1. Axiomatic quantum physics

This article provides a brief introduction to **axiomatic quantum physics**, which is founded on three core axioms.

First, we recall that a physical system is described by its state—a mathematical entity—and its time evolution, which is typically governed by a differential equation. In classical physics, the state is defined by generalized coordinates $q = \left(q_1, ..., q_f\right)$ and their corresponding generalized momenta $p = \left(p_1, ..., p_f\right)$. Then the time evolution is determined by **Hamilton function** and **Hamilton's canonical equation**.
We give the definition of the state of a quantum system as following axiom.

Axiom 1.1: The **state** of a quantum system is represented by a unit vector (referred to as the **state vector**) in a complex Hilbert space $\mathcal{H}$. For any state vector $\psi \in \mathcal{H}$ and any complex number α with absolute value 1, the vectors ψ and $\alpha\psi$ represent the same physical state.

Axiom 1.2: Any observable physical quantity is represented by a self-adjoint operator on $\mathcal{H}$.

Remark: Self-adjoint operators are used because their eigenvalues, which correspond to the possible measurement outcomes of a physical quantity, are always real numbers.
An observable quantity T of a state $\psi \in \mathcal{H}$ in quantum physics is stochastic. Then we define a random variable T_ψ. Here we add an axiom for T_ψ on a probability space$(\Omega, \mathcal{B}(\mathbb{R}), P)$.

Axiom 1.3: Let $B \in \mathcal{B}(\mathbb{R})$, then $P\left(T_\psi \in B\right) = \|E_{T(B)}\psi\|^2$, where E_T is the spectral measure of T.

Then we obtain the following useful thoerem.

Theorem 1.1: The expectation value of the random variable T_ψ is given by $E\left[T_\psi\right] = \langle\psi, T\psi\rangle$.

Proof: Let P_{T_ψ} the probability distribution of T_ψ.

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\begin{aligned}
E\left[T_\psi\right] = \int_{\mathbb{R}} \lambda d P_{T_\psi}(\lambda) = \int \lambda d \|E_{T(B)}\psi\|^2 \\
&= \int_{\mathbb{R}} \lambda d \langle E_{T(B)}\psi, E_{T(B)}\psi\rangle \\
&= \int_{\mathbb{R}} \lambda d \langle \psi, E_{T(B)}^2\psi\rangle \because E_{T(B)} \text{ is self adjoint operator} \\
&= \int_{\mathbb{R}} d \langle \psi, \lambda E_{T(B)}\psi\rangle \because E_{T(B)} \text{ is projection oprerator} \\
&= \langle\psi, T\psi\rangle \because E_{T(B)} \text{ is spectral measure}
\end{aligned}
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