AA 242A Homework 6 Solution

1 Problem 1

$\overleftrightarrow{a} = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 1 & 6 \\ 7 & 3 & 5 \end{bmatrix}, \overleftrightarrow{b} = \begin{bmatrix} 6 & 3 & 3 \\ 1 & 7 & 0 \\ 1 & 2 & 8 \end{bmatrix}, \overleftrightarrow{c} = \begin{bmatrix} 8 & 2 & 3 \\ 6 & 3 & 9 \\ 1 & 5 & 0 \end{bmatrix}$

a. Matrix Multiplication Associativity

$$\overleftrightarrow{a}\overleftrightarrow{b} = \begin{bmatrix} 16 & 29 & 14 \\ 31 & 31 & 60 \\ 50 & 52 & 61 \end{bmatrix}, \overleftrightarrow{b}\overleftrightarrow{c} = \begin{bmatrix} 69 & 36 & 45 \\ 50 & 23 & 66 \\ 28 & 48 & 21 \end{bmatrix}$$

$$(\overleftrightarrow{a}\overleftrightarrow{b})\overleftrightarrow{c} = \begin{bmatrix} 316 & 189 & 309 \\ 494 & 455 & 372 \\ 773 & 561 & 618 \end{bmatrix} = \overleftrightarrow{a}(\overleftrightarrow{b}\overleftrightarrow{c})$$

b. Invariant Trace under Similarity Transformation

We know Trace(A) $= \sum_{i=1}^{m} A_{ii}$, so:

$$\begin{aligned} \text{Trace}(\overleftrightarrow{a}) &= 2+1+5=8 \\ \text{Trace}(\overleftrightarrow{b}\overleftrightarrow{a}\overleftrightarrow{b}^{-1}) &= \text{Trace}\left(\begin{bmatrix} 7.0202 & 0.6364 & 2.2424 \\ 4.6633 & -1.6061 & 3.6263 \\ 10.7710 & -1.2121 & 2.5859 \end{bmatrix}\right) \\ &= 7.0202 - 1.6061 + 2.5859 \\ &= 8 \end{aligned}$$

c. Orthogonal Product

For a matrix $\overleftrightarrow{A}$ to be orthogonal, we must have $\overleftrightarrow{A}^T = \overleftrightarrow{A}^{-1}$, so it is enough to check if $\overleftrightarrow{A}^T * \overleftrightarrow{A} = \boldsymbol{I} = \overleftrightarrow{A} * \overleftrightarrow{A}^T$:

$$\begin{aligned} \overleftrightarrow{c} &= \overleftrightarrow{a}\overleftrightarrow{b} = \begin{bmatrix} \frac{\sqrt{6}}{4} & \frac{\sqrt{6}}{4} & \frac{1}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{3}}{2} \end{bmatrix} \\ \overleftrightarrow{c}^T &= \begin{bmatrix} \frac{\sqrt{6}}{4} & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{4} \\ \frac{\sqrt{6}}{4} & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{4} \\ \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{bmatrix} \\ \overleftrightarrow{c}^T\overleftrightarrow{c} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \overleftrightarrow{c}\overleftrightarrow{c}^T \end{aligned}$$

d. Antisymmetry under Similarity Transformation

$$\overleftrightarrow{b}\,\overleftrightarrow{a}\,\overleftrightarrow{b}^{-1} = \begin{bmatrix} 0 & 6.4608 & -3.5355 \\ -6.4608 & 0 & 7.1943 \\ 3.5355 & -7.1943 & 0 \end{bmatrix}$$

so antisymmetry is conserved.

2 Problem 2

a) Form the rotation matrices:

$$
\begin{aligned}
R_x &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \\
R_z &= \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

Here we have $\psi = 270^o$, so:

$$
\begin{aligned}
R &= R_z R_x \\
&= \begin{bmatrix} 0 & -\cos\theta & -\sin\theta \\ 1 & 0 & 0 \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}
\end{aligned}
$$

However, we want the rotation matrix expressing the orientation with respect to the original frame, so we must take R^T:

$$
R^T = \begin{bmatrix} 0 & 1 & 0 \\ -\cos\theta & 0 & -\sin\theta \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}
$$

b) We know that for a given transformation, the trace of that tranformation is independent of the sequence of operation. If it was carried out in one single rotation of angle θ about a certain axis $\overrightarrow{r}$, the trace would be $1 + 2\cos\theta$. Therefore:

$$
\begin{aligned}
\mathrm{Tr}(A) &= 1 + 2\cos\theta \\
\phi &= \cos^{-1}\frac{(\cos\theta - 1)}{2}
\end{aligned}
$$

The axis of rotation must stay invariant under such transformation A; this is equivalent of saying that $\overrightarrow{r}$ is the eigenvector of A corresponding to the eigenvalue $\lambda = 1$:

$$\begin{aligned}
[sI - A]\overrightarrow{r}|_{s=1} &= \overrightarrow{0} \\
\begin{bmatrix} 1 & -1 & 0 \\ \cos\theta & 1 & \sin\theta \\ \sin\theta & 0 & 1-\cos\theta \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} &= 0 \\
\rightarrow \overrightarrow{r} = \begin{bmatrix} 1 \\ 1 \\ -\frac{1+\cos\theta}{\sin\theta} \end{bmatrix}
\end{aligned}$$

Notice that with $\sin\theta$ at the bottom this expression is undefined (or at singularity) for $\theta = n\pi$, indicating that at these θ angles it is not possible to find a unique combination of rotation axis and angles.

3 Problem 3

a) Let the 3-2-1 rotation angles be ψ,θ,ϕ respectively. Then:

$$\begin{aligned}
\overleftrightarrow{C}_3 &= \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
\overleftrightarrow{C}_2 &= \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \\
\overleftrightarrow{C}_1 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}
\end{aligned}$$

and so the overall 3-2-1 rotation matrix:

$$\begin{aligned}
\overleftrightarrow{R} &= \overleftrightarrow{C}_1\overleftrightarrow{C}_2\overleftrightarrow{C}_3 \\
&= \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \cos\psi\sin\theta\sin\phi - \sin\psi\cos\phi & \sin\psi\sin\theta\sin\phi + \cos\psi\cos\phi & \cos\theta\sin\phi \\ \cos\psi\sin\theta\cos\phi + \sin\psi\sin\phi & \sin\psi\sin\theta\cos\phi - \cos\psi\sin\phi & \cos\theta\cos\phi \end{bmatrix}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\psi &= \text{atan2}(R_{12}, R_{11}) \\
\phi &= \text{atan2}(R_{23}, R_{33}) \\
\theta &= \text{atan2}(-R_{13}, \frac{R_{23}}{\sin\phi})
\end{aligned}$$

where the atan2 function is used to remove ambiguities. You should try avoid using atan directly since there exists sign ambiguity. This ambiguity must be resolved by comparing sin/cos to ensure that you are in the correct quardrant.

For 3-2-1 rotation, comparing this matrix to the given $\overleftrightarrow{R}$ matrix we find:

$$\begin{aligned}
\psi &= \text{atan2}(R_{12}, R_{11}) = -60^o \\
\phi &= \text{atan2}(R_{23}, R_{33}) = 30^o \\
\theta &= \text{atan2}(-R_{13}, \frac{R_{23}}{\sin\phi}) = -45^o
\end{aligned}$$

4 Problem 4

For a 3-2-1 rotation:

$$\begin{aligned}
\overleftrightarrow{\omega} &= \dot{\phi}\hat{z}' + \dot{\theta}\hat{y}'' + \dot{\psi}\hat{x} \\
&= \overleftrightarrow{C}_1(\theta_1)\overleftrightarrow{C}_2(\theta_2)\begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix} + \overleftrightarrow{C}_1(\theta_1)\begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + \begin{bmatrix} \dot{\psi} \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{bmatrix}\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix} + \ldots \\
&\quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{bmatrix}\begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + \begin{bmatrix} \dot{\psi} \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\psi & \cos\theta\sin\psi \\ 0 & -\sin\psi & \cos\theta\cos\psi \end{bmatrix}\begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} \\
\Rightarrow \overleftrightarrow{A} &= \frac{1}{\cos\theta}\begin{bmatrix} \cos\theta & \sin\psi\sin\theta & \cos\psi\sin\theta \\ 0 & \cos\psi\cos\theta & -\sin\psi\cos\theta \\ 0 & \sin\psi & \cos\psi \end{bmatrix} \quad \text{(multiplied together)}
\end{aligned}$$

Alternatively,

$$\overleftrightarrow{A} = \begin{bmatrix} 1 & \sin\psi\tan\theta & \cos\psi\tan\theta \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi\sec\theta & \cos\psi\sec\theta \end{bmatrix}$$