Attendance Code:
xxxx

Artificial Intelligence and Machine Learning (AIML)

2023–24

#Code

- **Last lecture**: Bayes' theorem, naive Bayes' classifier

posterior

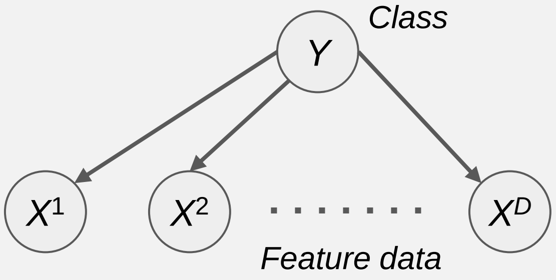
$$P(X|Y) = \frac{P(Y|X)P(X)}{P(Y)}$$ **(Bayes' Theorem)**

- If evidence distribution $P(Y)$ is unknown, can use instead: $P(Y) = \sum_{x \in \Omega_X} P(Y|X=x)P(X=x)$

- **Naïve Bayes' Classifier**

$$P(X|Y) = P(X^1|Y)P(X^2|Y) \cdots P(X^D|Y)$$

$$y^\star = \arg\max_{y \in \Omega_Y} P(X^1|Y)P(X^2|Y) \cdots P(X^D|Y)P(Y=y)$$



#Code

- **Last lecture**: Bayes' theorem, naive Bayes' classifier

likelihood

prior

posterior

$$P(X|Y) = \frac{P(Y|X)P(X)}{P(Y)}$$

(Bayes' Theorem)

evidence

- If evidence distribution $P(Y)$ is unknown, can use instead: $P(Y) = \sum_{x \in \Omega_X} P(Y|X=x)P(X=x)$

- **Naïve Bayes' Classifier**

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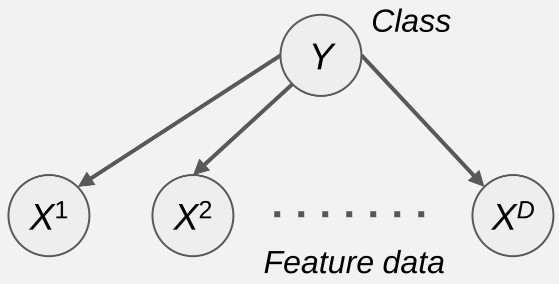

#Code

- **Last lecture**: Bayes' theorem, naive Bayes' classifier

- **New email** containing words "friend" and "thank"
- Is it likely to be a regular email or spam?
  - $y^\star = \arg\max_{y \in \Omega_Y} P(X^2|Y)P(X^3|Y)P(Y = y)$

Regular emails:
- dear: 8 out of 17 words - $P(X^1|Y = R) = \frac{8}{17} = 0.47$
- friend: 5 out of 17 words - $P(X^2|Y = R) = \frac{5}{17} = 0.29$
- thank: 3 out of 17 words - $P(X^3|Y = R) = \frac{3}{17} = 0.18$
- buy: 1 out of 17 words - $P(X^4|Y = R) = \frac{1}{17} = 0.06$

Spam emails:
- dear: 4 out of 17 words - $P(X^1|Y = S) = \frac{5}{17} = 0.24$
- friend: 2 out of 17 words - $P(X^2|Y = S) = \frac{2}{17} = 0.12$
- thank: 1 out of 17 words - $P(X^3|Y = R) = \frac{1}{17} = 0.06$
- buy: 10 out of 17 words - $P(X^4|Y = R) = \frac{10}{17} = 0.59$

- Many applications of ML involve **ordered data** that is, data for which the ordering matters
  - **natural language** (ordered sequences of words),
  - **appointment calendar entries** (date and time-ordered event names)
  - **electronic health records** (time-ordered sequences of medical system interactions)
  - **macroeconomic time series** (time-ordered sequences of GDP values)
  - **genomics** (base pairs in a genome sequence)

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  - $y^\star = \arg\max_{y \in \Omega_Y} P(X^2|Y)P(X^3|Y)P(Y=y)$
    - $Y = r$: $\mathrm{p}(Y = R|X) = 0.29 \times 0.18 \times 0.67 = 0.035$
    - $Y = s$: $\mathrm{p}(Y = S|X) = 0.12 \times 0.06 \times 0.33 = 0.002$
  - $y^\star = r$ (not spam)

Note that the result is the same regardless of the order of "thank" and "friend" in the email

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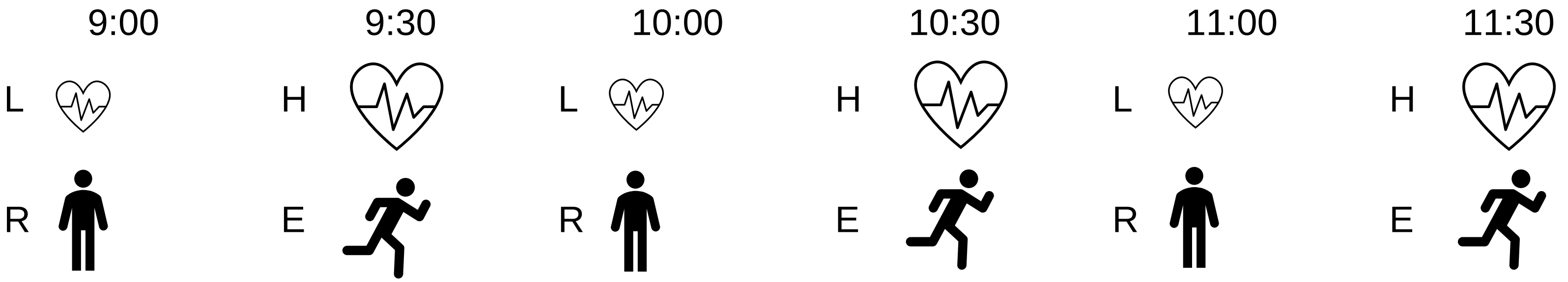
- **This lecture**: sequence modelling, hidden Markov models

#Code

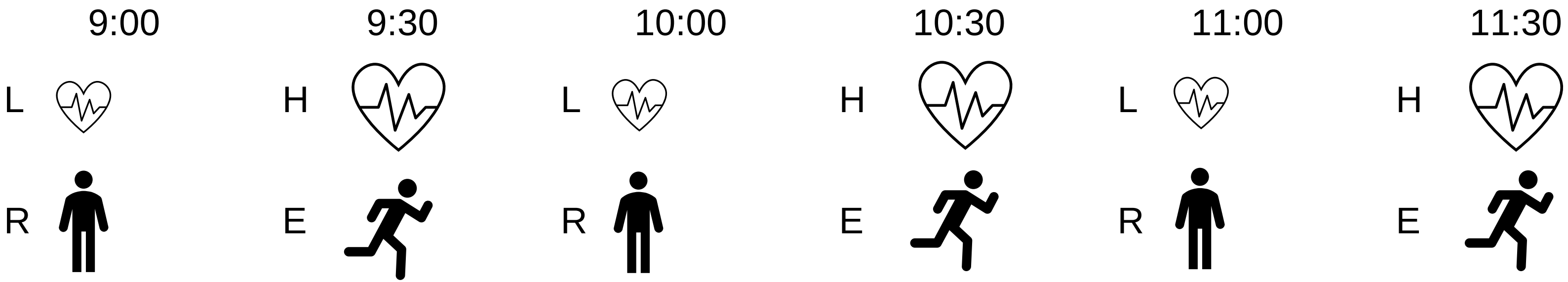
Sequence modelling: intuition

- **Problem**: Smartwatch-based Activity Monitoring System
- **Measured observation (X_t):** heart rate (high vs low)
  - $X_t \in \Omega_X = \{h, l\}$



Sequence modelling: intuition

- **Problem**: Smartwatch-based Activity Monitoring System
- **Measured observation (X_t):** heart rate (high vs low)
  - $X_t \in \Omega_X = \{h, l\}$
- **Inferred observation (Y_t):** activity (rest vs. exercise)
  - $Y_t \in \Omega_Y = \{r, e\}$
  - It is a **hidden state** (not directly observable)



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- **Assumption:**
  - we aren't randomly on rest or exercise;
  - If we are at rest at a given time, it's likely we will continue at rest

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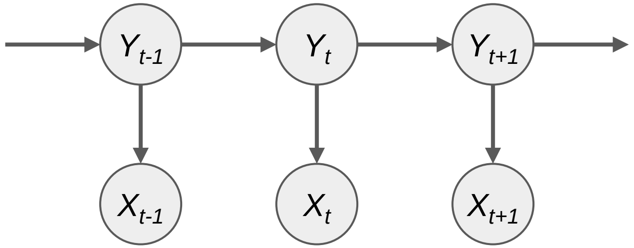
Sequence modelling: Hidden Markov Models (HMMs)

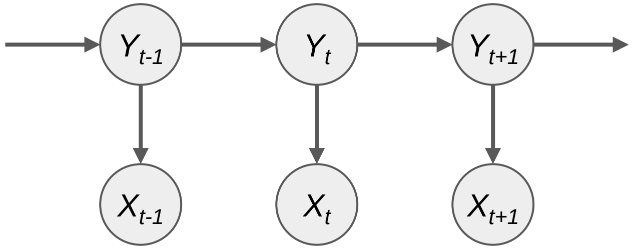
- The **hidden Markov model** (HMM) captures time-dependent RVs which are not directly measured



#Code

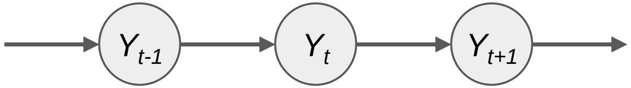
Sequence modelling: Hidden Markov Models (HMMs)

- The **hidden Markov model** (HMM) captures time-dependent RVs which are not directly measured
- Each **hidden states** $Y_t \in \Omega_Y$ with K distinct values, depends only upon the one before it in time, Y_{t-1} for all $t = 0,1,\ldots, T$
- The measured **observations** X_t depend only upon the associated hidden state, Y_t

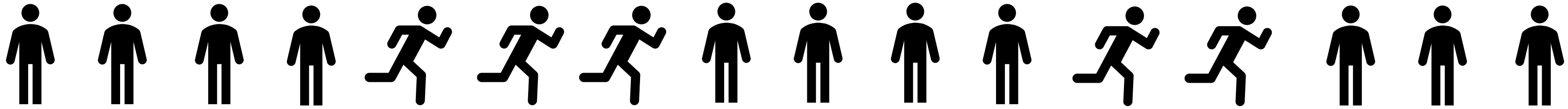


#Code

Sequence modelling: model fitting



- given observed data for $X_0, X_1, \ldots, X_T$, estimate the distribution functions $P(X_t|Y_t),\ P(Y_t|Y_{t-1})$
- Training data (**transition probabilities**)



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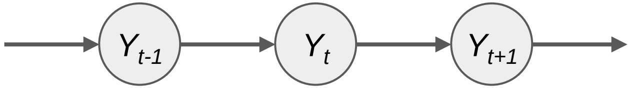

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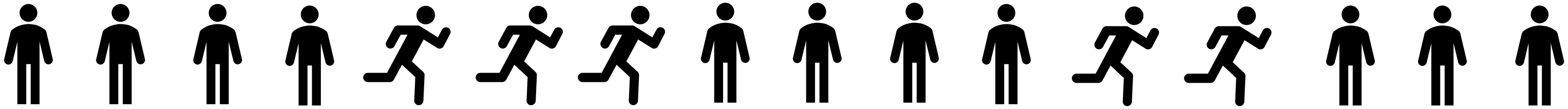
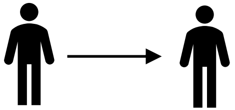
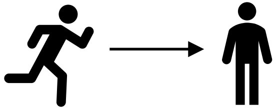
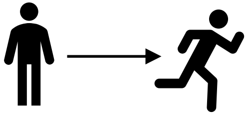




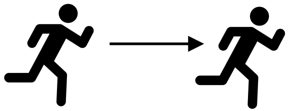
$P(Y_t = r|Y_{t-1} = r) = \frac{8}{10} = 0.8$



$P(Y_t = r|Y_{t-1} = e) = \frac{2}{5} = 0.4$



$P(Y_t = e|Y_{t-1} = r) = \frac{2}{10} = 0.2$

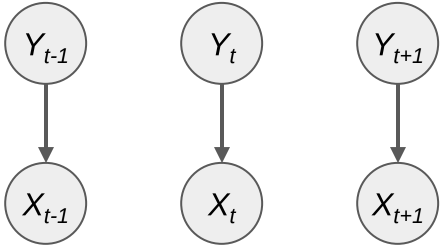


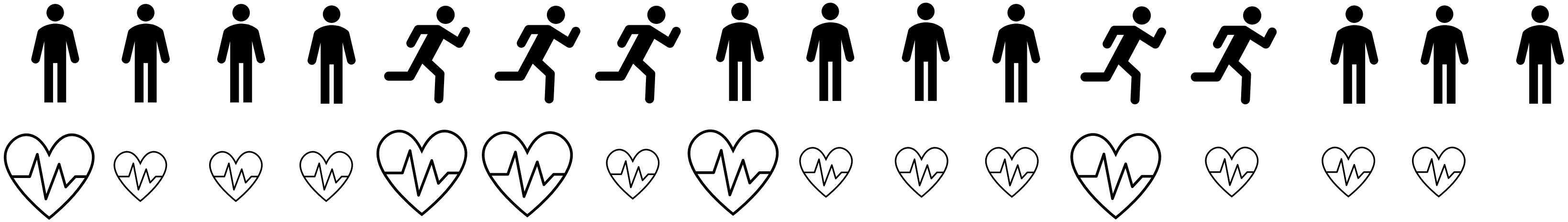
$P(Y_t = e|Y_{t-1} = e) = \frac{3}{5} = 0.6$

Sequence modelling: model fitting

#Code

- given observed data for $X_0, X_1, \ldots, X_T$, estimate the distribution functions $P(X_t|Y_t),\ P(Y_t|Y_{t-1})$
- Training data (**emission probabilities**)





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#Code

Sequence modelling: model fitting

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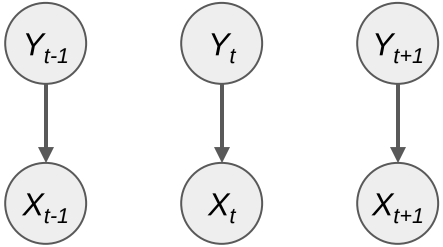

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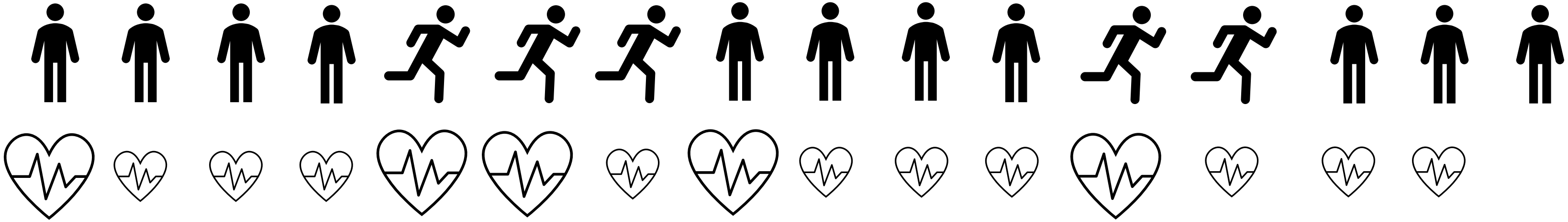

$$P(X_t = l|Y_t = r) = \frac{8}{10} = 0.8$$

$$P(X_t = h|Y_t = r) = \frac{2}{10} = 0.2$$

$$P(X_t = l|Y_t = e) = \frac{2}{5} = 0.4$$

$$P(X_t = h|Y_t = e) = \frac{3}{5} = 0.6$$

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Sequence modelling: model fitting

#Code

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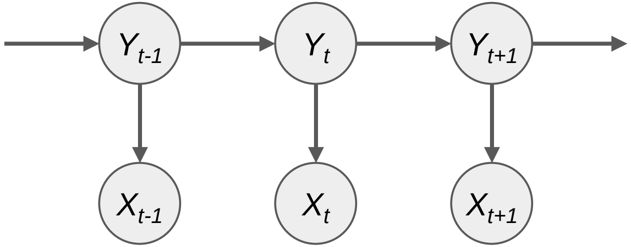

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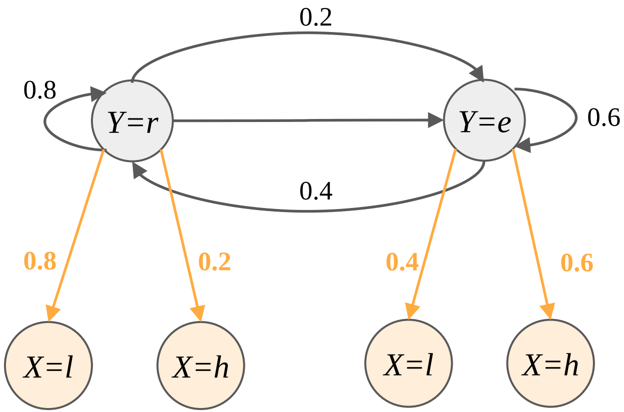
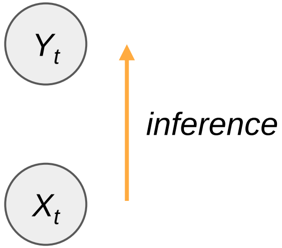


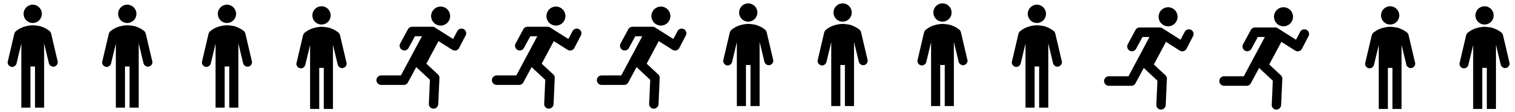
#Code

Sequence modelling: single evaluation

- Given fixed model parameters and observed data, compute the probability of the hidden state



- If we currently measured heart rate to be low, what's the probability that the user is at rest or exercising?



$$P(Y = r) = \frac{10}{15} = \frac{2}{3} = 0.67$$

$$P(Y = e) = \frac{5}{15} = \frac{1}{3} = 0.33$$

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Summary

#Code

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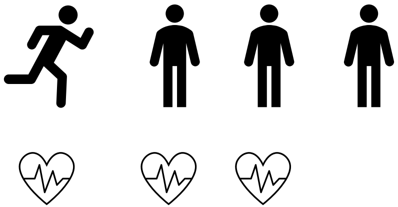

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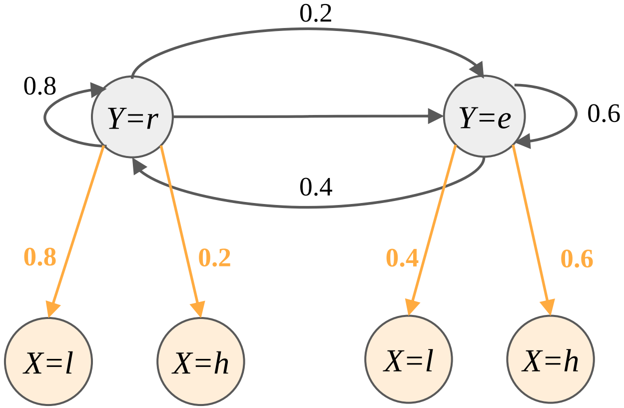

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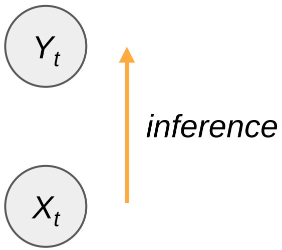
Sequence modelling: single evaluation

- given fixed model parameters and observed data, compute the probability of the hidden state



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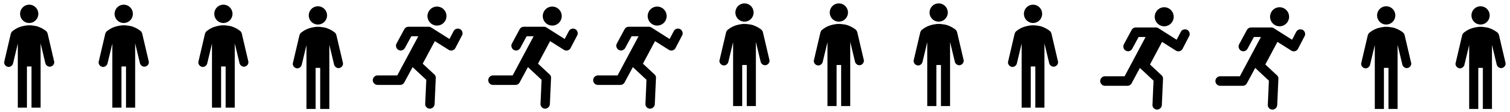

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$$P(Y = r|X = l)$$

$$P(Y = e|X = l)$$

Sequence modelling: single evaluation

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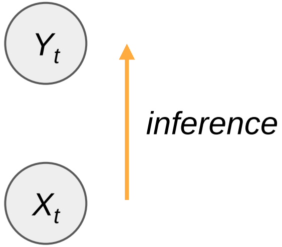

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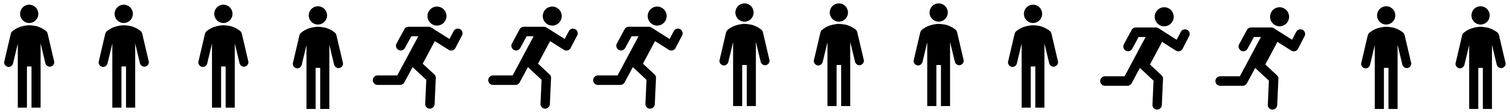

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Bayes' Theorem:

$$P(Y=r|X=l) \propto P(X=l|Y=r)P(Y=r) = 0.8 \times 0.67 = 0.536$$

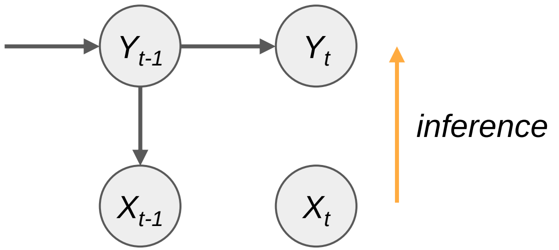
$$P(Y=e|X=l) \propto P(X=l|Y=e)P(Y=e) = 0.4 \times 0.33 = 0.132$$

Decision: $y^\star = \arg\max_{y \in \Omega_Y} P(X|Y)P(Y=y) = r$

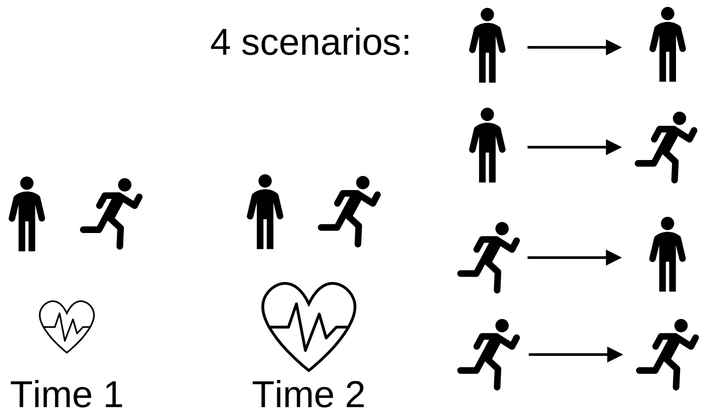
#Code

Sequence modelling: decoding

- given fixed model parameters and data, compute the most probable sequence of hidden states, $y = [y_0^\star, y_1^\star, \ldots, y_T^\star]$



- If, for two consecutive measurements, we get heart rate to be **low, high**, what was the most likely scenario for activity?



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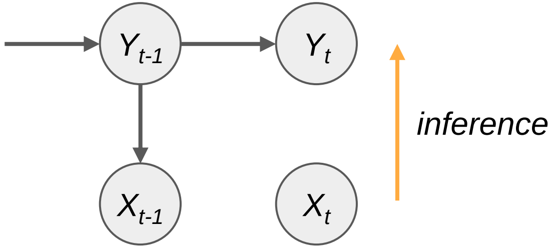

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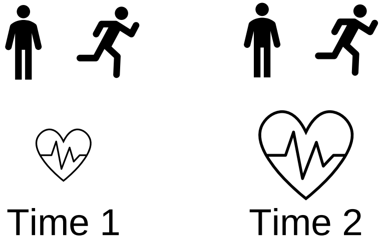
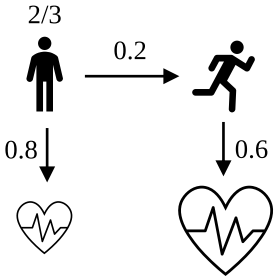
4 scenarios:





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$$= P(X_0 = l | Y_0 = r) P(Y_0 = r) P(Y_1 = e | Y_0 = r) P(X_1 = h | Y_1 = e)$$

$$= 0.8 \times 0.67 \times 0.2 \times 0.6 = 0.064$$

Sequence modelling: decoding

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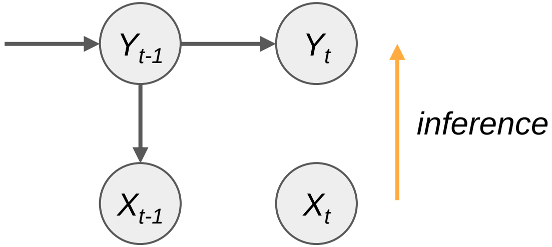

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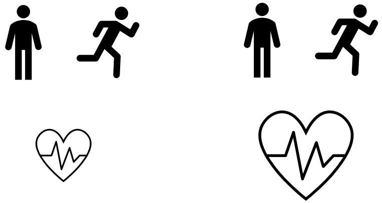
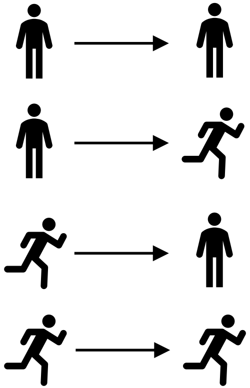


4 scenarios:



$P(y) = 0.085$ (most likely to happen)

$P(y) = 0.064$

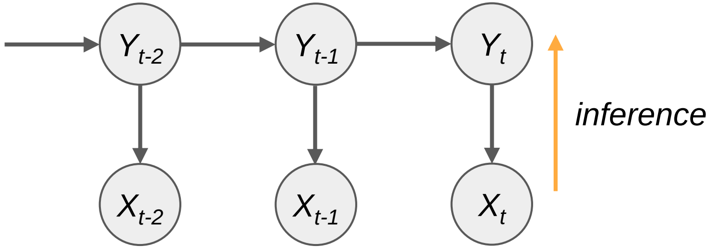
$P(y) = 0.011$

$P(y) = 0.048$

Sequence modelling: decoding

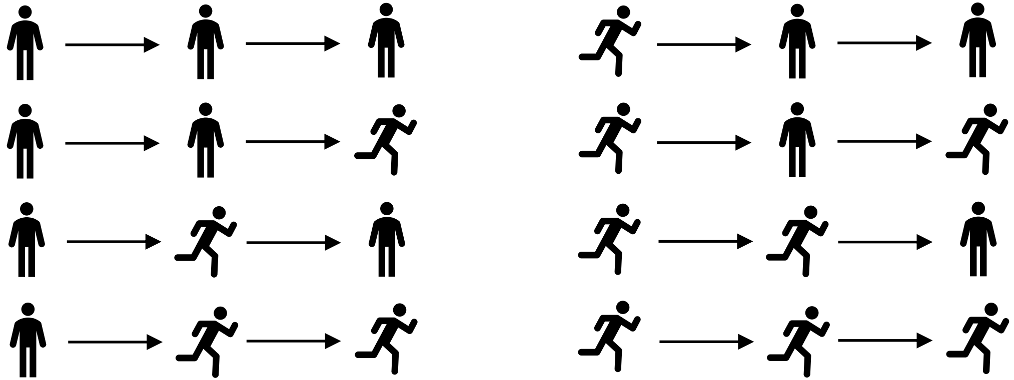
#Code

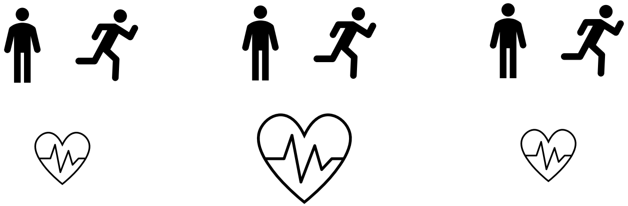
- given fixed model parameters and data, compute the most probable sequence of hidden states, $y = [y_0^\star, y_1^\star, \ldots, y_T^\star]$



- If, for three consecutive measurements, we get heart rate to be **low, high**, **low**, what was the most likely scenario for activity?

8 scenarios:





Time 1 Time 2 Time 3

#Code

HMM sequence modelling problems

- In applications of HMMs, typically need to solve the following problems
    - **Model fitting**: given observed data for $X_0, X_1, \ldots, X_T$, estimate the distribution functions $P(X_t|Y_t)$, $P(Y_t|Y_{t-1})$;
    - **Evaluation**: given fixed model parameters and observed data, compute the probability of the data, $P(X)$;
    - **Decoding**: given fixed model parameters and data compute the most probable sequence of hidden states $y = [y_0^*, y_1^*, y_2^*, \ldots, y_T^*]$.

#Code

HMM sequence modelling problems

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    - **Model fitting**: given observed data for $X_0, X_1, \ldots, X_T$, estimate the distribution functions $P(X_t|Y_t)$, $P(Y_t|Y_{t-1})$;
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- Solving these problems requires evaluating **all possible sequences of hidden states**; if there are K hidden states, this requires $O(K^T)$ (exponential complexity)

#Code

HMM sequence modelling problems

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  - **Model fitting**: given observed data for $X_0,X_1,\ldots,X_T$, estimate the distribution functions $P(X_t|Y_t)$, $P(Y_t|Y_{t-1})$;
  - **Evaluation**: given fixed model parameters and observed data, compute the probability of the data, $P(X)$;
  - **Decoding**: given fixed model parameters and data compute the most probable sequence of hidden states $y = [y_0*,y_1*,y_2*,\ldots,y_\mathrm{T}*]$.
- Solving these problems requires evaluating **all possible sequences of hidden states**; if there are K hidden states, this requires $O(K^T)$ (exponential complexity)
- Use of **dynamic programming** makes this tractable in order $O(TK^2)$.

AIML

#Code

Bellman recursion for optimal sequence probability

- Reading off PGM, at time step $t-1$, optimal sequence probability:

$$P^\star(X_0, \dots, X_{t-1}, Y_{t-1}) = \max_{y' \in \mathcal{Y}_{t-2}} P(X_0, \dots, X_{t-1}, Y_0 = y'_0, \dots, Y_{t-2}, Y_{t-1})$$

where $\mathcal{Y}_{t-2}$ is set of all possible state sequences, up to time $t-2$.

- Optimal sequence probability, as a function of y up to time t,

$$p_t^\star(y) = P^\star(X_0, \dots, X_t, Y_t = y)$$

is obtained using **Bellman recursion**,

$$p_t^\star(y) = \max_{y' \in \Omega_Y} [p_{t-1}^\star(y')\, P(Y_t = y | Y_{t-1} = y') P(X_t = x_t | Y_t = y)]$$

HMM Viterbi decoding: algorithm

- **Step 1**. *Initialization*: Compute the initial optimal probability function,

$$p_0^\star(y) = P(X_0 = x_0 | Y_0 = y)\, P(Y_0 = y)$$

- **Step 2**. *Forward recursion*: Sequence of optimal probability functions,

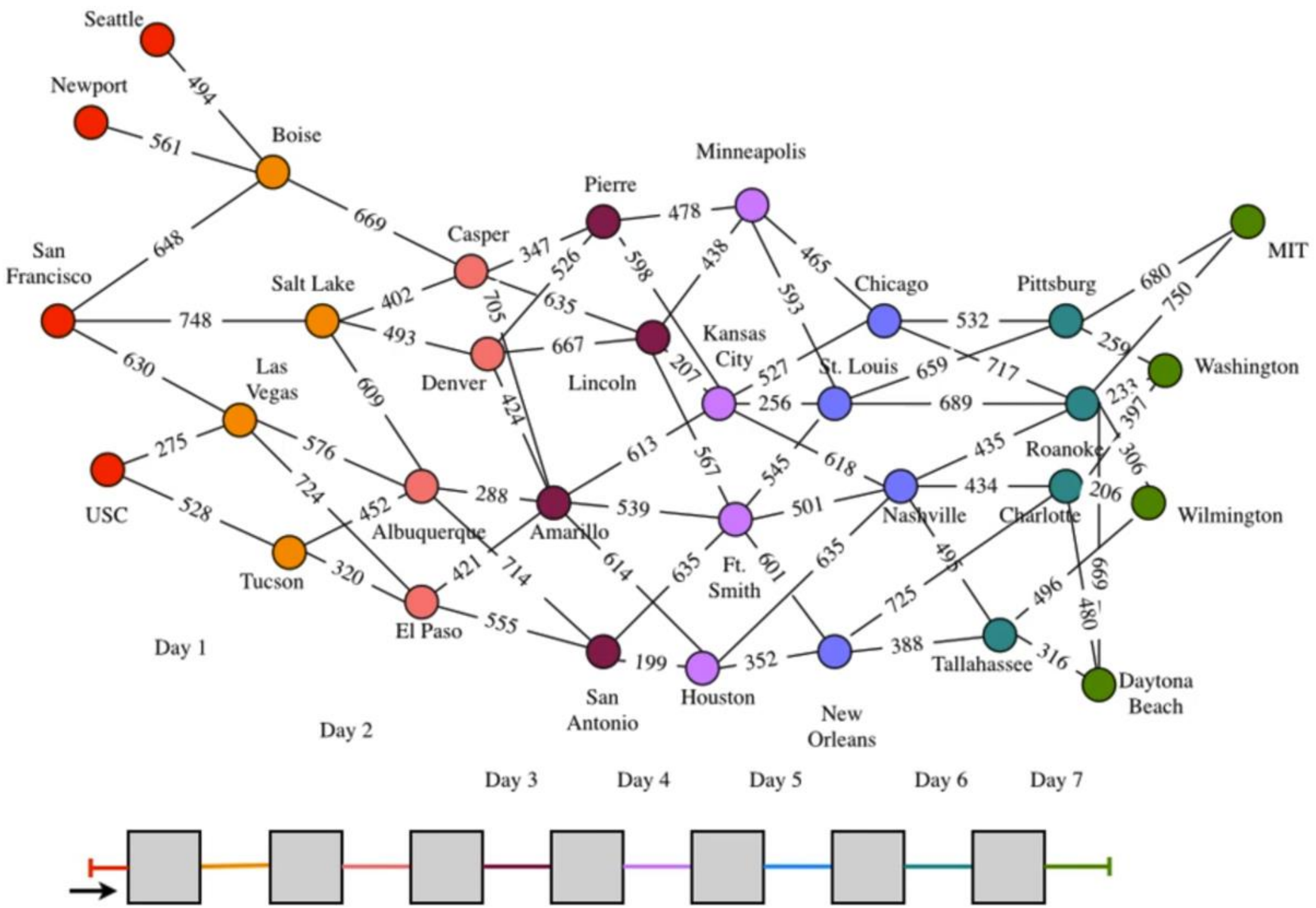
$$p_t^\star(y) = \max_{y' \in \Omega_Y} p_{t-1}^\star(y')\, P(Y_t = y | Y_{t-1} = y')\, P(X_t = x_t | Y_t = y)$$

  for $t = 1, 2, \ldots, T$, keeping track of the corresponding decision,

$$Y_t^\star(y) = \arg\max_{y' \in \Omega_Y} p_{t-1}^\star(y')\, P(Y_t = y | Y_{t-1} = y')$$

- **Step 3**. *Backtrack*: Find optimal sequence in reverse, for $t = T-1, T-2, \ldots, 1$,

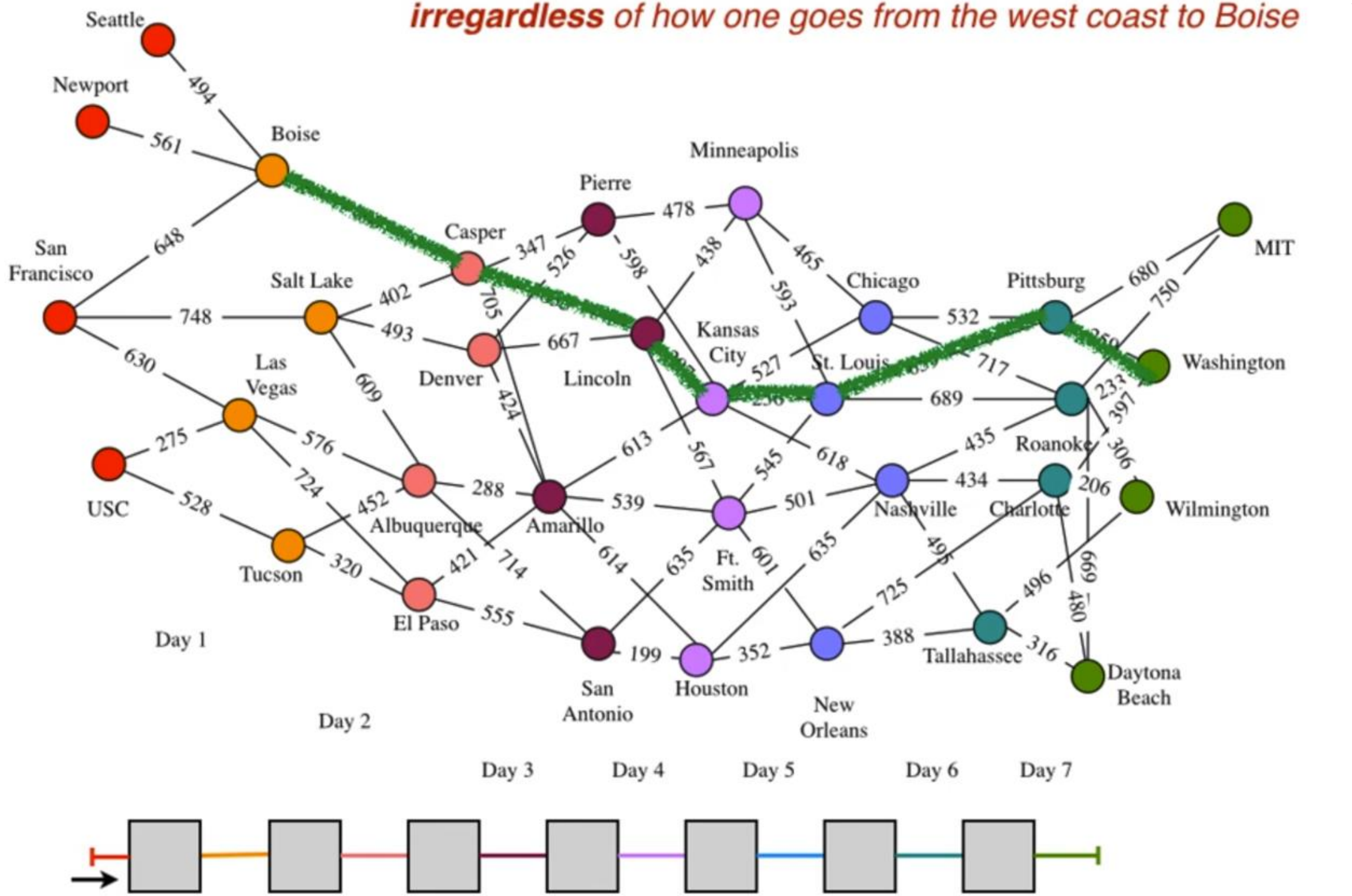
$$y_T^\star = \arg\max_{y \in \Omega_Y} p_T^\star(y), y_{t-1}^\star = Y_t^\star(y_t^\star)$$



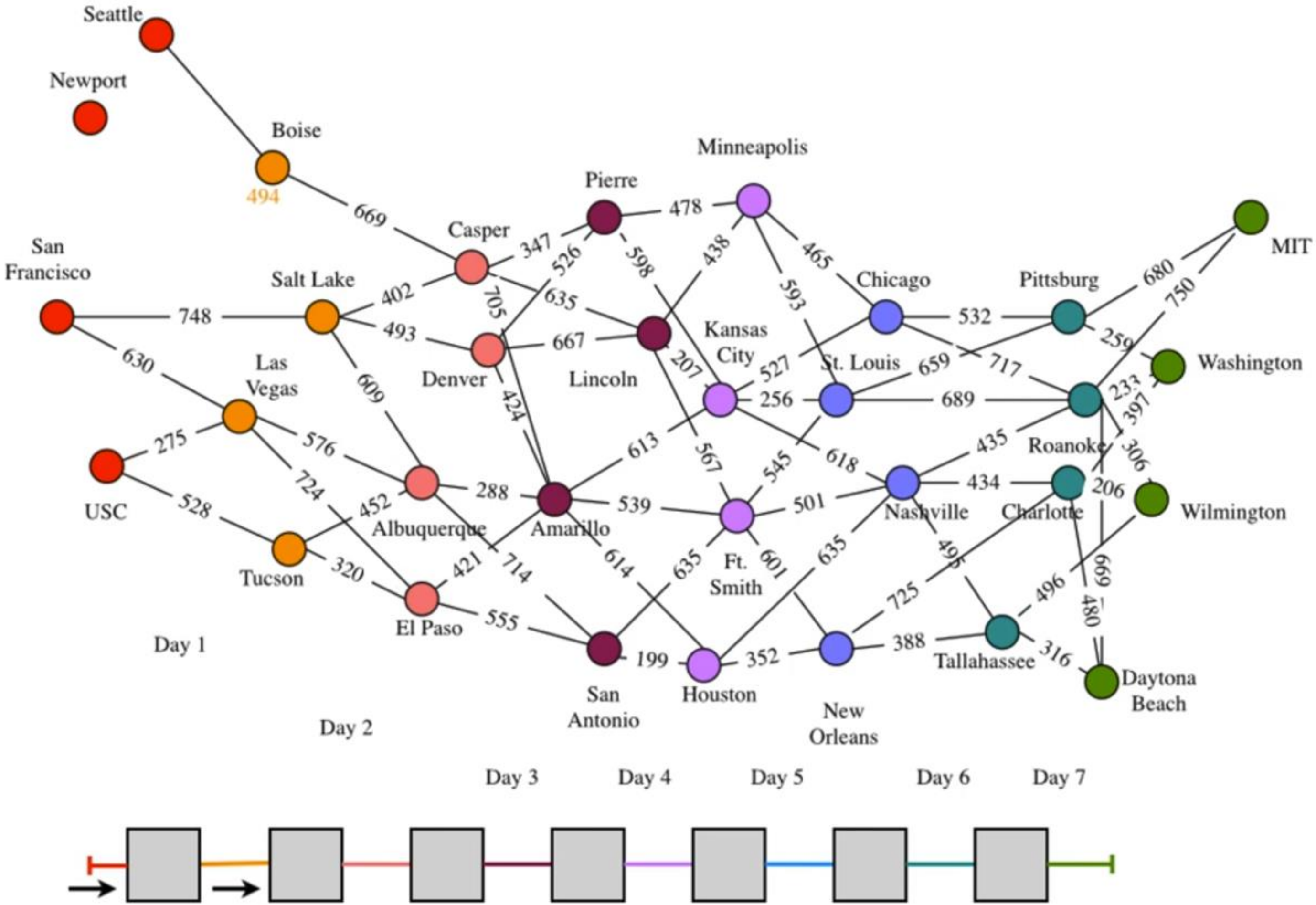
Seattle
Newport
Boise
San Francisco
Salt Lake
Las Vegas
USC
Tucson
Casper
Denver
Albuquerque
El Paso
Pierre
Lincoln
Amarillo
San Antonio
Minneapolis
Kansas City
Ft. Smith
Houston
Chicago
St. Louis
Nashville
New Orleans
Pittsburg
Roanoke
Charlotte
Tallahassee
MIT
Washington
Wilmington
Daytona Beach
Day 1
Day 2
Day 3
Day 4
Day 5
Day 6
Day 7

the best path between Boise and the east coast is 2685 miles

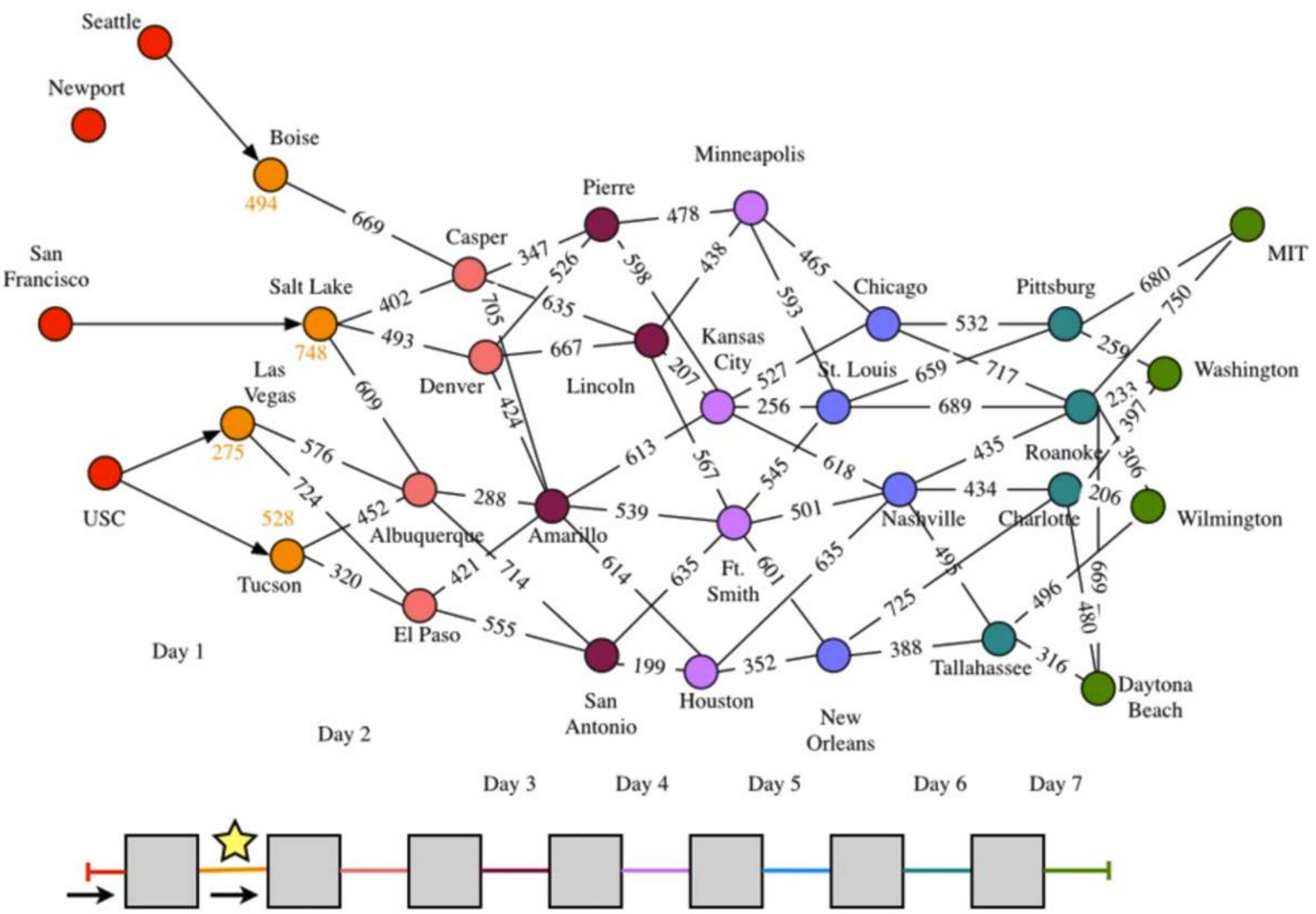
***irregardless** of how one goes from the west coast to Boise*



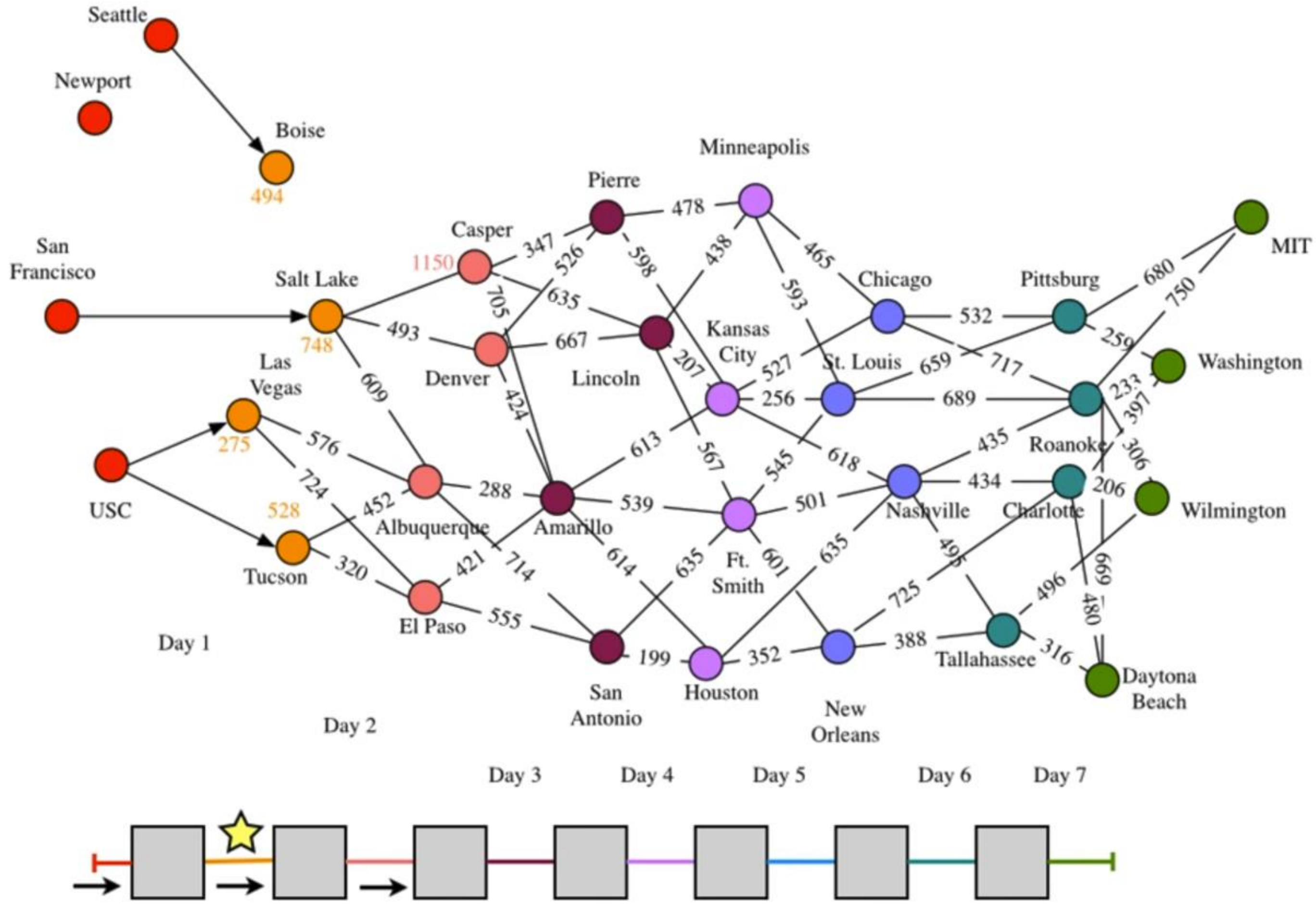
If best path from west coast to east coast, passes through Boise, it must coincide with best path from west coast to Boise



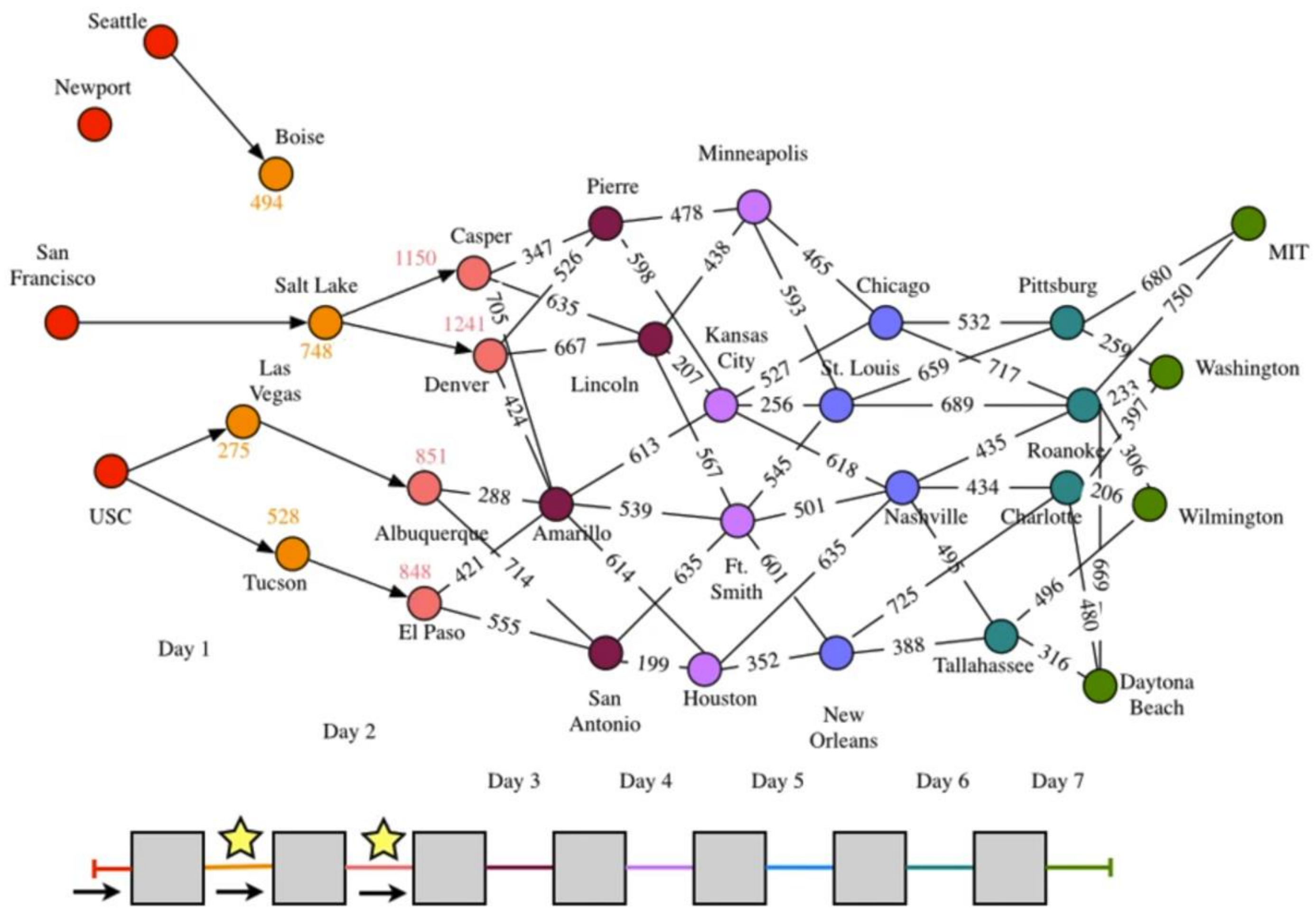
Repeat Boise-argument for Salt Lake, Las Vegas, Tucson

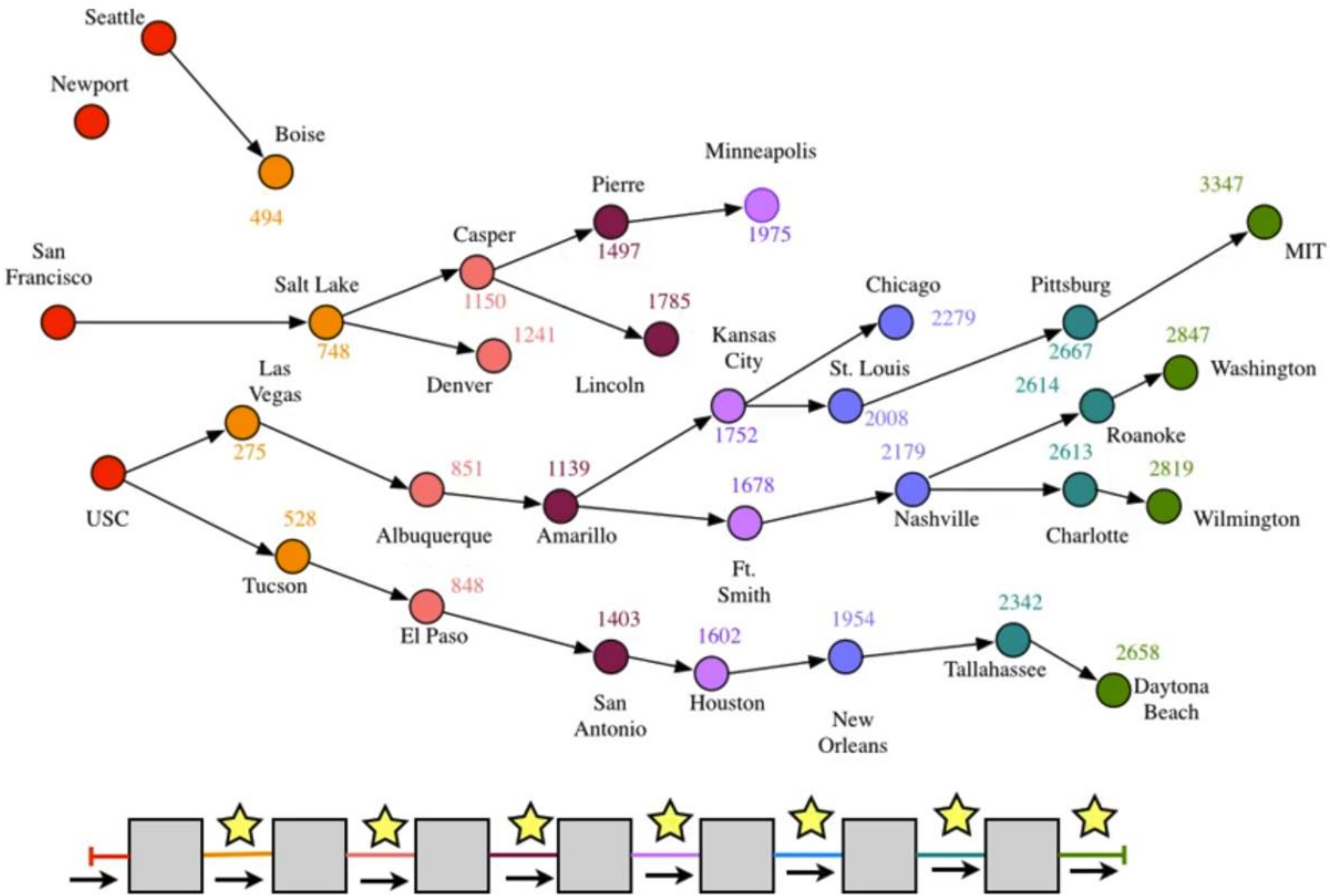


Repeat for Casper



Repeat for day-2 destination cities





Seattle
Newport
Boise
494
Minneapolis
Pierre
1975
1497
Casper
San Francisco
Salt Lake
1150
1785
748
1241
Denver
Lincoln
Las Vegas
275
851
Albuquerque
1139
Amarillo
USC
528
Tucson
848
El Paso
1403
San Antonio
Kansas City
1752
1678
Ft. Smith
1602
Houston
Chicago
2279
St. Louis
2008
2179
Nashville
1954
New Orleans
Pittsburg
2667
3347
MIT
2614
Roanoke
2847
Washington
2613
Charlotte
2819
Wilmington
2342
Tallahassee
2658
Daytona Beach

#Code

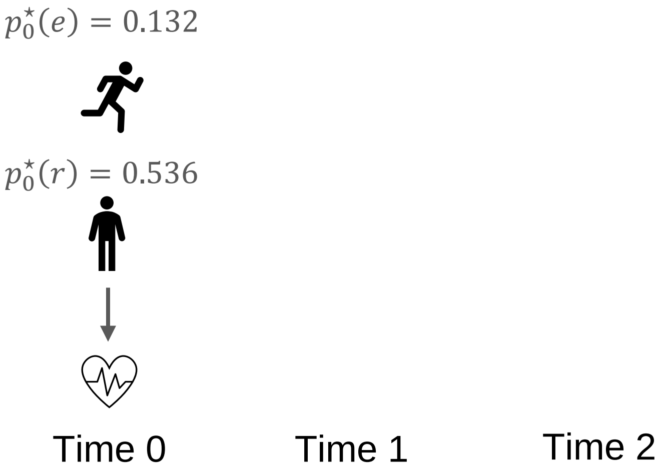
Viterbi algorithm for our example

- If, for three consecutive measurements, we get heart rate to be **low, high**, **low**, what was the most likely scenario for activity?

$$p_0^\star(y) = P(X_0 = l|Y_0 = y)P(Y_0 = y)$$

$$p_0^\star(r) = 0.8 \times 0.67 = 0.536$$

$$p_0^\star(e) = 0.4 \times 0.33 = 0.132$$



Viterbi algorithm for our example

- If, for three consecutive measurements, we get heart rate to be **low, high**, **low**, what was the most likely scenario for activity?

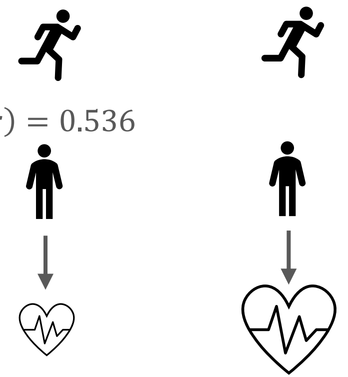
$$p_1^\star(y) = \max_{y'\in\Omega_Y} p_0^\star(y')P(Y_1 = y|Y_0 = y')P(X_1 = h|Y_1 = y)$$

$$p_1^\star(r) = \max\{0.536 \times 0.8 \times 0.2\ , 0.132 \times 0.4 \times 0.2\} = \max\{0.086, 0.011\}$$
$$= 0.086$$

$$p_1^\star(e) = \max\{\underbrace{0.536 \times 0.2 \times 0.6}_{y' = r}\ , \underbrace{0.132 \times 0.6 \times 0.6}_{y' = e}\} = \max\{0.064, 0.048\}$$
$$= 0.064$$

$p_0^\star(e) = 0.132$

$p_0^\star(r) = 0.536$



Time 0 Time 1 Time 2

Viterbi algorithm for our example

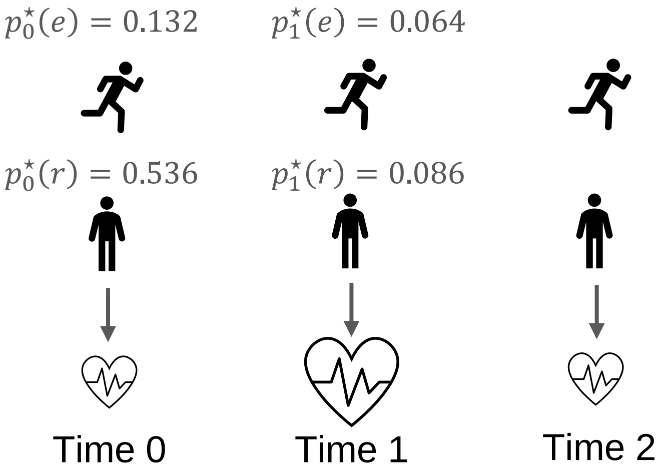
#Code

- If, for three consecutive measurements, we get heart rate to be **low, high**, **low**, what was the most likely scenario for activity?

$$p_2^\star(y) = \max_{y' \in \Omega_Y} p_1^\star(y')P(Y_2 = y | Y_1 = y')P(X_2 = l | Y_2 = y)$$

$$p_2^\star(r) = \max\{\overset{y'=r}{0.086 \times 0.8 \times 0.8}, \overset{y'=e}{0.064 \times 0.4 \times 0.8}\}$$
$$= \max\{0.055, 0.020\} = 0.055$$

$$p_2^\star(e) = \max\{0.086 \times 0.2 \times 0.4, 0.064 \times 0.6 \times 0.4\}$$
$$= \max\{0.0069, 0.015\} = 0.015$$



$$Y_1^\star(y) = [r, r]$$

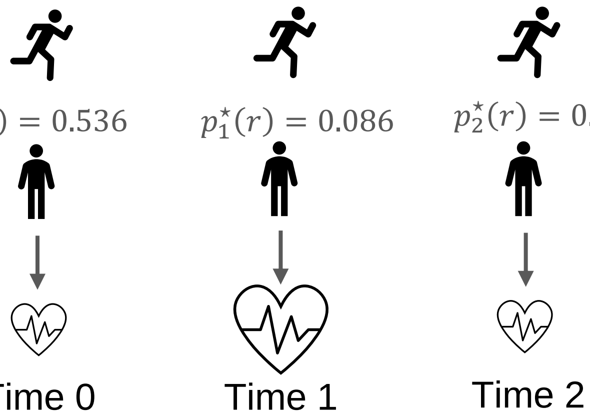
#Code

Viterbi algorithm for our example

- If, for three consecutive measurements, we get heart rate to be **low, high**, **low**, what was the most likely scenario for activity?

$p_0^\star(e) = 0.132$ $p_1^\star(e) = 0.064$ $p_2^\star(e) = 0.015$

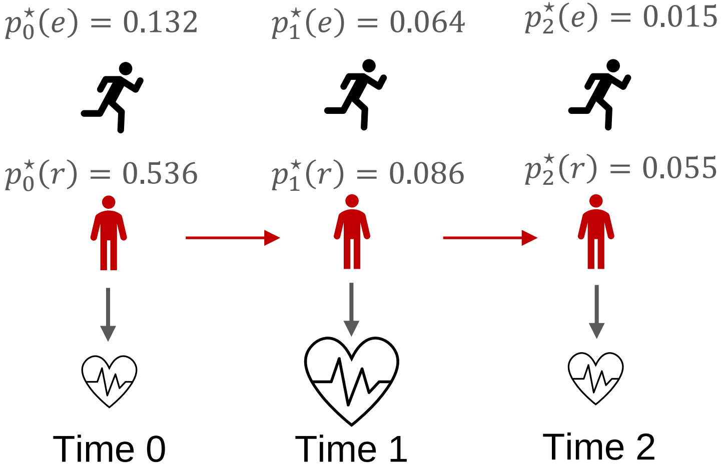
$p_0^\star(r) = 0.536$ $p_1^\star(r) = 0.086$ $p_2^\star(r) = 0.055$



$$Y_2^\star(y) = [r, r, r]$$

Viterbi algorithm for our example

- If, for three consecutive measurements, we get heart rate to be **low, high**, **low**, what was the most likely scenario for activity?

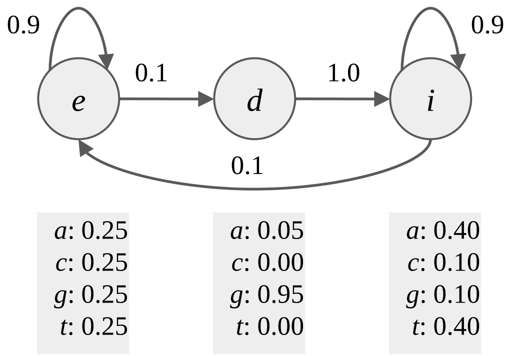


$$Y_2^\star(y) = [r, r, r]$$

#Code

Genome sequence region segmentation

- **Problem**: optimal segmentation of DNA sequences into exon (e), intron (i) donor site (d) sub-sequences
- **Distributions**: Y – regions, $\Omega_Y=\{e,d,i\}$, X – DNA sequences, $\Omega_X=\{a,c,g,t\}$
- **State transition distribution and observation distributions**: from molecular biology:



Stage	x_t	$p^*_t(y=e)$	$p^*_t(y=d)$	$p^*_t(y=i)$	$Y^*_t(y=e)$	$Y^*_t(y=d)$	$Y^*_t(y=i)$	y^*_t
t=0	g	0.2500	0.0000	0.0000				e
t=1	g	0.0563	0.0238	0.0000	e	e	e	e
t=2	g	0.0127	0.0053	0.0024	e	e	d	e
t=3	g	0.0028	0.0012	0.0005	e	e	d	d
t=4	t	0.0006	0.0000	0.0005	e	e	d	i
t=5	a	0.0001	0.0000	0.0002	e	e	i	i

Input data: $x=[g,g,g,g,t,a]$
Globally optimal sequence: $y^*=[e,e,e,d,i,i]$

#Code

To recap

- We discussed sequence modelling (data in which ordering matters)
  - **Hidden Markov Model**: sequences through a series of (discrete) hidden states that are not directly observable, but follows a certain probability distribution

- **Next**: Other sequential models (Kalman filter, Recurrent neural nets)

Further Reading

- **PRML**, Section 13.2
- **R&N**, Section 14.3 (matrix algebra approach)
- **MLSP**, Section 9.4