AA 242A Homework 6

Assigned: **Thursday, November 10th, 2022**

Due: **Thursday, November 17th, 2022**

1. Show the following matrix properties (either numerically or in the general case):
   a. For three matrices defined as

$$\boldsymbol{A} = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 1 & 6 \\ 7 & 3 & 5 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 6 & 3 & 3 \\ 1 & 7 & 0 \\ 1 & 2 & 8 \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} 8 & 2 & 3 \\ 6 & 3 & 9 \\ 1 & 5 & 0 \end{bmatrix}$$

   Show that matrix multiplication is associative (i.e. $\left(\boldsymbol{AB}\right)\boldsymbol{C} = \boldsymbol{A}\left(\boldsymbol{BC}\right)$, meaning the order which multiplication is performed is not important).

   b. Show that the trace of matrix $\boldsymbol{A}$ (given in part a) is unchanged under a similarity transformation with transformation matrix $\boldsymbol{B}$ (given in part a, you may also use mathematical properties of the trace).

   c. For two orthogonal matrices

$$\boldsymbol{A} = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{-1}{2} & 0 & \frac{\sqrt{3}}{2} \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ \frac{-\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   Show that the product is also orthogonal.

   d. Show that the anti-symmetry property of matrix

$$\boldsymbol{A} = \begin{bmatrix} 0 & 4 & -9 \\ -4 & 0 & 3 \\ 9 & -3 & 0 \end{bmatrix}$$

   is preserved under the orthogonal similarity transformation with transformation matrix

$$\boldsymbol{B} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\ 0 & \frac{\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} \\ \frac{-1}{2} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} \end{bmatrix}$$

Definitions that may help:

Orthogonal matrix: Square matrix such that

$$\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^T = \boldsymbol{I}$$

Where $\boldsymbol{I}$ is the identity matrix and therefore,

$$\left(a^{-1}\right)_{ij} = a_{ji} \Leftrightarrow \boldsymbol{A}^{-1} = \boldsymbol{A}^T$$

Similarity transformation: $\boldsymbol{A} = \boldsymbol{B}\boldsymbol{A}\boldsymbol{B}^{-1}$ where $\boldsymbol{B}$ is the transformation matrix, and where $\boldsymbol{A}, \boldsymbol{B} \in \boldsymbol{R}^{n\times n}$

Trace of a matrix: $tr\left(\boldsymbol{A}\right) \equiv \sum_i^n a_{ii}$, where $\boldsymbol{A} \in \boldsymbol{R}^{n\times n}$,

and the trace is invariant under cyclic permutations, that is,

$$tr\left(\boldsymbol{ABCD}\right) = tr\left(\boldsymbol{DABC}\right) = tr\left(\boldsymbol{CDAB}\right) = tr\left(\boldsymbol{BCDA}\right)$$

Antisymmetric (skew-symmetric) matrix: If $\boldsymbol{A}$ is antisymmetric, then the components of $\boldsymbol{A}$, represented by a_{ij}, satisfy

$$a_{ij} = -a_{ji} \Leftrightarrow \boldsymbol{A}^T = -\boldsymbol{A}, where\ \boldsymbol{A} \in \boldsymbol{R}^{n\times n}$$

2. A spacecraft rotates some angle θ about its own x-axis, then by 270° about its own z-axis.
    a. Find the rotation matrix expressing its orientation with respect to its original frame

b. If this was to be expressed as a rotation about an arbitrary axis, find the equivalent angle and axis.

3. Consider the 3-2-1 Euler angle sequence (Passive ZYX Tait-Bryan angle). The resulting rotation matrix is given by

$$\boldsymbol{A} = \begin{bmatrix} \frac{\sqrt{2}}{4} & -\frac{\sqrt{6}}{4} & \frac{\sqrt{2}}{2} \\ \frac{3}{4} - \frac{\sqrt{2}}{8} & \frac{(\sqrt{2}+2)\sqrt{3}}{8} & \frac{\sqrt{2}}{4} \\ -\frac{(\sqrt{2}+2)\sqrt{3}}{8} & \frac{3\sqrt{2}}{8} - \frac{1}{4} & \frac{\sqrt{6}}{4} \end{bmatrix}$$

Find the *Z-Y-X* Euler angles (ϕ, θ, ψ) for this rotation matrix (express your answer in degrees)

4. For a 3-2-1 rotation sequence, derive the kinematic relationship for the Euler angular rates $(\dot{\phi}, \dot{\theta}, \dot{\psi})$ (rotations about Z, Y, and X, respectively) as a function of the body axis angular rate $(\omega_x, \omega_y, \omega_z)$:

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = [\boldsymbol{A}] \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

Solve for $\boldsymbol{A}$.

5. We will begin with your "Creative Problem, Version 2" this week where you will identify and solve a classical dynamics problem **that is relevant to aeronautics and astronautics**. Once again, you will carry this problem through on the next few

homeworks (homework 6-8), and it will become your final project for the course. For this week, describe in words and draw a picture of a dynamics problem (e.g. aircraft, spacecraft, space station). Don’t feel limited to real aircraft or spacecraft; use sci-fi as inspiration if you wish. The problem will be a bit more complicated than your first problem and must be in 3D and involve rigid body dynamics (i.e. must use moments of inertia in your solution). What are you trying to solve? What is included in your system? Choose your reference frame and coordinate system and label it on your drawing. Explain why you chose what you did. Identify and label all of the relevant forces (i.e. create a free-body diagram, aka FBD). Do NOT go beyond this – this week is meant for you to carefully describe your problem visually and in words. Be creative and specific with your setup (do not just say “I am going to solve the EOM for an aircraft”.)