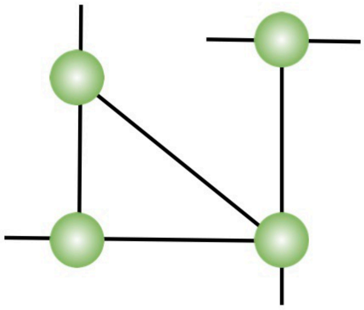
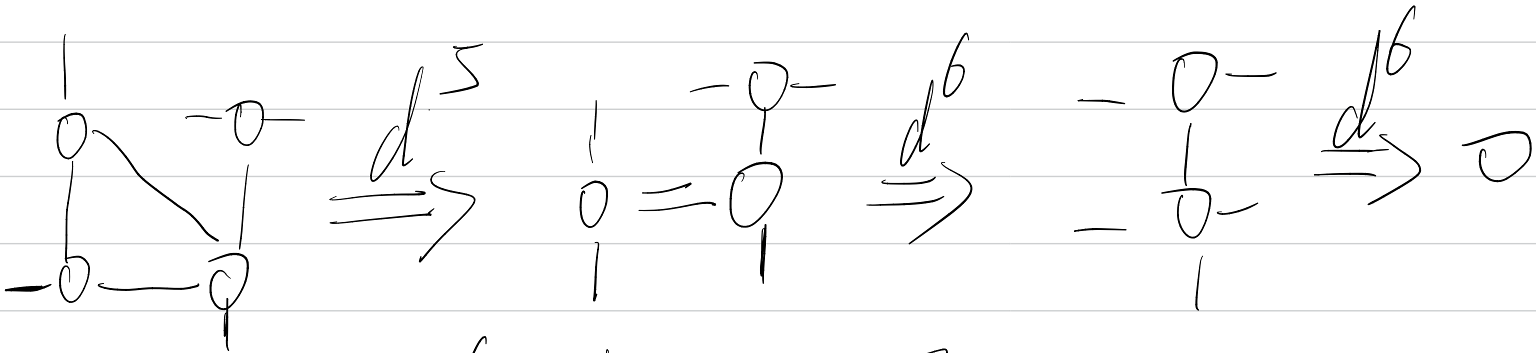
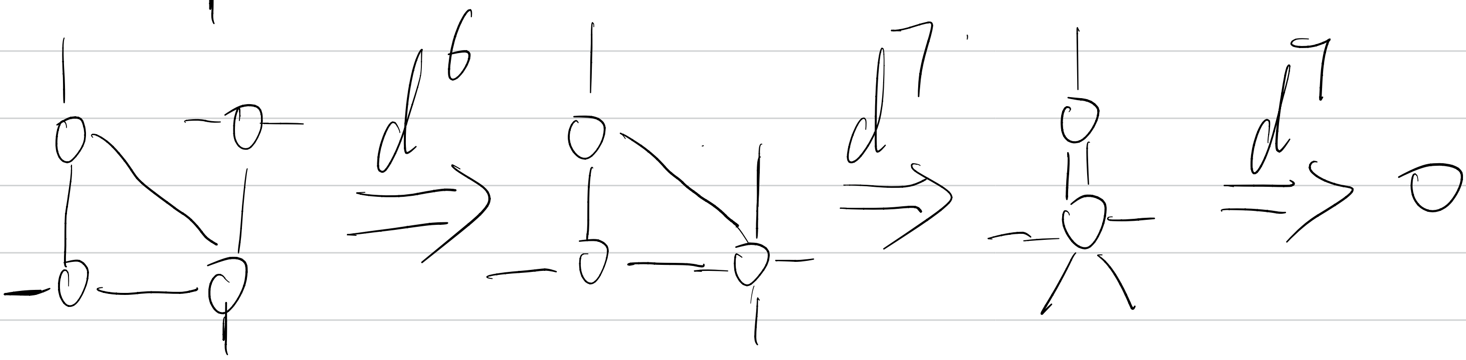
1. Order of tensor contraction (8 points)

Consider the following tensor network. Find two ways of contracting the network with different complexity. For each way, show the complexity of each contraction, and report the overall contraction complexity (4 points each way).







2. SVD of simple matrix (10 points)

Obtain manually the singular values (2 points), the left singular vectors (matrix U, 4 points) and the right singular vectors (matrix $V^\dagger$, 4 points) of the matrix

$$O = \begin{pmatrix} 2 & 4 \\ 1 & 3 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{1}$$

from the eigenvalue decomposition of $OO^\dagger$ and $O^\dagger O$.

$$OO^\dagger = \begin{pmatrix} 2 & 4 \\ 1 & 3 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 & 0 \\ 4 & 3 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 20 & 14 & 0 & 0 \\ 14 & 10 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$OO^\dagger = U\Lambda^2 U^\dagger$$

$$(20-\lambda)(10-\lambda) - 14^2 = 0$$

$$\lambda^2 - 30\lambda + 4 = 0 \qquad \lambda = 15 \pm \sqrt{221}$$

Singular value of O : $\sqrt{15+\sqrt{221}}$, $\sqrt{15-\sqrt{221}}$

$$U = \frac{1}{\sqrt{1+\frac{1}{196}(5+\sqrt{221})^2}} \begin{pmatrix} \frac{5+\sqrt{221}}{14} & 1 & 0 & 0 \\ \frac{5-\sqrt{221}}{14} & 1 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{1+\frac{1}{196}(\sqrt{221}+5)^2} \\ 0 & 0 & \sqrt{1+\frac{1}{196}(5+\sqrt{221})^2} & 0 \end{pmatrix}$$

$$O^\dagger O = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 4 & 3 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 1 & 3 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 11 \\ 11 & 25 \end{pmatrix} = V\Lambda^2 V^\dagger$$

$$V = \frac{1}{\sqrt{1+\frac{1}{121}(-10+\sqrt{221})^2}} \begin{pmatrix} \frac{1}{11}(-10-\sqrt{221}) & \frac{1}{11}(\sqrt{221}-10) \\ 1 & 1 \end{pmatrix}$$

$$V^\dagger = \frac{1}{\sqrt{1+\frac{1}{121}(-10+\sqrt{221})^2}} \begin{pmatrix} -\frac{11}{2\sqrt{221}} & \frac{1}{2} - \frac{5}{\sqrt{221}} \\ \frac{11}{2\sqrt{221}} & \frac{1}{2} + \frac{5}{\sqrt{221}} \end{pmatrix}$$

6. Mixed Canonical MPS (10 points)

Obtain analytically the mixed canonical MPS structure of a tensor C of order N, with elements $C_{n_1 n_2 \cdots n_N}$ and $n_j = 0, 1$. Describe the normalization condition of all the tensors that form the MPS. *Hint:* Check Section 4.1.3(iii) of the review of U. Schollwöck.

$$C_{n_1 \cdots n_N} = \sum_{n_\ell} (A^{n_1} \cdots A^{n_\ell}) S_{n_\ell, n_\ell} (V^\dagger)_{n_\ell \cdots n_N}$$

$$= \sum_{n_\ell} (A^{n_1} \cdots A^{n_\ell}) S_{n_\ell, n_\ell} \sum_{a_{\ell+1} \cdots a_N} B^{n_{\ell+1}}_{a_\ell, a_{\ell+1}} \cdots B^{n_N}_{a_{N-1}}$$

$$\delta_{a_\ell, a_\ell} = V^\dagger V = \left(\sum_{n_{\ell+1}} B^{n_{\ell+1}} B^{n_{\ell+1} \dagger} \right)_{a_\ell, a_\ell'}$$

$$\delta_{a_\ell, a_\ell} = U^\dagger U = \left(\sum_{n_{\ell+1}} A^{n_{\ell+1}} A^{n_{\ell+1} \dagger} \right)_{a_\ell, a_\ell'}$$

$$C_{n_1 \cdots n_N} = A^{n_1} \cdots A^{n_\ell} S B^{n_{\ell+1}} \cdots B^{n_N}$$

$$= \sum_{a_\ell} S_a |a_\ell\rangle_A |a_\ell\rangle_B$$

$$C_{n_1 \cdots n_N} C^\dagger_{n_1 \cdots n_N} = S S^\dagger = I$$

Normalized condition

$$S S^\dagger = I$$

$$\sum^{k} S_a S_a = 1$$

7. Spin 1/2 states as MPS (14 points)

a) Consider a system of two spins 1/2, and the *spin singlet* state $|\psi\rangle = |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$. Write this state as an MPS. *Hint:* The matrices are 1×2 and 2×1 (6 points).

b) Consider a system of four spins 1/2. The so-called W state, which is entangled, is given by

$$|W\rangle = |\uparrow\downarrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\downarrow\rangle + |\downarrow\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\downarrow\uparrow\rangle. \tag{2}$$

Show that this state has an MPS with the following matrices (when associating $\uparrow$ to the first matrix of each site, and $\downarrow$ to the second matrix):

$$A^1[j] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A^2[j] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad 1 < j < 4. \tag{3}$$

The first and second row vectors of site $j = 1$ correspond to the first and second rows of the bulk matrices; similarly for the first and second column vectors of site $j = 4$ (8 points).

(a) $|\psi\rangle = \sum_{n_1, n_2} A^1_{n_1} A^2_{n_2} |n_1, n_2\rangle$

$= A^1_\uparrow A^2_\uparrow |\uparrow\uparrow\rangle + A^1_\uparrow A^2_\downarrow |\uparrow\downarrow\rangle + A^1_\downarrow A^2_\uparrow |\uparrow\downarrow\rangle$

$+ A^1_\downarrow A^2_\downarrow |\downarrow\downarrow\rangle$

$$\begin{cases} A^1_\uparrow A^2_\uparrow = 0 & \Rightarrow (1,0) \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\ A^1_\uparrow A^2_\downarrow = 1 & \Rightarrow (1,0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ A^1_\downarrow A^2_\uparrow = -1 & \Rightarrow (0,1) \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\ A^1_\downarrow A^2_\downarrow = 0 & \Rightarrow (0,1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{cases}$$

$$\begin{cases} A^1_\uparrow = (1,0) \\ A^1_\downarrow = (0,1) \\ A^2_\uparrow = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\ A^2_\downarrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{cases}$$

(b) $|W\rangle = \sum_{n_1,n_2,n_3,n_4} A^1_{n_1} A^2_{n_2} A^3_{n_3} A^4_{n_4} |n_1 n_2 n_3 n_4\rangle$

$A^1_\uparrow = \begin{pmatrix} 0 & 1 \end{pmatrix}$ $\quad A^1_\downarrow = \begin{pmatrix} 1 & 0 \end{pmatrix}$

$A^2_\uparrow = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad A^2_\downarrow = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$A^3_\uparrow = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad A^3_\downarrow = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$A^4_\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad A^4_\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

For $|\uparrow\downarrow\downarrow\downarrow\rangle$: $A^1_\uparrow A^2_\downarrow A^3_\downarrow A^4_\downarrow = \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1$

For $|\downarrow\uparrow\downarrow\downarrow\rangle$: $A^1_\downarrow A^2_\uparrow A^3_\downarrow A^4_\downarrow = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1$

For $|\downarrow\downarrow\uparrow\downarrow\rangle$ $A^1_\downarrow A^2_\downarrow A^3_\uparrow A^4_\downarrow = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1$

For $|\downarrow\downarrow\downarrow\uparrow\rangle$ $A^1_\downarrow A^2_\downarrow A^3_\downarrow A^4_\uparrow = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$

All other combination,

like $A^1_\downarrow A^2_\downarrow A^3_\uparrow A^4_\uparrow = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$

Only 4 combination is 1

$|W\rangle = |\uparrow\downarrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\downarrow\rangle + |\downarrow\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\downarrow\uparrow\rangle$