Fourier Interpolation with Magnitude Only

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Abstract

In this note we prove that *all* even Schwartz functions $f : \mathbb{R} \to \mathbb{R}$ are uniquely determined by their values at $\{|\hat{f}(\sqrt{n})|, f(\sqrt{n/2})\}$ for $n \geq 0$ indexing the non-negative integers.

1 Introduction

Our motivation comes from phase retrieval type problems in signal processing, where in several imaging applications one can only measure the magnitude of the Fourier transform and not its phase (recall in general the Fourier transform of a function is complex-valued). Since we are dealing with 1D even signal on the real line $f(-x) = f(x)$ where $f : \mathbb{R} \to \mathbb{R}$, its Fourier transform

$$\hat{f}(s) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i s x}\, dx$$

is purely real and has even symmetry as well, therefore phase ambiguity reduces to the sign ambiguity of $\hat{f}(s)$. The question we address in this note is: how many samples does one have to take on its function value f and its Fourier magnitude $|\hat{f}|$ such that any even Schwartz function is *uniquely determined* from these measurements? We will discuss implications for recovery, but it is not our intention to give explicit reconstruction formula, rather we focus on well-posedness here.

For band-limited signal where $\text{supp}(\hat{f}(s)) \subset [-w/2, w/2]$, Shannon interpolation [3] prescribes a reconstruction formula on the integer grid with sinc basis:

$$f(x) = \sum_{n \in \mathbb{Z}} f(n/w) \cdot \text{sinc}(wx - n)\,,$$

which gives the famous Nyquist rate from sampling theory. Before proceeding, we record a definition. As a remark, the conclusion in the note is expected to apply to odd real-valued functions $f(-x) = -f(x)$ as well that have purely imaginary spectrum.

Definition 1. *We call a function $f : \mathbb{R} \to \mathbb{R}$ a Schwartz function if $f \in C^{\infty}(\mathbb{R})$ and*

$$\sup_{x \in \mathbb{R}} |x^{\alpha} f^{(\beta)}(x)| < \infty \;\forall \alpha, \beta \geq 0\,.$$

In words, Schwartz function is infinitely differentiable where the function itself and all its derivatives decay faster than any inverse power of x.

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For general radial Schwartz functions with unbounded Fourier support, the values at

$$\{f(n), f'(n), \hat{f}(n), \hat{f}'(n)\}_{n\geq 0}$$

simply aren't enough to uniquely determine the signal, as manifested by the following example.

Example 1. *The function and its Fourier pair*

$$f(x) = \sin^2(\pi x) \cdot (g(x+1) - 2g(x) + g(x-1))$$

$$\hat{f}(s) = \sin^2(\pi s) \cdot (\hat{g}(s+1) - 2\hat{g}(s) + \hat{g}(s-1))$$

vanish at all integer nodes to first order but one can choose two different even Schwartz functions g_1, g_2*, for example* $g_1(x) = \exp(-\alpha x^2), g_2(x) = \exp(-\beta x^2)$ *which give two functions that do not agree everywhere on the real line.*

Our starting point is the following mesmerizing interpolation formula on $\{\sqrt{n}\}_{n\geq 0}$ lattices due to Radchenko-Viazovska [2], which also played an important role in the construction of the "magic function" of the linear programming bound for the densest sphere packing problem.

Theorem 1. *There exists a collection of even Schwartz functions* $a_n : \mathbb{R} \to \mathbb{R}$ *with the property that for any even Schwartz function* $f : \mathbb{R} \to \mathbb{R}$ *and any* $x \in \mathbb{R}$ *we have*

$$f(x) = \sum_{n=0}^{\infty} a_n(x) f(\sqrt{n}) + \sum_{n=0}^{\infty} \hat{a}_n(x) \hat{f}(\sqrt{n}) \tag{1}$$

where the right-hand side converges absolutely. In particular, if a function and its Fourier transform vanish at all $\{\sqrt{n}\}_{n\geq 0}$ *indices, the function* f *is identically zero.*

A stronger isomorphic result can also be established: given any set of $\{g(\sqrt{n}), \hat{g}(\sqrt{n})\}_{n\geq 0}$, if they satisfy the Poisson summation formula

$$\sum_{n\in\mathbb{Z}} g(n) = \sum_{n\in\mathbb{Z}} \hat{g}(n)\,,$$

there must exist a corresponding even Schwartz function g. Theorem 1 therefore says that the Fourier pairs $f, \hat{f}$, which obey certain uncertainty principle, are perfectly constrained on these discrete sets. Compared to Shannon's classical result, the spacing is non-uniform and the gap between consecutive nodes decreases as n increases as $\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}} \sim \frac{1}{2\sqrt{n}}$, and there are roughly n^2 number of sampled points over an interval $[0, n]$. We also mention in passing that there exists Fourier interpolation formula [1, Theorem 1.7] for radial function in dimension 8 and 24 – this is the analogue of even/odd function in 1D where the function f is parameterized by $f(\|x\|)$ – using basis $\{f(\sqrt{2n}), f'(\sqrt{2n}), \hat{f}(\sqrt{2n}), \hat{f}'(\sqrt{2n})\}$ that collects twice as more information at each queried index but is spaced twice further apart.

2 Why isn't $\sqrt{n}$ enough?

It has to do with the interpolation basis $\hat{a}_n(x), a_n(x)$ constructed in (1), which can be shown to be sparse on $\{\sqrt{n}\}$ grid (in fact, Dirac delta's). We begin by stating a lemma proved in [2].

Lemma 1. *We have* $\hat{a}_n(x) = \frac{b_n^+(x) - b_n^-(x)}{2}$, $a_n(x) = \frac{b_n^+(x)+b_n^-(x)}{2}$, *where* $b_m^\epsilon \colon \mathbb{R} \to \mathbb{R}$ *is an even Schwartz function satisfying*

$$b_m^\epsilon(\sqrt{n}) = \delta_{n,m}, \quad n \geq 1, m \geq 0, \epsilon = \pm$$

and $b_m^+(0) = \delta_{m,0}$, $b_0^-(x) = 0$. *Moreover* $b_m^-(0) = -2$ *if* $m \geq 1$ *is a square and* $= 0$ *otherwise.*

With this property of the interpolation basis on hand, it is not hard to come up with counterexamples. In fact, the argument below also gives explicit ways to construct functions in the "equivalent class". We note that we haven't specified a_n properly but Lemma 1 suffices in all that follow.

Theorem 2. *Not all even Schwartz functions* f *are uniquely determined by their values at* $\{f(\sqrt{n}), |\hat{f}(\sqrt{n})|\}_{n\geq 0}$.

Proof. Take two real-valued even Schwartz functions $f(x) \neq g(x)$, suppose they differ at one Fourier frequency $\hat{f}(\sqrt{k}) = -\hat{g}(\sqrt{k}) \neq 0$ but agree on all other points. Using Theorem 1, we can write

$$f(x) = \sum_{n=0}^{\infty} a_n(x) f(\sqrt{n}) + \sum_{n=0}^{\infty} \hat{a}_n(x) \hat{f}(\sqrt{n})$$

$$g(x) = \sum_{n=0}^{\infty} a_n(x) f(\sqrt{n}) + \sum_{n\neq k} \hat{a}_n(x) \hat{f}(\sqrt{n}) - \hat{a}_k(x)\hat{f}(\sqrt{k})$$

which means $f(x) - g(x)$, as a function that vanishes at all $\sqrt{n}$ indices, has representation

$$f(x) - g(x) = 2\hat{a}_k(x)\hat{f}(\sqrt{k})$$

for some $\hat{f}(\sqrt{k}) \neq 0$, which implies there must exists a k such that $\hat{a}_k(\sqrt{n})$ vanishes for all $n \in \mathbb{Z}$, but $\hat{a}_k(x)$ as a function on the real line, of course doesn't vanish identically. This is possible by just picking for example $k = 2$ using Lemma 1. The Fourier transform of $f(x) - g(x)$ is given by

$$\hat{f}(s) - \hat{g}(s) = 2\hat{f}(\sqrt{k}) a_k(s)$$

and also vanishes at all $\{\sqrt{n}\}$ except at $s = \sqrt{2}$, where it is equal to $2\hat{f}(\sqrt{2})$, which also holds from Lemma 1 since $a_2(\sqrt{2}) = 1$ and it is equal to 0 everywhere else on the $\{\sqrt{n}\}$ grid. □

Before seeking ways for obtaining extra information, let us generalize the previous argument a little bit. We see that if we have sign dis-agreements at frequencies $(k_1, k_2, \cdots, k_t)$, we require the following linear system to be satisfied for $\hat{f}(\sqrt{k_1}), \ldots, \hat{f}(\sqrt{k_t}) \neq 0$:

$$\begin{bmatrix} \hat{a}_{k_1}(0) & \hat{a}_{k_2}(0) & \cdots & \hat{a}_{k_t}(0) \\ \vdots & \vdots & \cdots & \vdots \\ \hat{a}_{k_1}(\sqrt{n}) & \hat{a}_{k_2}(\sqrt{n}) & \cdots & \hat{a}_{k_t}(\sqrt{n}) \end{bmatrix} \begin{bmatrix} 2\hat{f}(\sqrt{k_1}) \\ \vdots \\ 2\hat{f}(\sqrt{k_t}) \end{bmatrix} = 0\,. \tag{2}$$

This is true if for example $k_1 = 0, k_2 = 1$ and we impose $\hat{f}(0) = -2\hat{f}(1)$ on the signal; or if $k_1 = 5, k_2 = 6$. We can see from Lemma 1 that (1) $\hat{a}_0(x)$ is 1-sparse with $\hat{a}_0(0) = 1/2$; (2) $\hat{a}_k(x)$ is 1-sparse if $k \geq 1$ is a square with $\hat{a}_k(0) = 1$; (3) all other k's vanish entirely on the $\{\sqrt{n}\}$ grid. Therefore the sparsity of the $\hat{a}_k(\sqrt{n})$ is the culprit for the abundance of signals satisfying such constraints. One could check that (2) implies the correct constraint on the Fourier coefficients $\hat{f}(s) - \hat{g}(s)$ at $\{\sqrt{n}\}$ as well.

3 Oversampling can make up for the loss in phase

We will have to look at finer grids on $\hat{a}_n(x), a_n(x)$ – one can see from Figure 1 below that they are much denser on $\{\sqrt{n/2}\}$ lattice. In some sense we lose 1 bit of information from each Fourier measurement, but may hope to make up the loss from better resolution on the sampled function value f. For example the counter-example in Theorem 2

$$f(x) - g(x) = 2\hat{a}_k(x)\hat{f}(\sqrt{k})$$

will have to vanish at all $x = \sqrt{n/2}$ points as well. In fact, only one more sample at point $x = \sqrt{1/2}$ will resolve the ambiguity in this case. Consequently it suffices to study when does the matrix

$$M := \begin{bmatrix} \hat{a}_{k_1}(\sqrt{1/2}) & \hat{a}_{k_2}(\sqrt{1/2}) & \cdots & \hat{a}_{k_t}(\sqrt{1/2}) \\ \hat{a}_{k_1}(\sqrt{3/2}) & \hat{a}_{k_2}(\sqrt{3/2}) & \cdots & \hat{a}_{k_t}(\sqrt{3/2}) \\ \vdots & \vdots & \cdots & \vdots \\ \hat{a}_{k_1}(\sqrt{(2n+1)/2}) & \hat{a}_{k_2}(\sqrt{(2n+1)/2}) & \cdots & \hat{a}_{k_t}(\sqrt{(2n+1)/2}) \end{bmatrix} \tag{3}$$

has a trivial kernel, in the worst case when $t = n+1$. Above $k_1, \cdots, k_t$ take values from $0, 1, 2, \cdots, n$.

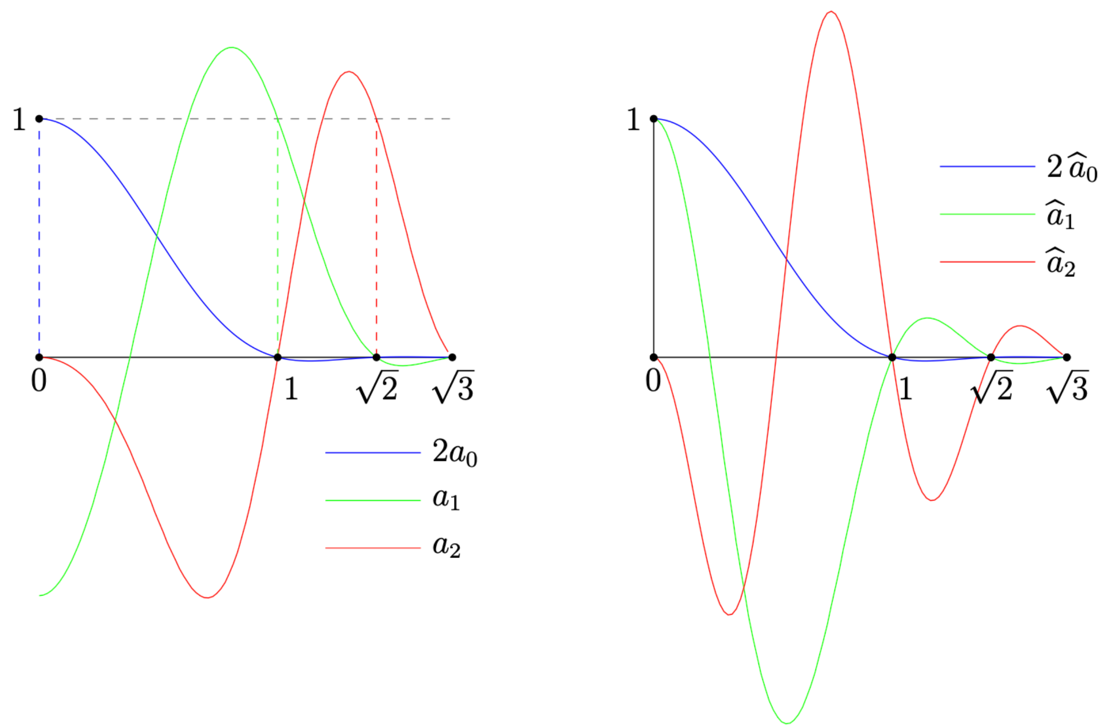

Figure 1: Plots of the first few $a_n, \hat{a}_n$ taken from [2]. Note that $\hat{a}_0 = a_0$.

The lemma below is the key observation behind our theorem.

Lemma 2. *The square matrix M defined in (3) is invertible, i.e., full-rank.*

Proof. Since the basis $a_n : \mathbb{R} \to \mathbb{R}$ are themselves even Schwartz functions as stated in Theorem 1, and from Lemma 3 below $a_n(\sqrt{2}x)$ are also even Schwartz functions, they admit the expansion

$$a_k(\sqrt{2}x) = \sum_{n=0}^{\infty} a_n(x)a_k(\sqrt{2n}) + \sum_{n=0}^{\infty} \hat{a}_n(x)\frac{1}{\sqrt{2}}\hat{a}_k(\sqrt{n/2}) \tag{4}$$

where we used that if $h(x) = f(ax)$, the Fourier transform $\hat{h}(s) = \frac{1}{|a|}\hat{f}(s/a)$.

Now for any $k = 1, 3, 5, 7, 9, \cdots 2n+1$, using Lemma 1, the left hand side of (4) at $x = \sqrt{1/2}, \sqrt{3/2}, \sqrt{5/2}, \sqrt{7/2}, \cdots \sqrt{(2n+1)/2}$ becomes the identity matrix ($n+1$ terms altogether). For these set of k's, the first term on the right hand side of (4) disappears since $a_k(\sqrt{2n})$ is 0 with n ranging over the positive integers, except we pick up $a_k(0) = -1$ when k is a square at $n = 0$, but these together with the second term yield $a_0(x)(\frac{1}{\sqrt{2}} - 1)$ since $\hat{a}_k(0) = 1$ at these points and $\hat{a}_0(x) = a_0(x)$, as shown in Lemma 4 below.

This allows us to rewrite the right hand side as (incorporating the extra $a_0(x)(\frac{1}{\sqrt{2}} - 1)$ term when k is a square, which is the same as a rank-1 outer product $(\frac{1}{\sqrt{2}} - 1)\hat{a}_0(x)\hat{a}_k(0)^\top$)

$$\begin{bmatrix} \hat{a}_0(\sqrt{1/2}) & \hat{a}_1(\sqrt{1/2}) & \hat{a}_2(\sqrt{1/2}) & \cdots & \hat{a}_n(\sqrt{1/2}) \\ \hat{a}_0(\sqrt{3/2}) & \hat{a}_1(\sqrt{3/2}) & \hat{a}_2(\sqrt{3/2}) & \cdots & \hat{a}_n(\sqrt{3/2}) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \hat{a}_0(\sqrt{(2n+1)/2}) & \hat{a}_1(\sqrt{(2n+1)/2}) & \hat{a}_2(\sqrt{(2n+1)/2}) & \cdots & \hat{a}_n(\sqrt{(2n+1)/2}) \end{bmatrix} \times$$

$$\frac{1}{\sqrt{2}} \times \begin{bmatrix} (1-\sqrt{2})\hat{a}_1(\sqrt{0/2}) & (1-\sqrt{2})\hat{a}_3(\sqrt{0/2}) & (1-\sqrt{2})\hat{a}_5(\sqrt{0/2}) & \cdots & (1-\sqrt{2})\hat{a}_{2n+1}(\sqrt{0/2}) \\ \hat{a}_1(\sqrt{1/2}) & \hat{a}_3(\sqrt{1/2}) & \hat{a}_5(\sqrt{1/2}) & \cdots & \hat{a}_{2n+1}(\sqrt{1/2}) \\ \hat{a}_1(\sqrt{2/2}) & \hat{a}_3(\sqrt{2/2}) & \hat{a}_5(\sqrt{2/2}) & \cdots & \hat{a}_{2n+1}(\sqrt{2/2}) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \hat{a}_1(\sqrt{n/2}) & \hat{a}_3(\sqrt{n/2}) & \hat{a}_5(\sqrt{n/2}) & \cdots & \hat{a}_{2n+1}(\sqrt{n/2}) \end{bmatrix}$$

The first matrix above is exactly the M we are after and we have therefore found its inverse. Since for two square matrices, $MB = I$ implies $BM = I$ therefore M is invertible with inverse B. $\square$

Theorem 3. *All even Schwartz functions $f\colon \mathbb{R} \to \mathbb{R}$ are uniquely determined by their values at $\{|\hat{f}(\sqrt{n})|, f(\sqrt{n/2})\}_{n\geq 0}$.*

Proof. We have proved in Lemma 2 that the square matrix M is invertible. Since this is the hardest case – the case with less sign flips will correspond to an over-constrained tall and skinny matrix and is more likely to have full column rank. This means that two different even Schwartz functions that agree on $\{f(\sqrt{n}), |\hat{f}(\sqrt{n})|\}$ must disagree on at least one point from the set $\{f(\sqrt{1/2}), \ldots, f(\sqrt{n/2})\}$. $\square$

We see that we never had to explicitly write out the analytical expressions for b_n^+ and b_n^-, since these are quite complicated objects and defining them with modular forms, theta group etc. will take us too far, we simply refer the reader to [2]. With the following technical lemmas, our proof is finished.

Lemma 3. *If $f(x)$ is even and Schwartz, $f(\sqrt{2}x)$ is an even Schwartz function as well.*

Proof. Since f is even and Schwartz, time re-scaling won't change its symmetry or differentiability, and for all $x \in \mathbb{R}$, $\forall \alpha, \beta \geq 0$, $|x^\alpha f^{(\beta)}(x)| < \infty$. Define $h(x) := f(\sqrt{2}x)$, we need to show $|x^\alpha h^{(\beta)}(x)| < \infty$ similarly holds. But it is evident since

$$|x^\alpha h^{(\beta)}(x)| = |x^\alpha (2)^{\beta/2} f^{(\beta)}(\sqrt{2}x)|$$

and $|(\sqrt{2}x)^\alpha f^{(\beta)}(\sqrt{2}x)| < \infty$ for all $\alpha, \beta \geq 0$. $\square$

Lemma 4. *It holds that $\hat{a}_0(s) = a_0(x)$.*

Proof. It is known from [2] that the closed form expression for a_0 is

$$a_0(x) = \frac{1}{4}\int_{-1}^{1} \theta^3(z) e^{i\pi z x^2}\, dz \quad \text{for} \quad \theta(z) = \sum_{n\in\mathbb{Z}} \exp(i\pi n^2 z)\,.$$

Using that Fourier transform of Gaussian $\mathcal{F}(e^{-ax^2})(s) = \sqrt{\pi/a}\cdot e^{-\pi^2 s^2/a}$, and $\theta(-1/z) = \sqrt{-iz}\theta(z)$ which can be seen from the Gaussian Fourier pair $\sqrt{-iz}\cdot \hat{e}_z(s) = e_{-1/z}(s)$ for $e_z(x) = e^{i\pi x^2 z}$, we calculate

$$\begin{aligned}
\hat{a}_0(s) &= \frac{1}{4}\int_{-1}^{1} \theta^3(z)\cdot\sqrt{\frac{i}{z}} e^{-\frac{i\pi s^2}{z}}\, dz \\
&= \frac{1}{4}\int_{1}^{-1} \theta^3(-\frac{1}{y})\sqrt{-iy} e^{i\pi s^2 y} y^{-2}\, dy \\
&= \frac{1}{4}\int_{1}^{-1} (\sqrt{-iy})^3\sqrt{-iy}\theta^3(y) e^{i\pi s^2 y} y^{-2}\, dy \\
&= \frac{1}{4}\int_{-1}^{1} \theta^3(y) e^{i\pi s^2 y}\, dy
\end{aligned}$$

where we performed a change of variable. □

Remark. One possible reconstruction strategy (although an inefficient one) is to loop over all the sign configurations, form the candidate function $\tilde{f}$ using the interpolation formula with sampled points on $\{\sqrt{n}\}$ lattice, and check if its value at $\{\sqrt{n/2}\}$ agrees with the provided information on f, for signal with compact support say. Or perhaps some variant of the Gerchberg-Saxton algorithm that alternatively projects onto the two sets of constraints.

4 Is it possible to oversample a bit less?

It is very natural to ask what if instead of extra information from the function value on a finer grid, we have additional information on the Fourier magnitude available. The answer, and the argument, turns out to be almost symmetric to those given in the previous section. The term $\hat{f}(s) - \hat{g}(s)$ at $s = \sqrt{1/2}, \sqrt{3/2}, \ldots, \sqrt{(2n+1)/2}$ can be written as

$$\begin{bmatrix} a_{k_1}(\sqrt{1/2}) & a_{k_2}(\sqrt{1/2}) & \cdots & a_{k_t}(\sqrt{1/2}) \\ a_{k_1}(\sqrt{3/2}) & a_{k_2}(\sqrt{3/2}) & \cdots & a_{k_t}(\sqrt{3/2}) \\ \vdots & \vdots & \cdots & \vdots \\ a_{k_1}(\sqrt{(2n+1)/2}) & a_{k_2}(\sqrt{(2n+1)/2}) & \cdots & a_{k_t}(\sqrt{(2n+1)/2}) \end{bmatrix} \begin{bmatrix} 2\hat{f}(\sqrt{k_1}) \\ \vdots \\ 2\hat{f}(\sqrt{k_t}) \end{bmatrix} =: M_1 b \tag{5}$$

which can take value either $2\hat{f}(\sqrt{s/2})$ or 0 for each element, where $s = 1, 3, 5, \ldots, 2n+1$. The goal is to show that whichever these two values each entry takes, when put into a vector c, either b must be 0 for the equality $M_1 b = c$ to hold, in which case there is no sign ambiguity to resolve in the first place; or the set $k_1, \ldots, k_t$ must be empty, implying that the two signals f and g agree on $\sqrt{n}$ lattice in terms of both function value and Fourier transform, which taken together with Theorem 1 concludes that $f = g$ everywhere.

Theorem 4. *The values at $\{|\hat{f}(\sqrt{n/2})|, f(\sqrt{n})\}_{n\geq 0}$ are not sufficient to completely specify all even Schwartz function $f\colon \mathbb{R}\to\mathbb{R}$.*

Proof. Argument largely similar to Lemma 2 by expanding on

$$a_k(x/\sqrt{2}) = \sum_{n=0}^{\infty} a_n(x) a_k(\sqrt{n/2}) + \sum_{n=0}^{\infty} \hat{a}_n(x)\sqrt{2}\hat{a}_k(\sqrt{2n})$$

shows that M_1^{-1} exists and is full-rank. Therefore the only way for $b = M_1^{-1}c = 0$ is when $c = 0$, which will introduce additional ambiguity in the recovery. In other words, any $c \neq 0$, which can be consistent with the $|\hat{f}(\sqrt{n/2})| = |\hat{g}(\sqrt{n/2})|$ constraint, can lead to b not identically 0, therefore disagreements on the $\hat{f}(\sqrt{n})$ lattice remain. □

5 Discussion

Several further questions come out of this investigation: (1) Do we have some flexibility in the location where we place the nodes? This is already studied to some degree in the context of Fourier interpolation. (2) What is the minimum number of measurements one could afford (with possibly random choices)? From [2] we know $\{f(\sqrt{n})\}_{n\geq 0}, \{\hat{f}(\sqrt{n})\}_{n\geq 1}$ is tight for Fourier interpolation. Our result seems to suggest a sparse spectrum may entail less oversampling.

References

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A Additional details on Example 1

Using trigonometric identity, we have

$$\sin^2(\pi x) = \frac{1-\cos(2\pi x)}{2} = \frac{1 - 1/2\cdot(\exp(-i2\pi x) + \exp(i2\pi x))}{2},$$

its Fourier spectrum is 3-sparse, with coefficient $1/2, -1/4, -1/4$ at $s = 0, 1, -1$.

Moreover, time translation results in extra phase factor when taking Fourier transform, we have

$$\mathcal{F}(g(x+1) - 2g(x) + g(x-1))(s) = (\exp(i2\pi s) + \exp(-i2\pi s) - 2)\hat{g}(s) = -4\sin^2(\pi s)\hat{g}(s).$$

Now since multiplication in time domain corresponds to convolution in frequency domain, and convolution with Dirac delta shifts the function

$$\hat{f}(s) = -4\sin^2(\pi s)(-1/4\hat{g}(s-1) - 1/4\hat{g}(s+1) + 1/2\hat{g}(s)) = \sin^2(\pi s)\cdot(\hat{g}(s+1) - 2\hat{g}(s) + \hat{g}(s-1))$$

as claimed. Since both $f(x)$ and $\hat{f}(s)$ have a multiplicative $\sin^2(\pi x)$ term (resp. $\sin^2(\pi s)$), it is clear that they have double zeros at equispaced integers $x, s \in \mathbb{Z}$ and vanish to first order at these points. In fact, this example also highlights the importance of the non-uniform grid $\{0, \sqrt{1}, \sqrt{2}, \sqrt{3}, \dots\}$.