Supplementary Example for Lecture 3: Combinatorial Optimization

January 24, 2024

Suppose a symbolic AI optimization problem is to select a *subset* of items from a given set, S. For instance, a publisher has to select a number of advertisements to fill out the available space in their publication; they might be able to use all the advertisements, or they may only have space for a selection of them. This is a subset selection problem. In the terminology of the course, the *configurations* in this problem are subsets of S.

To make this concrete, a specific example might be $S = \{a, b, c\}$, which has $N = 3$ items. One subset is $X = \{a, c\}$. This leads to the following possible selections, e.g. the *configuration space* $\mathcal{X}$ is

$$\mathcal{X} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\},$$

so the size of the configuration space to search is $|\mathcal{X}| = 2^3 = 8$ configurations. The problem to solve requires choosing an optimal configuration. In the context of the publisher's advertising problem, this might mean choosing the *least costly* configuration. To actually make this selection, we need to assign some kind of measure of quality to each configuration. This is the *objective function* $F(X)$, which takes a configuration, X, and outputs a number. For this specific example, the quality of each configuration is listed as,

$$\begin{aligned}
F(\emptyset) &= 2 \\
F(\{a\}) &= 4 \\
F(\{b\}) &= 7 \\
F(\{c\}) &= 5 \\
F(\{a, b\}) &= 1 \\
F(\{a, c\}) &= 4 \\
F(\{b, c\}) &= 8 \\
F(\{a, b, c\}) &= 3.
\end{aligned}$$

Now, we can solve the optimization problem to find $X^\star$ by evaluating each configuration in turn, and select the one which has the smallest objective function value. In this case, since for $X = \{a, b\}$ we evaluate $F(X) = 1$, this is the optimal solution so $X^\star = \{a, b\}$. Here, $X^\star$ is *the globally optimal* solution to equation (3.1) in the lecture notes,

$$X^\star = \underset{X' \in \mathcal{X}}{\arg\min}\, F\left(X'\right).$$

Implicitly, we found this by exhaustive search (it is a good exercise to satisfy yourself that condition (3.2) in the lecture notes holds for $X^\star$). There is a *unique* global optimum because no other configuration has the same objective function value. This may not always be the case, if, for instance, $F(\{b\}) = 1$, then there would be two optimal solutions, $X^\star = \{a, b\}$ and $X^\star = \{b\}$, but in practice it might not matter which one is used so we just select one of them.

Since in this case $N = 3$ and the problem size is very small, it is feasible to check each configuration one-by-one and then pick the best one, i.e. to carry out *exhaustive search.* But, when N gets large, $|\mathcal{X}| = 2^N$ which grows *exponentially* with N (to see this for yourself, try computing 2^N for $N = 10$ or $N = 20$) so the size of the configuration space quickly gets out of hand. This is called *combinatorial explosion.* So in practice, we have to find *smarter* ways than simple dumb checking of every possible configuration. Symbolic AI is, essentially, concerned with finding more intelligent ways of solving intractable problems like this.

Whereas an exact solution is globally optimal for the whole of $\mathcal{X}$ an *approximate* solution might only be guaranteed to be *locally optimal* in some *neighbourhood* $\mathcal{N} \subset \mathcal{X}$. For example, within the restricted space of configurations $\mathcal{N} = \{\{a, c\}, \{a, b, c\}, \{c\}\}$, then $X^\star = \{a, b, c\}$ is the (locally) optimal configuration with quality $F(X^\star) = 3$. Since this neighbourhood only has $|\mathcal{N}| = 3$ configurations, it is much easier to search this restricted space than it is the entire space.