Stat 201A, Fall 2024: Lab 5

Conceptual review

- When to use Poisson approximation instead of normal approximation for a Binomial distribution?
- How are the binomial and multinomials distributions related?

Problem 1 (Poisson approximation)

1. A large company has a large fleet of cars. On average, there are 3 accidents each week. What is the probability that at most 2 accidents happens next week?
2. Every evening Murdoc goes to the local casino. There is a 1% chances that he wins \$10000 and 99% he loses \$100. Define a random variable X_k representing the winning/losing outcome of Murdoc after each day k. After a full year passes, estimate the probability that Murdoc wins at least least \$1000.

Problem 2 (Exchangeability and multinomial distribution)

Suppose an urn contains 2 green, 3 red and 4 yellow balls. Six balls are chosen **with** replacement. Find the probability that green appeared 1 times, red 2 times, and yellow 3 times. Six balls are chosen **without** replacement. Find the probability that the 3rd ball chosen is green, given that the 5th ball chosen is yellow?

Problem 3 (Poisson distribution)

1. Let $X \sim \text{Geom}(1/3)$ and $Y \sim \text{Poisson}(2)$ be independent random variables. Calculate $\mathbb{P}(X = Y + 2)$.

2. Suppose that $X \sim \text{Poisson}(\lambda)$. Find the probability $\mathbb{P}(X \text{ is even})$.

3. Let $X \sim \text{Poisson}(\mu)$. Compute $\mathbb{E}\left(\frac{1}{1+X}\right)$.

Problem 4

Let $N \sim \text{Poisson}(\lambda)$, and let $X_1, X_2, \ldots$ be a sequence of i.i.d. geometric random variables with parameter p, where $X_i \sim \text{Geometric}(p)$. Define $S_N = X_1 + X_2 + \cdots + X_N$. N is independent of the X_i 's.

1. Find the probability generating function (PGF) of S_N
   Hint: Use the compounding theorem discussed in Lecture 11. The PGF of a Poisson random variable $N \sim \text{Poisson}(\lambda)$ is given by

   $$G_N(t) = e^{\lambda(t-1)}$$

   and the PGF of a geometric random variable $X \sim \text{Geometric}(p)$ is

   $$G_X(t) = \frac{p}{1-(1-p)t}, \quad |t| < \frac{1}{1-p}$$

2. Suppose $p = 0.5$, $\lambda = 1$. Calculate the probability $\mathbb{P}(S_N = 1)$.

3. (Bonus) Verify the PGF of a Poisson random variable and a geometric random variable through explicit calculation.

Problem 5

Recall: When $N \sim \text{Poisson}(\lambda)$ and $(X_1, \ldots, X_m) \mid N \sim \text{Multinomial}(N, p_1, \ldots, p_m)$, the joint distribution of $X_1, \ldots, X_m$ follows independent Poisson distributions, i.e., $X_j \sim \text{Poisson}(p_j \lambda)$. We can show this result through simulation.

1. Simulate 10,000 samples of the Poissonized Multinomial and Independent Poisson distributions with $\lambda = 10$ and $\mathbf{p} = (0.3, 0.5, 0.2)$.

2. Visualize and compare the joint distribution of X_1 and X_2 from two data simulation procedure using either a 2D density plot or an overlayed scatter plot.