Homework Seven, for Tue 3/8 CSE 250B

Your homework must be typeset, and the PDF file should be uploaded to Gradescope by midnight on the due date.

1. *Singular values versus eigenvalues.* Recall from class that any $p \times q$ matrix M (with $p \leq q$, say) can be written in the form:

$$M = \underbrace{\begin{pmatrix} \uparrow & & \uparrow \\ u_1 & \cdots & u_p \\ \downarrow & & \downarrow \end{pmatrix}}_{p \times p \text{ matrix } U} \underbrace{\begin{pmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_p \end{pmatrix}}_{p \times p \text{ matrix } \Lambda} \underbrace{\begin{pmatrix} \longleftarrow & v_1 & \longrightarrow \\ & \vdots & \\ \longleftarrow & v_p & \longrightarrow \end{pmatrix}}_{p \times q \text{ matrix } V^T}$$

where $u_1, \ldots, u_p$ are orthonormal vectors in $\mathbb{R}^p$, $v_1, \ldots, v_p$ are orthonormal vectors in $\mathbb{R}^q$, and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \geq 0$ are known as *singular values.* In this problem, we will try to understand these quantities by relating them to eigenvalues and eigenvectors of suitably defined matrices.

   (a) What is Mv_i (for $1 \leq i \leq p$)? Express the answer as simply as possible, in terms of the singular values and vectors of M.

   (b) What is $M^T u_i$?

   (c) What is $M^T M v_i$? And what is $MM^T u_i$?

   (d) Notice that MM^T is a symmetric $p \times p$ matrix and therefore has p real eigenvalues. What are its eigenvalues and eigenvectors?

   (e) How do the eigenvalues and eigenvectors of $M^T M$ relate to those of MM^T?

   (f) Suppose M has rank k. How would this be reflected in the singular values σ_i?

2. *Rank-1 matrices.*

   (a) Find the best rank-1 approximation to the matrix:

$$M = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}.$$

   If you use the SVD method in Python to solve this problem, you should use the setting `full_matrices` = 0 to get the kind of decomposition we've been discussing (sometimes called the "thin SVD").

   (b) In general, a rank-1 matrix of dimension $p \times q$ is a matrix that can be written in form uv^T, where $u \in \mathbb{R}^p$ and $v \in \mathbb{R}^q$. Do you think this decomposition is unique, or is it in general possible to find a different pair of vectors $a \in \mathbb{R}^P$ and $b \in \mathbb{R}^q$ such that $uv^T = ab^T$?

   (c) Let M be some $p \times q$ matrix whose singular value decomposition is as specified in problem 1. Notice that M can equally be written in the form

$$M = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_p u_p v_p^T,$$

   that is, as a sum of rank-1 matrices. For $k < p$, let $\widehat{M}$ be the best rank-k approximation to M. Express $\widehat{M}$ as a sum of rank-1 matrices.

3. *The Gram matrix.* Write down the 4×4 Gram matrix for the following data set of four points in $\mathbb{R}^3$:

$$(1, 0, 0),\ (1, 0, 1),\ (1, 1, 0),\ (1, 1, 1).$$

   Do you think there are other sets of four points with exactly the same Gram matrix? Why or why not?

4. *Classical multidimensional scaling.* In this problem, we'll use classical multidimensional scaling to reconstruct the locations of ten US cities given only the distances between them.

   (a) The 10×10 array of distances can be found in the file `distances.txt` (found on the course website). The city names are in `cities.txt` (but one of the cities has been called "mystery").

   (b) Use MDS to plot these cities on a 2-d grid, and make sure to annotate each point with the corresponding city name. Does the plot look right? What would you need to do to fix it?

   (c) The mystery city is a large metropolis. What do you think it is?