

1. Axiomatic quantum physics

This article provides a brief introduction to **axiomatic quantum physics**, which is founded on three core axioms.

First, we recall that a physical system is described by its state—a mathematical entity—and its time evolution, which is typically governed by a differential equation. In classical physics, the state is defined by generalized coordinates $q = (q_1, \dots, q_f)$ and their corresponding generalized momenta $p = (p_1, \dots, p_f)$. Then the time evolution is determined by **Hamilton function** and **Hamilton's canonical equation**.

We give the definition of the state of a quantum system as following axiom.

Axiom 1.1: The **state** of a quantum system is represented by a unit vector (referred to as the **state vector**) in a complex Hilbert space \mathcal{H} . For any state vector $\psi \in \mathcal{H}$ and any non-zero complex number α , the vectors ψ and $\alpha\psi$ represent the same physical state.

Axiom 1.2: Any observable physical quantity is represented by a self-adjoint operator on \mathcal{H} .

Remark: Self-adjoint operators are used because their eigenvalues, which correspond to the possible measurement outcomes of a physical quantity, are always real numbers.

An observable quantity T of a state $\psi \in \mathcal{H}$ in quantum physics is stochastic. Then we define a random variable T_ψ . Here we add an axiom for T_ψ on a probability space $(\Omega, \mathcal{B}(\mathbb{R}), P)$.

Axiom 1.3: Let $B \in \mathcal{B}(\mathbb{R})$, then $P(T_\psi \in B) = \|E_{T(B)}\psi\|^2$, where E_T is the spectral measure of T .

Then we obtain the following useful theorem.

Theorem 1.1: The expectation value of the random variable T_ψ is given by $E[T_\psi] = \langle \psi, T\psi \rangle$.

Proof: Let P_{T_ψ} the probability distribution of T_ψ .

$$\begin{aligned} E[T_\psi] &= \int_{\mathbb{R}} \lambda dP_{T_\psi}(\lambda) = \int \lambda d\|E_{T(B)}\psi\|^2 \\ &= \int_{\mathbb{R}} \lambda d\langle E_{T(B)}\psi, E_{T(B)}\psi \rangle \\ &= \int_{\mathbb{R}} \lambda d\langle \psi, E_{T(B)}^2\psi \rangle \because E_{T(B)} \text{ is self adjoint operator} \\ &= \int_{\mathbb{R}} \lambda d\langle \psi, \lambda E_{T(B)}\psi \rangle \because E_{T(B)} \text{ is projection operator} \\ &= \langle \psi, T\psi \rangle \because E_{T(B)} \text{ is spectral measure} \end{aligned}$$

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