x^2} dx = \frac{1}{2}\bigg(\ln|1+x| - \ln|1-x| \bigg) + C$$

This expression can be split into the cases where $|x| < 1$ (physically, thrust is greater than drag, and the aircraft speeds up to reach a steady-state) and $|x| > 1$ (thrust is less than drag, aircraft slows down to reach a steady-state). In this problem we consider $|x| < 1$, and this allows us to remove the absolute value signs,

$$\int \frac{1}{1-x^2} dx = \frac{1}{2}\bigg(\ln(1+x) - \ln(1-x) \bigg) + C = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) + C$$

Using this solution to evaluate the integral gives,

$$\frac{m}{2\sqrt{Tb}} \left[\ln\left(\frac{1+\sqrt{\frac{b}{T}}v}{1-\sqrt{\frac{b}{T}}v} \right) + C_1 \right] = t, \quad C_1 = -\ln\left(\frac{1+\sqrt{\frac{b}{T}}v_0}{1-\sqrt{\frac{b}{T}}v_0} \right)$$

Solving for v gives,

$$\boxed{v(t) = \sqrt{\frac{T}{b}} \left(\frac{C_2 \exp\left(\frac{2\sqrt{Tb}}{m}t\right) - 1}{C_2 \exp\left(\frac{2\sqrt{Tb}}{m}t\right) + 1} \right), \quad C_2 = \frac{\sqrt{T/b} + v_0}{\sqrt{T/b} - v_0}.}$$

This expression is equivalent to the other solution boxed above.

b. The function $\tanh(t) \to 1$ as $t \to \infty$, so $v = \sqrt{\frac{T}{b}(1)^2} = \sqrt{\frac{T}{b}}$. We can also solve this problem by equating the forces acting on the airplane, since $\frac{dv}{dt} = 0$ in steady-state: $T = F_D = bv^2$.

c. If this were a rocket plane, then instead of using $F = m\frac{dv}{dt}$, which assumes a constant mass, we must now perform a differential momentum balance with the equation $F = \frac{dp}{dt}$, similarly to how the rocket equation is derived (covered later in the course). It is also acceptable to say that since $p = mv$, $\frac{dp}{dt} = m\frac{dv}{dt} + v\frac{dm}{dt} = \Sigma F$, but one must be careful with this. When mass loss is involved, one must include the mass element dm that leaves the system, in the momentum balance at a given instant (again see the rocket equation derivation).

Frequent mistake: It is not sufficient to replace m with $m(t)$ in the ODE equation of motion from part a, since although this leads to a more complicated solution, it still assumes $F = ma$, which is **incorrect** when $m = m(t)$.

4 Problem 4

MATLAB primer: This last homework problem is intended so that you can begin to familiarize yourself with MATLAB. Please print out your MATLAB session, which should include the commands you gave and the response. Answers should be clearly indicated; this problem should be solved using MATLAB (even if it is easy for you to do by hand). a. Find the unit vector in the direction of (1, -2, 4). b. Find the angle (in radians) between the vectors (1, -2, 3) and (3, 1, 4). c. Print out a plot of the level curves for the function

$$f(x, y) = (x^3 - 3x)(2y^3 - y)$$

for $-2 \leq x \leq 2 and - 1 \leq y \leq 1$. On your plot of the level curves draw (by hand) vectors in the direction of the gradient of f at four points near the four corners.

Solution

a. We can find the unit vector in the direction of (1,-2,4) using the definition directly in MATLAB or using the norm command:

```
>> r = [1;-2;4]
r =
     1
    -2
     4
>> r_hat = r/sqrt(r(1)^2+r(2)^2+r(3)^2)
r_hat =
    0.2182
   -0.4364
    0.8729
>> r_hat = r/norm(r)
r_hat =
    0.2182
   -0.4364
    0.8729
```

We usually write vectors in column form to facilitate their use with transformation matrices. However, MATLAB can operate on row vectors as well.

b. Here we can find the angle θ between the two vectors using either the definition of the dot product or the cross product:

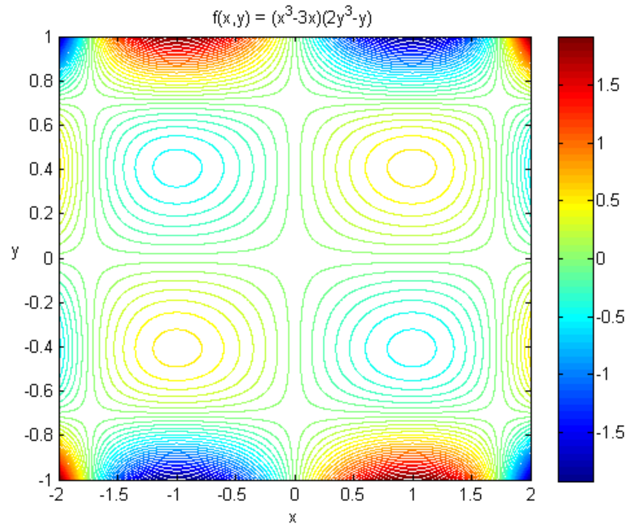
```
>> r_1 = [1;-2;3]
r_1 =
     1
    -2
     3
>> r_2 = [3;1;4]
r_2 =
     3
     1
     4
>> theta = acos(dot(r_1,r_2)/(norm(r_1)*norm(r_2)))
theta =
    0.8211
>> theta = asin(norm(cross(r_1,r_2))/(norm(r_1)*norm(r_2)))
theta =
    0.8211
```

The two angles match up so we know the angle must lie in the first quadrant. Since we have eliminated quadrant ambiguity, we have solved the problem. The angle between the two vectors is 0.8211 radians.

c. Using the following script:

```
>> x=linspace(-2,2,1000);
>> y = linspace(-1,1,1000);
>> [X,Y] = meshgrid(x,y);
>> f = (X.^3-3.*X).*(2.*Y.^3-Y);
>> figure;
>> contour(X,Y,f,50);
>> colorbar
>> xlabel('x')
>> ylabel('y','Rotation',0)
>> title('f(x,y) = (x^3-3x)(2y^3-y)');
```

We obtain:



In our contour plot, heights of the contours are indicated by each curve according to the color scale on the right (created using the "colorbar" command). The gradient can be found by computing the derivatives of f:

$$\begin{aligned}\frac{\partial f(x,y)}{\partial x} &= (3x^2-3)(2y^3-y)\\ \frac{\partial f(x,y)}{\partial y} &= (x^3-3x)(6y^2-1)\end{aligned}$$

These can be evaluated at any given x,y pair. Representative vectors pointing in the direction of the gradient are shown at each of the four corners of the plot. Remember that at any given coordinate location, the gradient points uphill and normal to the contours of f.

5 Problem 5

In this class, we will be deriving the equations of motion of many systems. Very often, these equations are second order, coupled and nonlinear Ordinary Differential Equations (ODEs) which are difficult (if not impossible) to solve analytically. Matlab's ODE45 and ODE113 are very useful packages that numerically integrate ODEs. This problem is intended for you to familiarize yourself with these packages which will be useful for future problem sets. Using ODE45 or ODE113, numerically integrate the following set of equations-

$$\begin{aligned} \ddot{x_1} &= -ax_1 + b(x_2 - x_1) \\ \ddot{x_2} &= -c(x_2 - x_1) \end{aligned}$$

where the dots indicate derivatives with respect to time. These are equations that may represent a coupled oscillatory system (like masses and springs). Solve these equations from t=0 to t=100 for the following initial conditions-

$$\begin{aligned} x_1(0) &= 1 \qquad & \dot{x_1}(0) &= 0 \\ x_2(0) &= 1 \qquad & \dot{x_2}(0) &= 0 \end{aligned}$$

Solve the equations for the two cases listed below and plot $x_1(t)$, $x_2(t)$ and $(x_1 + x_2)(t)$:

1. $a = 4, b = 1, c = 7.5$
2. $a = 1, b = 1, c = 1.5$

Notice anything interesting in the second case? Try changing initial conditions for this case. What happens?

Solution

The two second order equations need to be broken up into 4 first order equations for ODE45 or ODE113 as follows:

$$
\begin{aligned}
\dot{x_1} &= u \\
\dot{u} &= -ax_1 + b(x_2 - x_1) \\
\dot{x_2} &= v \\
\dot{v} &= -c(x_2 - x_1)
\end{aligned}
$$

Once we have them in this form, we can formulate the RHS function for ODE45 as follows:

```
%% PROBLEM 5 RHS FUNCTION

function [y_dot]=rhs_prob5(t,y)

a=1;
b=1;c=1.5;

y_dot=zeros(4,1);

y_dot(1)=y(2);
y_dot(2)=-a*y(1)+b*(y(3)-y(1));
y_dot(3)=y(4);
y_dot(4)=-c*(y(3)-y(1));


%% END PROBLEM 5 RHS FUNCTION
```

Here the vector y is $[x_1, u, x_2, v]^T$. The main code then applies the initial conditions and solves the equations using ODE45:

```
%% PROBLEM 5 Main Code
```

```
clear;

options=odeset('RelTol',1e-3,'AbsTol',1e-6); % Sets error tolerance

y_initial=[1 0 1 0]'; %Sets initial conditions

[t_out, y_out]=ode45(@rhs_prob5,[0:0.1:100]',y_initial,options);
% Solve ODEs


% Plot routine
subplot(2,1,1);
plot(t_out,y_out(:,1),t_out,y_out(:,3))
legend('X_1','X_2');
xlabel('t');
title('a=1 b=1 c=1.5');
subplot(2,1,2);
plot(t_out,y_out(:,3)+y_out(:,1))
xlabel('t');
ylabel('x_1 + x_2');

%% END PROBLEM 5 Main Code
```

The results are shown in the figures below:

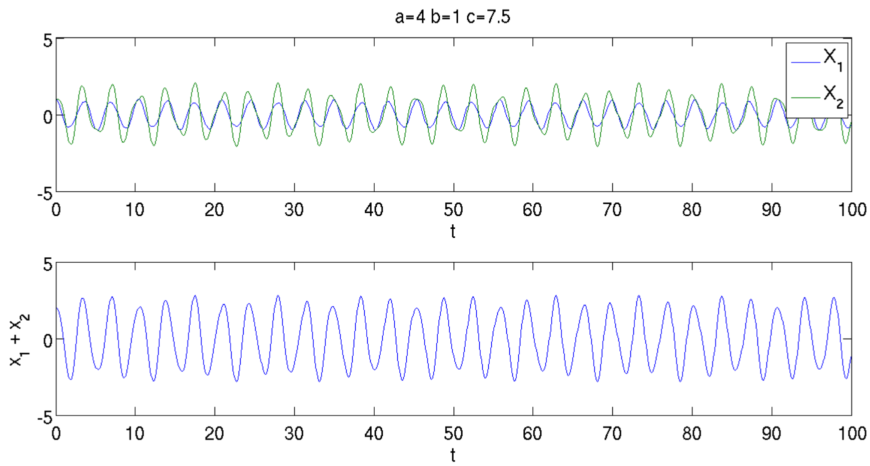


Figure 1: Case 1: a=4, b=1, c=7.5

In case 2, when the two equations are added, the combined equation for $x_1 + x_2$ is that of a simple harmonic oscillator. This is shown by the plot for case 2. On changing the initial conditions, the shape of the $x_1 + x_2$ curve does not change.

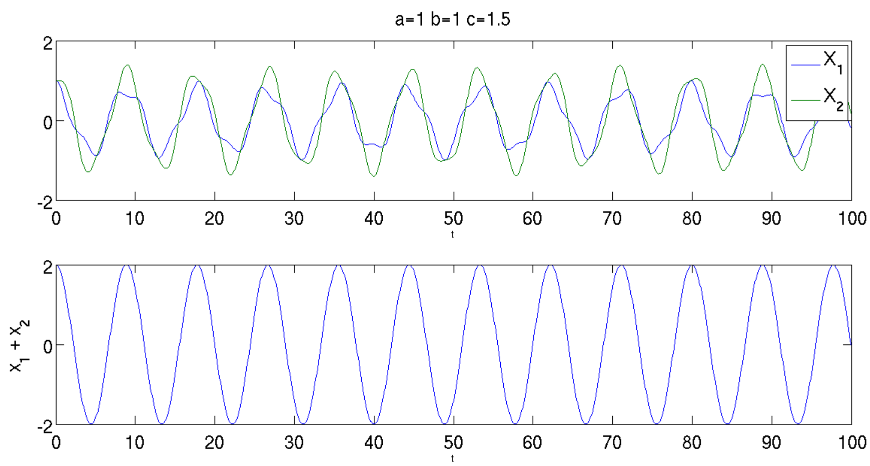


Figure 2: Case 2: a=1, b=1, c=1.5