Quantitative error analysis of Discontinuous Galerkin methods for Linear Convection Equation

CHEN Qifan

Summer Research Project at The Ohio State Univeristy

My email:chanqifan@qq.com

Mentor: Y. Xing

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Overview

Linear Convection Equation

Linear Convection Equation with periodic boundary conditions

$$u_t + u_x = 0, x \in [0, 2\pi], t > 0$$
$$u(x, 0) = \sin x$$

- Propagation of a wave without change of shape
- Most accessible equation in Computational Fluid Dynamics.
- Exact solution is $u(x, t) = \sin(x - t)$

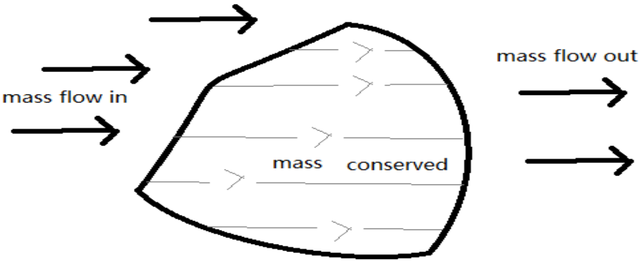
Examples for Linear Convection Equation

Conserved quantity: T with units of $\left[\text{stuff } m^{-3}\right]$:
Conservation equation:

$$\frac{\partial T}{\partial t} + \nabla \cdot \mathbf{F}_T = 0$$

where $\mathbf{F}_T$ is the vector flux of T. Flux is velocity $\times$ quantity: $\mathbf{F}_T = \mathbf{u}T$ with units $[\text{stuff } \mathrm{m/s}]$

In one-dimension: $\frac{\partial T}{\partial t} + \frac{\partial uT}{\partial x} = 0$



Numerical Solution to PDE

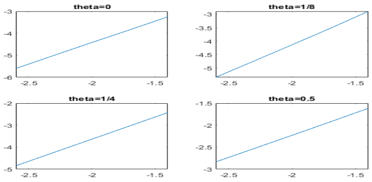


Figure: Order of accuracy: $\|u_{numerical} - u_{exact}\| = O\left(h^n\right)$

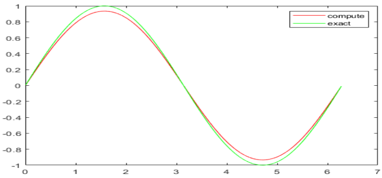


Figure: Structure Preserving:Energy conserving

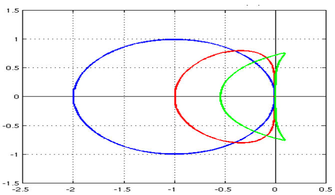


Figure: Stability region

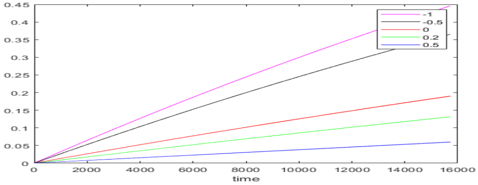


Figure: Exact error: coefficient on power function

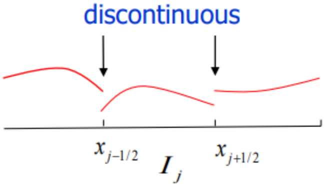
Discontinuous Galerkin method

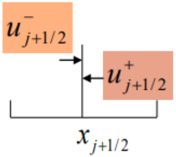
Solve PDE numerically: Finite difference, Finite volume,etc....

Finite element$\rightarrow$ **Discontinuous Galerkin method**

- $I_h = \{I_i\}_{i=1}^N$: uniform partition of the computational domain $[0, 2\pi]$ into N cells : $0 = x_1 < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = 2\pi.\Delta x = \frac{2\pi}{N}$.
- Approximation space: $V_h^k = \left\{ v : v|_{I_j} \in P^k(I_j) ; 1 \leqslant j \leqslant N \right\}$

  $P^k(I_j)$: polynomials of degree up to k defined on the cell I_j.
- Local basis of $P^k(I_j)$: Lagrangian polynomials on the $k+1$ equally spaced points $x_{j+\frac{2|-k|}{2|k+1|}} = x_j + \left(\frac{2l-k}{2(k+1)}\right)\Delta x, \quad l = 0, \ldots, k$





DG method and Fluxes

DG (semi-discrete) scheme

$$\int_{I_j} u_t v dx - \int_{I_j} u v_x dx + \hat{u}_{j+1/2} v^-_{j+1/2} - \hat{u}_{j-1/2} v^+_{j-1/2} = 0$$

$\hat{u}_{j\pm\frac{1}{2}}$ is the numerical flux, v is the test function in V_h^k.
In this section, flux is chosen to be :

$$\hat{u}_{j\pm\frac{1}{2}} = u^-_{j\pm\frac{1}{2}} + \theta\left(u^+_{j\pm\frac{1}{2}} - u^-_{j\pm\frac{1}{2}}\right), \theta \le \frac{1}{2}.$$

After substitution, finite difference representation of the DG method :

$$\frac{d\mathbf{u}_j}{dt} = \frac{1}{\Delta x}\left(A\mathbf{u}_{j-1} + B\mathbf{u}_j + C\mathbf{u}_{j+1}\right)$$

Fourier stability analysis(von Neumann stability analysis)

Assumption of uniform mesh sizes and periodic boundary conditions:

$$\boldsymbol{u}_j(t) = \hat{\boldsymbol{u}}(t)e^{imx_j}, m = 1, 2, 3, \ldots$$

Substitute into the matrix form :

$$\frac{d}{dt}\hat{\mathbf{u}}(t) = G\hat{\mathbf{u}}(t)$$

where G is the amplification matrix, given by

$$G = \frac{1}{\Delta x}\left(Be^{-i\xi} + A + Ce^{i\xi}\right), \quad \xi = m\Delta x$$

ODE system from Fourier stability analysis

PDE→ODE→Tools to analyze ODE

If G is diagonalizable, denote the eigenvalues of G as $\lambda_1, \dots, \lambda_{k+1}$ and the corresponding eigenvectors as $\bar{V}_1, \dots, \bar{V}_{k+1}$.

The general solution of the ODE system is

$$\hat{\mathbf{u}}(t) = C_1 e^{\lambda_1 t} \bar{V}_1 + \cdots + C_{k+1} e^{\lambda_{k+1} t} \bar{V}_{k+1}$$

where $C_1, \dots, C_{k+1}$ can be determined by the initial condition.

For convenience, write $C_i \bar{V}_i = V_i$

Implement of DG scheme and Fourier analysis when k=1

Choose the uniformly spaced points :

$$u_{j-\frac{1}{4}}, u_{j+\frac{1}{4}}, \quad j = 1, \ldots, N.$$

Representation of solution through the basis :

$$u(x) = u_{j-\frac{1}{4}}\phi_{j-\frac{1}{4}}(x) + u_{j+\frac{1}{4}}\phi_{j+\frac{1}{4}}(x) \text{ , where } \quad \phi_{j\pm\frac{1}{4}}(x) = \begin{cases} 1, & \text{at } \left(j \pm \frac{1}{4}\right) dx \\ 0, & \text{at } \left(j \mp \frac{1}{4}\right) dx \end{cases} .$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}_j = \frac{1}{\Delta x}\left(A\boldsymbol{u}_{j-1} + B\boldsymbol{u}_j + C\boldsymbol{u}_{j+1}\right) \text{ where } \boldsymbol{u}_j = \begin{pmatrix} u_{j-\frac{1}{4}}(t) \\ u_{j+\frac{1}{4}}(t) \end{pmatrix}.$$

For simplicity, in this project, we only consider the case for $m = 1$.

Eigenstructure and ODE system when k=1

Taylor expansions to eigenvalues:

$$\lambda_1 = -i + \frac{\mathrm{dx}^3}{72(2\theta-1)} + \frac{i\mathrm{dx}^4\left(6\theta^2-6\theta-1\right)}{270(1-2\theta)^2} + \frac{\mathrm{dx}^5\left(12\theta^2-12\theta-1\right)}{648(2\theta-1)^3} + +O\left(\mathrm{dx}^6\right), \theta < 0.5$$
$$\lambda_2 = \frac{6(2\theta-1)}{\mathrm{dx}} + 3i + \mathrm{dx}(1-2\theta) - \frac{i\mathrm{dx}^2}{3} + \frac{\mathrm{dx}^3\left(24\theta^2-24\theta+5\right)}{72(2\theta-1)} + O\left(\mathrm{dx}^4\right), \theta < 0.5$$

$$\lambda_1 = -i - \frac{i\mathrm{dx}^2}{48} + \frac{7i\mathrm{dx}^4}{15360} + \frac{599i\mathrm{dx}^6}{10321920} + O\left(\mathrm{dx}^8\right), \theta = 0.5$$
$$\lambda_2 = 3i - \frac{5i\mathrm{dx}^2}{16} + \frac{83i\mathrm{dx}^4}{5120} - \frac{313i\mathrm{dx}^6}{688128} + O\left(\mathrm{dx}^8\right), \theta = 0.5$$

$$V_1 + V_2 = U_{initial} = \begin{pmatrix} e^{i\left(-\frac{\mathrm{dx}}{4}\right)} \\ e^{i\left(\frac{\mathrm{dx}}{4}\right)} \end{pmatrix}, \quad U_{exact} = e^{-it}\begin{pmatrix} e^{i\left(-\frac{\mathrm{dx}}{4}\right)} \\ e^{i\left(\frac{\mathrm{dx}}{4}\right)} \end{pmatrix}, U_{compute} =$$
$$e^{\lambda_1 t}V_1 + e^{\lambda_2 t}V_2$$

Observations from eigenvalues when k=1

- When $\theta = \frac{1}{2}$, eigenvalues are purely imaginary. The scheme is energy conserving; when $\theta < \frac{1}{2}$, eigenvalues contain a negative real part. Tcheme is energy dissipating.

- The error could be decomposed into physical terms and nonphysical terms:

$$\begin{aligned} &|u_{exact} - u_{compute}| = |(V_1 + V_2)e^{-it} - e^{\lambda_2} bV_2 - e^{\lambda_1 t} V_1| \\ &= |V_2(1 - e^{\lambda_2 t + it}) + V_1(1 - e^{\lambda_1 t + it})| \end{aligned}.$$

  Physical terms and nonphysical terms have different behavior with time.

Physical terms when k=1

The Taylor expansion of physical terms:

$$|V_1(e^{-it}-e^{\lambda_1 t})| = \begin{pmatrix} \frac{\mathrm{dx}^3 t}{72-144\theta} + \frac{\mathrm{dx}^5\left(2304\theta^4-4608\theta^3+3936\theta^2-1782\theta-61\right)t}{129600(1-2\theta)^3} - \frac{\mathrm{dx}^6 t^2}{10368(1-2\theta)^2} + O\left(\mathrm{dx}^7\right) \\ \frac{\mathrm{dx}^3 t}{72-144\theta} + \frac{\mathrm{dx}^5\left(2304\theta^4-4608\theta^3+3936\theta^2-1482\theta-211\right)t}{129600(1-2\theta)^3} - \frac{\mathrm{dx}^6 t^2}{10368(1-2\theta)^2} + O\left(\mathrm{dx}^7\right) \end{pmatrix}, \theta \neq \frac{1}{2}$$

$$|V_1(e^{-it}-e^{\lambda_1 t})| = \begin{pmatrix} \frac{\mathrm{dx}^2 t}{48} - \frac{17\mathrm{dx}^4 t}{40960} - \frac{\mathrm{dx}^6\left(t\left(8960t^2+1422369\right)\right)}{23781703680} + O\left(\mathrm{dx}^7\right) \\ \frac{\mathrm{dx}^2 t}{48} - \frac{17\mathrm{dx}^4 t}{40960} - \frac{\mathrm{dx}^6\left(t\left(8960t^2+1422369\right)\right)}{23781703680} + O\left(\mathrm{dx}^7\right) \end{pmatrix}, \theta = \frac{1}{2}$$

- During different time intervals, the dominant terms are different, as observed [7].For intermediate time, the error is proportional to t. For longer time, another term with higher degree of t may dominate.

  This is caused by the expansion of $e^{(\lambda_1+i)t} \approx e^{\frac{dx^3 t}{72(2\theta-1)}}$.
- Bigger θ results in bigger difference between λ_1 and -i, thus a bigger error.

Nonphysical terms when k=1

λ_2 contains $\frac{1}{dx}$, so $e^{\lambda_2 t}$ could not be directly expanded around dx=0.

When $\theta < \frac{1}{2}$, λ_2 would contain a negative real part. $\rightarrow$ $e^{(\lambda_2+i)t}$ would decay exponentially $\rightarrow$ always dropped in the literature [2], [5]. $\rightarrow$ only V_2 is left.
Taylor expansion of V_2:

$$|-V_2| = \begin{pmatrix} \frac{\mathrm{dx}^2}{24-48\theta} + \frac{\mathrm{dx}^4\left(-1296\theta^4+2592\theta^3-2056\theta^2+888\theta+83\right)}{27648(2\theta-1)^3} + O\left(\mathrm{dx}^6\right) \\ \frac{\mathrm{dx}^2}{24-48\theta} + \frac{\mathrm{dx}^4\left(-1296\theta^4+2592\theta^3-2056\theta^2+632\theta+211\right)}{27648(2\theta-1)^3} + O\left(\mathrm{dx}^6\right) \end{pmatrix}, \theta \neq \frac{1}{2}.$$

$$|-V_2| = \begin{pmatrix} \frac{\mathrm{dx}}{16} - \frac{7\mathrm{dx}^3}{1536} + \frac{\mathrm{dx}^5}{491520} + \frac{3421\mathrm{dx}^7}{660602880} + \frac{8353\mathrm{dx}^9}{54358179840} + O\left(\mathrm{dx}^{10}\right) \\ \frac{\mathrm{dx}}{16} - \frac{7\mathrm{dx}^3}{1536} + \frac{\mathrm{dx}^5}{491520} + \frac{3421\mathrm{dx}^7}{660602880} + \frac{8353\mathrm{dx}^9}{54358179840} + O\left(\mathrm{dx}^{10}\right) \end{pmatrix}, \theta = \frac{1}{2}.$$

- When θ is bigger, this error would also be bigger.

Nonphysical terms when k=1

Based on formula to eigenvalue, when θ is bigger, the real part is bigger, thus $e^{(\lambda_2+i)t}$ would decay more slowly.

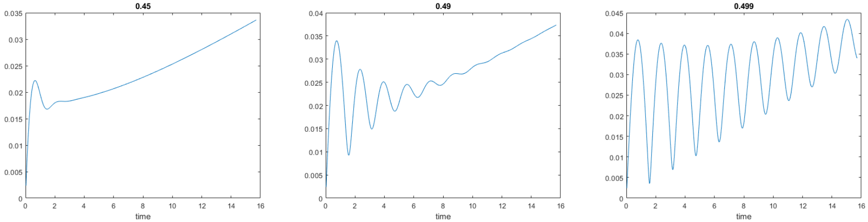


Figure: L-2 Error-time plot. exponential decay. $dx = \frac{\pi}{10}$; $s = 2\pi$; $dt = \frac{dx}{10}$; $t = 5\pi$

Other observations when k=1

- Nonphysical terms have stronger influence on short-time error. physical term will dominate in long time simulation.
- Optimal error estimate for energy dissipative scheme and only suboptimal error estimate for energy conservative scheme with central flux. Blow-up behavior caused by $(2\theta - 1)$ in the denominator.
- When θ is smaller, both short time error and long time would be smaller, while the stability region would also be smaller.

Numerical result for different θ when k=1

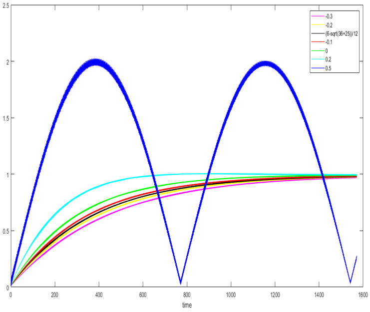


Figure: $L^\infty, dx = \frac{\pi}{5}, dt = \frac{dx}{10}, t = 500\pi$

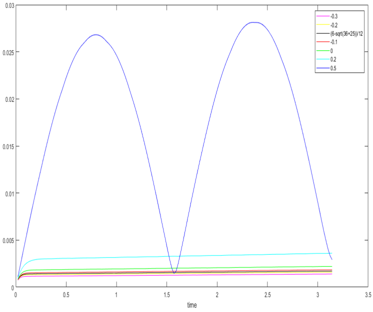


Figure: $L^\infty, dx = \frac{\pi}{10}, dt = \frac{dx}{10}, t = \pi$

Eigenstructure when k=2

Follow the same procedure, the eigenvalues to G :

$$
\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} -i + \frac{dx^5(2\theta-1)}{7200} - \frac{i\mathrm{dx}^6\left(14\theta^2-14\theta+1\right)}{42000} + \frac{\mathrm{dx}^7\left(-48\theta^3+72\theta^2-26\theta+1\right)}{120000} + O\left(dx^8\right) \\ \frac{-i\sqrt{-36\theta^2+36\theta+51}+6\theta-3}{\mathrm{dx}} + \frac{84\theta^2+\left(-84-14i\sqrt{-36\theta^2+36\theta+51}\right)\theta+7i\sqrt{-36\theta^2+36\theta+51}+1}{12i\theta^2+2\left(\sqrt{-36\theta^2+36\theta+51}-6i\right)\theta-\sqrt{-36\theta^2+36\theta+51}-17i} dx + O\left(\mathrm{dx}^2\right) \\ \frac{i\sqrt{-36\theta^2+36\theta+51}+6\theta-3}{\mathrm{dx}} + \frac{-84\theta^2+14\left(6-i\sqrt{-36\theta^2+36\theta+51}\right)\theta+7i\sqrt{-36\theta^2+36\theta+51}-1}{-12i\theta^2+2\left(\sqrt{-36\theta^2+36\theta+51}+6i\right)\theta-\sqrt{-36\theta^2+36\theta+51}+17i} dx + O\left(\mathrm{dx}^d\right) \end{pmatrix}
$$

The nonphysical error controls the short time error:

$$
V_2+V_3 = \begin{pmatrix} \frac{i\mathrm{dx}^3(18\theta+5)e^{-it}}{6480} + \frac{\mathrm{dx}^4\left(1296\theta^2-1728\theta+565\right)e^{-it}}{388800} + \frac{\mathrm{dx}^5 e^{-it}\left(-i\left(46656\theta^3-85536\theta^2+32994\theta+3875\right)\right)}{11664000} + O\left(\mathrm{dx}^6\right) \\ -\frac{1}{240} i\mathrm{dx}^3(2\theta-1)e^{-it} - \frac{\mathrm{dx}^4\left(\left(432\theta^2-432\theta+23\right)e^{-it}\right)}{43200} + \frac{\mathrm{dx}^5(2\theta-1)e^{-it}\left(3i\left(288\theta^2-288\theta+17\right)\right)}{144000} + O\left(\mathrm{dx}^6\right) \\ \frac{i\mathrm{dx}^3(18\theta-23)e^{-it}}{6480} + \frac{\mathrm{dx}^4\left(1296\theta^2-864\theta+133\right)e^{-it}}{388800} + \frac{\mathrm{dx}^5 e^{-it}\left(-i\left(46656\theta^3-54432\theta^2+1890\theta+2011\right)\right)}{11664000} + O\left(\mathrm{dx}^6\right) \end{pmatrix}
$$

Major observations when k=2

- Similarly, nonphysical terms have stronger influence on short-time error, while the physical term will dominate the error in long time simulation.
- $(2\theta - 1)$ is in the numerator, which would not blow-up when $\theta = \frac{1}{2}$. Hence optimal error estimate for both energy dissipative scheme($\theta < \frac{1}{2}$) and energy conservative scheme with central flux($\theta = \frac{1}{2}$).
- On the contrary, when θ is bigger, both short time error and long time would be smaller.

Numerical result for different θ for long time, k=2

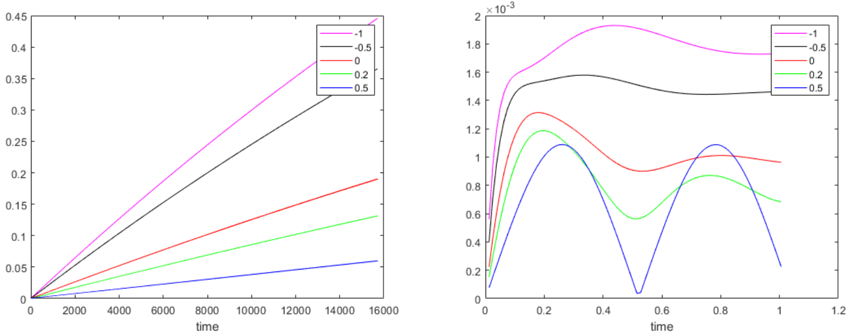
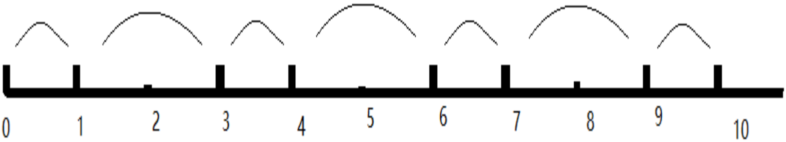


Figure: l^{∞} $error; dx = \frac{\pi}{5}, dt = \frac{dt}{10}, long\ \ time : 5000\pi, short\ \ time = 1$

"Nonuniform" mesh

Here the Quasi-uniform mesh region is specially constructed by two different mesh sizes. $\Delta x_{\text{left}} / \Delta x_{\text{right}} = 2$.



It reveals some property toward the nonuniform mesh.

Physical eigenvalue on nonuniform mesh

When k=1, physical eigenvalue :

$$-i+\frac{\mathrm{dx}^3}{432}\left(9(2\theta-1)+\frac{25}{2\theta-1}\right)-\frac{i\mathrm{dx}^4\left(90\theta^4-180\theta^3+94\theta^2-4\theta+11\right)}{270(1-2\theta)^2}+O\left(\mathrm{dx}^5\right),\theta\neq\frac{1}{2}$$

When k=2, physical eigenvalue :

$$-i+\frac{\mathrm{dx}^5}{43200}(81(2\theta-1)+\frac{49}{2\theta-1})-\frac{i\mathrm{dx}^6\left(1134\theta^4-2268\theta^3+1418\theta^2-284\theta+43\right)}{42000(1-2\theta)^2}+O\left(\mathrm{dx}^7\right)$$

.

- (1-2θ) occur in both numerator and denominator
  $\rightarrow\theta=\frac{1}{2}$ would blow-up
  $\rightarrow$suboptimal error estimate.

Convergence order on nonuniform mesh

number of cells(3dx long)	order,k=2,$\theta = 0.5$	order, k=2,$\theta = 0$	order,k=1,$\theta = 0.5$	order, k=1,$\theta = 0$
50($50 * 2 * 2$ points)				
100,t=1	2.1263	2.9239	0.9909	1.9995
200	2.1350	2.9623	0.9954	1.9996
400	2.0803	2.9748	0.9976	1.9996
600	2.0696	2.9789	0.9978	1.9997
800	2.0610	2.9811	0.9980	1.9997
1000	2.0576	2.9825	0.9983	1.9997

Table: convergence order on nonuniform mesh,k=2

- When k=1, convergence order is the same as on uniform mesh.
- When k=2 and $\theta = \frac{1}{2}$, convergence order would be suboptimal (second order) on nonuniform mesh.
  Recall on uniform mesh it remains optimal error estimate(third order).
  Proved theoretically by projection operator in [6].

Relationship between error and θ on nonuniform mesh

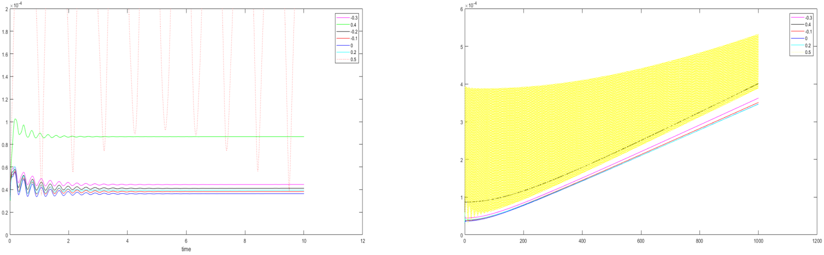


Figure: l^2 error for different θ on nonuniform mesh, k=2,3$dx = \frac{\pi}{10}, dt = \frac{dt}{10}$

- The error is not monotone with θ on nonuniform mesh.
- Same behavior is observed for k=1.
- Long time error behavior could be explained by eigenvalue.

Summary of results on nonuniform mesh

Comparing the results on uniform mesh and nonuniform mesh:

	upwind biase,k=1	upwind biase,k=2	central flux,k=1	central flux,k=2
uniform mesh	2	3	1	3
nonuniform mesh	2	3	1	2
error when θ increases	increase	decrease	indeterminate	indeterminate

Table: L2 error, order of accuracy and relationship between θ and error

Brief introduction to Fu's method

A new optimal, energy-conserving DG methods proposed in [4].

Double the unknowns and equations by introducing an auxiliary zero function

$$\begin{matrix} u_t + cu_x = 0, (x,t) \in I \times (0,T] \\ \phi_t - c\phi_x = 0, (x,t) \in I \times (0,T] \end{matrix}, u(x,0) = u_0(x), \phi(x,0) = 0$$

The flux used here : by

$$\widehat{u}_h|_{j-\frac{1}{2}} = \{u_h\}|_{j-\frac{1}{2}} + \frac{1}{2}\alpha\left[\phi_h\right]\Big|_{j-\frac{1}{2}}, \quad \widehat{\phi}_h|_{j-\frac{1}{2}} = \{\phi_h\}|_{j-\frac{1}{2}} + \frac{1}{2}\alpha\left[u_h\right]\Big|_{j-\frac{1}{2}}, \alpha \in R$$

Fourier Analysis

Follow the same procedure to get the amplification matrix G.
The expansion of eigenvalues

$$\lambda_1 = -i + \frac{i\left(6a^2-5\right)\mathrm{dx}^4}{1080a^2} + \frac{i\left(144a^4-357a^2+224\right)\mathrm{dx}^6}{217728a^4} + O\left(\mathrm{dx}^8\right)$$

$$\lambda_2 = i - \frac{i\left(6-\frac{5}{a^2}\right)\mathrm{dx}^4}{1080} - \frac{i\left(144a^4-357a^2+224\right)\mathrm{dx}^6}{217728a^4} + O\left(\mathrm{dx}^8\right)$$

$$\lambda_3 = \frac{6ia}{\mathrm{dx}} + \left(\frac{3i}{4a} - ia\right)\mathrm{dx} + \frac{i\left(48a^4-16a^2-27\right)\mathrm{dx}^3}{576a^3} - \frac{i\left(576a^6-432a^4+1060a^2-1215\right)\mathrm{dx}^5}{207360a^5} + O\left(\mathrm{dx}^6\right)$$

$$\lambda_4 = -\frac{6ia}{\mathrm{dx}} + \left(ia - \frac{3i}{4a}\right)\mathrm{dx} - \frac{i\left(48a^4-16a^2-27\right)\mathrm{dx}^3}{576a^3} + \frac{i\left(576a^6-432a^4+1060a^2-1215\right)\mathrm{dx}^5}{207360a^5} + O\left(\mathrm{dx}^6\right)$$

The numerical solution :

$$\begin{pmatrix} \hat{u}_{m,-\frac{1}{4}}(t) \\ \hat{u}_{m,\frac{1}{4}}(t) \\ \hat{\phi}_{m,\frac{-1}{4}}(t) \\ \hat{\phi}_{m,\frac{1}{4}}(t) \end{pmatrix} = e^{\lambda_1 t}V_1 + e^{\lambda_2 t}V_2 + e^{\lambda_3 t}V_3 + e^{\lambda_4 t}V_4.$$

Results from Fourier Analysis

- Eigenvalues are all purely imaginary$\rightarrow$ Scheme is energy conserving.
- When $\alpha = \sqrt{\frac{5}{6}}$, the physical eigenvalue would be closest to exact eigenvalue. Phase difference and long time error would be smaller. Consistent with the reuslt in [1].
- α occurs in the denominator of eigenvalue.When α=0, the scheme would blow and be suboptimal. Note that when $\alpha = 0$, the scheme is the same as central flux scheme.

Prediction by phase difference

In [1], it is observed different scheme could have similar phase error at different time.

physical eigenvalue$\rightarrow$phase difference$\rightarrow$shape of numerical solution

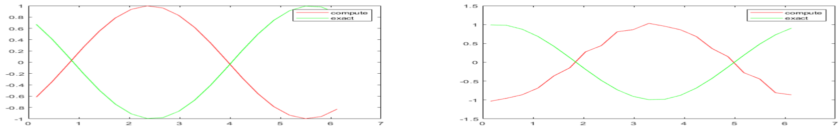


Figure: $dx = \frac{1}{5}\pi; s = 2\pi; dt = \frac{dx}{20}$. Fu's method at t=21769;central flux at t=382

Conclusion and future work

Conclusions

- We decomposed the error into physical and nonphysical components and discussed their different behavior with time.
- We explore the relationship between error and θ for the upwind biased DG scheme.
- We verified a few existing convergence results with the Fourier approach for the upwind biased DG scheme, the DG scheme with central flux, and the energy-conserving scheme in [4].

Ongoing and future work

- Relation ship between blow-up behavior caused by the parameter in denominator and suboptimal error estimate.
- The symbolic results which could hardly be computed now due to limited computation power.
- Superconvergence result through Fourier analysis.

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