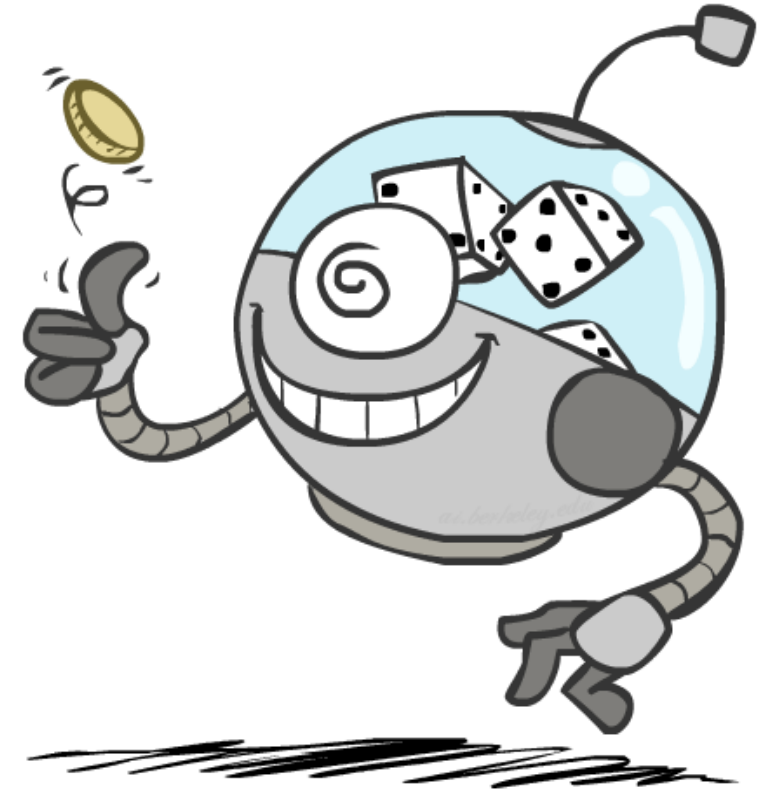


Random Variables

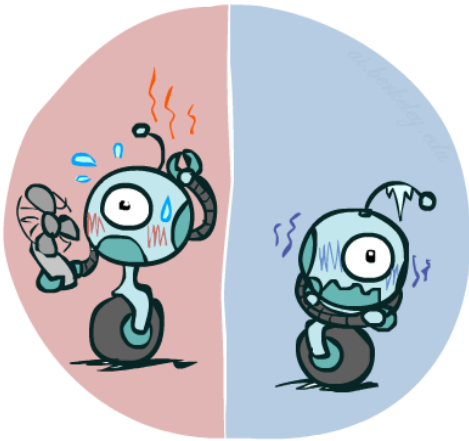
- A random variable is some aspect of the world about which we (may) have uncertainty
 - R = Is it raining?
 - T = Is it hot or cold?
 - D = How long will it take to drive to work?
 - L = Where is the ghost?
- We denote random variables with capital letters
 - **Constraint Satisfaction Problems (CSP)**
 - Like variables in a CSP, random variables have domains
 - R in $\{\text{true}, \text{false}\}$ (often write as $\{+r, -r\}$)
 - T in $\{\text{hot}, \text{cold}\}$
 - D in $[0, \infty)$
 - L in possible locations, maybe $\{(0,0), (0,1), \dots\}$



Probability Distributions

- Associate a probability with each value

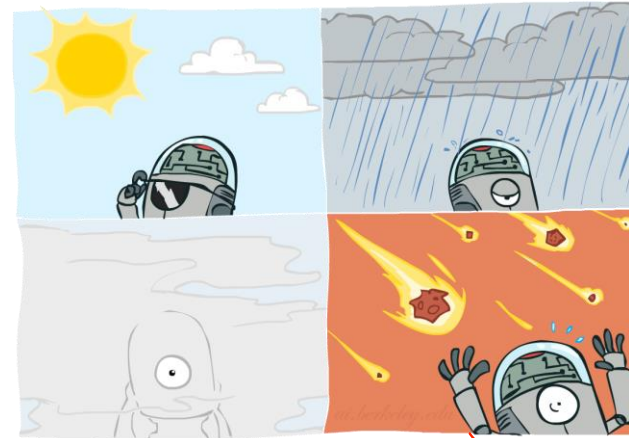
- Temperature:



$P(T)$

T	P
hot	0.5
cold	0.5

- Weather:



$P(W)$

W	P
sun	0.6
rain	0.1
fog	0.3
meteor	0.0

Probability Distributions

- Unobserved random variables have distributions

$$P(T)$$

T	P
hot	0.5
cold	0.5

$$P(W)$$

W	P
sun	0.6
rain	0.1
fog	0.3
meteor	0.0

Shorthand notation:

$$\begin{aligned}P(\textit{hot}) &= P(T = \textit{hot}), \\P(\textit{cold}) &= P(T = \textit{cold}), \\P(\textit{rain}) &= P(W = \textit{rain}), \\&\dots\end{aligned}$$

OK if all domain entries are unique

- A distribution is a TABLE of probabilities of values
- A probability (lower case value) is a single number

$$P(W = \textit{rain}) = 0.1$$

- Must have: $\forall x \ P(X = x) \geq 0$ and $\sum_x P(X = x) = 1$
For any probability, it is greater or equal to zero The sum of all probabilities in an event is 1

Joint Distributions

- A *joint distribution* over a set of random variables: X_1, X_2, \dots, X_n specifies a real number for each assignment (or *outcome*):

$$P(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n)$$

$$P(x_1, x_2, \dots, x_n)$$

- Must obey: $P(x_1, x_2, \dots, x_n) \geq 0$

$$\sum_{(x_1, x_2, \dots, x_n)} P(x_1, x_2, \dots, x_n) = 1$$

$$P(T, W)$$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

- Size of distribution if n variables with domain sizes d ?
 - For all but the smallest distributions, impractical to write out!

Probabilistic Models

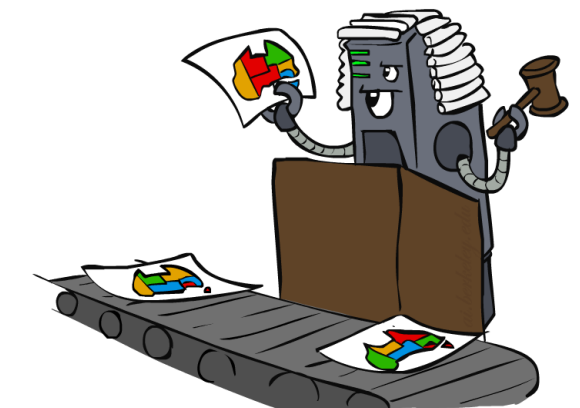
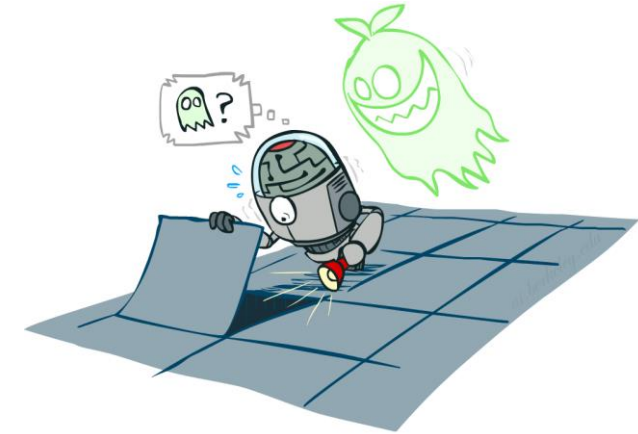
- A probabilistic model is a joint distribution over a set of random variables
- Probabilistic models:
 - (Random) variables with domains
 - Assignments are called *outcomes*
 - Joint distributions: say whether assignments (outcomes) are likely
 - *Normalized*: sum to 1.0
 - Ideally: only certain variables directly interact
- Constraint satisfaction problems:
 - Variables with domains
 - Constraints: state whether assignments are possible
 - Ideally: only certain variables directly interact

Distribution over T,W

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Constraint over T,W

T	W	P
hot	sun	T
hot	rain	F
cold	sun	F
cold	rain	T



Events

- An *event* is a set E of outcomes

$$P(E) = \sum_{(x_1 \dots x_n) \in E} P(x_1 \dots x_n)$$

- From a joint distribution, we can calculate the probability of any event
 - Probability that it's hot AND sunny?
 - Probability that it's hot?
 - Probability that it's hot OR sunny?
- Typically, the events we care about are *partial assignments*, like $P(T=\text{hot})$

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Quiz: Events

- $P(+x, +y)$?

0.2

- $P(+x)$?

$0.2 + 0.3 = 0.5$

- $P(-y \text{ OR } +x)$?

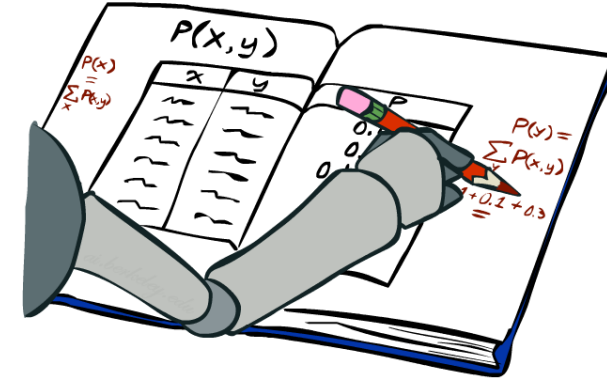
$0.2 + 0.1 = 0.3$

$P(X, Y)$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

Marginal Distributions

- Marginal distributions are sub-tables which eliminate variables
- Marginalization (summing out): Combine collapsed rows by adding



$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3



$$P(t) = \sum_s P(t, s)$$

$P(T)$

T	P
hot	0.5
cold	0.5



$$P(s) = \sum_t P(t, s)$$

$P(W)$

W	P
sun	0.6
rain	0.4

$$P(X_1 = x_1) = \sum_{x_2} P(X_1 = x_1, X_2 = x_2)$$

Quiz: Marginal Distributions

$P(X, Y)$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1



$$P(x) = \sum_y P(x, y)$$



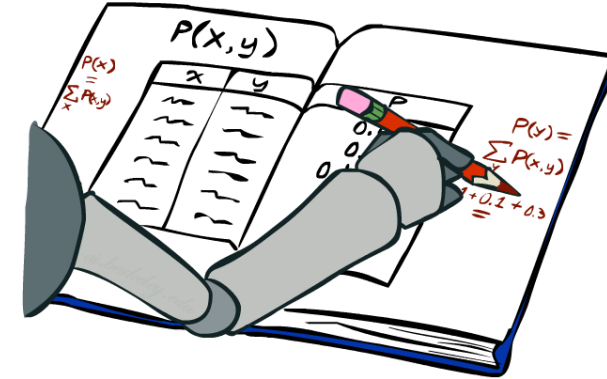
$$P(y) = \sum_x P(x, y)$$

$P(X)$

X	P
+x	0.2 + 0.3
-x	0.1 + 0.4

$P(Y)$

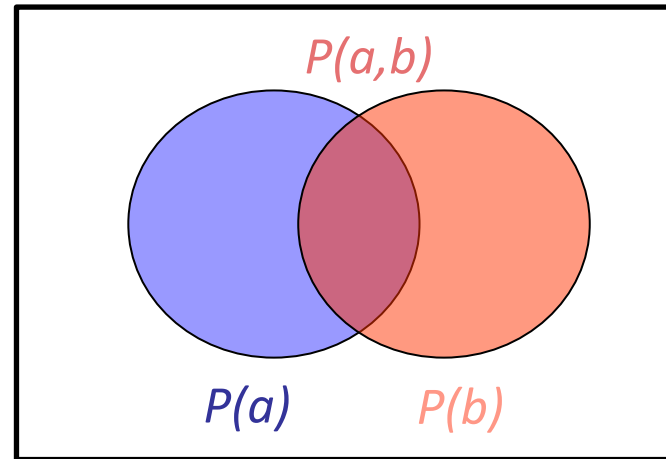
Y	P
+y	0.2 + 0.4
-y	0.3 + 0.1



Conditional Probabilities

- A simple relation between joint and conditional probabilities
 - In fact, this is taken as the *definition* of a conditional probability

$$P(a|b) = \frac{P(a, b)}{P(b)}$$



The probability of A given B, which is the probability of event A happen after event B happened. In this case, if the temperature is cold, then the probability of the weather is sunny is $0.2/0.5 = 0.4$

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$P(W = s|T = c) = \frac{P(W = s, T = c)}{P(T = c)} = \frac{0.2}{0.5} = 0.4$$

$$\begin{aligned} &= P(W = s, T = c) + P(W = r, T = c) \\ &= 0.2 + 0.3 = 0.5 \end{aligned}$$

Quiz: Conditional Probabilities

$P(X, Y)$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

- $P(+x \mid +y) ?$

$$P(+x, +y)/P(+y) = 0.2/(0.2+0.4) = 0.2/0.6 = 1/3$$

- $P(-x \mid +y) ?$

$$P(-x, +y)/P(+y) = 0.4/(0.2+0.4) = 0.4/0.6 = 2/3$$

- $P(-y \mid +x) ?$

$$P(-y, +x)/P(+x) = 0.3/(0.2+0.3) = 0.3/0.5 = 3/5$$

Conditional Distributions

- Conditional distributions are probability distributions over some variables given fixed values of others

Conditional Distributions

$P(W T)$	$P(W T = hot)$	
	W	P
	sun	0.8
	rain	0.2
	$P(W T = cold)$	
	W	P
	sun	0.4
	rain	0.6

Joint Distribution

$P(T, W)$		
T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Normalization Trick

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$\begin{aligned}P(W = s|T = c) &= \frac{P(W = s, T = c)}{P(T = c)} \\&= \frac{P(W = s, T = c)}{P(W = s, T = c) + P(W = r, T = c)} \\&= \frac{0.2}{0.2 + 0.3} = 0.4\end{aligned}$$



$P(W|T = c)$

W	P
sun	0.4
rain	0.6

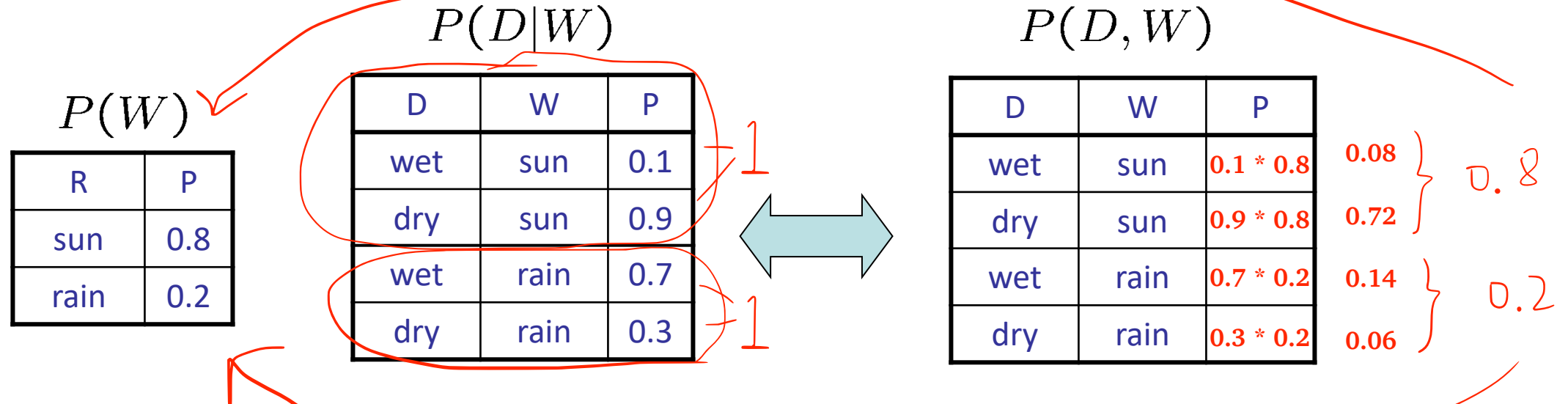
$$\begin{aligned}P(W = r|T = c) &= \frac{P(W = r, T = c)}{P(T = c)} \\&= \frac{P(W = r, T = c)}{P(W = s, T = c) + P(W = r, T = c)} \\&= \frac{0.3}{0.2 + 0.3} = 0.6\end{aligned}$$

The Product Rule

$$P(y)P(x|y) = P(x, y)$$

$$P(y) * P(x,y) \setminus P(y) = P(x,y)$$

- Example:



The Chain Rule

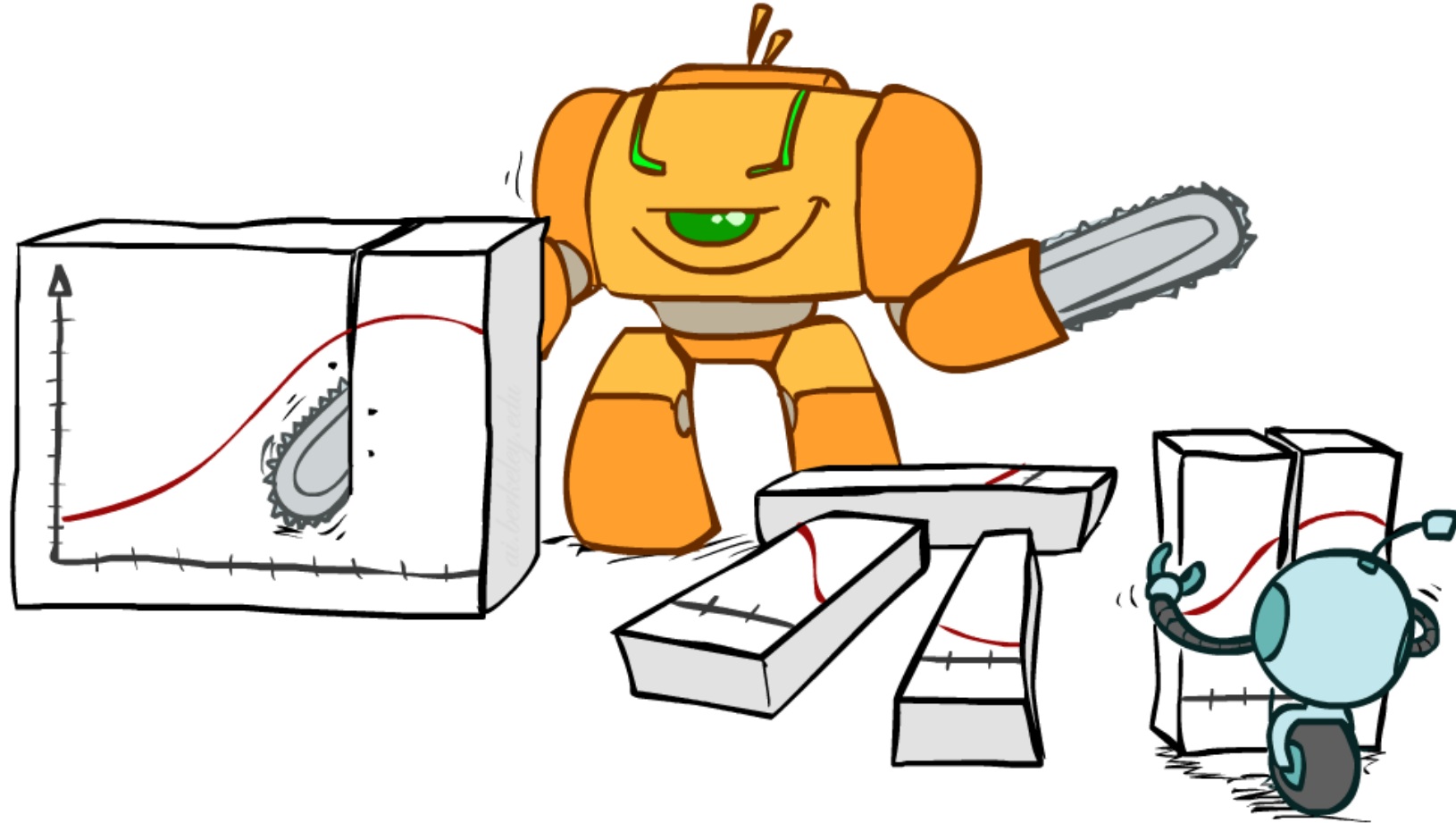
- More generally, can always write any joint distribution as an incremental product of conditional distributions

$$P(x_1, x_2, x_3) = P(x_1)P(x_2|x_1)P(x_3|x_1, x_2)$$

$$P(x_1, x_2, \dots, x_n) = \prod_i P(x_i|x_1 \dots x_{i-1})$$

- Why is this always true?

Bayes Rule



Bayes' Rule

- Two ways to factor a joint distribution over two variables:

$$P(x, y) = P(x|y)P(y) = P(y|x)P(x)$$

That's my rule!

- Dividing, we get:

$$P(x|y) = \frac{P(y|x)P(x)}{P(y)}$$

- Why is this at all helpful?
 - Lets us build one conditional from its reverse
 - Often one conditional is tricky but the other one is simple
 - Foundation of many systems we'll see later (e.g. ASR, MT)
- In the running for most important AI equation!

