Stat 201A, Fall 2024: Lab 3

Conceptual review

- How are the LLN and CLT different? How are they related?
- How are the exponential and gamma distributions related?

Problem 1

Noodle decide to improve her ability to calculate integrals. Each day she flips a coin until she gets tails. If she gets tails in 3 or less flips, she will calculate 10 integrals. If she needs strictly more than 3 flips to get tails she will calculate 60 integrals. After a full year passes, estimate the probability that Noodle has solved more than 6000 integrals.

Problem 2

Here is a limit theorem that one can prove without complicated tools. Suppose that $X_1, X_2, \ldots$ are i.i.d. random variables with distribution $Exp(1)$, and let $M_n = \max(X_1, \ldots, X_n)$. Show that for any $x \in \mathbb{R}$ we have

$$\lim_{n\to\infty} \mathbb{P}(M_n - \ln(n) \leq x) = \exp(-e^{-x}).$$

This is called the *Gumbel distribution*.

Problem 3

Prove that the Exponential Distribution is the only distribution on $(0, \infty)$ that satisfies:

$$\mathbb{P}(X > a + b | X > b) = \mathbb{P}(X > a),\ \forall\, a, b > 0.$$

Problem 4

In the lectures, we used the MGF to prove the following results regarding convergence in distribution.

- Let $G_n \sim \text{Geometric}\left(\frac{\lambda}{n}\right)$, where $\lambda > 0$, and $n = 1, 2, 3, \ldots$. Define $X_n = \frac{G_n}{n}$. As $n \to \infty$, X_n converges in distribution to an Exponential distribution with rate λ:

$$X_n \xrightarrow{d} \text{Exp}(\lambda)$$

- Let $F_{r,n} \sim \text{NB}\left(r, \frac{\lambda}{n}\right)$, where $\lambda > 0$ and $n = 1, 2, 3, \ldots$. Define $X_n = \frac{F_{r,n}}{n}$. As $n \to \infty$, X_n converges in distribution to a Gamma distribution with shape r and rate λ:

$$X_n \xrightarrow{d} \text{Gamma}(r, \lambda)$$

1. Apply MGF and use similar strategies discussed in the lecture to prove the following.
   Let $W_n \sim \text{Bin}\left(n, \frac{\lambda}{n}\right)$ represent a binomial random variable with probability $\frac{\lambda}{n}$, then W_n converges in distribution to a Poisson distribution with parameter λ:

$$W_n \xrightarrow{d} \text{Poi}(\lambda)$$

   .

2. Conduct simulations to show the convergence in distribution for three results above.

Problem 5

The MGF of a random variable X is given by

$$M_X(t) = \frac{c}{1-t} - \frac{2}{1-t} \quad \text{for } |t| < 1$$

1. Find the value of c.
2. Find $\mathbb{E}[X]$.
3. Find $\mathbb{E}[X^2]$.