Artificial Intelligence and Machine Learning (AIML)

2023–24

- **Last lecture**: combinatorial optimization in AI
- **This lecture**: exact SDP methods for combinatorial optimization

Exact methods: overview

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- We will concentrate on **sequential decision process (SDP)** methods as they encompass many practical AI algorithms

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- Can construct a "computational configuration graph", and special kinds of graphs arise due to the type of exact SDP algorithm: brute-force (**full tree**), greedy (**tree with a single optimal branch at each stage**), dynamic programming (**incomplete tree**)

Exact SDP methods: algorithm

- **Step 1**. *Initialization*: Start with $n = 0$, generate the "root" configuration(s) in the set of candidate configurations, S.
- **Step 2**. *Extension*: Set $n = n + 1$, and using input data item x_n, extend all candidate configurations in S, and append these to S.
- **Step 3**. *Reduction*: Remove any candidate configurations from S which cannot be extended to an optimal configuration.
- **Step 4**. *Iteration*: if $n < N$, go back to Step 2.
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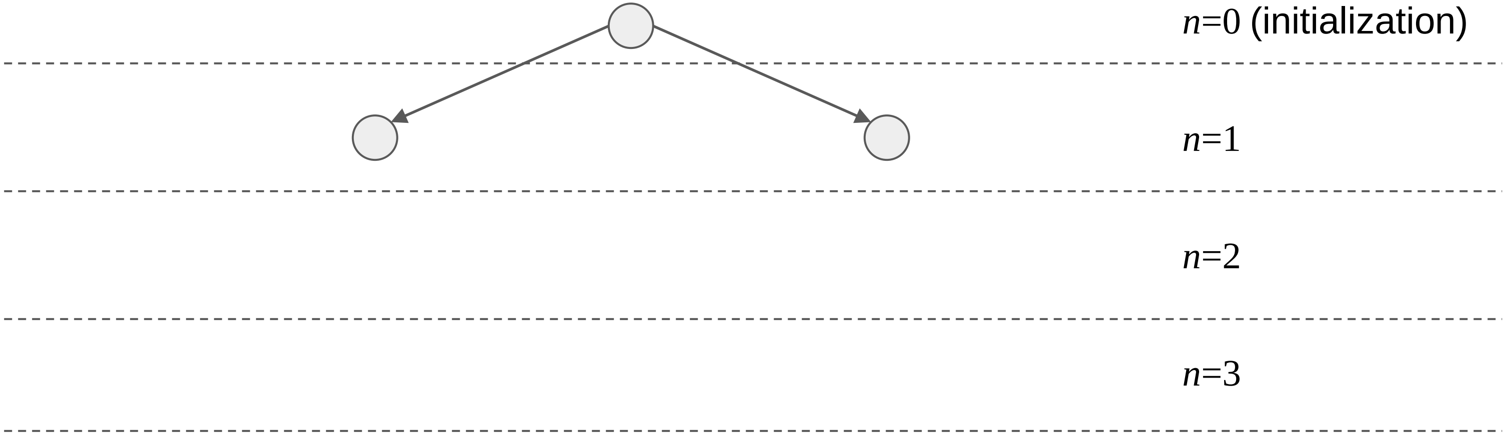

n=0 (initialization)

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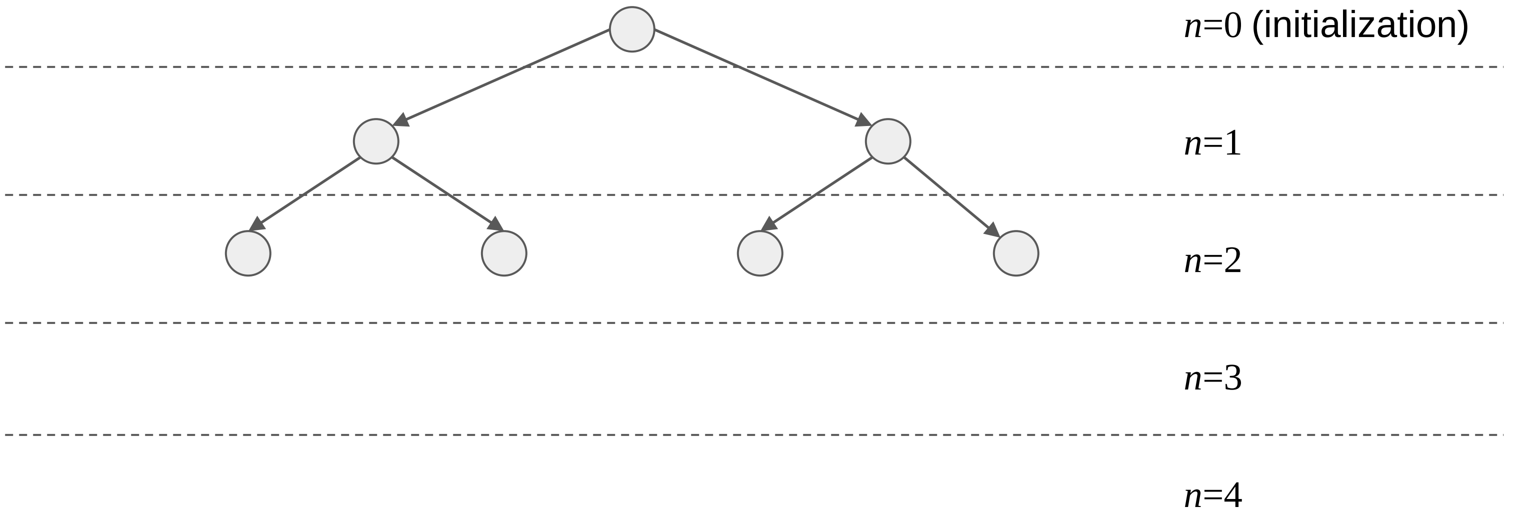
n=2

n=3

n=4

SDP exact: typical exhaustive computation graph



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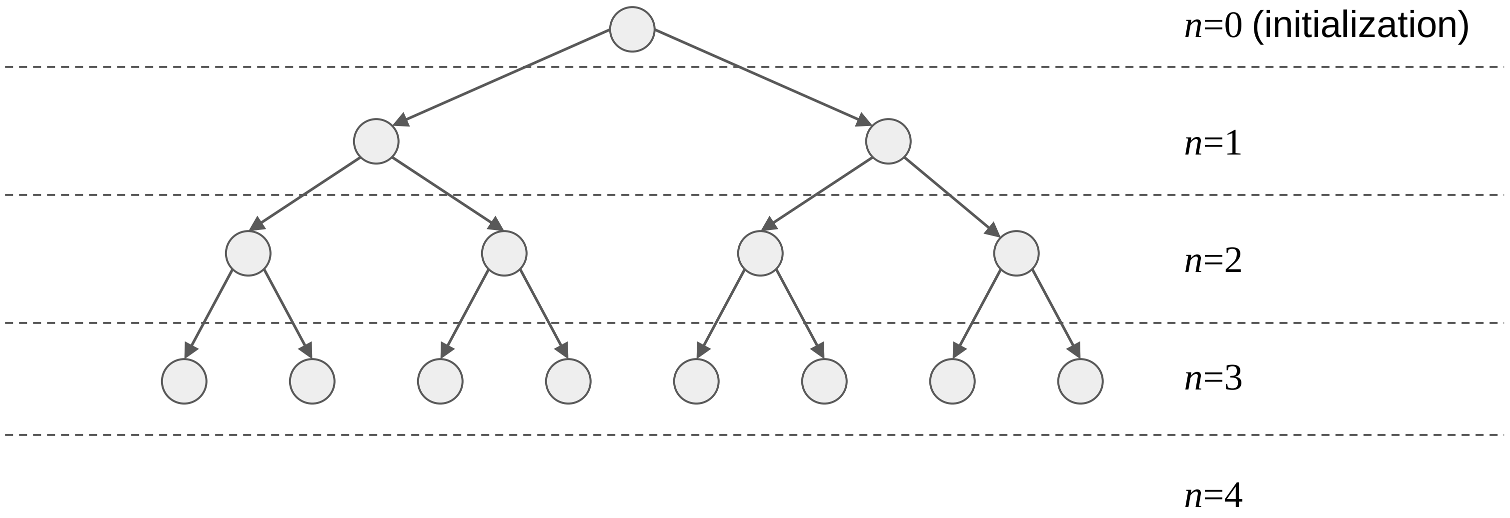

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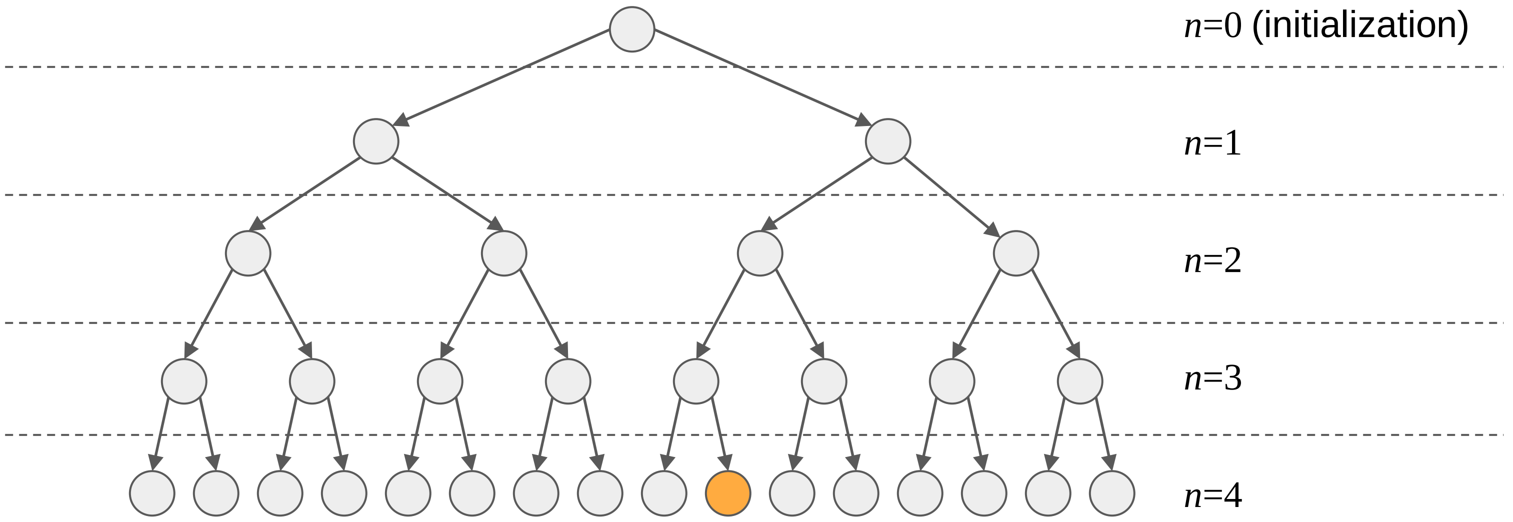

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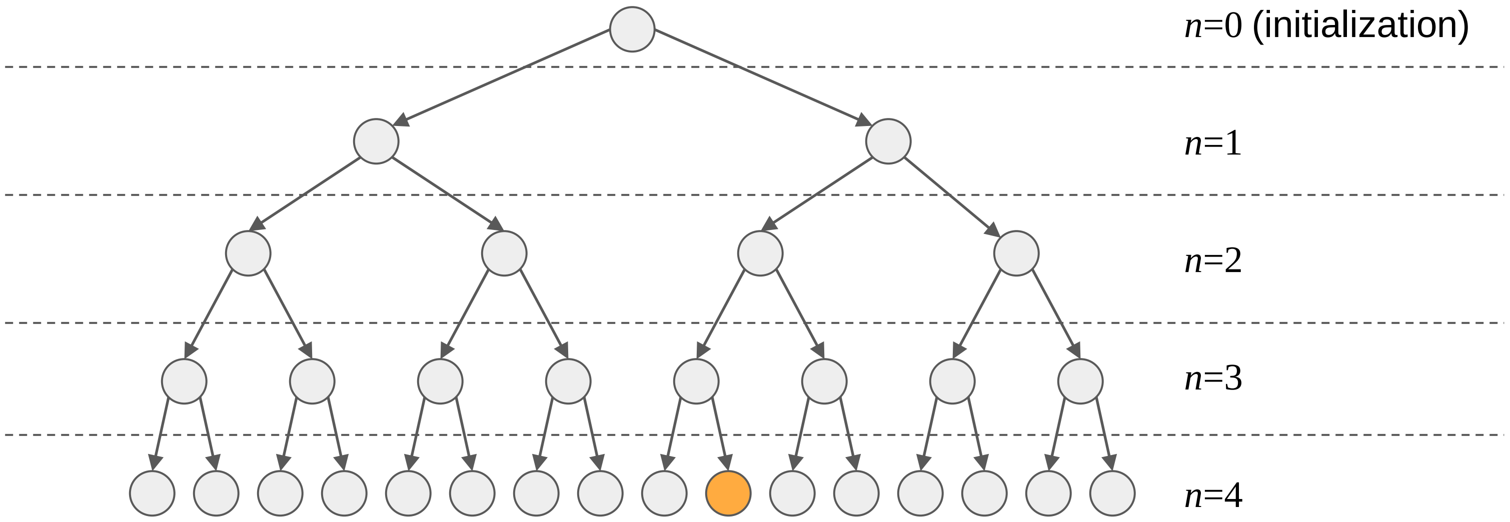

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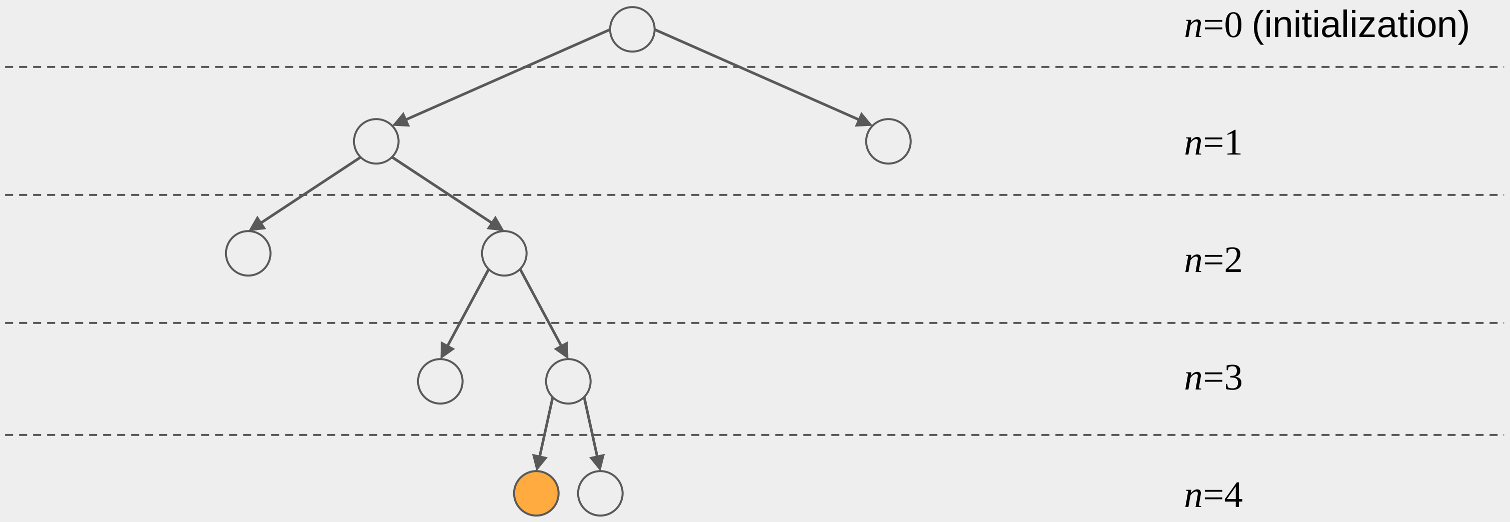
- **Complexity**: typically $O(Nk)$

SDP exact: typical greedy computation graph



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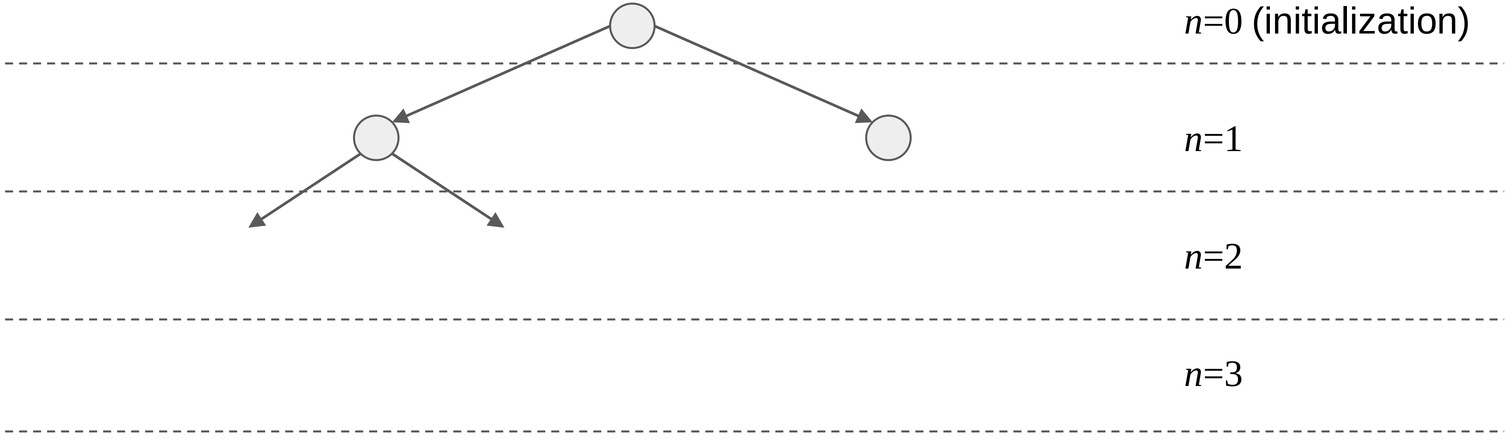

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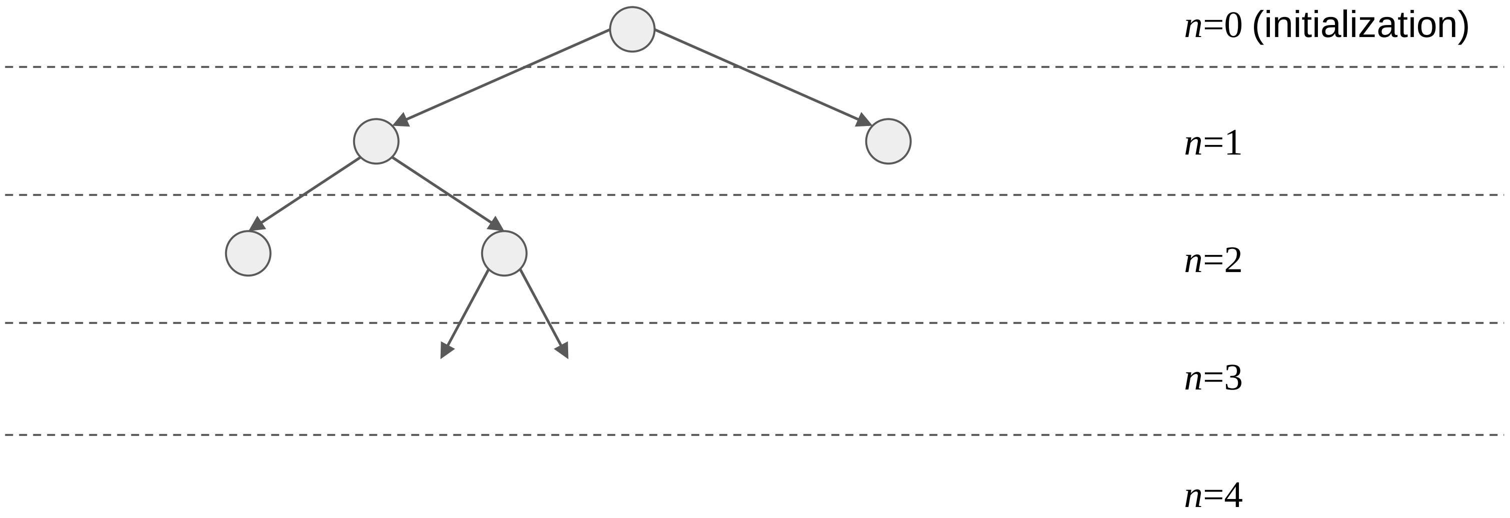

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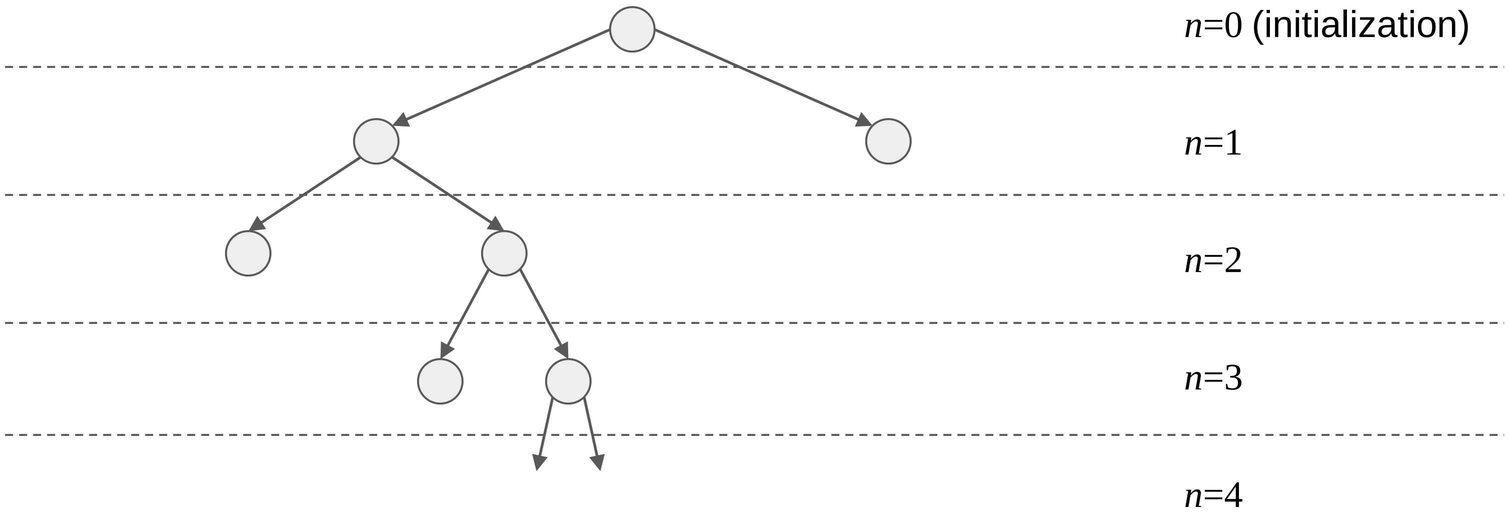

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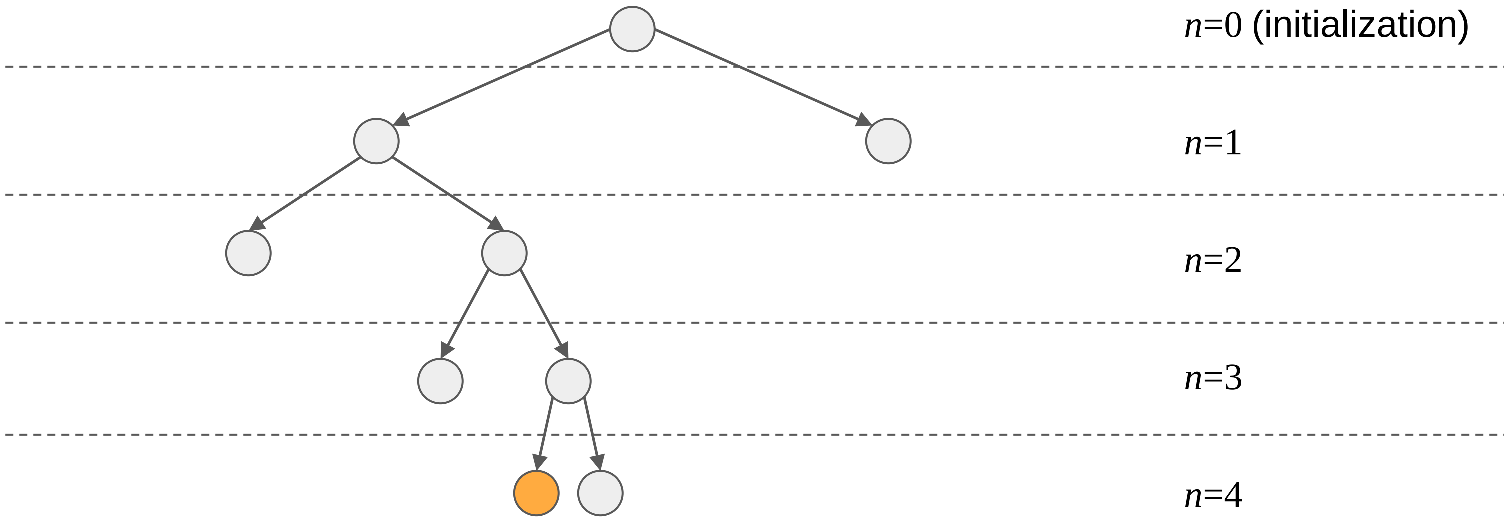

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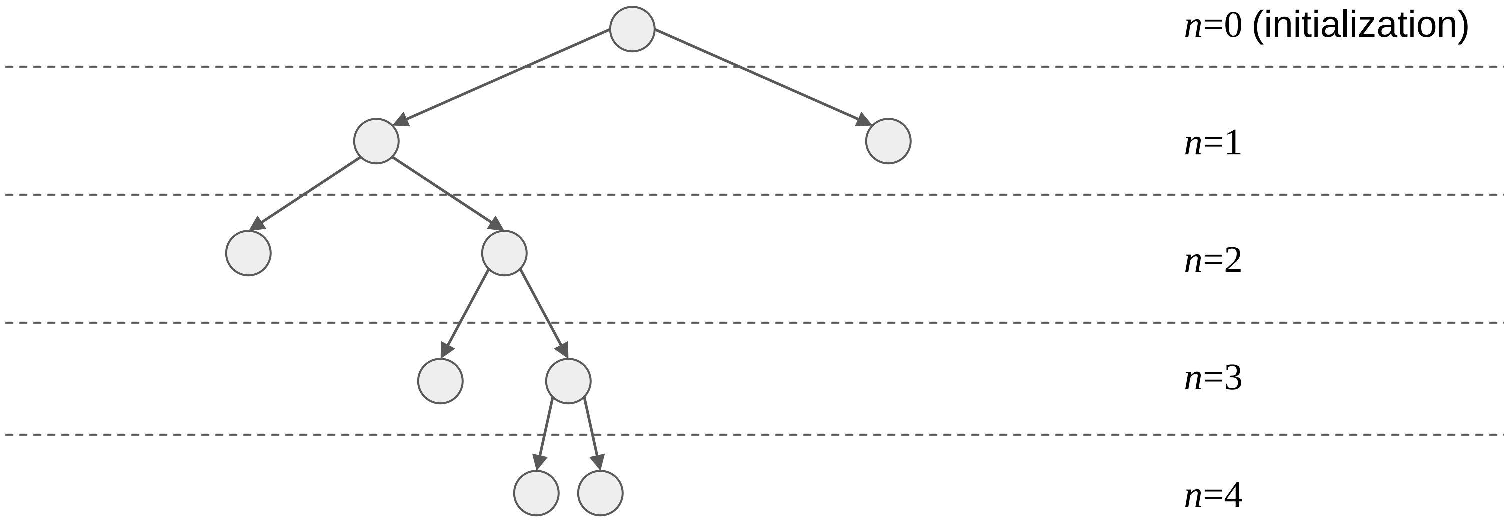
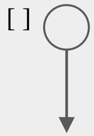


Insertion sort

- A sorting algorithm, like selection sort
- Let’s say you want to sort an array A with four elements
- A = [9, 1, 7, 3]

SDP exact: O(N^2) insertion sort



(initialization, n=0)

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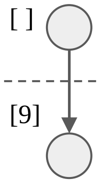

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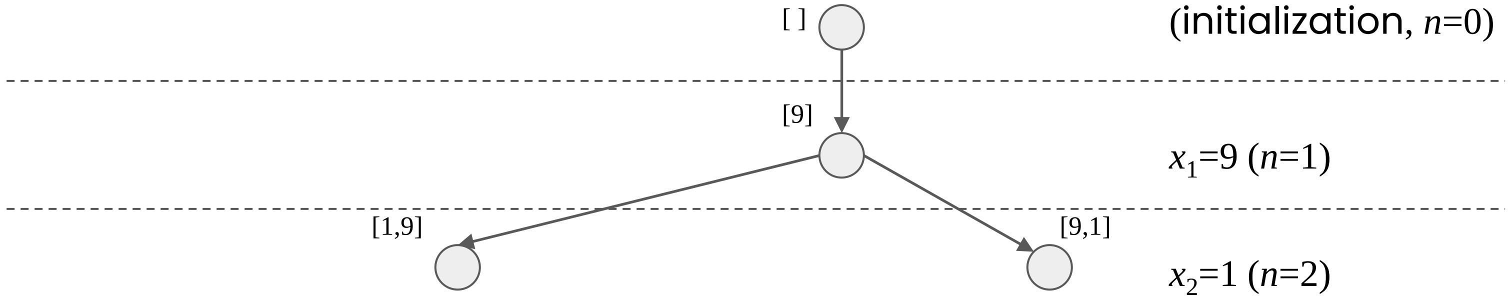
x_1=9 (n=1)

SDP exact: O(N^2) insertion sort



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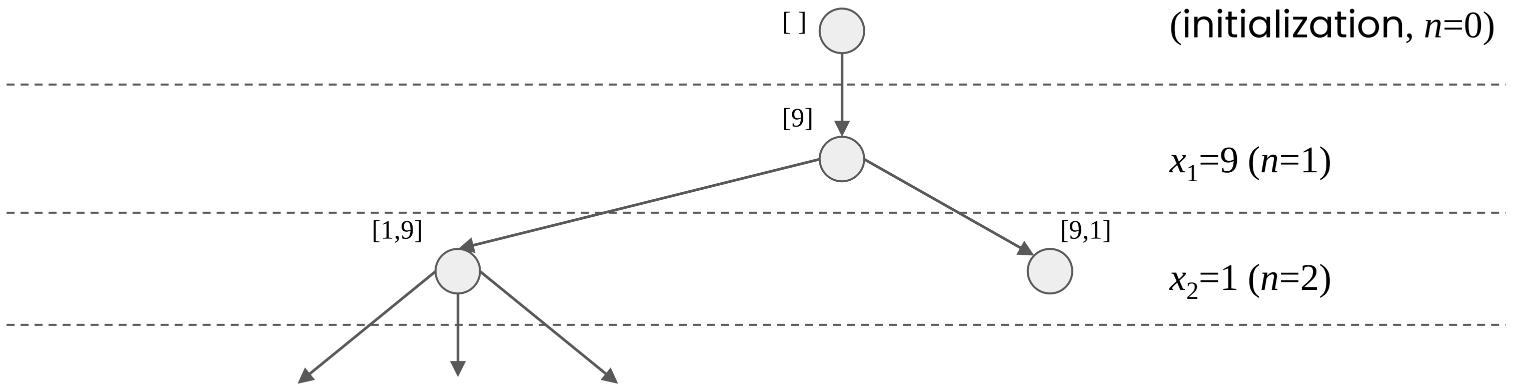

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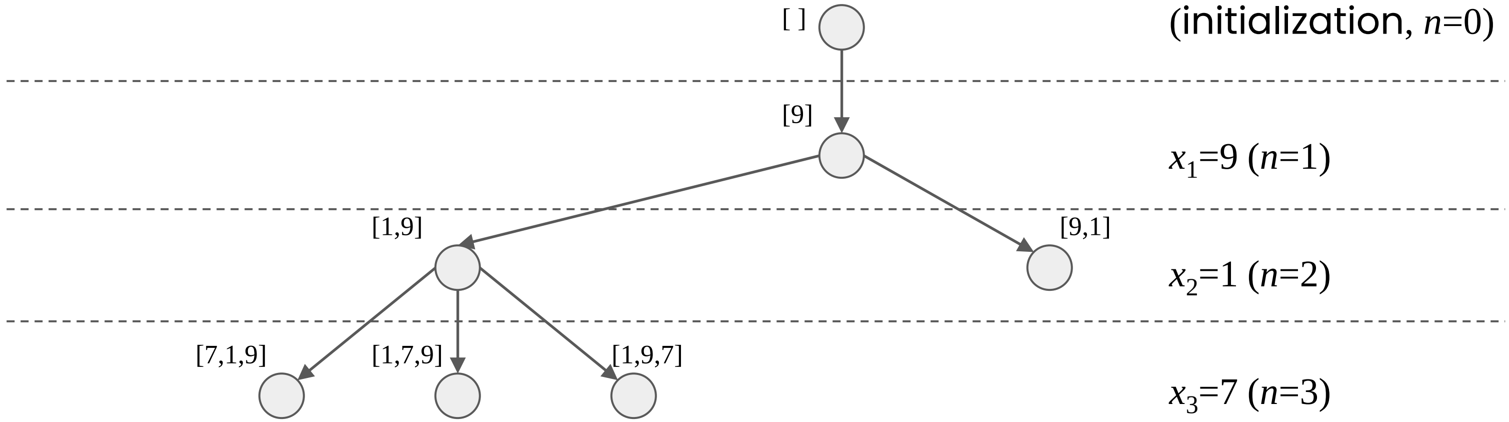

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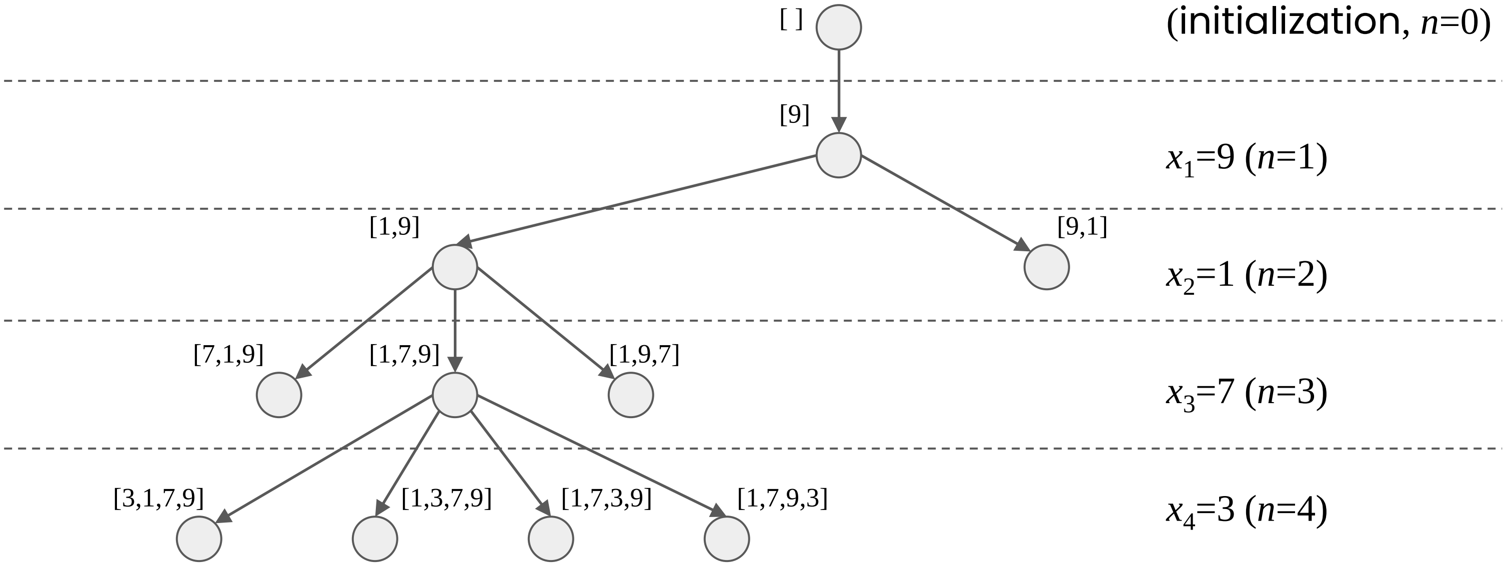

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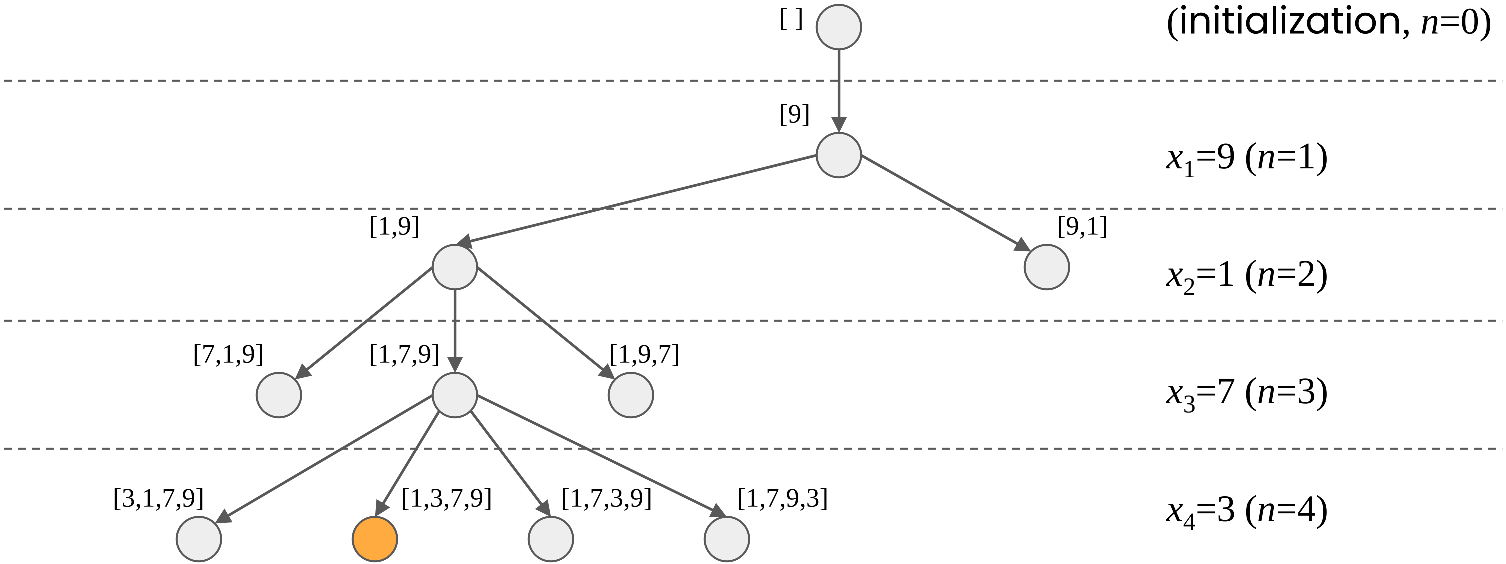

References and further reading

- **MLSP**, Section 2.6
- **CLRS**, Chapter 21