CS/ECE/ME 532
Homework 1: Vectors and Matrices

1. Matrix multiplication. The local factory makes widgets and gizmos. Making one widget requires 3 lbs of materials, 4 parts, and 1 hour of labor. Making one gizmo requires 2 lbs of materials, 3 parts, and 2 hours of labor.

a) Write the information above in a matrix. What do the rows represent? What do the columns represent?

b) Suppose materials cost \$1/lb, parts cost \$10 each, and labor costs \$100/hr. Write this information in a vector. Write out a matrix-vector multiplication that calculates the total cost of making widgets and gizmos.

c) Suppose the factory receives an order for 3 widgets and 4 gizmos. Again using matrix multiplication, find the total material, parts, and labor required to fill the order.

d) Calculate the total cost for the order (using, you guessed it, matrix multiplication)

e) Get up and running with either Matlab or Python. In your language of choice, write a script that computes the matrix multiplications in the previous parts of this problem.

2. Let $\boldsymbol{X} = [\boldsymbol{x}_1 \ \boldsymbol{x}_2 \ \cdots \ \boldsymbol{x}_n] \in \mathbb{R}^{p\times n}$, where $\boldsymbol{x}_i \in \mathbb{R}^p$ is the ith column of $\boldsymbol{X}$. Consider the matrix

$$\boldsymbol{C} \ = \ \frac{\boldsymbol{X}\boldsymbol{X}^T}{n} \ .$$

a) Express $\boldsymbol{C}$ as a sum of rank-1 matrices (i.e., columns of $\boldsymbol{X}$ times rows of $\boldsymbol{X}^T$).

b) Assuming $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ are linearly independent, what is the rank of $\boldsymbol{C}$?

3. Define the mapping $\Phi : \mathbb{R}^n \to \mathbb{R}$ as follows. For every $\boldsymbol{x} \in \mathbb{R}^n$ let

$$\Phi(\boldsymbol{x}) \ = \ \sum_{i=1}^{n} \sum_{j=i+1}^{n} \max(|x_i|, |x_j|) \ ,$$

where x_i and x_j are the ith and jth entries in $\boldsymbol{x}$. Is $\Phi(\boldsymbol{x})$ a norm?

4. Equivalence of norms. For each case below find positive constants a and b (possibly different in each case) so that for every $\boldsymbol{x} \in \mathbb{R}^n$

(i) $\quad a\|\boldsymbol{x}\|_1 \ \le \ \|\boldsymbol{x}\|_2 \ \le \ b\|\boldsymbol{x}\|_1$

(ii) $\quad a\|\boldsymbol{x}\|_1 \ \le \ \|\boldsymbol{x}\|_\infty \ \le \ b\|\boldsymbol{x}\|_1$

(iii) $\quad a\|\boldsymbol{x}\|_1 \ \le \ \Phi(\boldsymbol{x}) \ \le \ b\|\boldsymbol{x}\|_1$

where $\Phi(\boldsymbol{x})$ is defined in problem 3 above. Find the tightest constants in each case.

5. Let $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ be a non-singular matrix. Suppose that $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$.

a) Given $\boldsymbol{A}$ and $\boldsymbol{y}$, write an expression for $\boldsymbol{x}$.

b) Bound the 2-norm of $\boldsymbol{x}$ in terms of $\|\boldsymbol{y}\|_2$ and a function of the matrix $\boldsymbol{A}$.

6. Let

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

a) What is the rank of $\boldsymbol{A}$?

b) Suppose that $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$. Derive an explicit formula for $\boldsymbol{x}$ in terms of $\boldsymbol{y}$.

7. Let

$$\boldsymbol{X} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

a) What is the rank of $\boldsymbol{X}$?

b) What is the rank of $\boldsymbol{X}\boldsymbol{X}^T$?

c) Find a set of linearly independent columns in $\boldsymbol{X}$.