Phonons and Electron-Phonon Couplings

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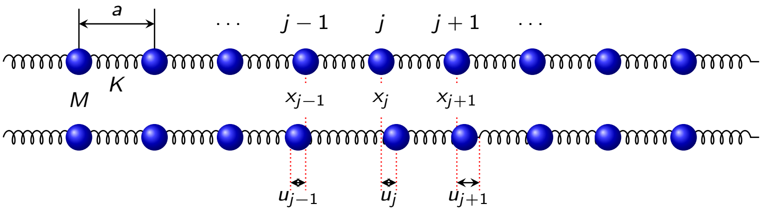
A 1D chain of N equally spaced atoms at $R_j(t) = x_j + u_j(t)$



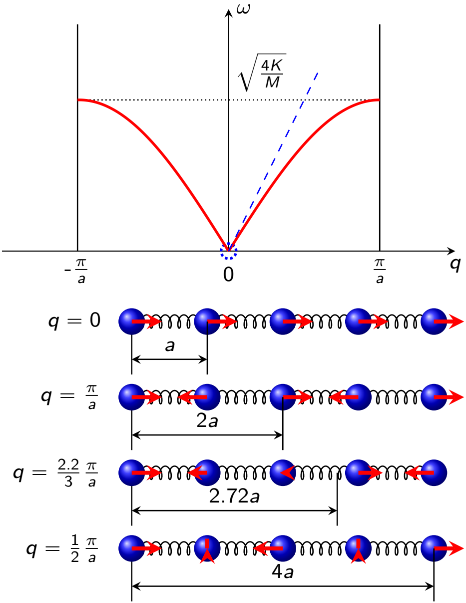
The Newton's Equation

$$M\frac{\mathrm{d}^2 u_j}{\mathrm{d}t^2} = K(u_{j+1} + u_{j-1} - 2u_j) \qquad j = 1, \dots, N$$

Assume the solution has the form $u_j(t) = \frac{A_q}{\sqrt{M}} e^{i(qx_j - \omega t)}$, then [1]

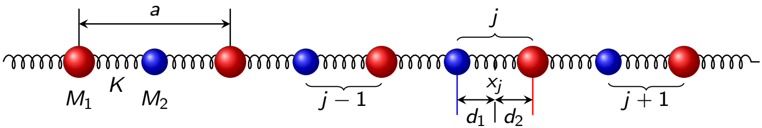
$$\begin{aligned} \omega^2 &= \frac{K}{M}(2 - e^{iqa} - e^{-iqa}) \\ &= \frac{2K}{M}(1 - \cos qa) \\ \Rightarrow \quad \omega &= \sqrt{\frac{4K}{M}}\left|\sin\frac{qa}{2}\right| \end{aligned}$$

[1] $u_j(t)$ here is complex. In practice, take the real part, i.e. $\mathrm{Re}[u_j(t)]$.



ω
$\sqrt{\frac{4K}{M}}$
$-\frac{\pi}{a}$
0
$\frac{\pi}{a}$
q
$q = 0$
a
$q = \frac{\pi}{a}$
2a
$q = \frac{2.2}{3}\frac{\pi}{a}$
2.72a
$q = \frac{1}{2}\frac{\pi}{a}$
4a

A 1D chain with 2 atoms in each unit: $R_s^j(t) = x_j + d_s + u_s^j(t); \quad s = 1, 2$



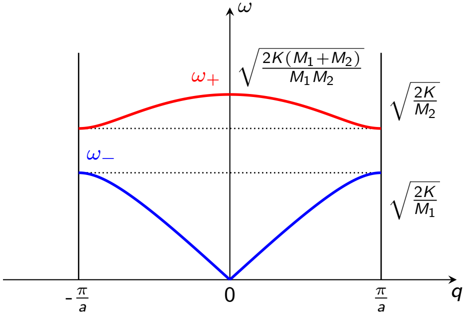
The Newton's Equation

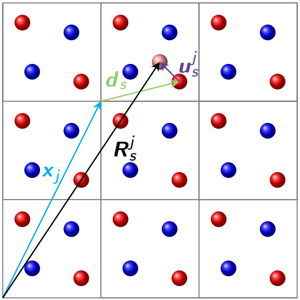
$$\begin{aligned} M_1 \frac{\mathrm{d}^2 u_1^j}{\mathrm{d}t^2} &= K(u_2^j + u_2^{j-1} - 2u_1^j) \\ M_2 \frac{\mathrm{d}^2 u_2^j}{\mathrm{d}t^2} &= K(u_1^j + u_1^{j+1} - 2u_2^j) \end{aligned} \quad \Longrightarrow \quad \begin{cases} u_j^1(t) = \dfrac{A_q}{\sqrt{M_1}} e^{i(qx_j - \omega t)} \\ u_j^2(t) = \dfrac{B_q}{\sqrt{M_2}} e^{i(qx_j - \omega t)} \end{cases}$$

We then have

$$\begin{pmatrix} \frac{2K}{M_1} & \frac{-K}{\sqrt{M_1 M_2}}(1 + e^{-iqa}) \\ \frac{-K}{\sqrt{M_1 M_2}}(1 + e^{iqa}) & \frac{2K}{M_2} \end{pmatrix} \begin{pmatrix} A_q \\ B_q \end{pmatrix} = \omega^2 \begin{pmatrix} A_q \\ B_q \end{pmatrix}$$

$$\Longrightarrow \quad \omega_\pm^2 = \frac{K}{M_1 M_2} \left((M_1 + M_2) \pm \sqrt{M_1^2 + M_2^2 + 2M_1 M_2 \cos qa} \right)$$


ω
ω_+
$\sqrt{\frac{2K(M_1+M_2)}{M_1M_2}}$
$\sqrt{\frac{2K}{M_2}}$
ω_-
$\sqrt{\frac{2K}{M_1}}$
$-\frac{\pi}{a}$
0
$\frac{\pi}{a}$
q



- $\boldsymbol{x}_j$: the position of unit cell j
- $\boldsymbol{d}_s$: the equilibrum position of the atom s in the cell
- $\boldsymbol{u}_s^j$: displacement from the equilibrum positon for the atom s in the cell j
- $\boldsymbol{R}_s^j$: the position of the atom s in the cell j

$$
\begin{aligned}
\boldsymbol{R}_s^j(t) &= \boldsymbol{x}_j + \boldsymbol{d}_s + \boldsymbol{u}_s^j(t) \\
&= \boldsymbol{r}_s^j + \boldsymbol{u}_s^j(t) \\
R_{s\alpha}^j(t) &= r_{s\alpha}^j + u_{s\alpha}^j(t) \quad (\alpha = x, y, z)
\end{aligned}
$$

The total energy can be written as

$$
E_{\text{tot}}\left(\{\boldsymbol{R}_s^j(t)\}\right) = E_{\text{tot}}^0\left(\{\boldsymbol{r}_s^j\}\right) + \sum_{js\alpha} \frac{\partial E_{\text{tot}}^0}{\partial u_{s\alpha}^j} u_{s\alpha}^j + \frac{1}{2} \sum_{\substack{js\alpha \\ kt\beta}} \frac{\partial^2 E_{\text{tot}}^0}{\partial u_{s\alpha}^j \partial u_{t\beta}^k} u_{s\alpha}^j u_{t\beta}^k + \ldots
$$

- The expression is exact if we take all the orders in the expansion.
- All the derivatives are taken at the equilibrium positions $\{\boldsymbol{r}_s^j\}$, i.e. $\frac{\partial E_{\text{tot}}^0}{\partial u_{s\alpha}^j} = 0$.
- Harmonic approximation: truncated at *second* order.

Within the harmonic approximation, the Newton's equation for the atom s in cell j

$$M_s \frac{\mathrm{d}^2 u_{s\alpha}^j(t)}{\mathrm{d}t^2} = -\frac{\partial E_{\mathrm{tot}}}{\partial u_{s\alpha}^j} = -\sum_{kt\beta} \frac{\partial^2 E_{\mathrm{tot}}^0}{\partial u_{s\alpha}^j \partial u_{t\beta}^k} u_{t\beta}^k = -\sum_{kt\beta} C_{s\alpha,t\beta}(j,k)\, u_{t\beta}^k \tag{1}$$

The ansatz of the solution

$$u_{s\alpha}^j(t) = \frac{\eta_\alpha^s(\boldsymbol{q})}{\sqrt{M_s}}\, e^{i\boldsymbol{q}\boldsymbol{x}_j}\, e^{-i\omega t} \tag{2}$$

Substitute Eq. 2 into Eq. 1

$$\omega^2(\boldsymbol{q})\, \eta_\alpha^s = \sum_{t\beta}\left[\sum_k \frac{1}{\sqrt{M_s M_t}} \frac{\partial^2 E_{\mathrm{tot}}^0}{\partial u_{s\alpha}^j \partial u_{t\beta}^k} e^{i\boldsymbol{q}(\boldsymbol{x}_k - \boldsymbol{x}_j)}\right] \eta_\beta^t = \sum_{t\beta} D_{s\alpha,t\beta}(\boldsymbol{q})\, \eta_\beta^t$$

In matrix form

$$\underbrace{\begin{pmatrix} \ddots & & \\ & D_{s\alpha,t\beta}(\boldsymbol{q}) & \\ & & \ddots \end{pmatrix}}_{3N_a \times 3N_a} \begin{pmatrix} \vdots \\ \eta_\beta^t(\boldsymbol{q}) \\ \vdots \end{pmatrix} = \omega^2(\boldsymbol{q}) \underbrace{\begin{pmatrix} \vdots \\ \eta_\beta^t(\boldsymbol{q}) \\ \vdots \end{pmatrix}}_{3N_a}$$

where N_a is the number of atoms in a cell.

The Interatomic Force Constants (IFC)

$$\frac{\partial^2 E^0_{\text{tot}}}{\partial u^j_{s\alpha} \partial u^k_{t\beta}} = C_{s\alpha,t\beta}(j,k)$$

- Symmetric

$$\frac{\partial^2 E^0_{\text{tot}}}{\partial u^j_{s\alpha} \partial u^k_{t\beta}} = \frac{\partial^2 E^0_{\text{tot}}}{\partial u^k_{t\beta} \partial u^j_{s\alpha}} \tag{3}$$

- Translation invariance, depend on the difference between j and k

$$\frac{\partial^2 E^0_{\text{tot}}}{\partial u^j_{s\alpha} \partial u^k_{t\beta}} = \frac{\partial^2 E^0_{\text{tot}}}{\partial u^0_{s\alpha} \partial u^{(k-j)}_{t\beta}} \tag{4}$$

- Acoustic Sum Rule (ASR): if we displace the whole solid by an arbitrary uniform displacement, the forces acting on the atoms must be zero.

$$F^j_{s\alpha} = -\sum_{\beta} \left[\sum_{kt} \frac{\partial^2 E^0_{\text{tot}}}{\partial u^j_{s\alpha} \partial u^k_{t\beta}} \right] \delta_\beta = 0 \qquad \Rightarrow \qquad \sum_{kt} \frac{\partial^2 E^0_{\text{tot}}}{\partial u^j_{s\alpha} \partial u^k_{t\beta}} = 0 \tag{5}$$

The Dynamical Matrix

$$D_{s\alpha,t\beta}(\boldsymbol{q}) = \frac{1}{\sqrt{M_s M_t}} \sum_k C_{s\alpha,t\beta}(j,k) e^{i\boldsymbol{q}(\boldsymbol{x}_k - \boldsymbol{x}_j)}$$
$$= \frac{1}{\sqrt{M_s M_t}} \sum_l C_{s\alpha,t\beta}(0,l) e^{i\boldsymbol{q}\boldsymbol{x}_l}$$

- Dynamical matrix is Hermitian

$$D_{s\alpha,t\beta}(\boldsymbol{q}) = D^*_{t\beta,s\alpha}(\boldsymbol{q})$$

Proof

$$D_{s\alpha,t\beta}(\boldsymbol{q}) = \frac{1}{\sqrt{M_s M_t}} \sum_l C_{s\alpha,t\beta}(0,l) e^{i\boldsymbol{q}\boldsymbol{x}_l}$$
$$= \frac{1}{\sqrt{M_s M_t}} \sum_l C_{s\alpha,t\beta}(-l,0) e^{i\boldsymbol{q}\boldsymbol{x}_l}$$
$$= \frac{1}{\sqrt{M_s M_t}} \sum_l C_{t\beta,s\alpha}(0,-l) e^{i\boldsymbol{q}\boldsymbol{x}_l}$$

$$D^*_{s\alpha,t\beta}(\boldsymbol{q}) = \frac{1}{\sqrt{M_s M_t}} \sum_l C_{t\beta,s\alpha}(0,-l) e^{-i\boldsymbol{q}\boldsymbol{x}_l}$$
$$= \frac{1}{\sqrt{M_s M_t}} \sum_l C_{t\beta,s\alpha}(0,l) e^{i\boldsymbol{q}\boldsymbol{x}_l}$$
$$= D_{t\beta,s\alpha}(\boldsymbol{q})$$

Thank you!