ASSIGNMENT 1

PSTAT 160A– Summer 2023
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Release date: **Monday, August 7th**
Due date: **Wednesday, August 16th at 11:59 pm**

Instructions for the homework: Solve all homework problems. Your reasoning has to be comprehensible and complete. To receive full credit sufficient explanations need to be provided. Submit your work as a PDF on Gradescope. Please write legibly and do not crowd your solution on the page.

Problem 1.1 (10 points) Let X and Y be two continuous positive random variables with joint density function given by

$$f_{X,Y}(x,y) = \begin{cases} c \cdot e^{-2x-4y}, & 0 < x < \infty,\ 0 < y < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

(a) (2 points) Determine the value of the constant c.

(b) (3 points) Find the marginal densities $f_X(x)$ and $f_Y(y)$

(c) (3 points) Compute the probability that $X < Y$.

(d) (2 points) Are X and Y independent? Justify your answer.

Problem 1.2 (10 points) A bird lays B eggs in her nest. Assume that B is Poisson distributed with parameter 8 and suppose that, independently from all the others, each egg will hatch and produce a healthy offspring with probability $p \in [0,1]$.

(a) (5 points) Compute the distribution of the number of hatched healthy bird offspring A.

(b) (5 points) Conversely, given the number of hatched offspring A, how is the original number of eggs B distributed?

Hint: Both answers can be phrased in terms of Poisson distributions. Recall the fact that

$$\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{\lambda} \qquad (\lambda \in \mathbb{R}).$$

Problem 1.3 (10 points) We chose a natural number from the set $\{1, 2, 3, \ldots, 100\}$ uniformly at random and denote this number by X. For each of the following choices decide whether the two events in question are independent or not.

(a) (4 points) $A = \{X \text{ is even}\}, B = \{X \text{ is divisible by } 7\}$

(b) (3 points) $C = \{X \text{ has two digits}\}, D = \{X \text{ is divisible by } 5\}$

(c) (3 points) $E = \{X \text{ is prime}\}, F = \{X \text{ has a digit } 7\}$. Note that 1 is not considered a prime number.

Problem 1.4 (10 points) Let the joint density of two random variables X and Y be given by

$$f(x,y) = \begin{cases} 2xe^{x^2-y}, & \text{if } 0 < x < 1 \text{ and } y > x^2 \\ 0, & \text{else.} \end{cases}$$

(a) (4 points) Find the conditional density function $f_{Y|X}(y|x)$

(b) (3 points) Find the conditional probability $\mathbb{P}(Y \geqslant 1/3|X)$ as a function of X.

(c) (3 points) Verify the averaging identity

$$\mathbb{P}(Y \geqslant 1/3) = \int_{-\infty}^{\infty} \mathbb{P}(Y \geqslant 1/3|X = x) f_X(x) dx.$$

Problem 1.5 (10 points) Consider the following game of chance with a monetary payoff. First, a *real* number U is chosen uniformly at random from the interval $[1, 10]$. Next, an integer X is sampled according to the Poisson distribution with parameter U. The player receives a reward of $\$X$.

What would be the fair price charged for playing this game? That is, how much should it cost to play so that expected net gain is zero?