Stationarity and Unit Roots

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BUTE

2016 Autumn

Stationary Processes: Wold Expansion

- Wold Decomposition theorem:
- Any mean zero covariance stationary process x_t can be represented in the form:

$$x_t = \sum_{j=0}^{\infty} \theta_j \varepsilon_{t-j} + \gamma_t$$

Where

1. $\varepsilon_t = x_t - P(x_t | x_{t-1}, x_{t-2}, \dots), \mathrm{P}(* \mid *)\ means\ projection$.
2. $P(\varepsilon_t | x_{t-1}, x_{t-2}, \dots) = 0, E\left(\varepsilon_t x_{t-j}\right) = 0, E(\varepsilon_t) = 0,$
$E({\varepsilon_t}^2) = {\sigma_\varepsilon}^2, E(\varepsilon_t \varepsilon_s) = 0\ for\ all\ t \neq s.$

All roots of θ(L) are on or outside the unit circle so the MA polynomial is *invertible*.

$$\sum_{j=0}^{\infty} {\theta_j}^2 < \infty, \theta_0 = 1$$

$$\{\theta_j\}\ and\ \{\varepsilon_s\}\ are\ unique\ and\ \gamma_t\ is\ linearly\ deterministic:$$
$$\gamma_t = P(\gamma_t | x_{t-1}, \dots)$$

Stacionary and Unit Root Processes

Random walk:

$$x_t = x_{t-1} + \varepsilon_t; \quad E_{t-1}(\varepsilon_t) = 0$$

Popular model for equity prices:

$$E_t(x_{t+1}) = x_t$$

Important features:

- Impulse response of the random walk 1 at all horizons.
- The forecast variance grows linearly with the forecast horizon:

$$var(x_{t+k}|x_t) = var(x_{t+k} - x_t) = k{\sigma_\varepsilon}^2$$

- It does not have a regular autocorrelation function.

Motivation for Unit Roots

- How to represent trend in time series?
  - Linear deterministic trends versus stochastic trends.
- Permanence of shocks:
  - Are business cycles short term, stationary deviations around the trend?
  - Or might the shocks to GDP resemble the permanent shocks of random walk?
  - Are stock returns less than perfect random walks?
- Statistical issues:
  - Distribution of AR(1) estimates
  - Inappropriate detrending
  - Spurious regressions

Statistical Issues

- Distribution of AR(1) estimates:

  Suppose the process is a random walk:

$$x_t = x_{t-1} + \varepsilon_t; \quad E_{t-1}(\varepsilon_t) = 0$$

  You might test by OLS, testing whether $\varphi = 1$:

$$x_t = \mu + \varphi x_{t-1} + \varepsilon_t;$$

  The asymptotic distribution theory for OLS estimates violated, since $XX^T/_T$ does not converge in probability.

  Solution: Dickey- Fuller simulated error bounds for the test statistics.

Inappropriate Detrending

- Things get more difficult with a trend in the model:

$$x_t = \mu + x_{t-1} + \varepsilon_t;$$

Suppose to detrend by OLS, then estimate an AR(1), so fit:

$$x_t = bt + (1 - \varphi L)^{-1}\varepsilon_t$$

This is equivalent with:

$$(1 - \varphi L)x_t = (1 - \varphi L)bt + \varepsilon_t = bt - \varphi b(t-1) + \varepsilon_t$$
$$= \varphi b + b(1 - \varphi)t + \varepsilon_t$$

Or

$$x_t = \alpha + \gamma t + \varphi x_{t-1} + \varepsilon_t.$$

Random walk likely to drift and drift could be modeled as linear trend.

Spurious Regressions

- Suppose we have two independent random walks:

$$x_t = x_{t-1} + \varepsilon_t$$

$$y_t = y_{t-1} + \delta_t, \qquad \mathrm{E}(\varepsilon_t \delta_t) = 0$$

- Suppose we run a regression of y_t on x_t by OLS:

$$y_t = \alpha + \beta x_t + \vartheta_t$$

- Assumptions of the usual distribution theory violated: you will tend to see significant β more often than OLS formula say.

- Campbell and Perron (1991) give a survey on this.

Difference- and Trend Stationary Processes

- Random walk with serially correlated disturbances: unit root or difference stationary (DS):

$$(1 - L)y_t = \mu + a(L)\varepsilon_t$$

- Simplest version, if a(L)=1: random walk with a drift:

$$y_t = \mu + y_{t-1} + \varepsilon_t$$

- Alternatively, if stationary around a linear trend:

$$y_t = \mu t + b(L)\varepsilon_t$$

This is called trend stationary (TS) process.

Difference- and Trend Stationary Processes

The TS model is a special case of the DS model, if $a(L)$ contains a unit root, we can write the DS model as:

$$y_t = \mu t + b(L)\varepsilon_t \rightarrow (1-L)y_t = \mu t + (1-L)b(L)\varepsilon_t$$

$$= \mu + a(L)\varepsilon_t$$

$$a(L) = (1-L)b(L).$$

- Thus if TS model is correct then DS model is still valid and stationary, but it has a noninvertible MA root.
- DS model is a perfect model for differences: unit roots are are non-stationary processes like GDP which is stationary in differences.

Random Walk Components and Stochastic Trends

- Every DS process can be written as a sum of a random walk and a stationary component:
- Beveridge- Nelson decomposition:

$$\text{If } (1-L)y_t = \mu + a(L)\varepsilon_t$$

$$\text{Then } y_t = c_t + z_t$$

$$\text{Where } z_t = \mu + z_{t-1} + a(1)\varepsilon_t$$

$$\text{And } c_t = a^*(L)\varepsilon_t;\ \ a_j{}^* = -\textstyle\sum_{k=j+1}^{\infty} a_k.$$

There are many ways to decompose a unit root into stationary and random walk component, the B-N decomposition is special. The random walk component can be interpreted as stochastic trend: z_t is a limiting forecast of future y_t. If GDP is below trend, it is forecasted to grow and vica-versa.

Empirical Work on Unit Roots

- Unit root tests: Nelson-Plosser 1982, Dickey-Fuller 1979, Phillips-Perron 1988
  - ➢ The first such test tested the unit root property of GDP
  - ➢ Difficulties to differenciate between a unit root process with tiny a(1) and a stationary process: unit root tests are just tests for the size of a(1).
- Investigation of a(1) by parametric measures: Campbell, Mankiw 1988
  - ➢ In this tests they fit a parametric (ARMA) model for the series and investigate the spectral density estimate of the parametric model
- Non-parametric estimates of a(1): Cochrane 1988, Lo and MacKinley 1988, Poterba and Summers 1988
  - ➢ Directly estimate the spectral density in 0 and thus a(1), ignoring rest of the process

Literature

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