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My Research Report

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This article provides a brief introduction to axiomatic quantum physics.

1. Introduction

First, we recall that a physical system is described by its state—a mathematical entity—and its time evolution, which is typically governed by a differential equation. In classical physics, the state is defined by generalized coordinates $q = (q_1, ..., q_f)$ and their corresponding generalized momenta $p = (p_1, ..., p_f)$. Then the time evolution is determined by **Hamilton function** and *Hamilton's canonical equation*. We give the definition of the state of a quantum system as following axiom.

> **Axiom 1.1.**
>
> The **state** of a quantum system is represented by a unit vector (referred to as the **state vector**) in a complex Hilbert space $\mathcal{H}$. For any state vector $\psi \in \mathcal{H}$ and any complex number α with absolute value 1, the vectors ψ and $\alpha\psi$ represent the same physical state. Moreover, Any observable physical quantity is represented by a self-adjoint operator on $\mathcal{H}$.

> **Remark**
>
> Self-adjoint operators are used because their eigenvalues, which correspond to the possible measurement outcomes of a physical quantity, are always real numbers.

An observable quantity T of a state $\psi \in \mathcal{H}$ in quantum physics is stochastic. Then we define a random variable T_ψ. Here we add an axiom for T_ψ on a probability space $(\Omega, \mathcal{B}(\mathbb{R}), P)$.

> **Axiom 1.2.**
>
> Let $B \in \mathcal{B}(\mathbb{R})$, then $P(T_\psi \in B) = \|E_{T(B)}\psi\|^2$, where E_T is the spectral measure of T.

> **Theorem 1.3. (From [Som22])**
>
> The expectation value of the random variable T_ψ is given by $E[T_\psi] = \langle \psi, T\psi \rangle$.

Proof. Let P_{T_ψ} the probability distribution of T_ψ.

$$
\begin{aligned}
E[T_\psi] = \int_{\mathbb{R}} \lambda dP_{T_\psi}(\lambda) &= \int \lambda d\|E_{T(B)}\psi\|^2 = \int_{\mathbb{R}} \lambda d\langle E_{T(B)}\psi, E_{T(B)}\psi \rangle \\
&= \int_{\mathbb{R}} \lambda d\langle \psi, E^2_{T(B)}\psi \rangle \because E_{T(B)} \text{ is self adjoint operator} \\
&= \int_{\mathbb{R}} d\langle \psi, \lambda E_{T(B)}\psi \rangle \because E_{T(B)} \text{ is projection oprerator} \\
&= \langle \psi, T\psi \rangle \because E_{T(B)} \text{ is spectral measure}
\end{aligned}
\tag{1}
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$\square$

Bibliography

[Som22] Someone, "A article," *arXiv?*, 2022.