Formal optimisation problem

- Recall the canonical form of an optimisation problem:

$$\begin{array}{rll} \text{maximise/minimise} & f(\mathbf{x}) & \\ \text{subject to} & g_i(\mathbf{x}) \leq 0, & i = 1, \ldots, m \\ & h_j(\mathbf{x}) = 0, & j = 1, \ldots, n \end{array}$$

- $\mathbf{x}$ is the vector of *design variables*.
- f is the *objective function*, e.g. the cost or quality of a solution.
- $g_1, \ldots, g_m$ are the *inequality constraints* and $h_1, \ldots, h_n$ are the *equality constraints*.
- In a *multi-objective* optimisation problem, there are more than one objective functions, e.g. $f_1, f_2, \ldots, f_k$.

Formal optimisation problem (continued)

Some more definitions:

- Each value of $\mathbf{x}$ is a *solution* to the optimisation problem.
- The *search space* consists of all possible solutions.
- A solution that satisfies the constraints is called *feasible*. A solution that does not satisfy the constraints is called *infeasible*.

Exercise 1

Consider the following problem:

- A company makes square boxes and triangular boxes. Square boxes take 2 minutes to make and sell for a profit of 4. Triangular boxes take 3 minutes to make and sell for a profit of 5. No two boxes can be created simultaneously. A client wants at least 25 boxes including at least 5 of each type in one hour. What is the best combination of square and triangular boxes to make so that the company makes the most profit from this client?
- Formalize this problem as a canonical optimisation problem, but consider it is ok for the objective to be a function to be maximised instead of minimised. Identify the design variables, the objective function and the constraints.

Exercise 1: Solution

- Let $x_1 \in \mathbb{N}$ be the number of square boxes and $x_2 \in \mathbb{N}$ be the number of triangular boxes. These are the design variables. Write $\mathbf{x} = (x_1, x_2)$. The formal optimisation problem is the following:

$$\begin{array}{rl} \text{maximise} & 4x_1 + 5x_2 \\ \text{subject to} & 2x_1 + 3x_2 \leq 60 \\ & x_1 \geq 5 \\ & x_2 \geq 5 \\ & x_1 + x_2 \geq 25 \end{array}$$

- Let us find the objective function and the constraints so that they follow the canonical formulation.

Exercise 1: Solution (continued)

- Objective function:

$$f(\mathbf{x}) = 4x_1 + 5x_2$$

- Constraints:

$$\begin{aligned} g_1(\mathbf{x}) &= 2x_1 + 3x_2 - 60 \\ g_2(\mathbf{x}) &= 5 - x_1 \\ g_3(\mathbf{x}) &= 5 - x_2 \\ g_4(\mathbf{x}) &= 25 - x_1 - x_2 \end{aligned}$$

There are no equality constraints, so no $h_1, h_2, \ldots$ functions.

- With these definitions, the canonical problem can be written

$$\begin{array}{ll} \text{maximise} & f(\mathbf{x}) \\ \text{subject to} & g_i(\mathbf{x}) \leq 0, \qquad i = 1, 2, 3, 4 \end{array}$$

Exercise 2

Consider the following problem:

- A woman makes pins and earrings. Each pin takes 1 hour to make and sells for a profit of 8. Each earring takes 2 hours to make and sells for a profit of 20. She wants to make exactly as many pins as earrings. She has 40 hours and wants to have made at least 20 items, including at least 4 of each item. How many each of pins and earrings should the woman make to maximise her profit?
- Formalize this problem as a canonical optimisation problem, but consider it is ok for the objective to be a function to be maximised instead of minimised. Identify the design variables, the objective function and the constraints.

Exercise 2: Solution

- Let $x_1 \in \mathbb{N}$ be the number of pins and $x_2 \in \mathbb{N}$ be the number of earrings. These are the design variables. Write $\mathbf{x} = (x_1, x_2)$. The formal optimisation problem is the following:

$$\begin{array}{rl} \text{maximise} & 8x_1 + 20x_2 \\ \text{subject to} & x_1 + 2x_2 \leq 40 \\ & x_1 + x_2 \geq 20 \\ & x_1 \geq 4 \\ & x_2 \geq 4 \\ & x_1 = x_2 \end{array}$$

- Let us find the objective function and the constraints so that they follow the canonical formulation.

Exercise 2: Solution (continued)

- Objective function:

$$f(\mathbf{x}) = 8x_1 + 20x_2$$

- Constraints:

$$\begin{aligned}
g_1(\mathbf{x}) &= x_1 + 2x_2 - 40 \\
g_2(\mathbf{x}) &= 20 - x_1 - x_2 \\
g_3(\mathbf{x}) &= 4 - x_1 \\
g_4(\mathbf{x}) &= 4 - x_2 \\
h_1(\mathbf{x}) &= x_1 - x_2
\end{aligned}$$

- With these definitions, the canonical problem can be written

$$\begin{array}{lll}
\text{maximise} & f(\mathbf{x}) & \\
\text{subject to} & g_i(\mathbf{x}) \leq 0, & i = 1, 2, 3, 4 \\
 & h_j(\mathbf{x}) = 0, & j = 1
\end{array}$$