AA 242A Homework 8 Solution

December 5, 2021

1 Problem 1

a) It should be obvious that Lagrangian approach should be used here. In this solution I will present a concise approach which requires some intuitions and insights into the problem (as shall be seen later); a more complete approach can be found in a separate file in the homework section.

First we express the location of the center of mass of the disk:

$$\begin{aligned}
x_{disk} &= x\cos\theta + r\sin\theta \\
y_{disk} &= -x\sin\theta + r\cos\theta \\
\Rightarrow \dot{x}_{disk} &= \dot{x}\cos\theta - x\dot{\theta}\sin\theta + r\dot{\theta}\cos\theta \\
\dot{y}_{disk} &= -\dot{x}\sin\theta - x\dot{\theta}\cos\theta - r\dot{\theta}\sin\theta \\
\Rightarrow T_{trans} &= \frac{1}{2}m(\dot{x}_{disk}^2 + \dot{y}_{disk}^2) \\
&= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}mx^2\dot{\theta}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + mr\dot{x}\dot{\theta}
\end{aligned}$$

Next is the rotational kinetic energy of the disk. Now here comes the little bit of insight: the rate of rotation of the disk in inertial frame is not only $\frac{\dot{x}}{r}$, but $\frac{\dot{x}}{r} + \dot{\theta}$ since it also drops with the rod.

$$\begin{aligned}
T_{rotation} &= \frac{1}{2}I_{center}\left(\frac{\dot{x}}{r} + \dot{\theta}\right)^2 \\
&= \frac{1}{4}m\dot{x}^2 + \frac{1}{2}mr\dot{x}\dot{\theta} + \frac{1}{4}mr^2\dot{\theta}^2
\end{aligned}$$

The rest is just routine work:

$$
\begin{aligned}
T_{rod} &= \frac{1}{2} I_{end} \dot{\theta}^2 \\
&= \frac{8}{3} m r^2 \dot{\theta}^2 \\
V_{rod} &= -2rmg \sin\theta \\
V_{disk} &= -mgx \sin\theta + mgr \cos\theta \\
L &= \frac{3m\dot{x}^2}{4} + \frac{mx^2\dot{\theta}^2}{2} + \frac{41mr^2\dot{\theta}^2}{12} + \frac{3mr\dot{x}\dot{\theta}}{2} - (mgr\cos\theta - mgx\sin\theta - 2rmg\sin\theta)
\end{aligned}
$$

Because no non-conservative forces are acting on the system (frictionless pivot), we shall use the form of Lagrangian equations with no generalized forces:

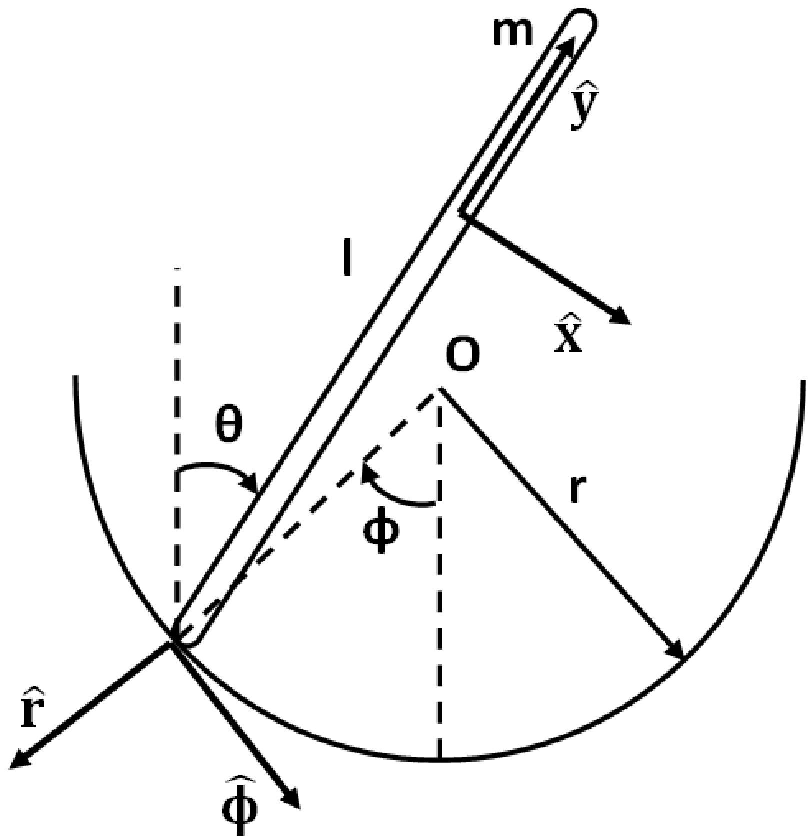
$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \\
&\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \\
&\Rightarrow \text{x-EOM: } 0 = \frac{3}{2} m\ddot{x} + \frac{3}{2} mr\ddot{\theta} - mx\dot{\theta}^2 - mg\sin\theta \\
&\quad \theta\text{-EOM: } 0 = \frac{41}{6} mr^2\ddot{\theta} + \frac{3}{2} mr\ddot{x} + mx^2\ddot{\theta} + 2mx\dot{x}\dot{\theta} - mg(x+2r)\cos\theta - mgr\sin\theta
\end{aligned}
$$

b) at $t = 0^+$, we know $\theta(0) = 0$, $x(0) = 2r$, $\dot{\theta}(0) = 0$, $\dot{x}(0) = 0$:

$$
\begin{aligned}
\text{x-EOM} \rightarrow \ddot{x} &= -\ddot{\theta} r \\
\theta\text{-EOM} \rightarrow \left(\frac{41}{6} r^2 + 4r^2 - \frac{3}{2} r^2\right) \ddot{\theta} &= 4gr \\
\Rightarrow \ddot{\theta} &= \frac{3}{7}\frac{g}{r} \\
\Rightarrow \frac{M_{total}}{I_{end}} = \ddot{\theta} &= \frac{3}{7}\frac{g}{r} \\
\frac{2mgr + 2F_{disk} r}{\frac{16}{3} mr^2} &= \frac{3g}{7r} \\
\Rightarrow F_{disk} &= \frac{1}{7} mg
\end{aligned}
$$

Notice that we have implicitly defined the downward direction as the positive direction of force. In inertial frame we would have $\vec{F}_{disk} = \left(-\frac{1}{7} mg\right) \hat{y}$.

2 Problem 2



Find position and velocity vectors for COM of rod:

$$\vec{r}_{CM} = r\hat{r} + \frac{l}{2}\hat{y}$$
$$\vec{v}_{CM} = -r\dot{\phi}\hat{\phi} + \frac{1}{2}\dot{\theta}\hat{x}$$

Form Lagrangian:

$$
\begin{aligned}
T &= \frac{1}{2}m\vec{v}_{CM}\cdot\vec{v}_{CM} + \frac{1}{2}I\omega^2 \\
&= \frac{1}{2}m\left(r^2\dot{\phi}^2 + \frac{1}{4}l^2\dot{\theta}^2 - rl\dot{\theta}\dot{\phi}\cos(\phi-\theta)\right) + \frac{1}{2}\frac{1}{12}ml^2\dot{\theta}^2 \\
&= \frac{1}{6}ml^2\dot{\theta}^2 + \frac{1}{2}mr^2\dot{\phi}^2 - \frac{1}{2}mrl\dot{\theta}\dot{\phi}\cos(\phi-\theta)
\end{aligned}
$$

$$
\begin{aligned}
V &= mg\left(-r\cos\phi + \frac{l}{2}\cos\theta\right) \\
L &= \frac{1}{6}ml^2\dot{\theta}^2 + \frac{1}{2}mr^2\dot{\phi}^2 - \frac{1}{2}mrl\dot{\theta}\dot{\phi}\cos(\phi-\theta) + mgr\cos\phi - mg\frac{l}{2}\cos\theta
\end{aligned}
$$

Lagragian Math:

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) &= \frac{d}{dt}\left(\frac{1}{3}ml^2\dot{\theta} - \frac{1}{2}mrl\dot{\phi}\cos(\phi-\theta)\right) \\
&= \frac{1}{3}ml^2\ddot{\theta} - \frac{1}{2}mrl\left(\ddot{\phi}\cos(\phi-\theta) - \dot{\phi}(\dot{\phi}-\dot{\theta})\sin(\phi-\theta)\right) \\
\frac{\partial L}{\partial\theta} &= -\frac{1}{2}mrl\dot{\theta}\dot{\phi}\sin(\phi-\theta) + mg\frac{l}{2}\sin\theta
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) &= \frac{d}{dt}\left(mr^2\dot{\phi} - \frac{1}{2}mrl\dot{\theta}\cos(\phi-\theta)\right) \\
&= mr^2\ddot{\phi} - \frac{1}{2}mrl\ddot{\theta}\cos(\phi-\theta) + \frac{1}{2}mrl\dot{\theta}(\dot{\phi}-\dot{\theta})\sin(\phi-\theta) \\
\frac{\partial L}{\partial\phi} &= \frac{1}{2}mrl\dot{\theta}\dot{\phi}\sin(\phi-\theta) - mgr\sin\phi
\end{aligned}
$$

Hence the final equations are:

$$
\begin{aligned}
\frac{1}{3}ml^2\ddot{\theta} - \frac{1}{2}mrl\ddot{\phi}\cos(\phi-\theta) + \frac{1}{2}mrl\dot{\phi}^2\sin(\phi-\theta) - mg\frac{l}{2}\sin\theta = 0 \\
mr^2\ddot{\phi} - \frac{1}{2}mrl\ddot{\theta}\cos(\phi-\theta) - \frac{1}{2}mrl\dot{\theta}^2\sin(\phi-\theta) + mgr\sin\phi = 0
\end{aligned}
$$

3 Problem 3

On the y-direction:

$$L_{disk} = Iw$$
$$L_{balls} = 2m(r + l\sin\theta)^2 w$$

with no external torque acting on the system, the y angular momentum is conserved:

$$Iw_0 + 2mr^2 w_0 = Iw + 2m(r + l\sin\theta)^2 w$$
$$w = \frac{(I + 2mr^2)w_0}{I + 2m(r + l\sin\theta)^2}$$

When solving for $\dot{\theta}$ in terms of θ it is important to realize that as the balls swing up and down, the disk itself, due to conservation of y linear momentum, is moving as well! If we describe the location of the disk as $[0;\ y]^T$ we have the following:

$$|\dot{x}_{ball}| = \dot{\theta} l \cos\theta$$
$$|\dot{y}_{ball}| = \dot{y} + \dot{\theta} l \sin\theta$$
$$\text{conservation of } \vec{p}_y \Rightarrow 0 = 2m\dot{y} + 2m(\dot{y} + l\dot{\theta}\sin\theta)$$
$$\dot{y} = -\frac{1}{2} l\dot{\theta}\sin\theta$$

Finally we apply conservation of energy to the entire system:

$$\frac{1}{2}Iw_0^2 + mr^2 w_0^2 = m\dot{y}^2 + \frac{1}{2}Iw^2 + m[(\dot{y} + l\dot{\theta}\sin\theta)^2 + (l\dot{\theta}\cos\theta)^2 + (r + l\sin\theta)^2 w^2]$$

substituting in the previous expression for w and isolate $\dot{\theta}$:

$$\dot{\theta} = \omega_0 \sqrt{\frac{I + 2mr^2}{ml^2(1 + \cos^2\theta)} \left[1 - \frac{I + 2mr^2}{I + 2m(r + l\sin\theta)^2}\right]}$$

4 Problem 4

a) We just need to find the impulsive moment imparted to the gimbal to make it rotate with the same angular velocity as the rocket:

$$\begin{aligned}
\overleftrightarrow{I_g} &= \begin{bmatrix} 0.05 & 0 & 0 \\ 0 & 0.05 & 0 \\ 0 & 0 & 0.1 \end{bmatrix} \\
\vec{H}_g &= \overleftrightarrow{I_g}\,(\vec{\omega}_f - \vec{\omega}_i) \\
&= \begin{bmatrix} 0.05 & 0 & 0 \\ 0 & 0.05 & 0 \\ 0 & 0 & 0.1 \end{bmatrix} \left(\begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 10^4 \end{bmatrix} \right) \\
&= \begin{bmatrix} 0.5 \\ 0 \\ -1000 \end{bmatrix}
\end{aligned}$$

The rocket experiences the negative of this, so:

$$\Delta\vec{H} = -0.5\hat{x} + 1000\hat{z}$$

b) Formulate this the same way as part a:

$$\begin{aligned}
\overleftrightarrow{I_r} &= \begin{bmatrix} 10^3 & 0 & 0 \\ 0 & 10^4 & 0 \\ 0 & 0 & 10^4 \end{bmatrix} \\
\vec{H}_{rf} &= \vec{H}_{ri} + \Delta H \\
&= \begin{bmatrix} 10^3 & 0 & 0 \\ 0 & 10^4 & 0 \\ 0 & 0 & 10^4 \end{bmatrix} \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -0.5 \\ 0 \\ 1000 \end{bmatrix} \\
&= \begin{bmatrix} 9999.5 \\ 0 \\ 1000 \end{bmatrix}
\end{aligned}$$

We can then find the angle between the original angular momentum and the new angular momentum:

$$\begin{aligned}
\beta &= \tan^{-1}\frac{H_z}{H_x} = \tan^{-1}\frac{1000}{9999.5} \\
&= 5.71^\circ \\
2\beta &= 11.42^\circ
\end{aligned}$$

c) We want to use the equation $\dot{\psi} = -\frac{H}{I_t}$ to solve for the precession rate. With this in mind we need:

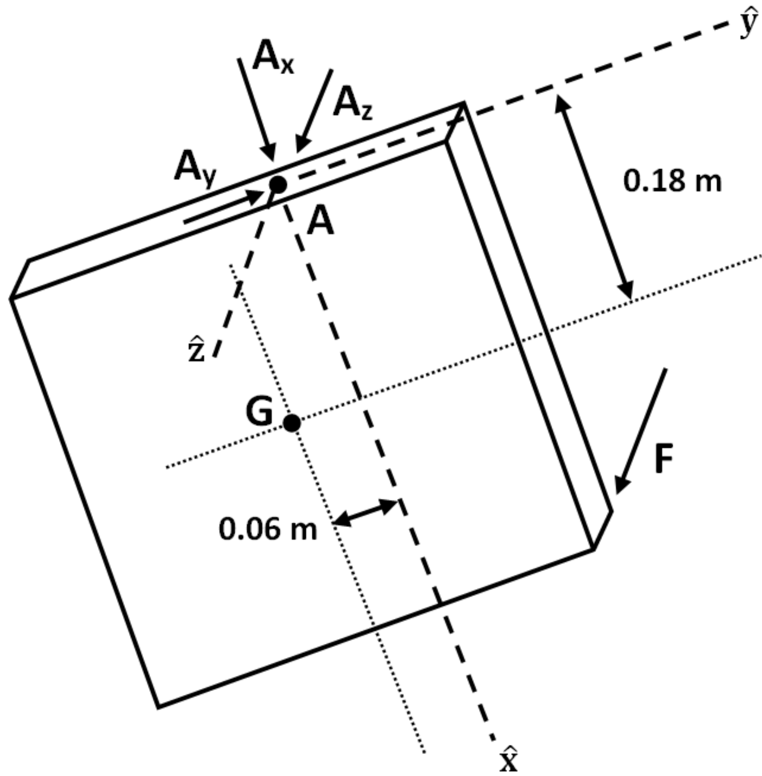
$$H = \sqrt{(9999.5)^2 + (1000)^2} = 10050$$

So then we have:

$$\dot{\psi} = -\frac{10050}{10000} = -1.005\frac{rad}{s}$$
$$T = \frac{2\pi}{|\dot{\psi}|} = 6.25s$$

5 Problem 5

Original problem from Ginsberg: Advanced Engineering Dynamics 2nd edition



a) The force $\vec{F}$ is much larger than the weight of the plate, so the latter is omitted from the free-body diagram. In contrast, the reaction exerted by the ball-and-socket joint is impulsive, because it must be as large as necessary to prevent movement of point A. We place the origin of xyz at point A in order to eliminate the angular impulse of this reaction. The coordinates of point A relative to parallel centroidal axes are (-0.18, 0.06, 0) meters, so the inertia properties are

$$
\begin{aligned}
I_{xx} &= \frac{1}{12}(10)(0.36^2) + 10(0.06^2) = 0.144 \\
I_{yy} &= \frac{1}{12}(10)(0.36^2) + 10(0.18^2) = 0.432 \\
I_{zz} &= \frac{1}{12}(10)(0.36^2 + 0.36^2) + 10(0.18^2 + 0.06^2) = 0.576 \\
I_{xy} &= 0 + 10(-0.18)(0.06) = -0.108 \\
I_{xz} &= I_{yz} = 0
\end{aligned}
$$

The angular velocity is initially zero. Let $\vec{\omega_2} = \omega_x\hat{x} + \omega_y\hat{y} + \omega_z\hat{z}$ denote the angular velocity at the termination of the impulsive action. Then the corresponding velocity of the center of mass is

$$\begin{aligned}(\vec{v}_G)_2 &= \vec{\omega}_2 \times \vec{r}_{G/A} \\ &= 0.06\omega_z\hat{x} + 0.18\omega_z\hat{y} - (0.06\omega_x + 0.18\omega_y)\hat{z}\end{aligned}$$

The final angular momentum about pivot A is

$$(\vec{H}_A)_2 = (0.144\omega_x + 0.108\omega_y)\hat{x} + (0.432\omega_y + 0.108\omega_x)\hat{y} + 0.576\omega_z\hat{z}$$

Applying the angular impulse-momentum principle to the 4-ms interval of the force leads to

$$\begin{aligned}(\vec{H}_A)_2 &= (\vec{r}_{F/A} \times \vec{F})\Delta t = [(0.36\hat{x} + 0.12\hat{y}) \times 5000\hat{z}](0.004) \\ &= 2.4\hat{x} - 7.2\hat{z}\end{aligned}$$

The result of matching like components of $(\vec{H}_A)_2$ is

$$\begin{aligned}(\vec{H}_A)_2 \cdot \hat{x} &= 0.144\omega_x + 0.108\omega_y = 2.4 \\ (\vec{H}_A)_2 \cdot \hat{y} &= 0.432\omega_y + 0.108\omega_x = -7.2 \\ (\vec{H}_A)_2 \cdot \hat{z} &= 0.576\omega_z = 0\end{aligned}$$

from which we obtain

$$\vec{\omega_2} = 35.90\hat{x} - 25.64\hat{y}\ rad/s$$

b) Form the linear impulse-momentum principle in order to determine the reaction. Using the earlier expression for $(\vec{v_G})_2$ leads to

$$\begin{aligned}m(\vec{v_G})_2 &= (\vec{A} + \vec{F})\Delta t \\ 10[0.06\omega_z\hat{x} + 0.18\omega_z\hat{y} - (0.06\omega_x + 0.18\omega_y)\hat{z}] &= [A_x\hat{x} + A_y\hat{y} + (A_z + F)\hat{z}]\Delta t\end{aligned}$$

After substitution of the result for the angular velocity $\vec{\omega_2}$, the components of this equation yield

$$\begin{aligned}A_x &= A_y = 0 \\ A_z &= -F - \frac{10(0.06\omega_x + 0.18\omega_y)}{\Delta t} = 1153\ N\end{aligned}$$