ARMA Models

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BUTE

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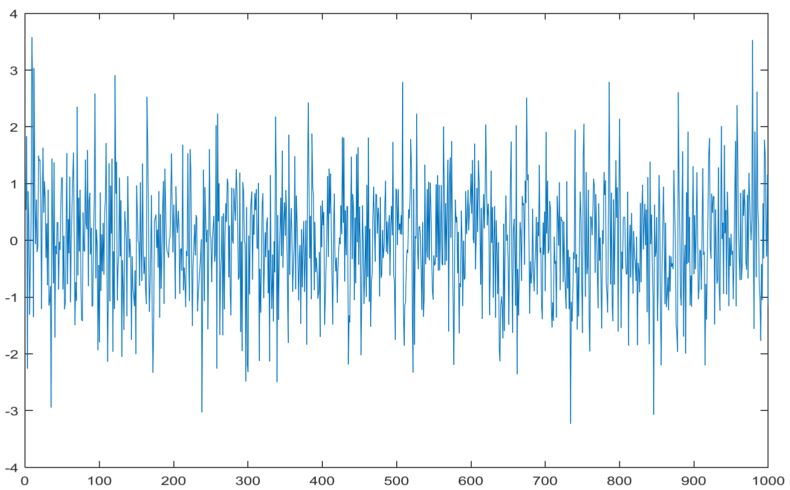
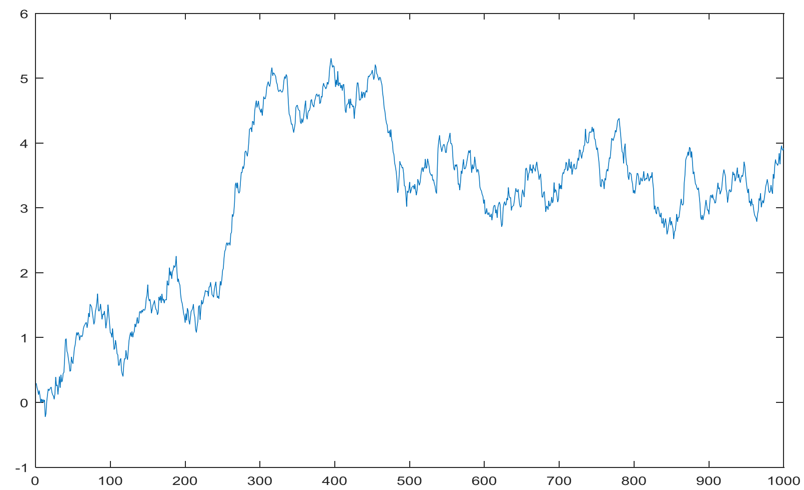
What is a time series?

Notation: $\{x_1, x_2, \dots, x_T\}$ or $\{x_t\}$, t= 1,2,…,T

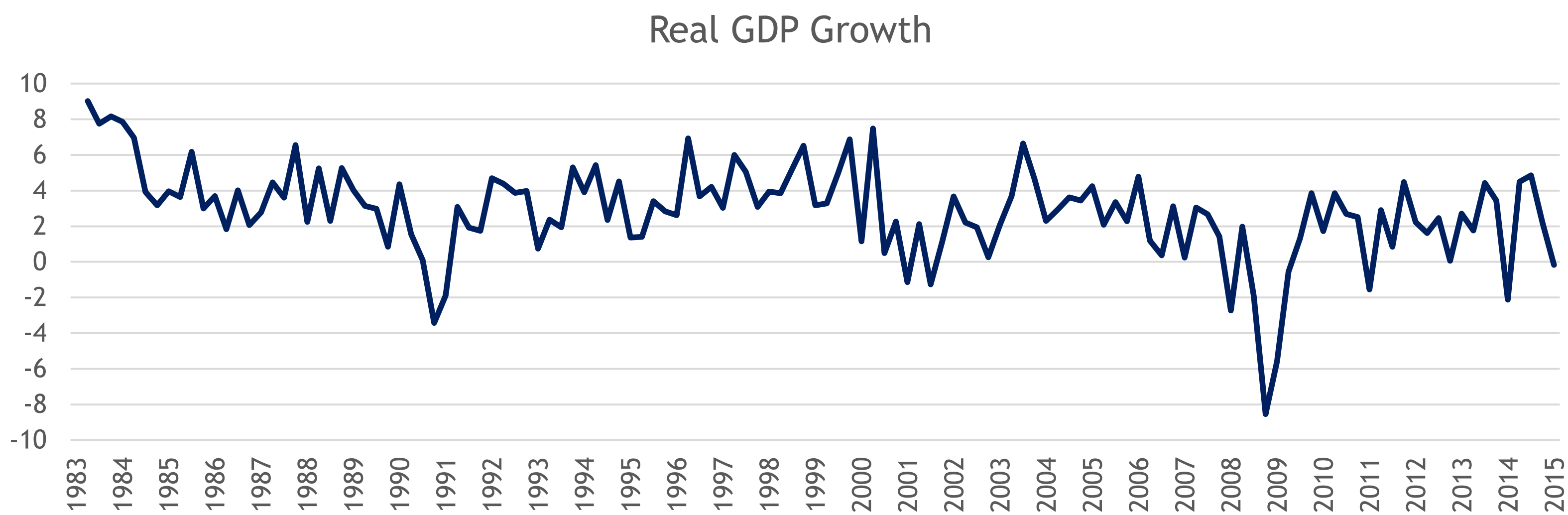
Example 1.: $y_t = \beta x_t + \epsilon_t,\ \mathrm{E}(\epsilon_t | x_t) = 0$

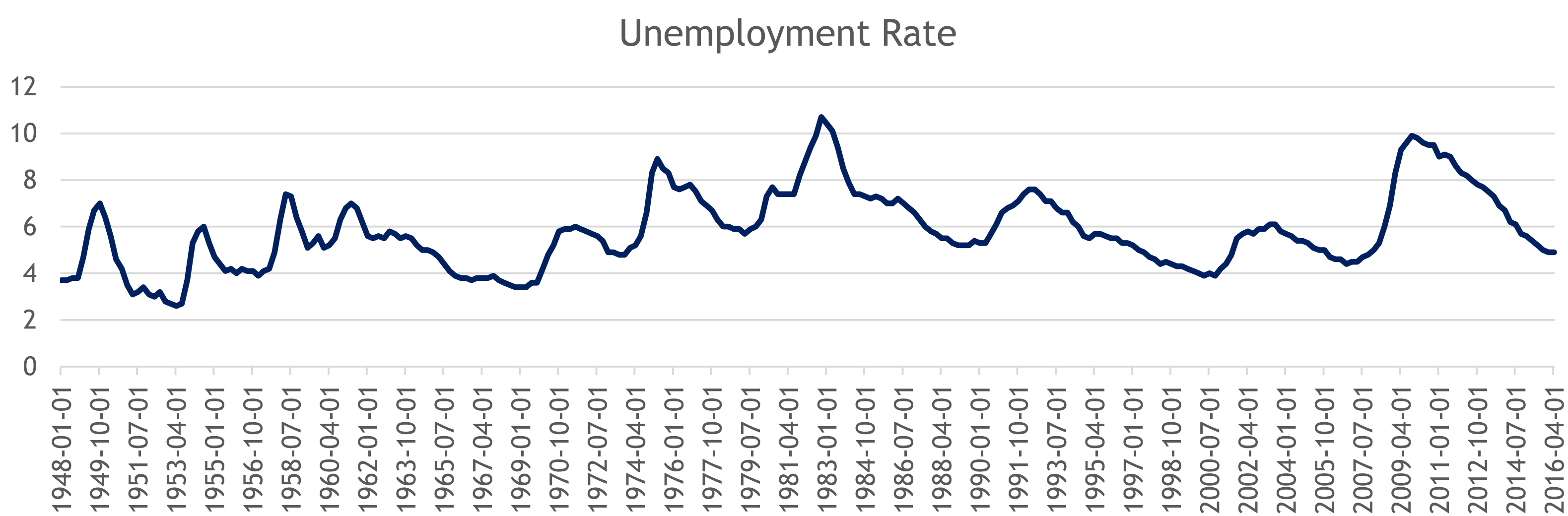
Example 2.: $x_t = \epsilon_t, \quad \epsilon_t \sim \mathrm{N}(0, \sigma^2)$





Real GDP Growth and Unemployment Rate of the USA





Linear Time Series Models

- 1. Building block:

White noise: $x_t = \epsilon_t, \quad \epsilon_t \sim i.i.d. \mathrm{N}(0, \sigma^2)$

1. $E(\epsilon_t) = \mathrm{E}(\epsilon_t | \epsilon_{t-1}, \epsilon_{t-2}, \dots) =$
   $E(\epsilon_t | all\ information\ at\ t-1) = 0$
2. $E(\epsilon_t \epsilon_{t-j}) = cov(\epsilon_t \epsilon_{t-j}) = 0$
3. $var(\epsilon_t) = var(\epsilon_t | \epsilon_{t-1}, \epsilon_{t-2}, \dots) =$
   $var(\epsilon_t | all\ information\ at\ t-1) = \sigma^2$

- 2. ARMA models:

AR(1): $x_t = \varphi x_{t-1} + \epsilon_t$

AR(p): $x_t = \varphi_1 x_{t-1} + \varphi_2 x_{t-2} + \cdots + \varphi_p x_{t-p} + \epsilon_t$

MA(1): $x_t = \epsilon_t + \theta \epsilon_{t-1}$

MA(q): $x_t = \epsilon_t + \theta_1 \epsilon_{t-1}$+…+$\theta_q \epsilon_{t-q}$

ARMA(p,q):$x_t = \varphi_1 x_{t-1} + \cdots + \epsilon_t + \theta_1 \epsilon_{t-1}$+…

Lag Operators and Lag Polynomials

- Lag operator moves the index back one time unit:

$$Lx_t = x_{t-1} \rightarrow L^2 x_t = x_{t-2} \rightarrow L^{-j} x_t = x_{t+j}$$

- So we can define lag polynomials eg.:

$$a(L)x_t = (a_0 L^0 + a_1 L^1 + a_2 L^2)x_t = a_0 x_t + a_1 x_{t-1} + a_2 x_{t-2}$$

- Using this notation we can rewrite ARMA models:

AR: $a(L)x_t = \epsilon_t$
MA: $x_t = b(L)\epsilon_t$
ARMA: $a(L)x_t = b(L)\epsilon_t$

AR-MA Substitution

AR(1): $x_t = \varphi x_{t-1} + \epsilon_t$

$$x_t = \varphi x_{t-1} + \epsilon_t = \varphi(\varphi x_{t-2} + \epsilon_{t-1}) + \epsilon_t = \varphi^2 x_{t-2} + \varphi\epsilon_{t-1} + \epsilon_t$$

$$x_t = \varphi^k x_{t-k} + \varphi^{k-1}\epsilon_{t-k+1} + \cdots + \varphi^2 x_{t-2} + \varphi\epsilon_{t-1} + \epsilon_t$$

If $|\varphi| < 1$ then $\lim_{k\to\infty} \varphi^k x_{t-k} = 0$, so $x_t = \sum_{j=0}^{\infty} \varphi^j \epsilon_{t-j}$.

It means that AR(1) can be expressed as MA(∞).

The Autocovariance Function

- Definition: The autocovariance of the series x_t is defined as

$$\gamma_j = cov(x_t, x_{t-j})$$

We assume that E$(x_t) = 0$ for all our models, since we ae specializing to ARMA models without constant terms.

$$\gamma_j = E(x_t, x_{t-j})$$

Note that $\gamma_0 = var(x_t)$, the autocorrelation function is

$$\rho_j = {}^{\gamma_j}/_{var(x_t)} = {}^{\gamma_j}/_{\gamma_0}$$

Autocovariance of ARMA Processes

- White Noise: since $x_t = \epsilon_t, \;\; \epsilon_t \sim i.i.d.\mathrm{N}(0, \sigma^2)$, it is obvious

$$\gamma_0 = {\sigma_\epsilon}^2, \gamma_j = 0 \; for \; j \neq 0$$
$$\rho_0 = 1, \rho_j = 0 \; for \; j \neq 0$$

- MA(1): $x_t = \epsilon_t + \theta\epsilon_{t-1}$, the autocovariance

$$\gamma_0 = var(\, x_t) = var(\epsilon_t + \theta\epsilon_{t-1}) = (1 + \theta^2){\sigma_\epsilon}^2$$
$$\gamma_1 = E(\, x_t \, x_{t-1}) = var(\epsilon_t + \theta\epsilon_{t-1})(\epsilon_{t-1} + \theta\epsilon_{t-2}) = \theta{\sigma_\epsilon}^2$$
$$\gamma_2 = var(\, x_t \, x_{t-2}) = var(\epsilon_t + \theta\epsilon_{t-1})(\epsilon_{t-2} + \theta\epsilon_{t-3}) = 0$$
$$\gamma_3 = 0$$

$$\rho_1 = {}^{\theta}/_{(1+\theta^2)}\, 1, \qquad \rho_{2,\dots} = 0$$

Autocovariance of ARMA Processes

- AR(1): $x_t = \varphi x_{t-1} + \epsilon_t$ the autocovariance

$$(1-\varphi L)x_t = \epsilon_t \rightarrow x_t = (1-\varphi L)^{-1}\epsilon_t = \sum_{j=0}^{\infty} \varphi^j \epsilon_{t-j}$$

$$\gamma_0 = \left(\sum_{j=0}^{\infty} \varphi^{2j}\right) {\sigma_\epsilon}^2 = \frac{1}{1-\varphi^2} {\sigma_\epsilon}^2, \rho_0 = 1$$

$$\gamma_1 = \left(\sum_{j=0}^{\infty} \varphi^j \varphi^{j+1}\right) {\sigma_\epsilon}^2 = \varphi \left(\sum_{j=0}^{\infty} \varphi^{2j}\right) {\sigma_\epsilon}^2 = \frac{\varphi}{1-\varphi^2} {\sigma_\epsilon}^2, \rho_1 = \varphi$$

$$\gamma_1 = E(\, x_t\, x_{t-1}) = \mathrm{E}\big((\varphi x_{t-1} + \epsilon_t)(\, x_{t-1})\big) = \varphi {\sigma_x}^2, \rho_1 = \varphi$$

$$\gamma_2 = E(\, x_t\, x_{t-2}) = \mathrm{E}\big((\varphi^2 x_{t-2} + \varphi\epsilon_{t-1} + \epsilon_t)(\, x_{t-2})\big) = \varphi^2 {\sigma_x}^2, \rho_2 = \varphi^2$$

$$\gamma_k = E(\, x_t\, x_{t-k}) = \mathrm{E}\left(\left(\varphi^k x_{t-k} + \epsilon_t\right)(\, x_{t-k})\right) = \varphi^k {\sigma_x}^2, \rho_k = \varphi^k$$

Strong and Weak Stationarity

- We used that the moments do not depend on time:

$$E(x_t) = E(x_s)\ for\ all\ t\ and\ s$$
$$E\left(x_t x_{t-j}\right) = E\left(x_s x_{s-j}\right)\ for\ all\ t\ and\ s$$

- This property is true for invertible ARMA models. But reflects a much more important property:

- Definitions:

  - A process is strongly or strictly stationary if the joint probabaility $\{x_{i_1+k}, x_{i_2+k}, \ldots, x_{i_n+k}\}$ is independent of k.
  - A process is weakly or covariance stationary if $E(x_t)$ and $E({x_t}^2)$ are finite and $E\left(x_t x_{t-j}\right)$ does not depend on t.

Conditions for Stationary ARMA's

1. Strong stationarity does not → weak stationarity. $E({x_t}^2)$ must be finite.
2. Strong stationarity plus $E(x_t)$ and $E({x_t}^2) < \infty$ → weak stationarity.
3. Weak stationarity does not → strong stationarity.
4. Weak stationarity plus normality → strong stationarity.

Conditions for stationary ARMA:

1. MA process: $x_t = \sum_{j=0}^{\infty} \theta_j \epsilon_{t-j}, var(x_t) = \sum_{j=0}^{\infty} {\theta_j}^2 \sigma^2$, so second moments exist iff the MA coefficents are square summable:

$$stationary\ MA \leftrightarrow \sum_{j=0}^{\infty} {\theta_j}^2 < \infty$$

If second moments exist it is easy to see that they are independent of t.

2. AR process: for the variance to be finite the AR lag polynomial must be invertible:

$$A(L)x_t = (1 - \mu_1 L)(1 - \mu_2 L) \dots x_t = \epsilon_t$$

Conditions for Stationary ARMA's

For the variance to be finite, the AR lag polynomial must be invertible or $|\mu_i| < 1$ for all i. Common way to say this:

$$A(L) = constant(L - \rho_1)(L - \rho_2) \ldots$$

Where ρ_i are the roots of the lag polynomial, rewriting the above equation:

$$A(L) = constant(-\rho_1)\left(1 - \frac{1}{\rho_1}L\right)(-\rho_2)\left(1 - \frac{1}{\rho_2}L\right)\ldots$$

Thus $\frac{1}{\mu_i} = \rho_i$, hence the rule $|\mu_i| < 1$ means $\rho_i > 1$, or since the roots of the lag polynomial can be complex:

AR stationary if all the roots of the lag polynomial lie outside the unit circle, ie. The lag polynomial is invertible.

Literature

1. John H. Cochrane: Time Series for Macroeconomics and Finance
2. G. Koop, D. Korobilis: Bayesian Multivariate Time Series Methods
3. F. Schorfheide, M. Del Negro: Bayesian Macroeconomics
4. Bolla, Krámli: Statisztikai következtetések elmélete
5. Durbin, Koopman: Time Series Analysis by State Space Methods
6. Tusnády, Ziermann: Idősorok analízise

Websites with Data Sources

1. US data:

    - http://www.bea.gov/
    - https://fred.stlouisfed.org/

2. World data:

    - http://www.imf.org/en/data
    - https://www.bis.org/

3. European data:

    - https://www.ecb.europa.eu/stats/keyind/html/index.en.html