Lecture 1

Consider two coin tosses.

Possible outcomes $\{HH, HT, TH, TT\} =: \Omega$

Some events we may want to consider:

$E_1 :=$ An event that the first toss is $H = \{HH, HT\} \subset \Omega$

$E_1^c :=$ " " " " " " " $T = \{TT, TH\} \subset \Omega$

$E_2 :=$ " " " " second toss is $H = \{HH, TH\} \subset \Omega$

$E_2^c :=$ " " " " " " " $T = \{TT, HT\} \subset \Omega$

$E_1 \cap E_1^c = \phi$, $E_1 \cup E_1^c = \Omega$

$E_1 \cup E_2 = \{HH, HT, TH\}$

$E_1 \cap E_2 = \{HH\}$

$(E_1 \cup E_2)^c = \{TT\} = E_1^c \cap E_2^c$ De Morgan's law

Probability of events

$\mathbb{P}[E_1] = p$, $\mathbb{P}[E_2] = q$ Identically distributed if $p = q$

$\mathbb{P}[E_1^c] = 1-p$ $\mathbb{P}[E_2^c] = 1-q$

$\mathbb{P}[E_1 \cap E_2] = r$ $r = pq$ if independent ($\perp\!\!\!\perp$)

In general, $0 \le r \le \min\{p, q\}$ since $E_1 \cap E_2 \subset E_1, E_2$

Probability space $(\Omega, \mathcal{F}, \mathbb{P})$

Ω = Sample space, the set of all outcomes.

$\mathcal{F}$ = a σ-algebra (aka σ-field) on Ω

A set of subsets of Ω satisfying certain properties

$\mathbb{P}$ = probability measure

$\boxed{\text{Def}}$: (σ-algebra of Measurable sets)

Given a set S, $\mathcal{F} \subseteq \mathcal{P}(S)$ is called a σ-alg on S if

power set of S

a) $\phi, S \in \mathcal{F}$

b) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

c) $A_i \in \mathcal{F}, i \in \mathbb{N} \implies \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$ Closed under countable union

$A \in \mathcal{F}$ is called $\mathcal{F}$-measurable e.g. 1) $\mathcal{F} = \{\phi, S\}$ the smallest σ-alg

2) $\mathcal{F} = \mathcal{P}(S)$. the largest σ-alg

$\boxed{\text{Def}}$ Measurable space $(S, \mathcal{F})$

a set ↗ ↖ a σ-alg on S

$\boxed{\text{Def}}$ (Measure)

Let $(S,\mathcal{F})$ be a measurable space. A non-negative set function $\mu : \mathcal{F} \to [0,\infty]$ is called a

$\mathbb{R}_{\geq} \cup \{\infty\}$

measure on $(S,\mathcal{F})$ if

a) $\mu(\phi) = 0$

b) $\forall (A_i \in \mathcal{F}, i \in \mathbb{N})$ s.t. $A_i \cap A_j = \phi$ if $i \neq j$,

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$ countably additive or σ-additive

<u>Remarks:</u>

1) $(S, \mathcal{F}, \mu)$ is called a measure space.

2) If $\mu(S) = 1$, μ is called a probability measure, often denoted by $\mathbb{P}$.

3) Specifying $(S,\mathcal{F})$ constrains the possible measure that can be defined on it.

<u>e.g.)</u> Consider a measure λ on $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$ satisfying

i) $\lambda([a,b]) = b-a$, for $b>a$

ii) $\lambda(x+A) = \lambda(A)$, for $x \in \mathbb{R}$, $A \in \mathcal{P}(\mathbb{R})$

$\exists$ a subset $V \in \mathcal{P}(\mathbb{R})$ for which $\lambda(V)$ cannot be defined consistently. (e.g. Vitaly set)

(Not Lebesgue measurable)

<u>e.g.)</u> Consider a unit ball $B \subset \mathbb{R}^3$ and drop a pt x u.a.r. on B. For <u>any</u> subset $A \subset B$, can we define $\mathbb{P}[x \in A] = \dfrac{\text{Volume}(A)}{\frac{4}{3}\pi}$?

<u>No!</u> (Banach-Tarski)

Some $A \in \mathcal{P}(B)$ are not Lebesgue measurable.