Fourier Interpolation with Magnitude Only

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It all start with sphere packing

A geometric optimization problem very simple to state:

> *How many equal-sized non-overlapping balls call be packed in $\mathbb{R}^d$? Optimality defined as volume fraction of space occupied by the balls.*

but only settled cases are dimension 1 (trivial), 2 (Thue 1892), 3 (Hales 1998), 8 (Viazovska 2016), 24 (CKMRV 2016) for developments spanning 2 centuries. Upper/lower bound quite far.

Figure 1: Hexagonal density: $\pi/\sqrt{12} \approx 0.91$, Pyramid density: $\pi/\sqrt{18} \approx 0.74$

In high dimension, we have no clue what the optimal configuration look like (not even in the physics sense of knowing) $\leftarrow$ understanding this will lead to progress in error-correcting codes

The magic of 8 and 24, 13 years later

Cohn and Elkies conjectured in 2003 that the (infinite-dimensional) Linear Programming bound can be used to prove sphere packing optimality when $d = 1, 2, 8, 24$: it amounts to solve for a radial Schwartz function $f : \mathbb{R}^d \to \mathbb{R}$

$$\begin{aligned} \max_f \; & f(0) \\ \text{s.t. } & \hat{f}(0) = 1 \\ & f(x) \leq 0 \text{ for } \|x\| \geq 1 \\ & \hat{f}(y) \geq 0 \text{ for all } y \end{aligned}$$

implying density upper bound $2^{-d}\mathrm{Vol}(\mathbb{B}_d) f(0)$.

The magic of 8 and 24, 13 years later

Numerical searches came extremely close, until 2016 Maryna Viazovska came up with a stunningly beautiful construction!

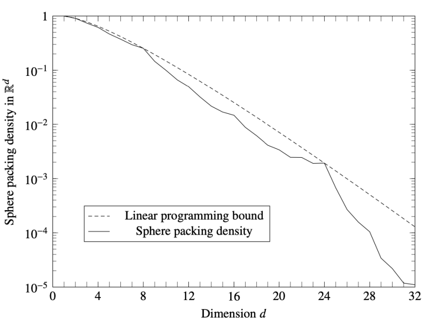
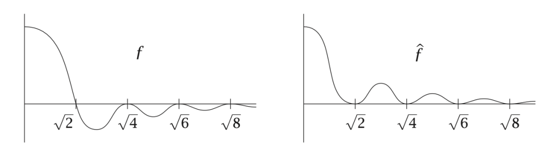


Figure 2: Upper bound from LP vs. best known packing density.

It revealed deep connection between metric geometry, number theory and harmonic analysis – I can't do full justice here but will consider myself successful if I piqued your interest in this result.

Every breakthrough begins with a bold conjecture

For the sphere packing LP bound to be tight, we in fact know where the roots are for the optimal $f, \hat{f}$ pair: $\{\sqrt{2n}\}$ for the E_8 lattice.



Since $f(x) \le 0$ for $\|x\| \ge \sqrt{2}$ and $\hat{f}(y) \ge 0$ most of the roots are double roots (derivative$=0$). The only function that can work actually works:

These information suffice for reconstruction [CKMRV '19]

Every radial Schwartz function in $\mathbb{R}^8, \mathbb{R}^{24}$ is uniquely determined by values at $f(\sqrt{2n}), f'(\sqrt{2n}), \hat{f}(\sqrt{2n}), \hat{f}'(\sqrt{2n})$ for $n \ge 1$ in $\mathbb{R}^8$, $n \ge 2$ in $\mathbb{R}^{24}$.

Why hard? Uncertainty principle. This implies the magic function f is essentially unique (up to scaling).

Fourier interpolation and uniqueness pair

When is f uniquely determined by the restriction $f|_A, \hat{f}|_B$? Some examples of Fourier interpolation you already know:

1. One of set A, B is everything
2. Shannon interpolation: $A = \mathbb{Z}, B = \mathbb{R} \backslash [-1/2, 1/2]$ when $\mathrm{supp}(\hat{f}) \subseteq [-1/2, 1/2]$ using sinc basis

Interpolation formula for bandlimited function can solve 1D case

Recall (1) $\hat{f}(0) = 1$; (2) $f(x) \leq 0$ for $\|x\| \geq 1$; (3) $\hat{f}(y) \geq 0$ for all y

$$\text{Take } f(x) = (1 - |x|)\mathbb{1}_{[-1,1]} = \mathbb{1}_{[-1/2,1/2]} * \mathbb{1}_{[-1/2,1/2]}$$

$$\text{therefore } \hat{f}(y) = \left(\frac{\sin(\pi y)}{\pi y}\right)^2 = \mathrm{sinc}(y)^2$$

So $f(0) = 1 \Rightarrow$ density $= 1$. We are implicitly using the set $A = \mathbb{R} \backslash [-1, 1]$, $B = \mathbb{Z}$.

In most known cases, one set is big (continous), the other small (discrete).

Fourier interpolation and uniqueness pair

The set $A = B = \mathbb{Z}$, together with their derivatives, are not enough for 1D Schwartz functions. But ...

First order interpolation [RV '17]

There exist Schwartz function $a_n : \mathbb{R} \to \mathbb{R}$ such that for every Schwartz function $f : \mathbb{R} \to \mathbb{R}$ and $x \in \mathbb{R}$,

$$f(x) = \sum_{n\in\mathbb{Z}} f(\sqrt{n}) a_n(x) + \sum_{n\in\mathbb{Z}} \hat{f}(\sqrt{n}) \hat{a}_n(x)\,.$$

Moreover, the set $A = \{\sqrt{n}\}, B = \{\sqrt{n}\}\backslash 0$ is tight and $\{f(\sqrt{n}), \hat{f}(\sqrt{n})\}$ completely characterize the information in f (i.e., isomorphism).

[RS, KNS '19-'20] show one can perturb the (non-uniform) nodes a little bit. Basis $a_n(x), \hat{a}_n(x)$ are explicit but quite complicated functions built from modular forms.

Numerical evidence suggests that interpolation formula exist for even more dimensions (if not all) at $\{\sqrt{2n}\}$ for radial functions, although no known consequence for sphere packing.

Variations on a theme (what this result is about)

Phase retrieval problem, slightly generalized

Given some knowledge on $f(x)$ and $|\hat{f}(y)|$, uniquely recover f. Various results exist when one of A or B is continuous and the other discrete.

Focusing on 1D even case, can take $A = \{\sqrt{n/2}\}, B = \{\sqrt{n}\}$.

1. The analytical expressions of $\{a_n\}, \{\hat{a}_n\}$ allow us to characterize the ambiguity on $f(x)$ induced by only knowing $|\hat{f}(\sqrt{n})|, f(\sqrt{n})$
2. The additional information will have to be "incoherent" to this set of functions from (1) $\leftarrow$ the interpolation formula can help (certain operator has to be invertible)

We lose 1 bit of info from $|\hat{f}(\sqrt{n})|$, but make up for the loss from twice oversampling on $f(\sqrt{n/2})$. Amount of oversampling should scale with sparsity of the spectrum.

Is mathematics invented or discovered?

More broadly,

- LP bound not tight generally, SDP bound conjectured to prove optimality in dimension 4! Another magic function? [CLS '22]
- $d = 2$ is open for LP. The root locations are (provably) not enough to uniquely determine $f \to$ need more knowledge to constrain f.
- Higher d beyond radial function?

The mysterious connection b/w Fourier analysis and sphere packing whisper hints that otherwise would be very hard to even conjecture.

Figure 3: One of the interpolation basis $a_n(x)$.

Thanks!