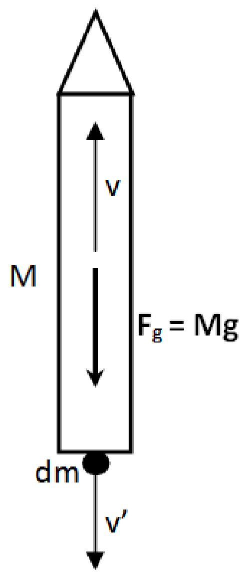
AA 242A Homework 4 Solution

2021.10.26

1 Problem 1



Let's first form the linear momentum equation:

$$
\begin{aligned}
L_1 &= (M + dm)v \\
L_2 &= M(v + dv) + dm(v - v') \\
L_2 - L_1 &= Mdv - dmv' \\
F = \dot{L} &= M\frac{dv}{dt} + v'\frac{dM}{dt} \\
M\frac{dv}{dt} &= -Mg - v'\frac{dM}{dt}
\end{aligned}
$$

It is important to note that $dM + dm = 0$ (dM is the mass differential of rocket mass, and dm is the mass differential of the escaping gas.) due to conservation of mass. Then, we use a little trick here:

$$\frac{dv}{dt} = \frac{dv}{dM}\frac{dM}{dt}$$

So going on:

$$M\frac{dv}{dM}\frac{dM}{dt} = -Mg - v'\frac{dM}{dt}$$

We know that $\frac{dM}{dt} = -\frac{1}{60}m_0$. Using this we can go on:

$$\begin{aligned}
dv &= -\frac{v'}{M}dM + \frac{60g}{m_0}dM \\
\int_0^{v_f} \mathrm{d}v &= -v' \int_{m_0}^{m_e} \frac{1}{M}\,\mathrm{d}M + \int_{m_0}^{m_e} \frac{60g}{m_0}\,\mathrm{d}M \\
v_f &= -v' ln\left(\frac{m_e}{m_0}\right) + 60g\left(\frac{m_e}{m_0} - 1\right)
\end{aligned}$$

We have to make an approximation here that the fuel mass, m_f, is much greather than that of the empty rocket, m_e ($m_f \gg m_e$). We also use the fact that $m_f + m_e = m_0$.

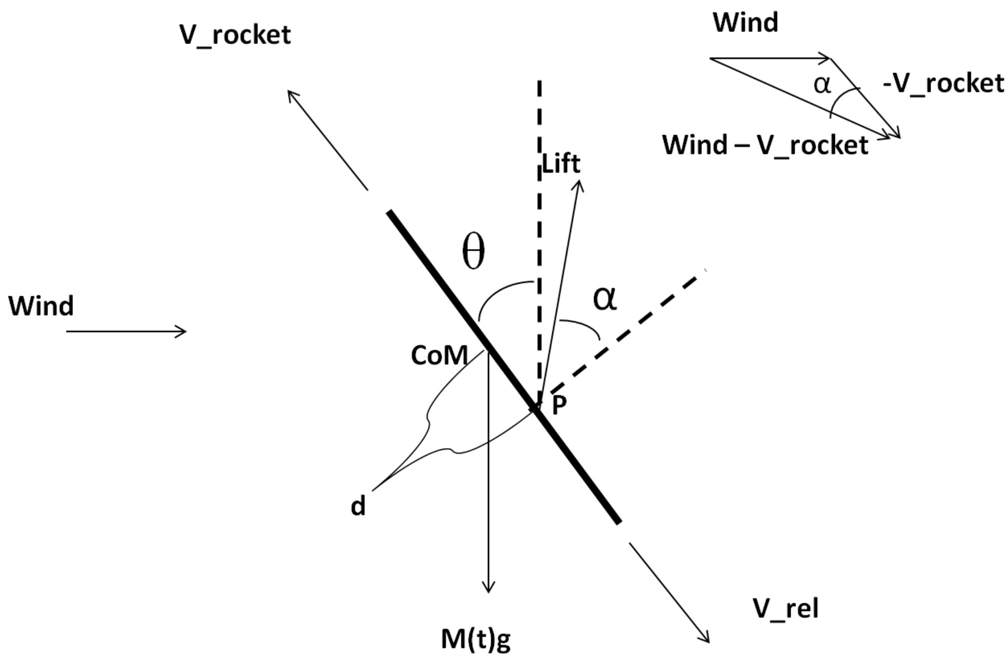
$$\begin{aligned}
v &= v' ln\left(\frac{m_0}{m_e}\right) + 60g\left(\frac{m_e - m_0}{m_0}\right) \\
&= v' ln\left(1 + \frac{m_f}{m_e}\right) + 60g\left(-\frac{m_f}{m_e + m_f}\right) \\
&\approx v' ln\left(\frac{m_f}{m_e}\right) - 60g\left(\frac{m_f}{m_f}\right) \\
\frac{m_f}{m_e} &= exp\left(\frac{v + 60g}{v'}\right)
\end{aligned}$$

Plug in $v = 11200\frac{m}{s}$, $g = 9.8\frac{m}{s^2}$, and $v' = 2100\frac{m}{s}$ and we get:

$$\frac{m_f}{m_e} = 274$$

Hence the ratio of the fuel mass to that of the empty rocket is almost 300!

2 Problem 2



There are essentially three unknowns in this problem - the x and y components of rocket velocity and the flight path angle θ. We need to come up with differential equations for these three quantities. In previous homeworks we looked at the rocket equation in one dimension. Since Newton's law is a vector equation, the two dimensional extension becomes rather straightforward:

$$F_x = \frac{dp_x}{dt} = \frac{p_x(t+dt) - p_x(t)}{dt}$$
$$\Rightarrow m(t)\frac{dv_x}{dt} - v_{rel_x}\frac{dm}{dt} = Lsin(\beta)$$
$$\Rightarrow \frac{dv_x}{dt} = \frac{1}{m(t)}\left(Lsin(\beta) - cv_{rel}sin(\theta)\right)$$

Similarly, we write the y-component of the same equation:

$$
\begin{aligned}
F_y = \frac{dp_y}{dt} &= \frac{p_y(t+dt) - p_y(t)}{dt} \\
\Rightarrow m(t)\frac{dv_y}{dt} - v_{rel_y}\frac{dm}{dt} &= Lcos(\beta) - m(t)g \\
\Rightarrow \frac{dv_y}{dt} = \frac{1}{m(t)}&\left(Lcos(\beta) + cv_{rel}cos(\theta)\right) - g
\end{aligned}
$$

Here, c is the constant mass loss rate so that $m(t) = m_0 - ct$. Also, β is the angle that $(\vec{w} - \vec{v_r})$, that is the wind velocity with respect to the rocket makes with the horizontal. It can be calculated easily by either using Matlab's $atan2$ function or as

$$\beta = arccos(\frac{w - v_x}{|\vec{w} - \vec{v_r}|})$$

From the figure it should be clear that

$$\alpha = \frac{\pi}{2} - \beta - \theta$$

The sign for the v_{rel} term in the above equations can sometimes cause confusion. The easy way to check whether this sign is correct or not is to see whether the thrust produced, that is the sign of $\dot{v}_x$ etc. due to this term is correct. Another mistake is to try and find α directly as the angle between the relative wind and the rocket velocity vector. Remember that at the initial time $\vec{v_r}$ is 0 and therefore has no defined direction. If you fix α to 90 degrees in the beginning, you end up with a divide by zero in the lift equation.

The equation for lift is given as:

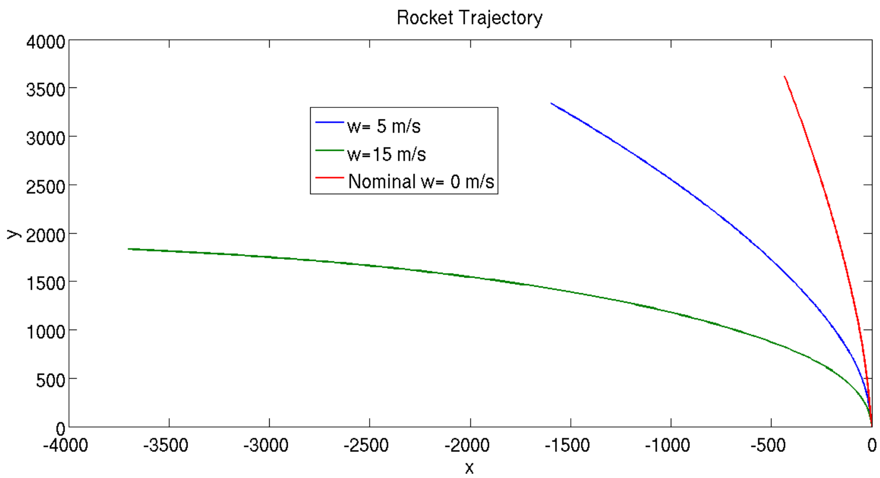
$$L = 0.09621 * \frac{\alpha}{cos(\alpha)}\rho_{air}((w - v_x)^2 + {v_y}^2)$$

Finally the equation for θ is

$$\ddot{\theta} = \frac{Lcos(\alpha)d}{I}$$

Here too, sometimes the sign causes confusion. It is important to check that the torque causes θ to change in a manner that is consistent with one's coordinate frame. I is taken as a constant, which is an assumption since mass is changing.

Now we need to plug these equations into ODE45, like we did in HW1 and produce results (check code in the hw folder on coursework). The following trajectories are created:



From this plot we can clearly see the effect of the wind on the rocket. The rocket flies into the wind because of the lift created by the wind and deviates from its nominal no wind trajectory towards the west. Expectedly, in the 5m/s case the rocket is still within the "tolerance" zone and the launch is good. This cannot be said for the 15 m/s wind case. This was a highly reduced order model that leaves out a lot of the complexities, but does capture the dominant effect of surface winds on a rocket. For a video of a rocket getting (not severely) weather-cocked, check out the video posted in the homework folder. This was taken from a camera on the launch rail at the launch of the hybrid rocket built by Stanford's AA284 series team last year.

Bonus

Weather-cocking is an essential design consideration in designing the thrust profile of rocket engines (especially for small rockets). Typically surface winds are confined to a few hundred meters above the ground. Therefore two things that are usually done to prevent wind effects are-

1. Have high initial thrust so as to exit the wind area quickly.

2. Spin the rocket about its long axis to provide gyroscopic stabilization.

3 Problem 3

It is most convenient (and safe) to break down the positional components back into inertial frame, and work on the derivatives from there. Defining the pivot point O as the origin we have the following ($\sin x = S_x$, $\cos x = C_x$ for convenience):

$$
\begin{aligned}
\vec{r_1} &= lS_\theta \hat{\mathbf{x}} - lC_\theta \hat{\mathbf{y}} \\
\dot{\vec{r_1}} &= l\dot{\theta}C_\theta \hat{\mathbf{x}} + l\dot{\theta}S_\theta \hat{\mathbf{y}} \\
\vec{r_2} &= (lS_\theta + lS_{\phi+\theta})\hat{\mathbf{x}} - (lC_\theta + lC_{\phi+\theta})\hat{\mathbf{y}} \\
\dot{\vec{r_2}} &= (l\dot{\theta}C_\theta + l(\dot{\theta}+\dot{\phi})C_{\phi+\theta})\hat{\mathbf{x}} + (l\dot{\theta}S_\theta + l(\dot{\phi}+\dot{\theta})S_{\theta+\phi})\hat{\mathbf{y}} \\
T &= \frac{1}{2}m(\dot{\vec{r}}_1^2 + \dot{\vec{r}}_2^2) \\
T &= \frac{1}{2}ml^2(3\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi} + 2\dot{\theta}^2 C_\phi + 2\dot{\phi}\dot{\theta}C_\phi) \\
V &= -2mglC_\theta - mglC_{\theta+\phi} \\
L &= T - V
\end{aligned}
$$

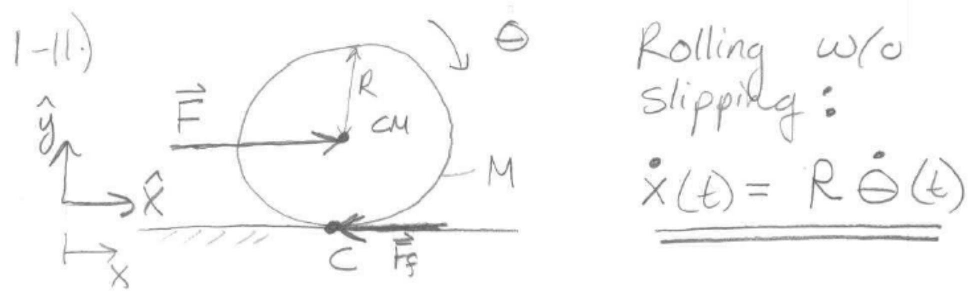
The process of differentiation on θ and ϕ is tedious, and special attentions needs to be paid to the $\cos\phi$ terms for they are not constant under $\frac{d}{dt}$; but the general steps are straight forward:

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} &= 0 \\
\Rightarrow 0 &= (3+2C_\phi)\ddot{\theta} - 2S_\phi\dot{\theta}\dot{\phi} + (1+C_\phi)\ddot{\phi} - S_\phi\dot{\phi}^2 + \frac{g}{l}(S_{\phi+\theta} + 2S_\theta) \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} &= 0 \\
\Rightarrow 0 &= \ddot{\phi} + (1+C_\phi)\ddot{\theta} + S_\phi\dot{\theta}^2 + \frac{g}{l}S_{\phi+\theta}
\end{aligned}
$$

b) for small angles x, $\cos x \approx 1$, $\sin x \approx x$ and the product of any two displacements and velocities is also small:

$$
\begin{aligned}
5\ddot{\theta} + 2\ddot{\phi} + \frac{g}{l}(3\theta + \phi) &= 0 \\
\ddot{\phi} + 2\ddot{\theta} + \frac{g}{l}(\theta + \phi) &= 0
\end{aligned}
$$

4 Problem 4



It should be notice that even though the constraint here is can be integrated to form a holonomic constraint on position ($x(t) = R\theta(t) + constant$), in its original form it is a rate constraint on $\dot{x}$ and $\dot{\theta}$.

a) Form kinetic energy:

$$\begin{aligned} T &= T_{trans} + T_{rot} \\ T &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2 \end{aligned}$$

I = moment of inertia of disk rotating around its central axis

$$\begin{aligned} I &= \frac{1}{2}MR^2 \\ \Rightarrow T &= \frac{1}{2}M\dot{x}^2 + \frac{1}{4}MR^2\dot{\theta}^2 \end{aligned}$$

Invoking the constraint equation, we have $T = \frac{3}{4}M\dot{x}^2$. The motion is completely horizontal, therefore it is convenient to define $V = 0$ at the center of the disk for simplification, and we have $L = T = \frac{3}{4}M\dot{x}^2$. Now differentiate with respect to the x coordinate:

$$\begin{aligned} \frac{d}{dt}(\frac{\partial L}{\partial \dot{x}}) &= \frac{3}{2}M\ddot{x} \\ \frac{\partial L}{\partial x} &= 0 \\ Q_x &= \sum_i \vec{F} \cdot \frac{\partial \vec{r}_i}{\partial x} = \vec{F} \cdot \frac{\partial (x\hat{x})}{\partial x} \end{aligned}$$

There is another force in the system: the frictional force. Although it's magnitude is non-zero, it applied to the point of contact, whose velocity is constrained to zero by the no-slip condition.
Thus we have the following: and so the complete Lagrange's equation:

$$\frac{3}{2}M\ddot{x} = F$$

It is possible to choose θ as your generalized coordinate. Following the same procedure outlined above, you should arrive at the following equation:

$$\frac{3}{2}MR^2\ddot{\theta} = Q_\theta = FR$$

b) If we apply an out-of-plane force, we would expect the motion to not be simple 2D motion any longer. There could be tipping, wobbling, etc. The simple relation exploited in part a) stops being valid.

5 Problem 5

The kinetic energy of the mass-rod system can be broken down into kinetic energy describing the motion of the center of mass, T_{COM}, and kinetic energy of the motion of the two mass points around the center of mass, T_{around}. Notice that this decomposition can only be applied to the center of mass; applying such scheme to a non-COM point would result in coupling terms. The motion of COM is easy to find:

$$\begin{aligned}
\vec{r}_{COM} &= a\hat{r} \\
\dot{\vec{r}}_{COM} &= a\dot{\psi}\hat{\psi}
\end{aligned}$$

We set up an x-y-z frame at the COM, with x pointing radially out, z pointing vertically up, and y completing the right-handed system. We use the angle θ to indicate the angle between the rod and $\hat{z}$, and ϕ for the angle between the projection of mass position and $\hat{x}$.

$$\begin{aligned}
\vec{r}_1 &= \frac{l}{2}(S_\theta C_\phi \hat{x} + S_\theta S_\phi \hat{y} + C_\theta \hat{z}) \\
\dot{\vec{r}}_1 &= \frac{l}{2}[(C_\theta C_\phi \dot{\theta} - S_\theta S_\phi \dot{\phi})\hat{x} + (C_\theta S_\phi \dot{\theta} + S_\theta C_\phi \dot{\phi})\hat{y} - S_\theta \dot{\theta}\hat{z})
\end{aligned}$$

Finding v^2 requires the summation of the squares of all components, which is a rather tedious but straight-forward process. With caution we would arrive at the following:

$$v_1^2 = \frac{l^2}{4}(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)$$

Using symmetry argument, we can conclude that $v_2^2 = v_1^2$; therefore:

$$\begin{aligned}
T &= T_{COM} + Taround \\
T &= \frac{1}{2} * 2m * (a\dot{\psi})^2 + 2 * \frac{1}{2}m\frac{l^2}{4}(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) \\
T &= ma^2\dot{\psi}^2 + \frac{1}{4}ml^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta)
\end{aligned}$$