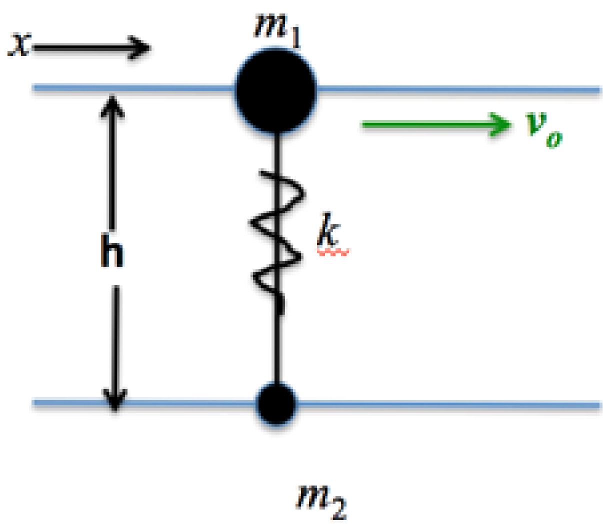
AA 242A Midterm 2020 Solutions

October 30, 2020

Problem 1

Janeway and Torres are fixing the warp core. They are sliding without friction on parallel fixed horizontal wires, separated by a distance h. Approximate Janeway as a particle of mass $m_1 = 2m$, and Torres as a particle of mass $m_2 = m$. A spring of stiffness k and unstretched length h connects Janeway and Torres. Define their positions on the wires as x_1 and x_2, respectively. Assume Janeway has an initial velocity v_0 and that Torres is initially motionless; ignore gravity.



a) What is and is not conserved in this system. Justify your answer.

Defining the system as both Janeway and Torres:

- Energy is *conserved*. There are external constraint forces from the wires that act on both Janeway and Torres, but they do no work. Additionally, there is an internal force from the spring, but this is conservative. No acting forces are dissipative since the wires are frictionless.

- Linear momentum in the x direction is *conserved*. No external forces act in the x direction and the spring is internal to the system. Note that linear momentum in the x direction is *not conserved* for Janeway and Torres individually.

- Linear momentum in the y direction is *conserved*. Although external forces act on the system in y, they balance each other such that the system has no motion in this coordinate. Note that it *is conserved* for Janeway and Torres individually as well.

- Since linear momentum of the system is conserved, the reference frame attached to the center of mass is an inertial frame. Angular momentum about the center of mass, however, is *not conserved* since the constraint forces do not act radially and can therefore exert a moment on the system. This is true for Janeway and Torres individually as well.

b) Find the maximum velocity of Torres.
We can use conservation of energy and linear momentum in x to solve this problem. Since m_1 is the only mass moving and the spring is unstretched initially, the kinetic energy, potential energy, and momentum at $t = 0$ are

$$T_0 = \frac{1}{2}m_1 v_0^2 = m v_0^2, \qquad V_0 = 0, \qquad p_0 = m_1 v_0 = 2m v_0$$

At some other point in time, the energies and momentum for the system are

$$T = m v_1^2 + \frac{1}{2}m v_2^2, \qquad V = \frac{1}{2}k\delta^2, \qquad p = 2m v_1 + m v_2$$

where v_1 and v_2 are the velocities of Janeway and Torres, respectively, and δ is the stretch in the spring (such that $\delta + h$ is the total length of the spring. Using conservation,

$$\begin{aligned} v_0^2 &= v_1^2 + \frac{1}{2}v_2^2 + \frac{k}{2m}\delta^2 \\ 2v_0 &= 2v_1 + v_2 \end{aligned}$$

We have two equations (from conservation) and 3 unknowns. The third equation comes from the condition for maximum velocity of Torres:

$$\frac{d}{dt}v_2 = 0$$

We can take this derivative directly, or we can use the conservation equations to eliminate a variable, say v_1, and then find the value of the other variable δ that maximizes v_2. Using momentum,

$$v_1 = v_0 - \frac{1}{2}v_2$$

Plugging into energy conservation

$$\begin{aligned} v_0^2 &= \left(v_0 - \frac{1}{2}v_2\right)^2 + \frac{1}{2}v_2^2 + \frac{k}{2m}\delta^2 \\ 0 &= \frac{3}{4}v_2^2 - v_0 v_2 + \frac{k}{2m}\delta^2 \\ v_2 &= \frac{2}{3}\left(v_0 \pm \sqrt{v_0^2 - \frac{3k}{2m}\delta^2}\right) \end{aligned}$$

The last equation gives v_2 as a function of δ. The value of δ that gives the maximum stretch can be found by inspection; the discriminant decreases in value for any $\delta \neq 0$, so $\delta = 0$ gives the extreme value. More formally, we can solve for the stretch that gives the maximum velocity by taking the derivative and setting to zero.

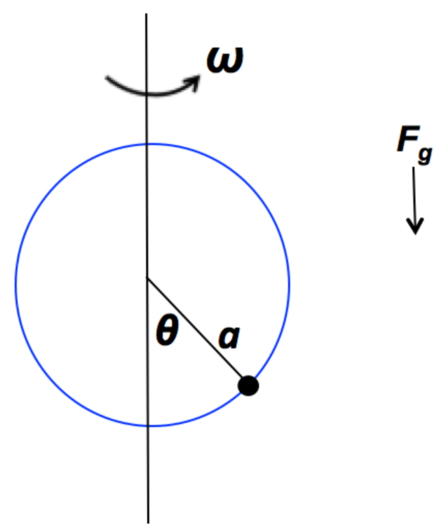
$$\frac{dv_2}{d\delta} = \mp \frac{\frac{k}{m}\delta}{\sqrt{v_0^2 - \frac{3k}{2m}\delta^2}} = 0$$

This is trivially given by $\delta = 0$. We have two solutions for this particular value of the stretch, $v_2^* = 0$ and $v_2^* = \frac{4}{3}v_0$. The former value gives the minimum while the latter value gives the maximum.

$$\boxed{v_2^* = \frac{4}{3}v_0}$$

Problem 2

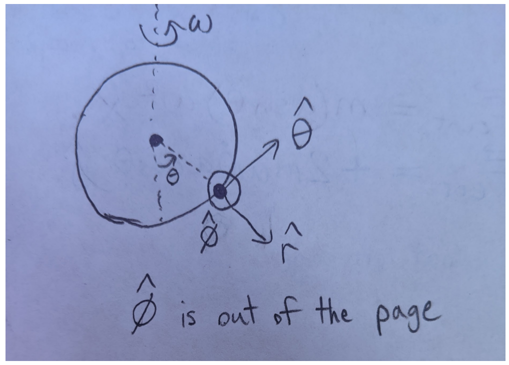
Tuvok has developed a new *effective* weapon to battle Species 8472. His weapon can be approximated as a point particle of mass m that slides without friction under the influence of gravity on a rotating massless wire hoop of radius a. The hoop rotates with a constant angular velocity ω about its vertical axis.



First, let's define our coordinate system. The motion of the particle on the hoop, coupled with the hoop's rotation, lends itself well to a spherical coordinate system - see image below.

a) Write down the constraint equation(s).

There are 2 constraints: the mass being fixed at constant radius a on the hoop, and the hoop rotating with constant angular velocity ω. Let's define the particles position using a spherical coordinate system, with radius r, azimuth angle ϕ, and elevation angle θ (shown



in problem statement).

First, the particle is constrained to slide on the wire-hoop. The constraint equation for this condition is

$$\boxed{f_1 = r - a = 0}$$

Second, the hoop rotates with constant angular velocity ω.

$$\boxed{f_2 = \phi - \omega t = 0}$$

b) Write down the generalized coordinate(s). How many degrees of freedom are there? It is natural to use the following set of generalized coordinates to describe the particle's position: $\boxed{r, \theta, \phi}$.

Since we have 3 generalized coordinates, and 2 constraints, there are $3 - 2 = \boxed{1 \text{ DOF}}$.

c) Write an expression for the angular velocity of the particle with respect to the inertial frame.
The particle has two "layers" of rotation: its rotation due to the hoop's rotational motion about the vertical $\hat{z}$-axis, and it's rotation about the hoop's central axis (out of/into the page) due to its sliding **on** the hoop.

Rotation of hoop with respect to inertial surroundings:

$$\vec{\omega}_{H/I} = \omega \hat{z}$$

$\hat{z}$ is not a spherical coordinate, so it'll be easiest to get it in terms of our spherical unit vector basis. Using some trig,

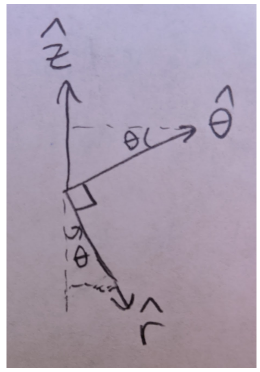
$$\hat{z} = -\cos\theta \hat{r} + \sin\theta \hat{\theta}$$

So,

$$\vec{\omega}_{H/I} = -\omega \cos\theta \hat{r} + \omega \sin\theta \hat{\theta}$$

Rotation of the mass sliding on the hoop:

$$\vec{\omega}_{P/H} = \dot{\theta} \hat{\phi}$$



Summing the two rotations to get the particle's total rotation vector,

$$\boxed{\vec{\omega}_{P/I} = \vec{\omega}_{P/H} + \vec{\omega}_{H/I} = \dot{\theta}\hat{\phi} - \omega\cos\theta\hat{r} + \omega\sin\theta\hat{\theta}}$$

d) Write down the Lagrangian of the system. The Lagrangian is defined as

$$L = T - V$$

To find T, we need the inertial velocity of the particle. First we need the position vector,

$$\vec{r} = a\hat{r}$$

Taking an inertial time-derivative to get velocity, and applying the **Golden Rule**,

$$^{I}\vec{v} = {}^{I}\dot{\vec{r}} = a\,{}^{I}\dot{\hat{r}} = a(\vec{\omega}_{P/I} \times \hat{r})$$

$$\vec{\omega}_{P/I} \times \hat{r} = (\dot{\theta}\hat{\phi} - \omega\cos\theta\hat{r} + \omega\sin\theta\hat{\theta}) \times \hat{r} = \dot{\theta}\hat{\theta} - \omega\sin\theta\hat{\phi}$$

$$\Rightarrow \quad {}^{I}\vec{v} = a\dot{\theta}\hat{\theta} - a\omega\sin\theta\hat{\phi}$$

Finding kinetic energy of the particle,

$$T = \frac{1}{2}m({}^{I}\vec{v} \cdot {}^{I}\vec{v}) = \frac{1}{2}m(a^2\dot{\theta}^2 + a^2\omega^2\sin^2\theta)$$

Finding potential energy of the particle, defining our datum as the particle's height when $\theta = 0$,

$$V = mga(1 - \cos\theta)$$

The Lagrangian is,

$$\boxed{L = \frac{1}{2}m(a^2\dot{\theta}^2 + a^2\omega^2\sin^2\theta) - mga(1 - \cos\theta)}$$

e) Find the EoM(s). Since only one generalized coordinate remains in our Lagrangian, we must find only one EoM, for θ.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

Plugging in the Lagrangian,

$$ma^2\ddot{\theta} - ma^2\omega^2 \sin\theta \cos\theta - mga \sin\theta = 0$$

$$\boxed{a\ddot{\theta} - a\omega^2 \sin\theta \cos\theta + g \sin\theta = 0}$$

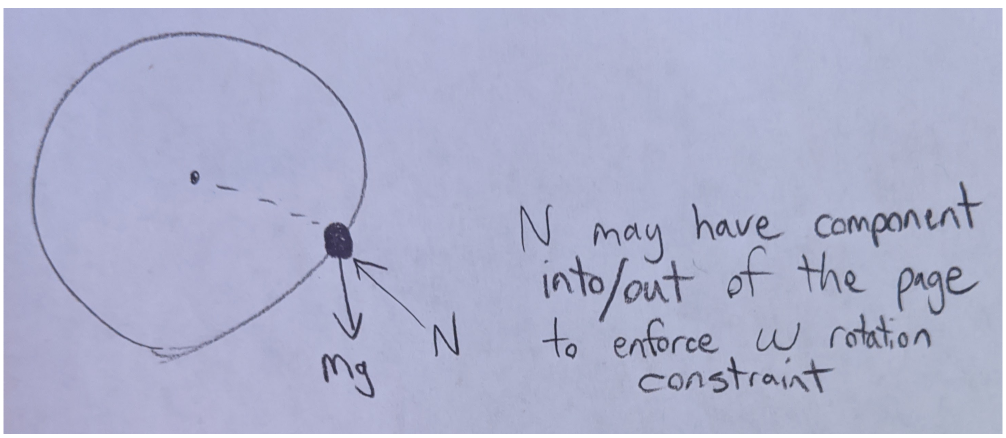
f) In two sentences or less, explain what happens to your EoM if ω is 0, and therefore what the term with ω represents. Setting $\omega = 0$ and simplifying,

$$\ddot{\theta} + \frac{g}{a} \sin\theta = 0$$

This is the equation of motion for a simple pendulum.
With finite ω, a centrifugal force is introduced to the system.

Problem 3

g) Draw a FBD in the inertial frame.



h) Find the EoM(s) using Force Balance. (Use your result from part c). The relevant result from part c is the angular velocity of the particle wrt inertial. The result for the particle's inertial velocity is also useful here.

For the force-balance method, we apply Newton's 2nd law,

$$\sum \vec{F} = m^I\vec{a}$$

We need the inertial acceleration. From problem 2 we have already solved for inertial velocity,

$${}^I\vec{v} = a\dot{\theta}\hat{\theta} - a\omega \sin\theta \hat{\phi}$$

Taking another inertial derivative to get acceleration,

$${}^I\vec{a} = {}^I\dot{\vec{v}} = a\ddot{\theta}\hat{\theta} + a\dot{\theta}\,{}^I(\dot{\hat{\theta}}) - a\omega\dot{\theta}\cos\theta\hat{\phi} - a\omega \sin\theta\,{}^I(\dot{\hat{\phi}})$$

Using the **Golden Rule** to find time-derivatives of unit vectors,

$$^{I}\dot{(\hat{\theta})} = \vec{\omega}_{P/I} \times \hat{\theta} = (\dot{\theta}\hat{\phi} - \omega\cos\theta\hat{r} + \omega\sin\theta\hat{\theta}) \times \hat{\theta} = -\dot{\theta}\hat{r} - \omega\cos\theta\hat{\phi}$$

$$^{I}\dot{(\hat{\phi})} = \vec{\omega}_{P/I} \times \hat{\phi} = (\dot{\theta}\hat{\phi} - \omega\cos\theta\hat{r} + \omega\sin\theta\hat{\theta}) \times \hat{\phi} = \omega\cos\theta\hat{\theta} + \omega\sin\theta\hat{r}$$

Plugging these in to find the acceleration vector,

$$^{I}\vec{a} = a\ddot{\theta}\hat{\theta} + a\dot{\theta}(-\dot{\theta}\hat{r} - \omega\cos\theta\hat{\phi}) - a\omega\dot{\theta}\cos\theta\hat{\phi} - a\omega\sin\theta(\omega\cos\theta\hat{\theta} + \omega\sin\theta\hat{r})$$

Simplifying,

$$^{I}\vec{a} = (a\omega^2\sin^2\theta - a\dot{\theta}^2)\hat{r} + (a\ddot{\theta} - a\omega^2\sin\theta\cos\theta)\hat{\theta} - (2a\dot{\theta}\omega\cos\theta)\hat{\phi}$$

Taking force balance,

$$\sum \vec{F} = N_r\hat{r} + N_\phi\hat{\phi} - mg\hat{z} = (N_r + mg\cos\theta)\hat{r} - mg\sin\theta\hat{\theta} + N_\phi\hat{\phi} = m^{I}\vec{a}$$

Note that the normal force $\vec{N}$ exerted on the mass by the hoop has components in the $\hat{r}$ and $\hat{\phi}$ directions, N_r and N_ϕ, respectively, since these are the directions in which the mass is constrained by the hoop. The mass is free to slide in the $\hat{\theta}$ direction, so $N_\theta = 0$.

Splitting the force balance vector equation into $\hat{r}$, $\hat{\theta}$, and $\hat{\phi}$ components gives us our EoMs in each direction,

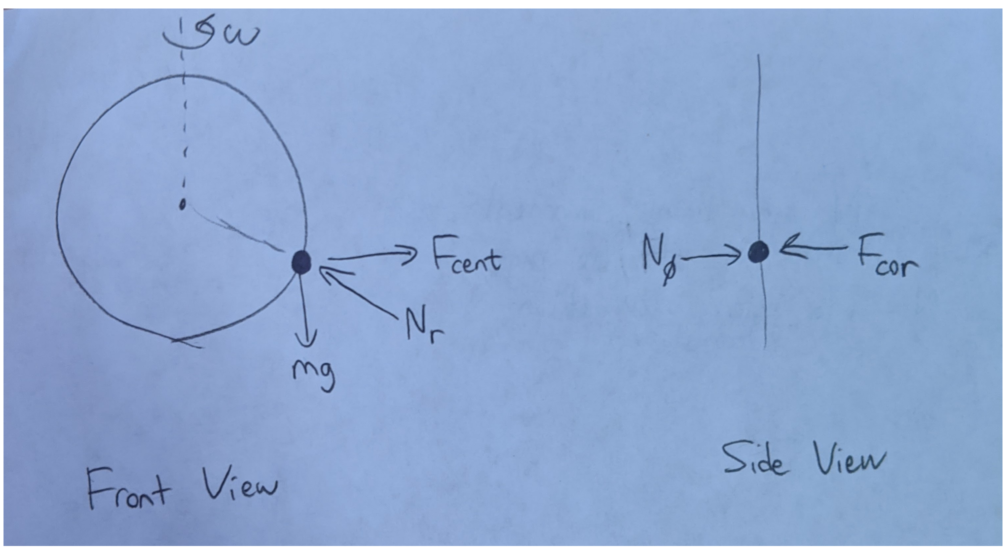
$$\boxed{\begin{aligned} \hat{r} :&\quad N_r + mg\cos\theta = ma\omega^2\sin^2\theta - ma\dot{\theta}^2 \\ \hat{\theta} :&\quad -mg\sin\theta = ma\ddot{\theta} - ma\omega^2\sin\theta\cos\theta \\ \hat{\phi} :&\quad N_\phi = -2ma\dot{\theta}\omega\cos\theta \end{aligned}}$$

The normal forces N_r and N_ϕ are unknown, but since we have 3 equations, we can solve for them in terms of θ and its time-derivatives. Manipulating the $\hat{\theta}$ equation gives

$$\boxed{a\ddot{\theta} + g\sin\theta - a\omega^2\sin\theta\cos\theta = 0}$$

which is the same EoM as our result from Lagrange. For this problem, both the force balance and Lagrange methods have their own advantages. Force balance is advantageous because it gives us a direct result for all of the constraint forces exerted by the hoop on the mass. Lagrange is advantageous because it is simpler mathematically, and requires only one inertial time-derivative involving the **Golden Rule** to get velocity, instead of two to get acceleration.

i) Draw an FBD in the rotating frame. In this case, we must imagine the forces that an observer in a frame of reference rotating with the hoop would observe. This includes fictitious forces due to the hoop's rotation. See figure below.


ω
Fcent
Nr
mg
Front View
Nφ
Fcor
Side View