Greedy Algorithms: Main Ideas

Michael Levin

Higher School of Economics

Algorithmic Toolbox
Data Structures and Algorithms

Outline

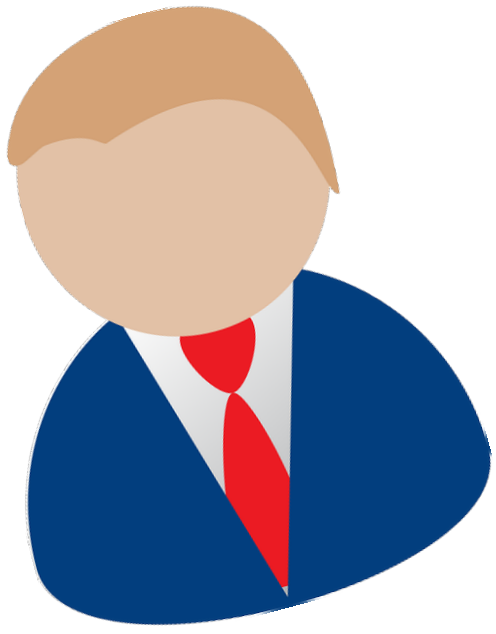
1. Largest Number
2. Car Fueling
3. Implementation and Analysis
4. Main Ingredients

Learning objectives

- Come up with a greedy algorithm yourself

Job Interview

Job Interview



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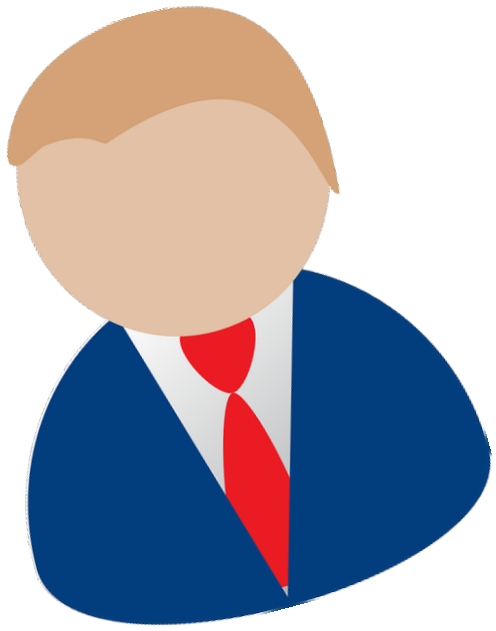

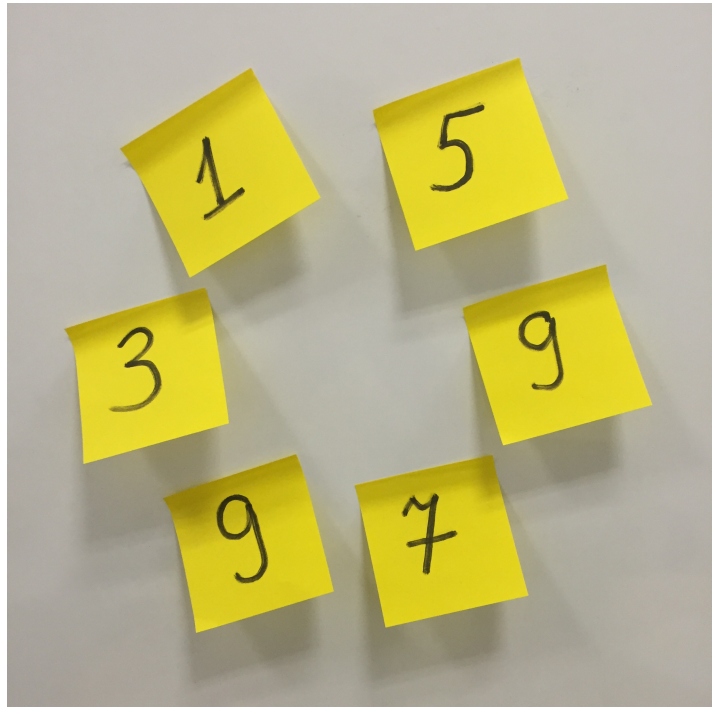

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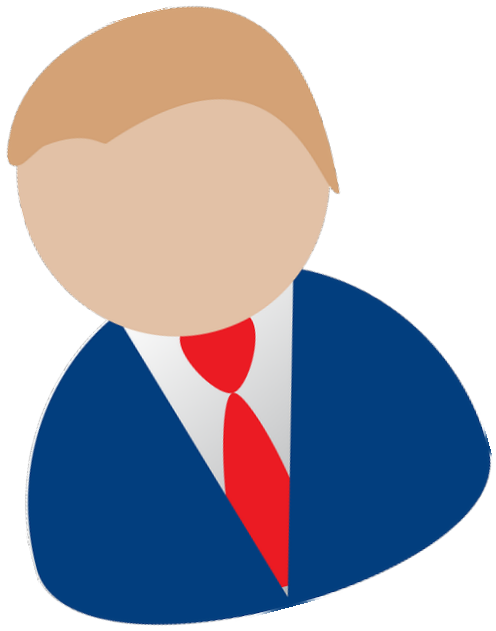
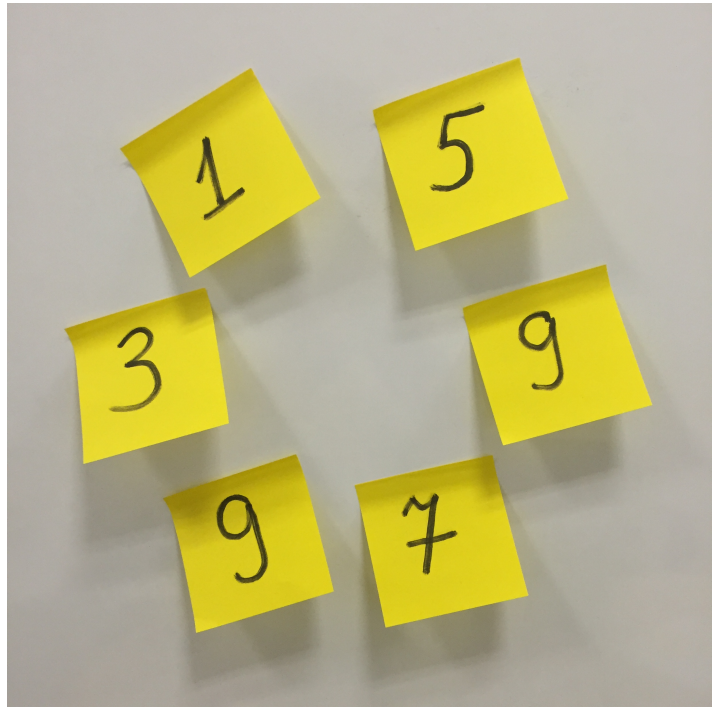
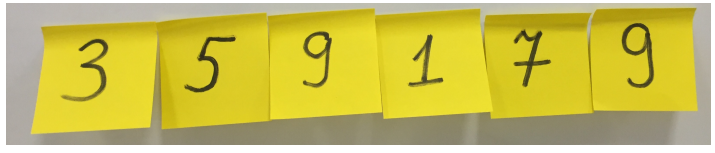






Largest Number

Toy problem

What is the largest number that consists of digits 3, 9, 5, 9, 7, 1? Use all the digits.

Largest Number

Toy problem

What is the largest number that consists of digits 3, 9, 5, 9, 7, 1? Use all the digits.

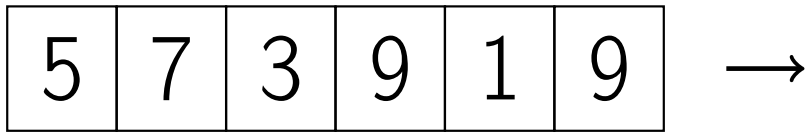
Examples

$$359179, 537991, 913579, \ldots$$

Correct answer

997531

Greedy Strategy



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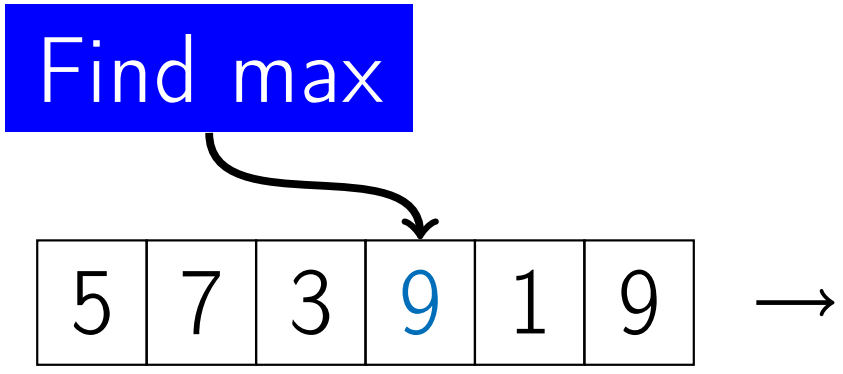
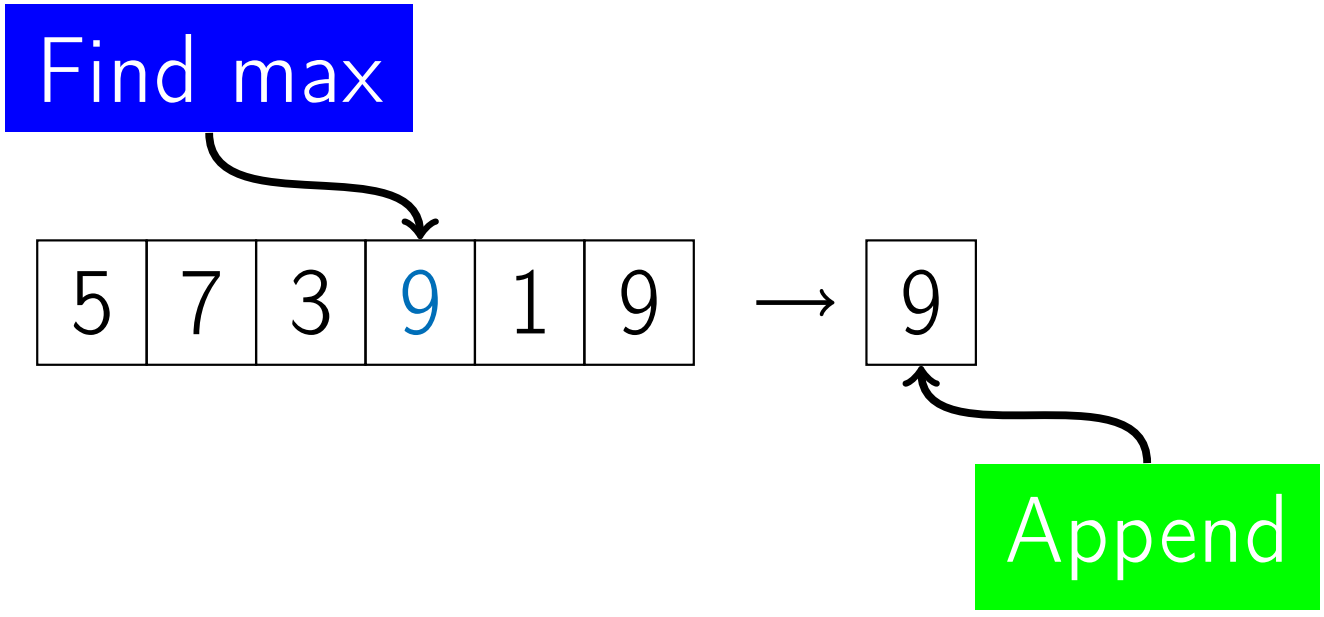


- Find max digit

Greedy Strategy



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- Append it to the number

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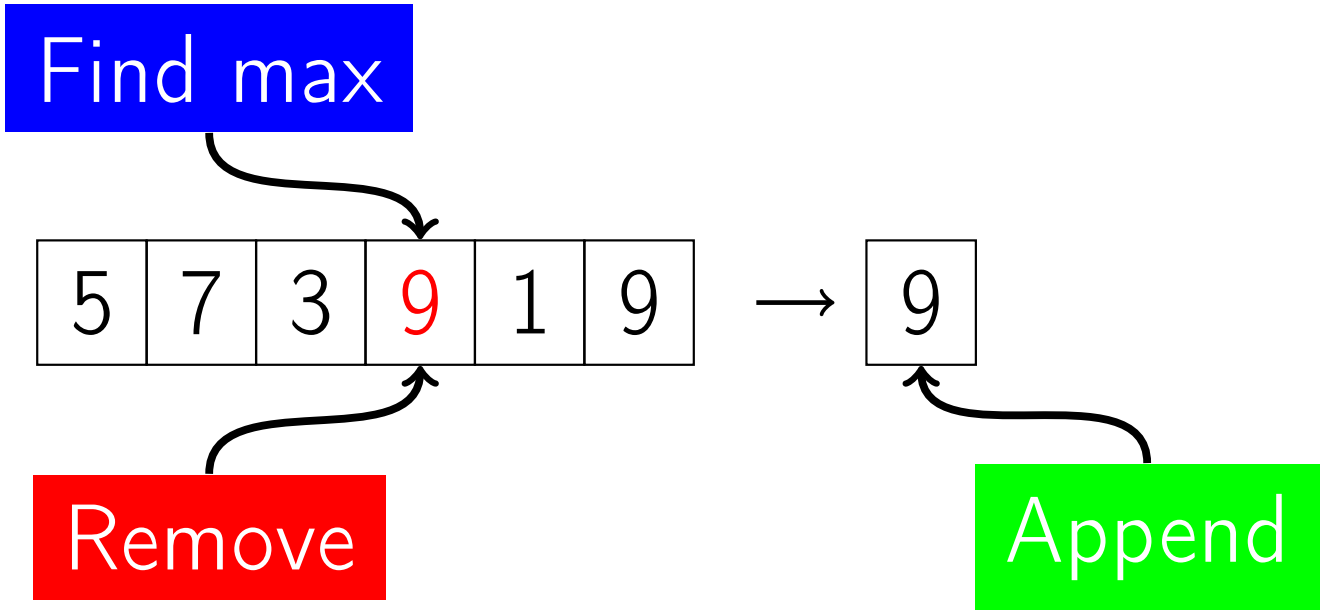

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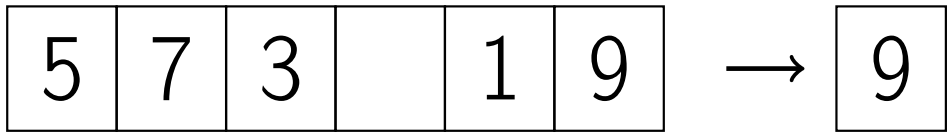

Remove

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Greedy Strategy

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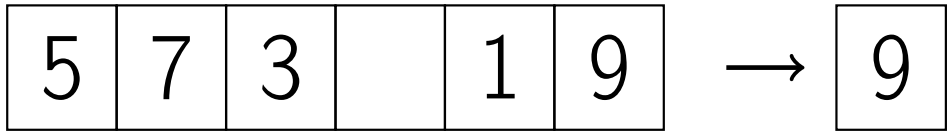

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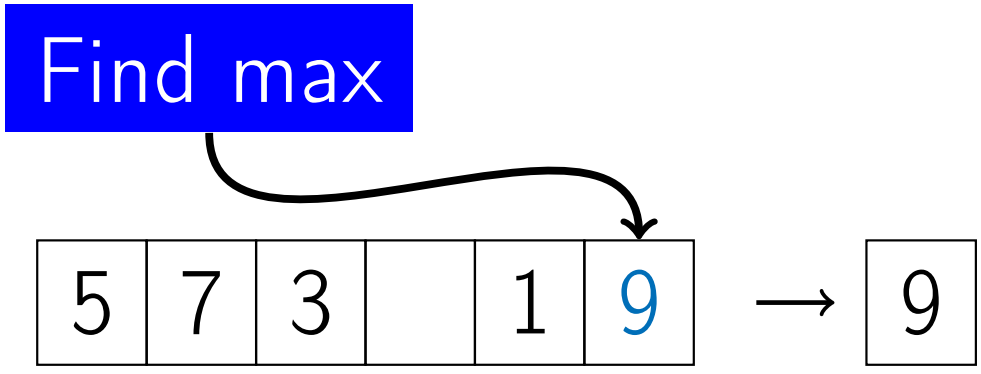

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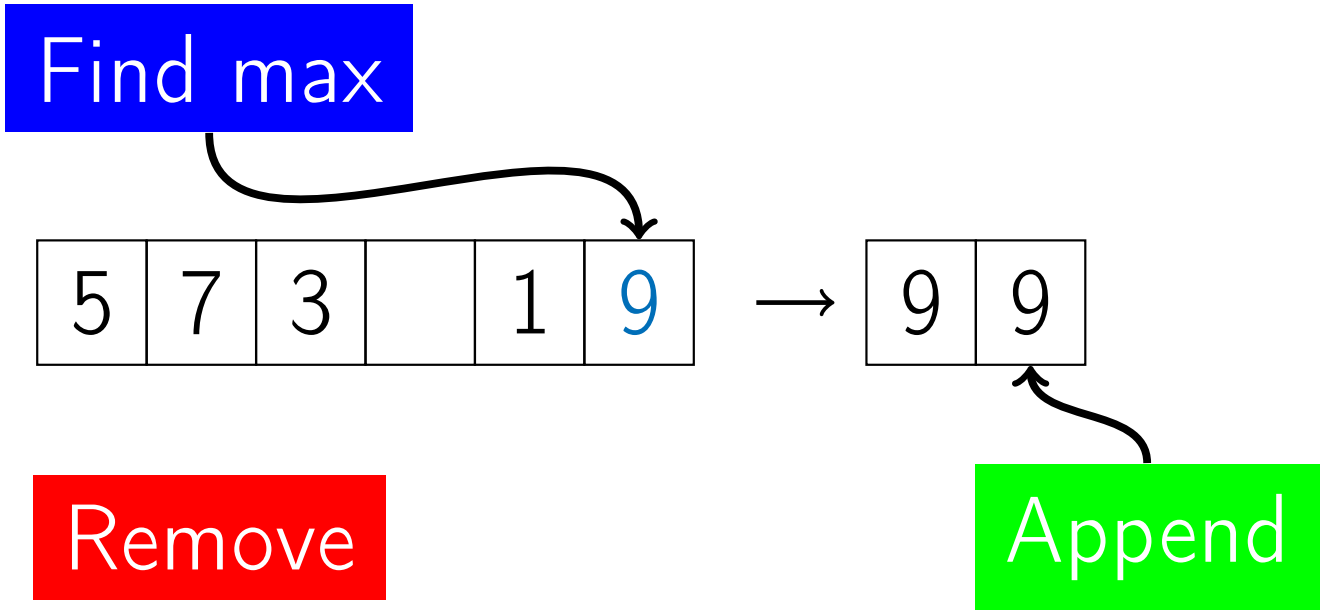

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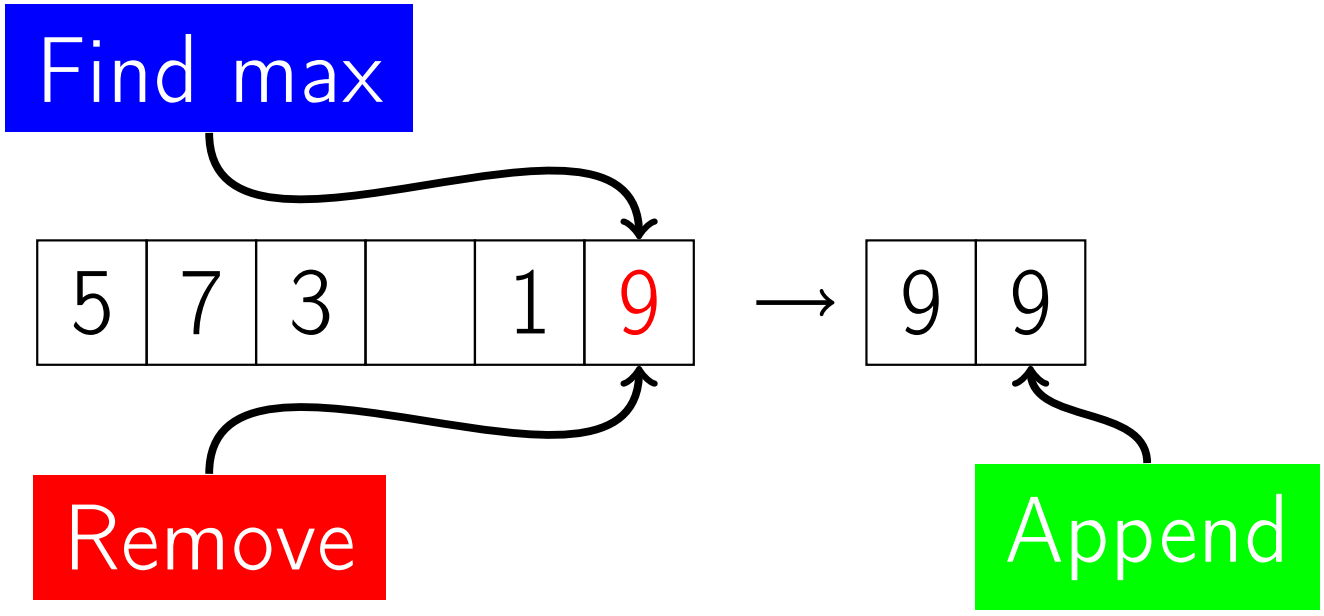

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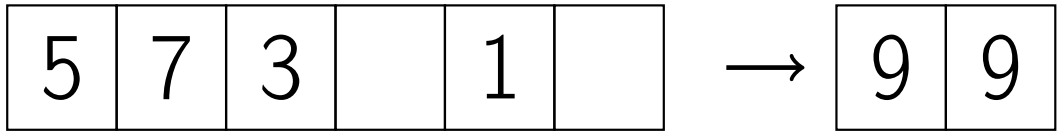

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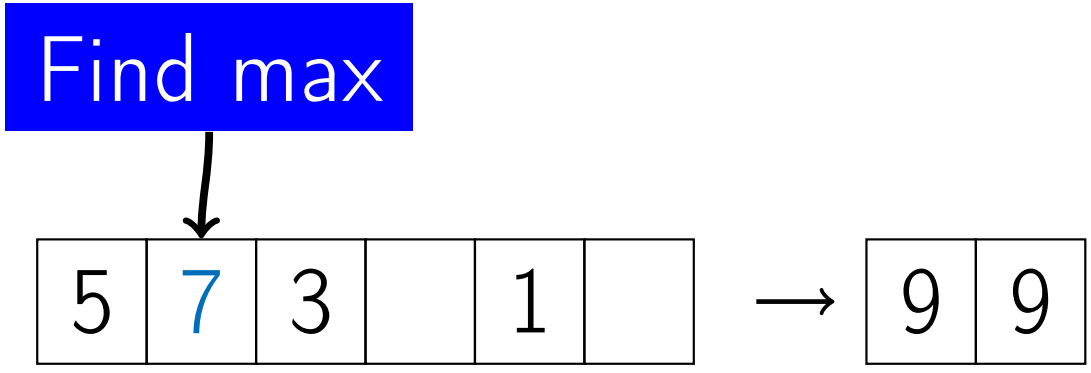

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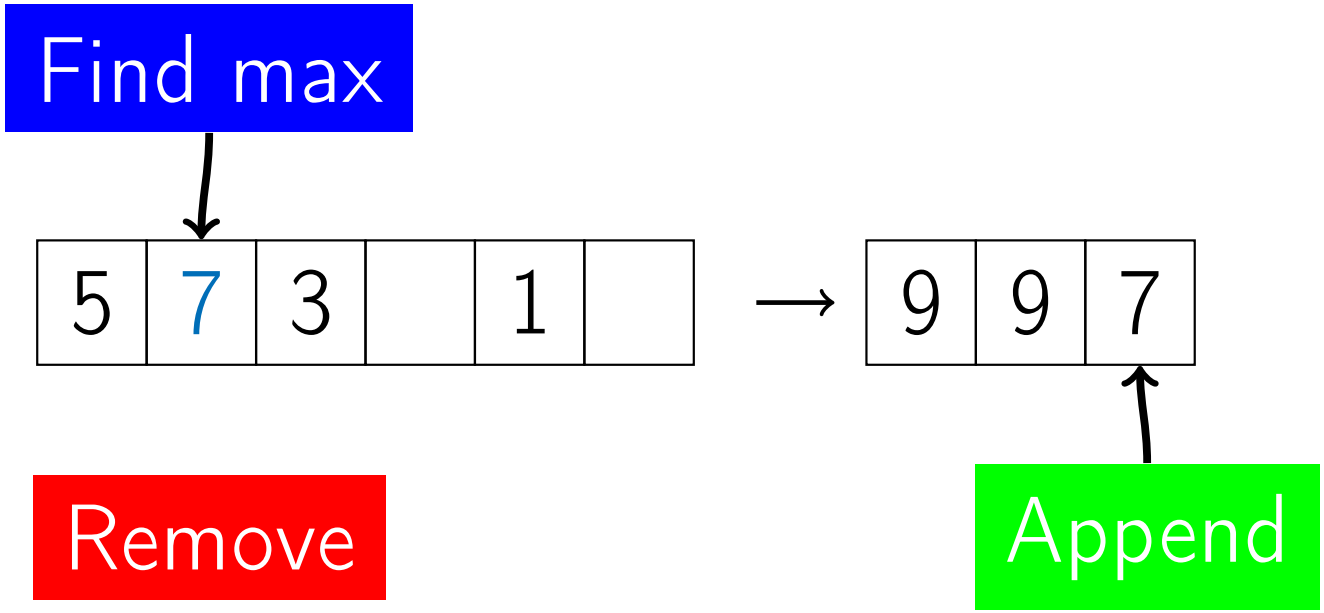

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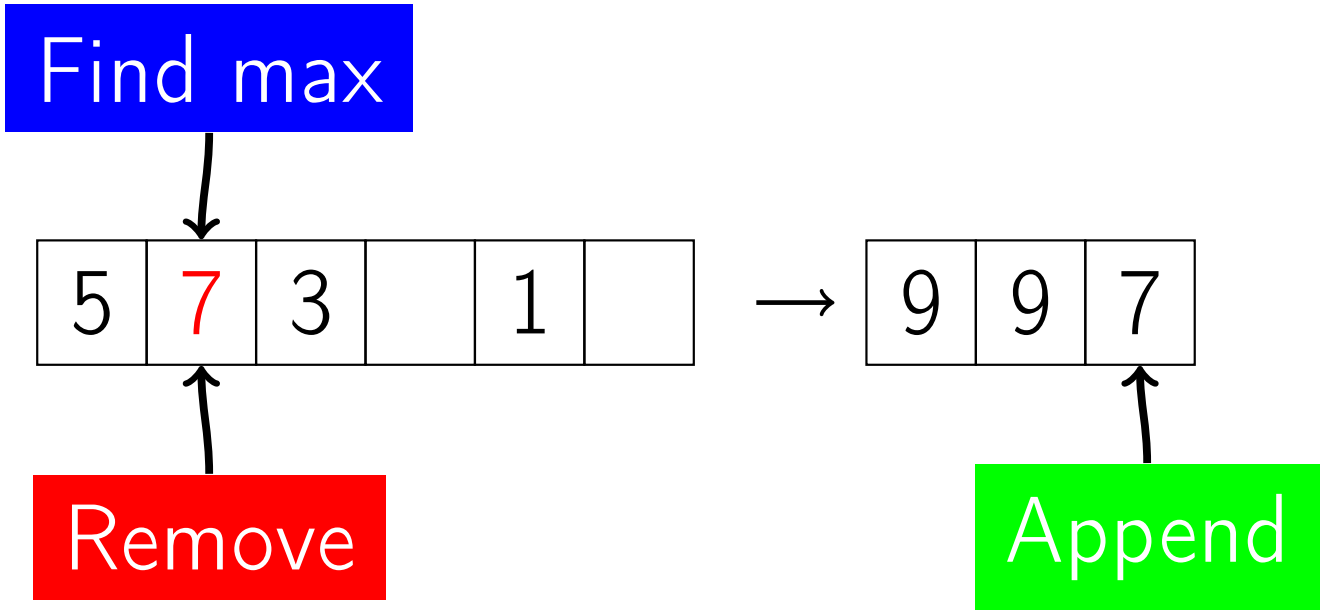

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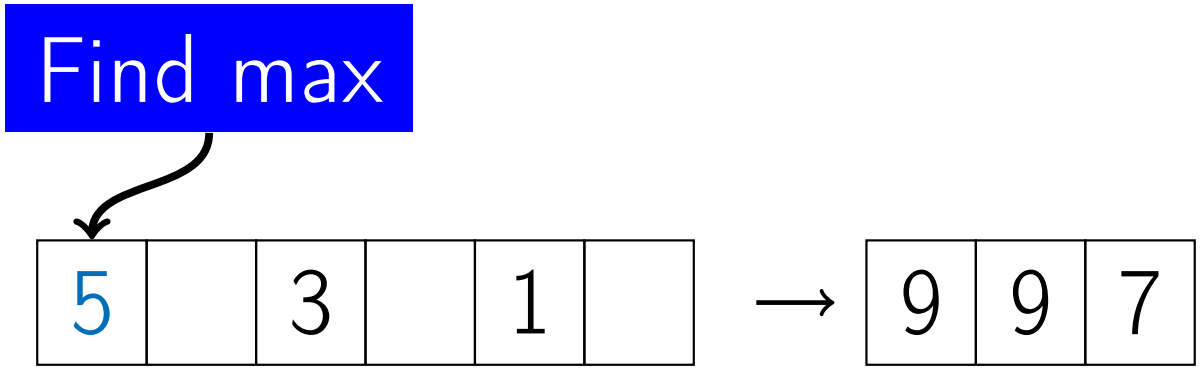

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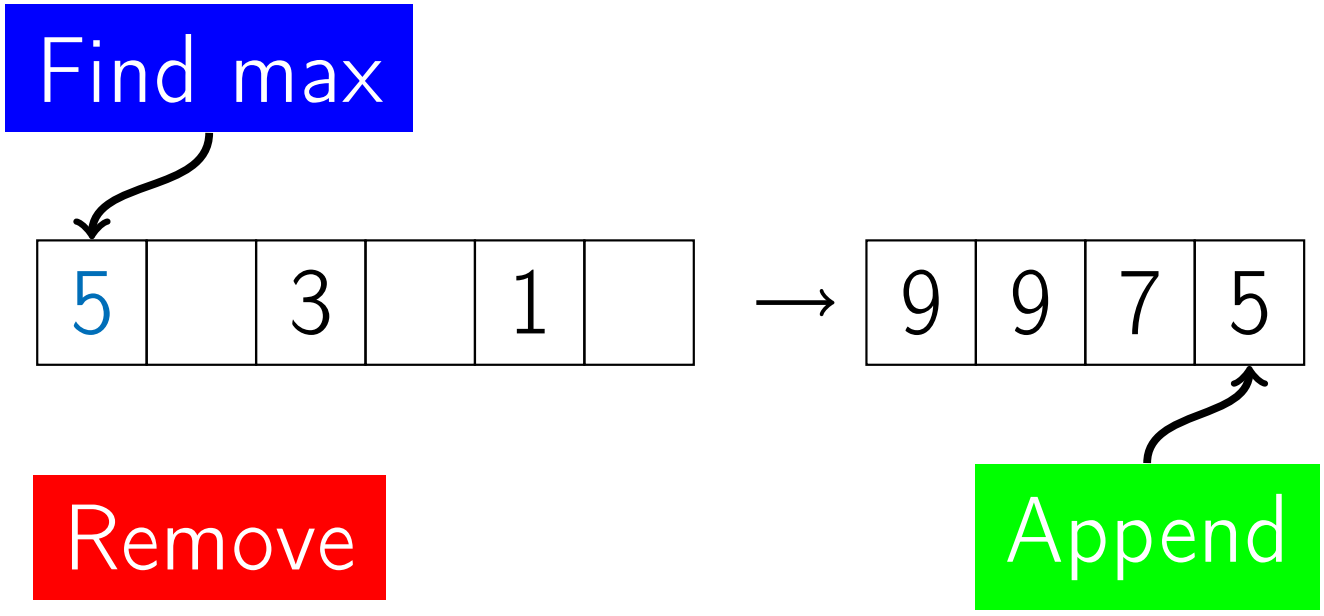

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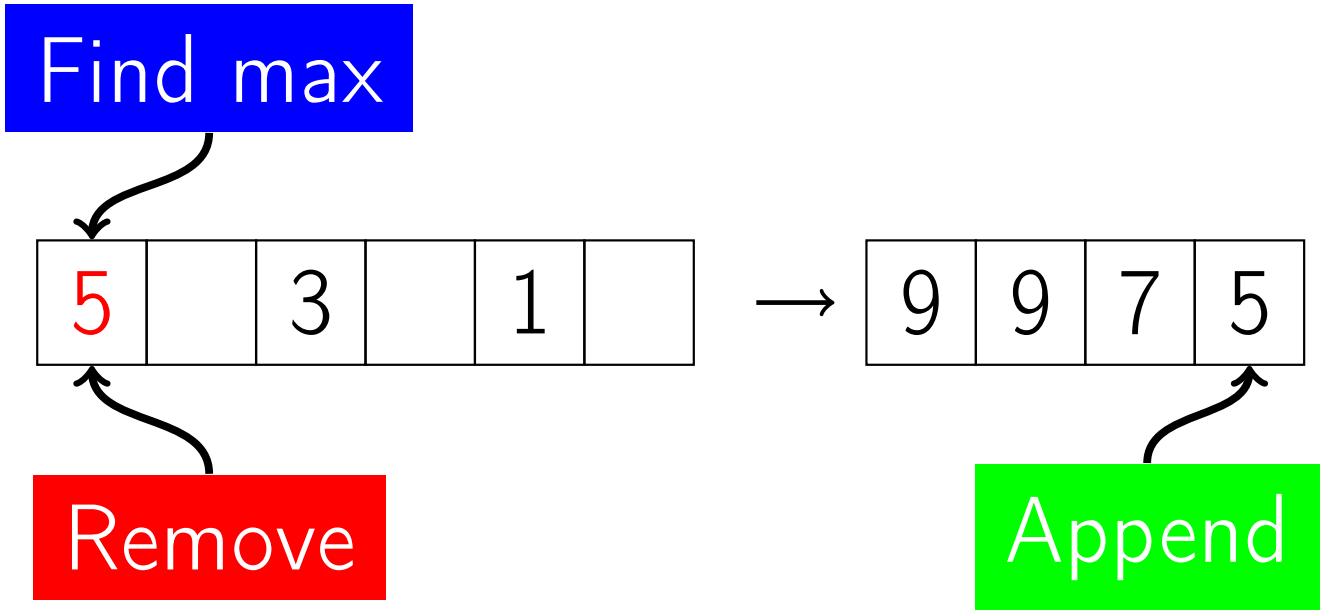

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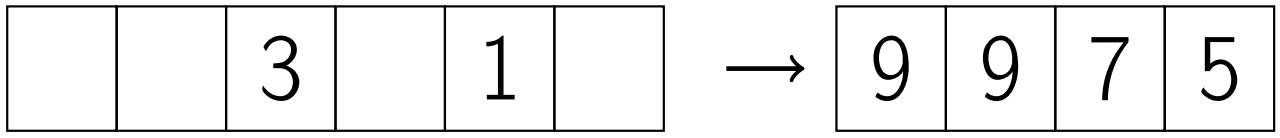

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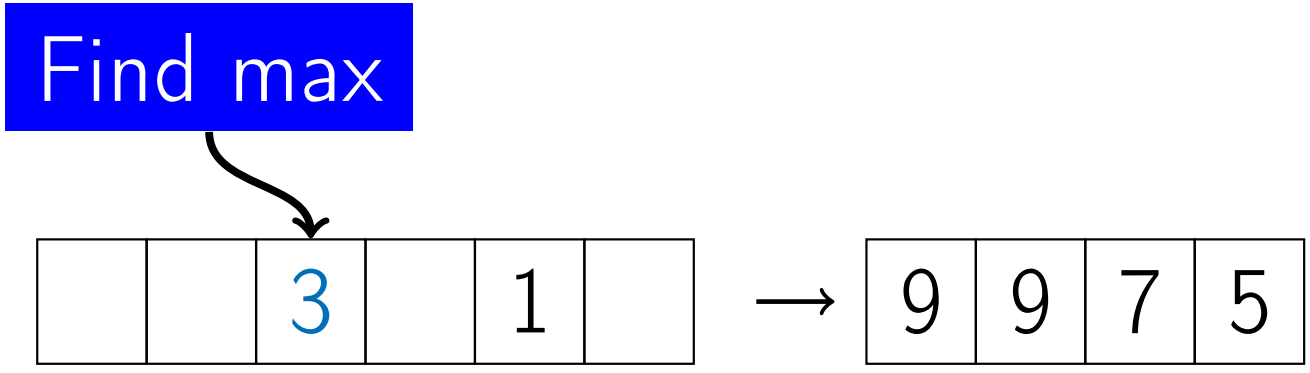

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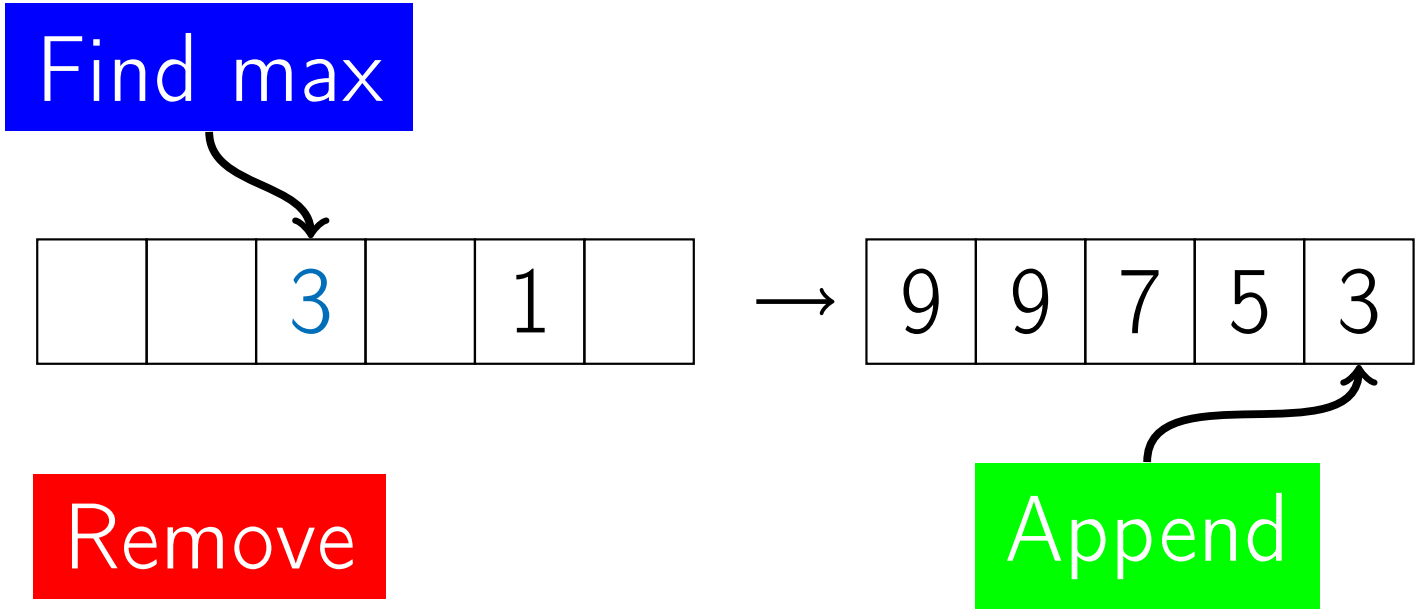

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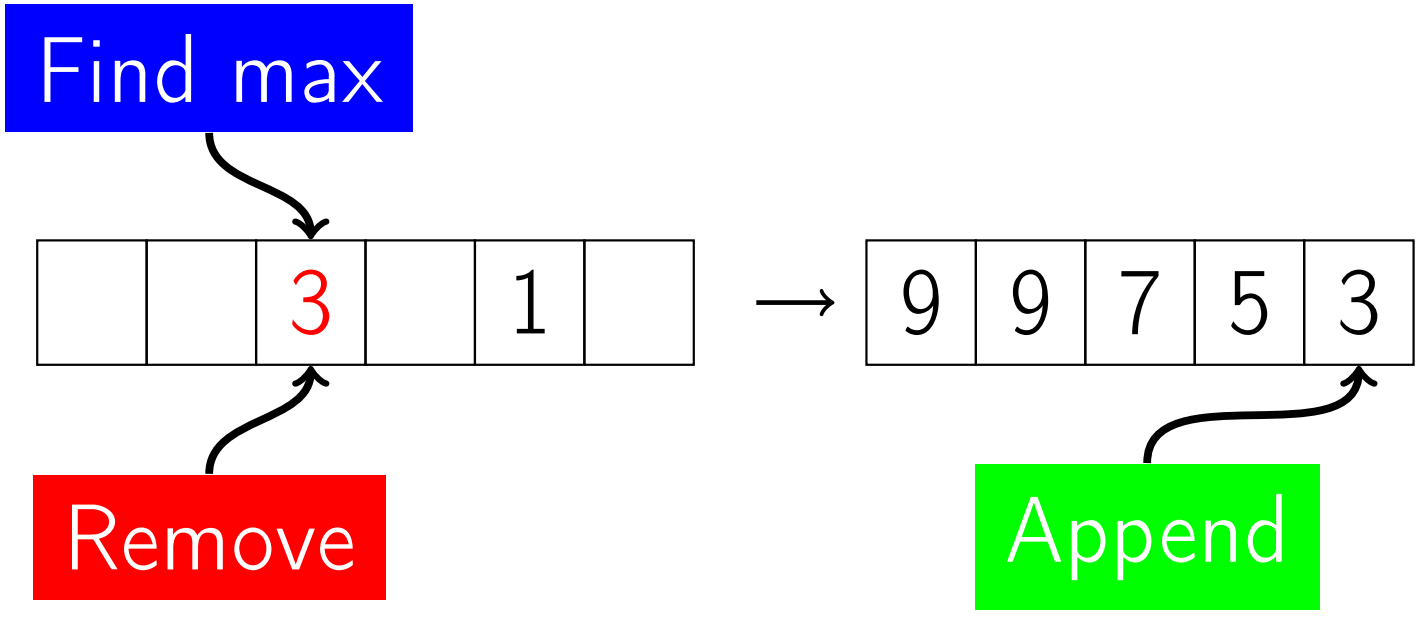

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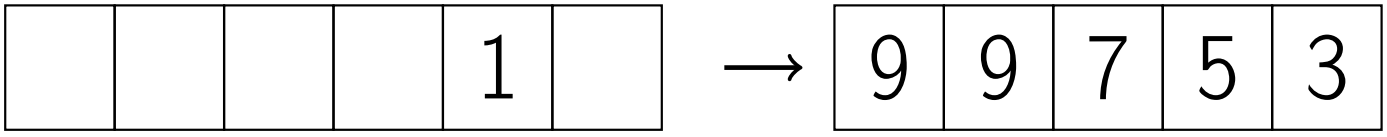

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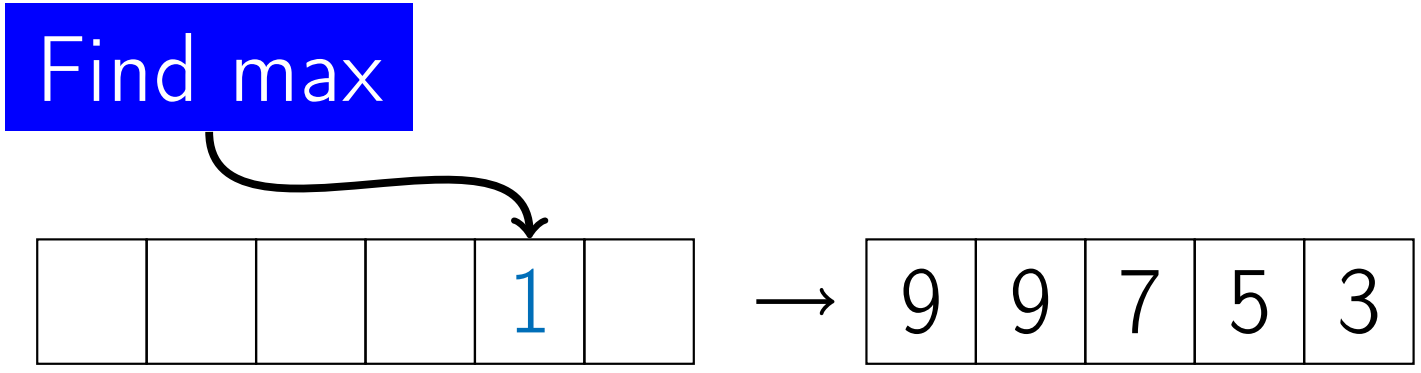

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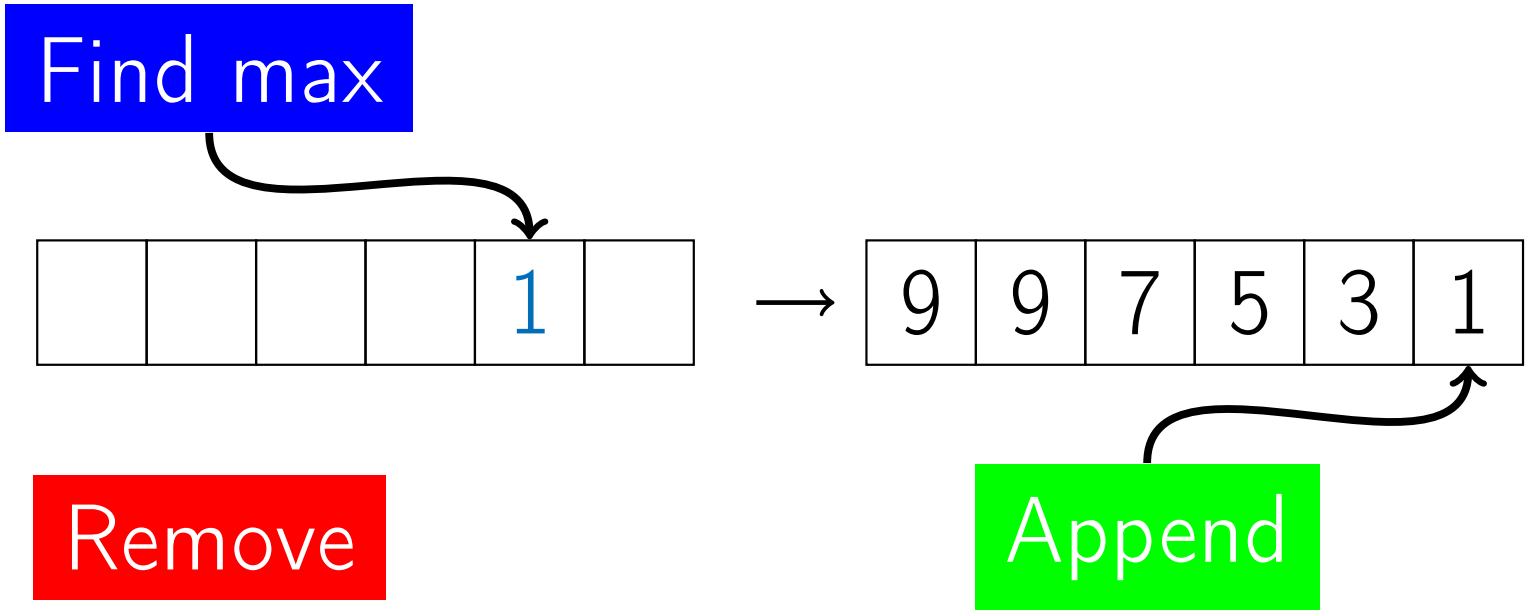

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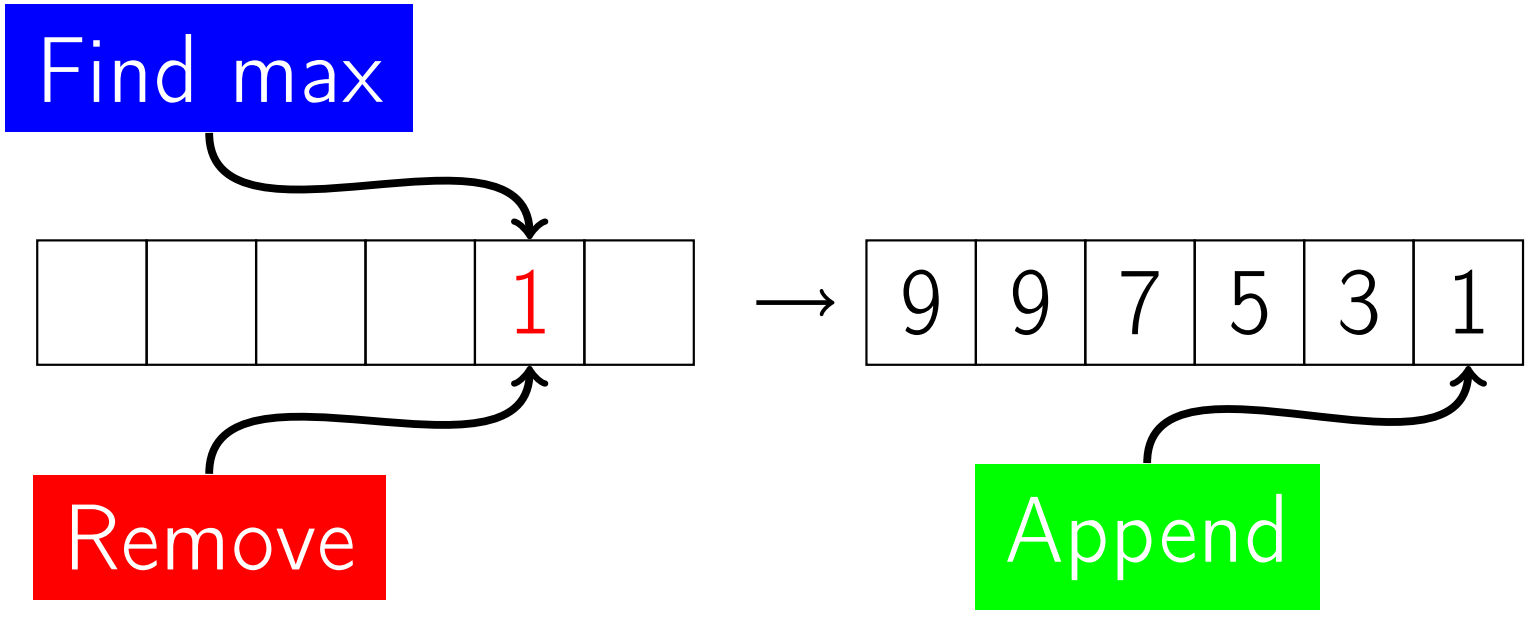

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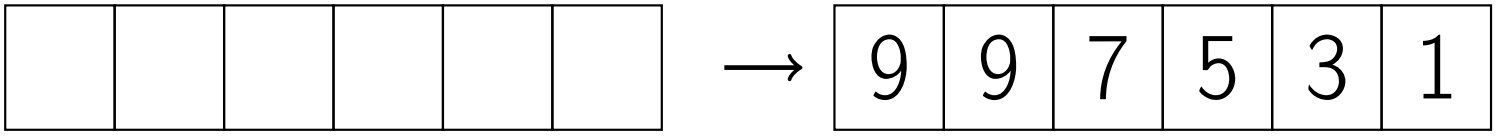

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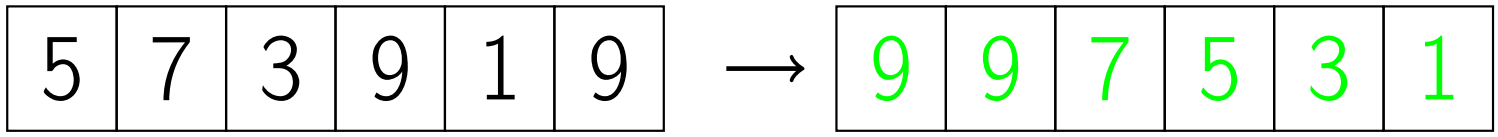

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Outline

1. Largest Number
2. Car Fueling
3. Implementation and Analysis
4. Main Ingredients

Car Fueling

Distance with full tank = 400km

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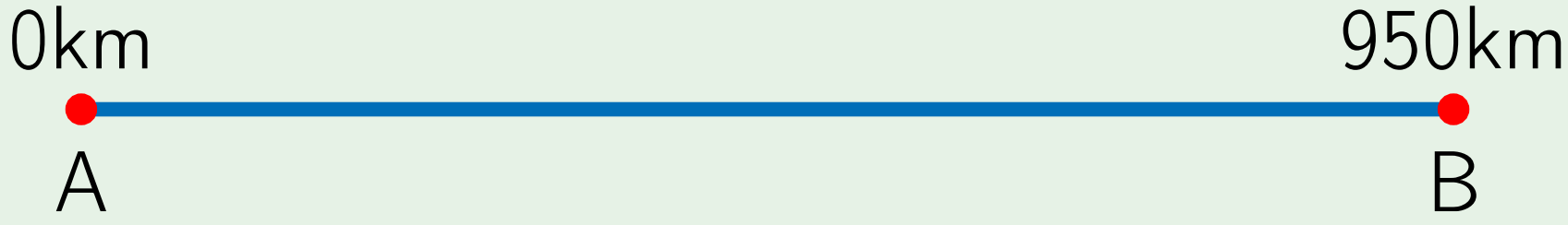

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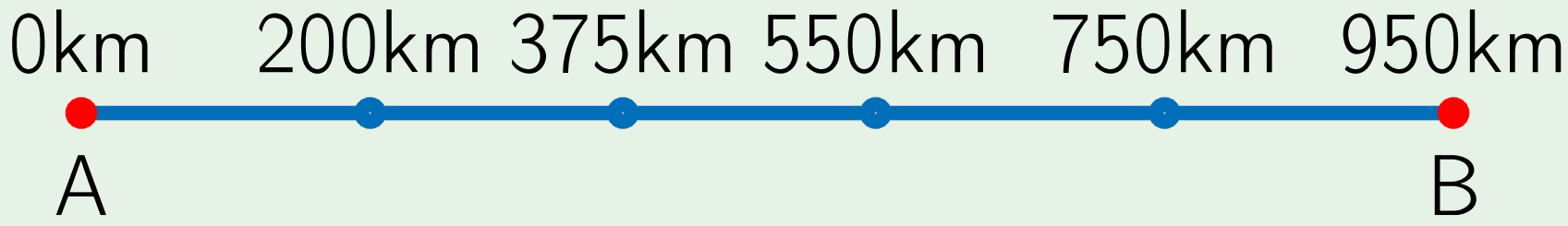

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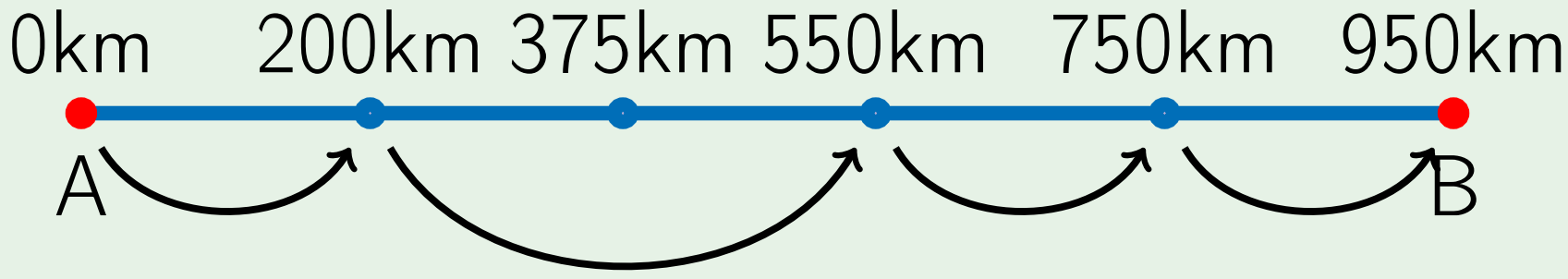

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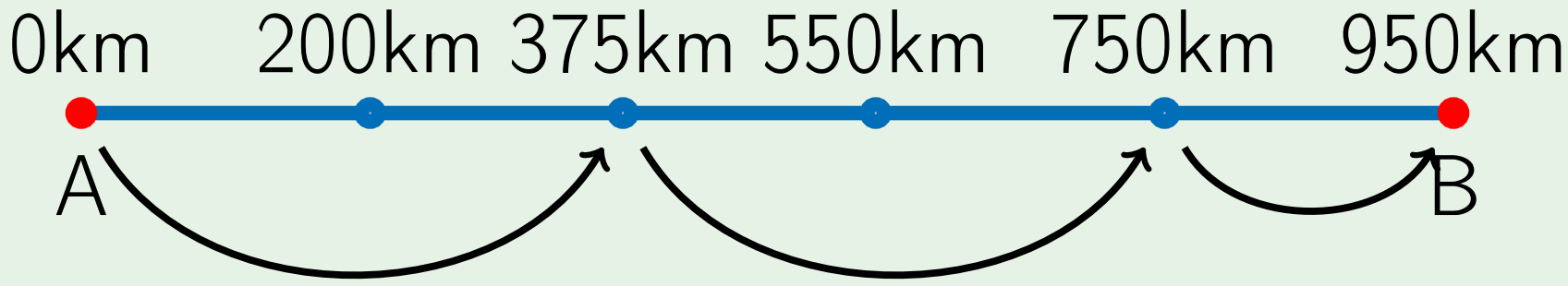

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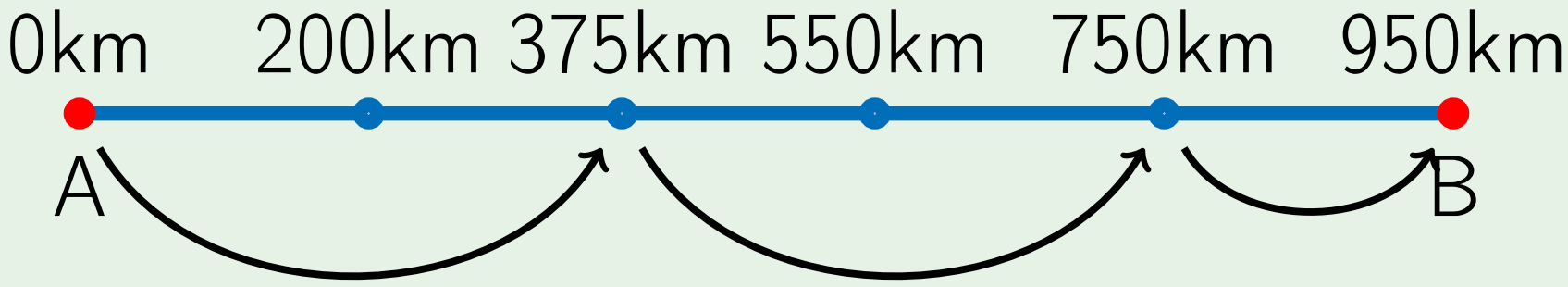

Minimum number of refills = 2

Car Fueling

Input: A car which can travel at most L kilometers with full tank, a source point A, a destination point B and n gas stations at distances $x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_n$ in kilometers from A along the path from A to B.

Output: The minimum number of refills to get from A to B, besides refill at A.

Greedy Strategy

- Make some greedy choice
- Reduce to a smaller problem
- Iterate

Greedy Choice

- Refill at the the closest gas station
- Refill at the farthest reachable gas station
- Go until there is no fuel

Greedy Choice

- Refill at the the closest gas station
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Greedy Algorithm

- Start at A

Greedy Algorithm

- Start at A
- Refill at the farthest reachable gas station G

Greedy Algorithm

- Start at A
- Refill at the farthest reachable gas station G
- Make G the new A

Greedy Algorithm

- Start at A
- Refill at the farthest reachable gas station G
- Make G the new A
- Get from new A to B with minimum number of refills

Definition

Subproblem is a similar problem of smaller size.

Subproblem

Examples

- $\texttt{LargestNumber}(3, 9, 5, 9, 7, 1) =$

Subproblem

Examples

- $\texttt{LargestNumber}(3, 9, 5, 9, 7, 1) =$ “9” $+$

Subproblem

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- $\texttt{LargestNumber}(3, 9, 5, 9, 7, 1) =$
  $\text{“9”} + \texttt{LargestNumber}(3, 5, 9, 7, 1)$

Subproblem

Examples

- $\texttt{LargestNumber}(3, 9, 5, 9, 7, 1) =$
  “9” $+ \texttt{LargestNumber}(3, 5, 9, 7, 1)$
- Min number of refills from A to $B =$

Subproblem

Examples

- $\texttt{LargestNumber}(3, 9, 5, 9, 7, 1) =$ “9” $+ \texttt{LargestNumber}(3, 5, 9, 7, 1)$
- Min number of refills from A to $B =$ first refill at G $+$

Subproblem

Examples

- $\texttt{LargestNumber}(3, 9, 5, 9, 7, 1) =$ “9” $+ \texttt{LargestNumber}(3, 5, 9, 7, 1)$
- Min number of refills from A to $B =$ first refill at G + min number of refills from G to B

Safe Move

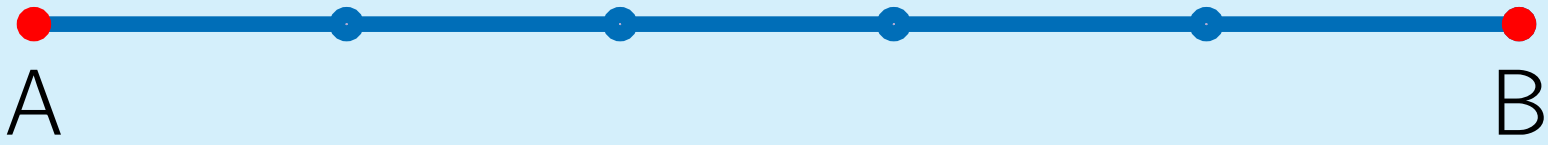
Definition

A greedy choice is called safe move if there is an optimal solution consistent with this first move.

Lemma

To refill at the farthest reachable gas station is a safe move.

Proof



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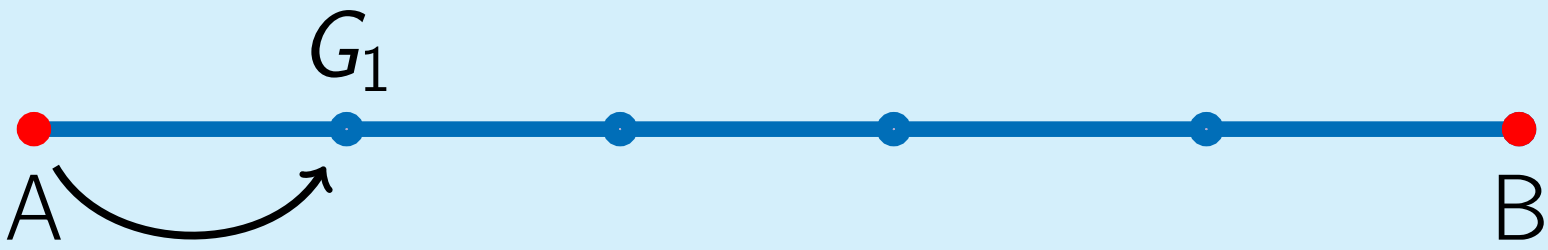

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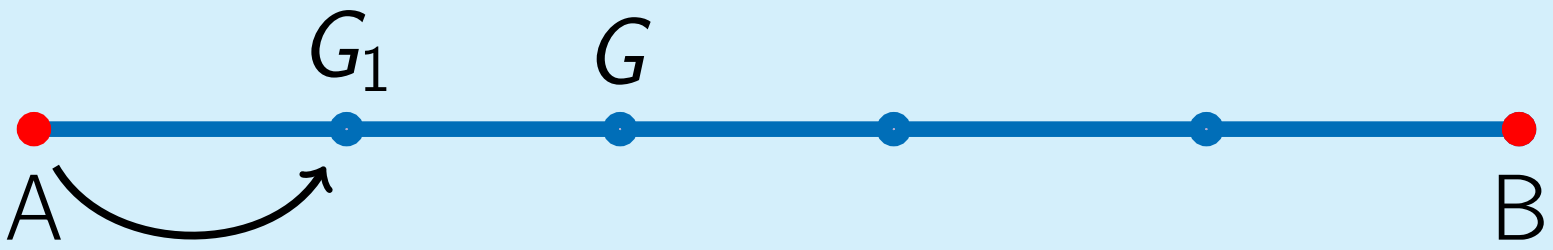

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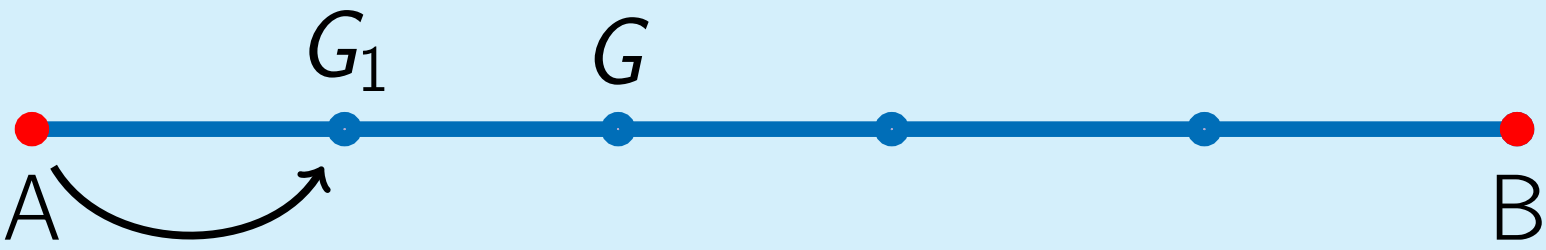
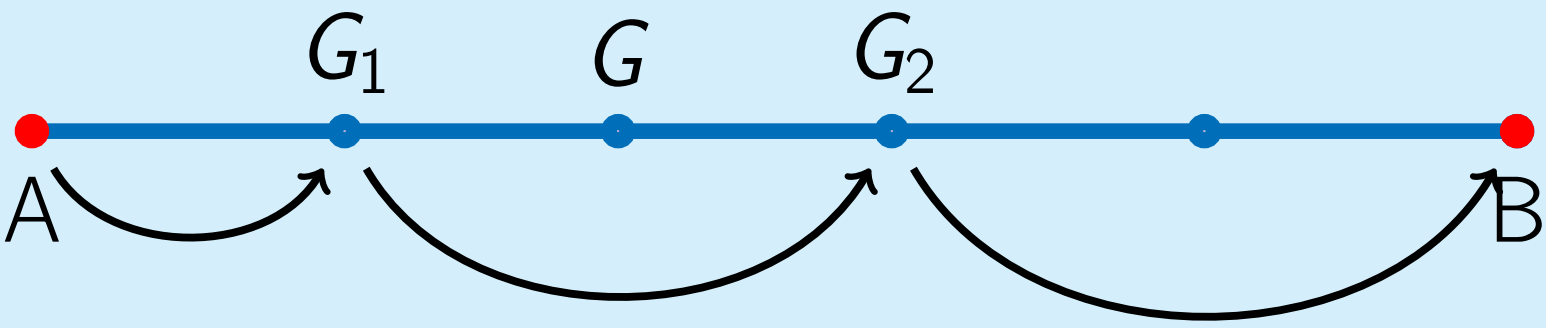


First case: G is closer than G_2

Proof



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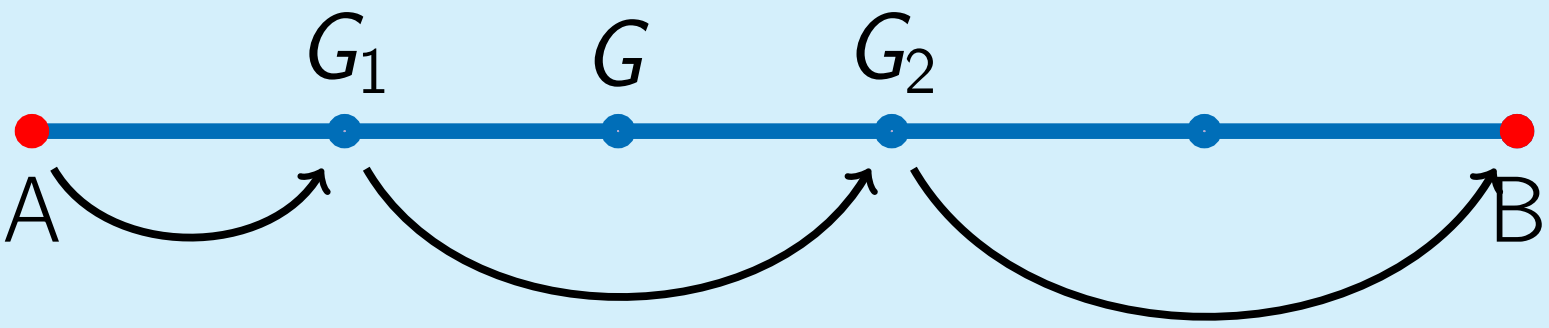
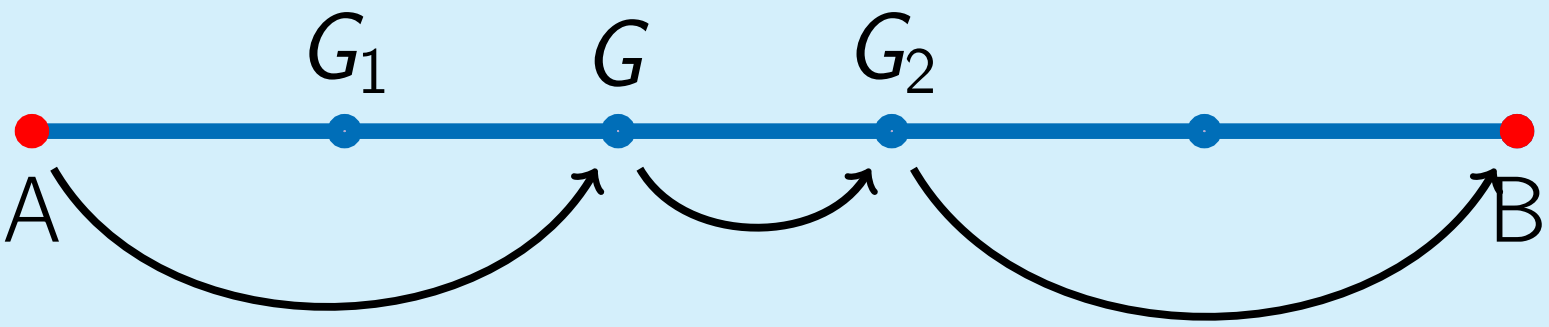


First case: G is closer than G_2

Refill at G instead of G_1

Proof



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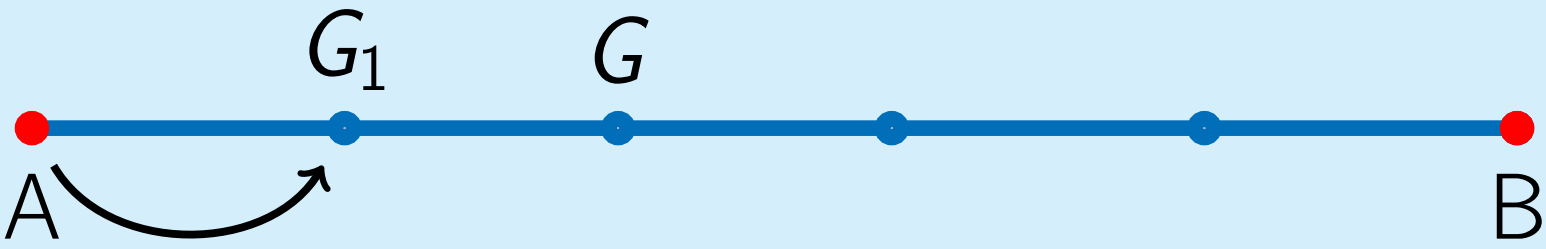
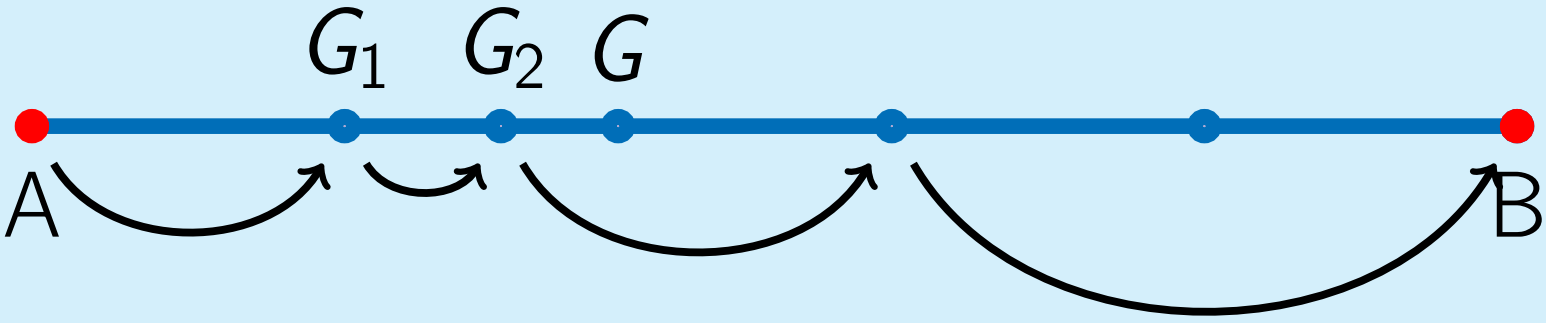


Second case: G_2 is closer than G

Proof



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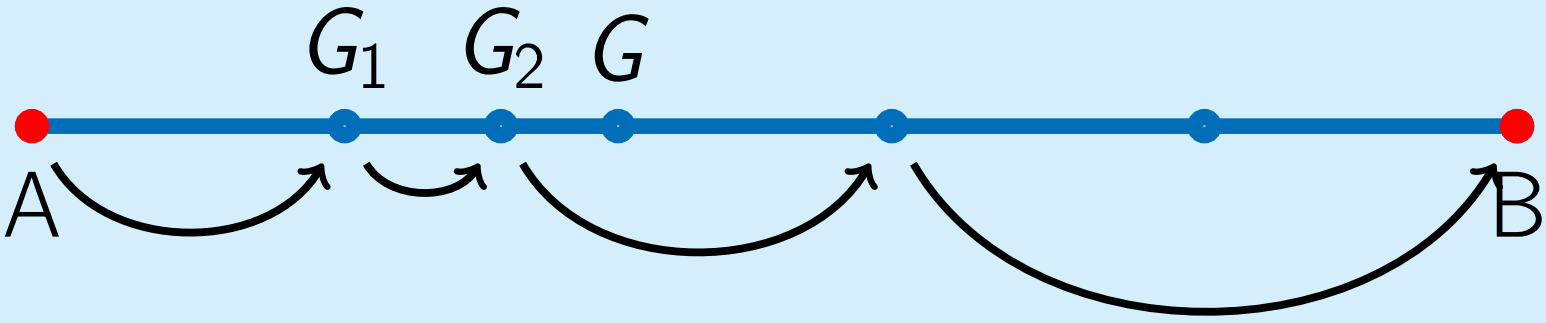

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Avoid refill at G_1

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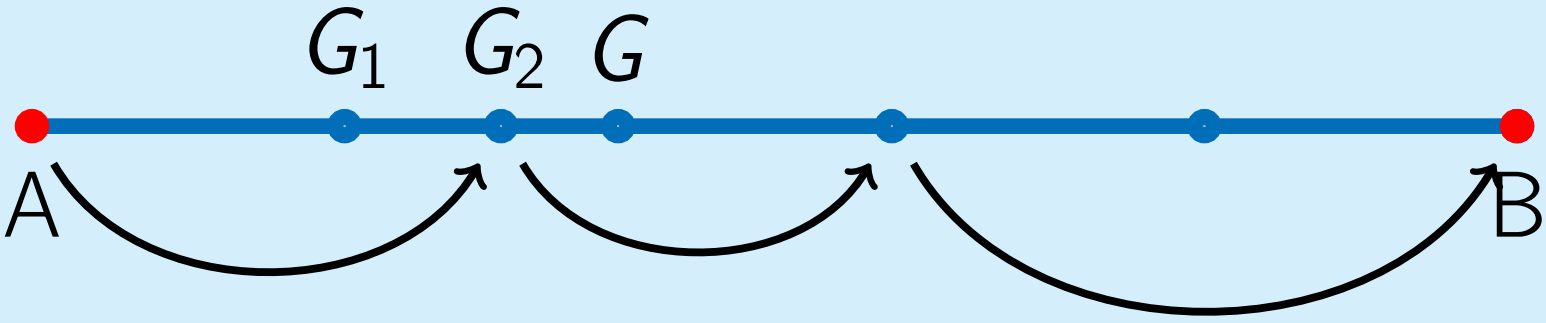

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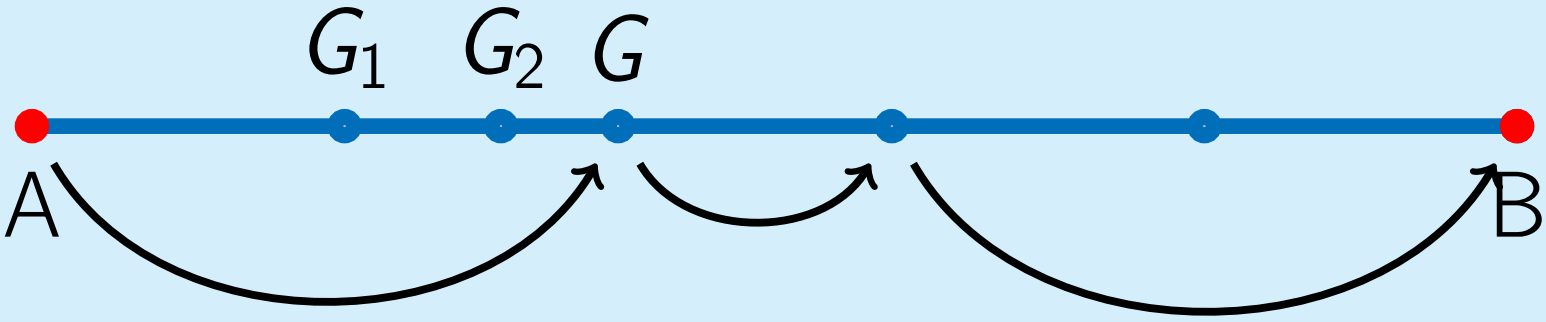

Second case: G_2 is closer than G

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- Otherwise, avoid refill at G_1 □

Outline

1. Largest Number
2. Car Fueling
3. Implementation and Analysis
4. Main Ingredients

$$A = x_0 \le x_1 \le x_2 \le \cdots \le x_n \le x_{n+1} = B$$

MinRefills(x, n, L)

```
numRefills ← 0,  currentRefill ← 0
while currentRefill ≤ n:
  lastRefill ← currentRefill
  while (currentRefill ≤ n and
         x[currentRefill + 1] − x[lastRefill] ≤ L):
    currentRefill ← currentRefill + 1
  if currentRefill == lastRefill:
    return IMPOSSIBLE
  if currentRefill ≤ n:
    numRefills ← numRefills + 1
return numRefills
```

Lemma

The running time of `MinRefills`(x, n, L) is $O(n)$.

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- Thus, $O(n)$ iterations □

Outline

1. Largest Number
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Reduction to Subproblem

- Make a first move
- Then solve a problem of the same kind
- Smaller: fewer digits, fewer fuel stations
- This is called a "subproblem"

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- Often greedy moves are not safe

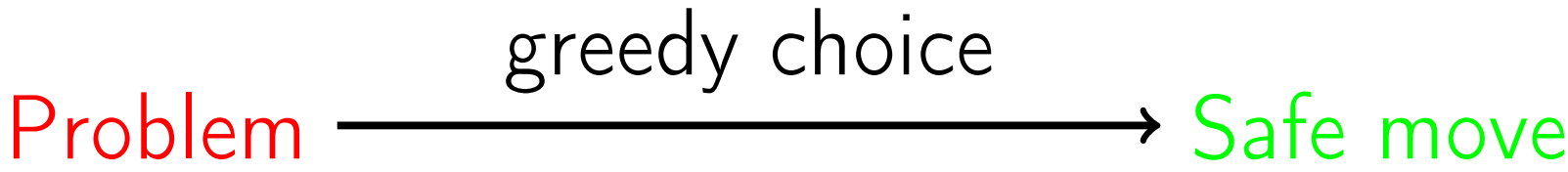
General Strategy

Problem

General Strategy

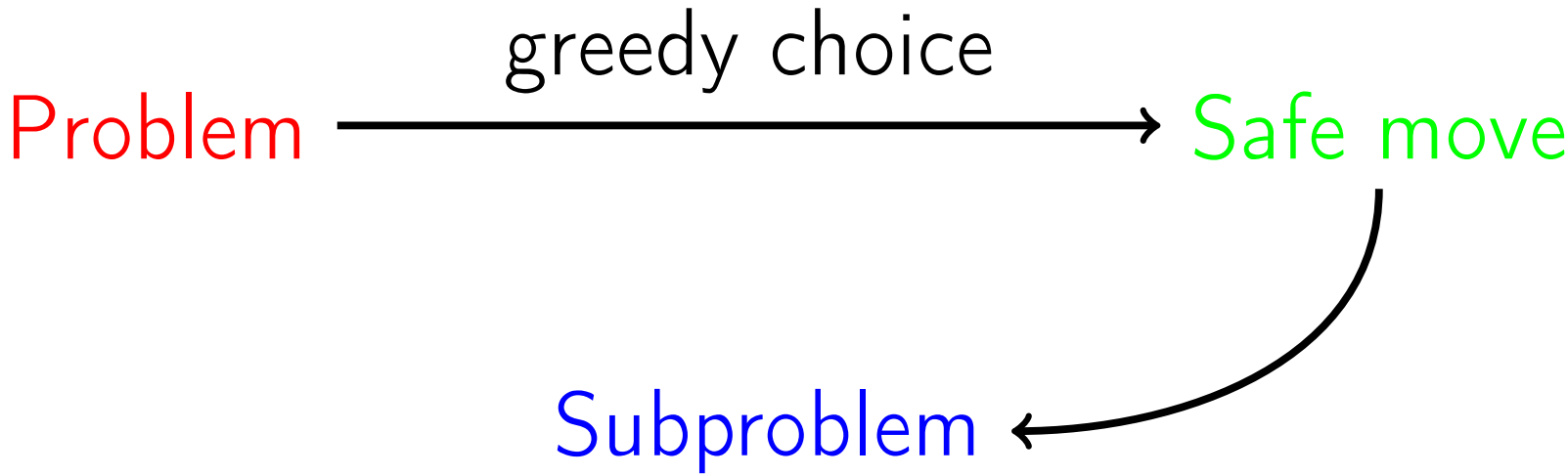
- Make a greedy choice

General Strategy



- Make a greedy choice
- Prove that it is a safe move

General Strategy



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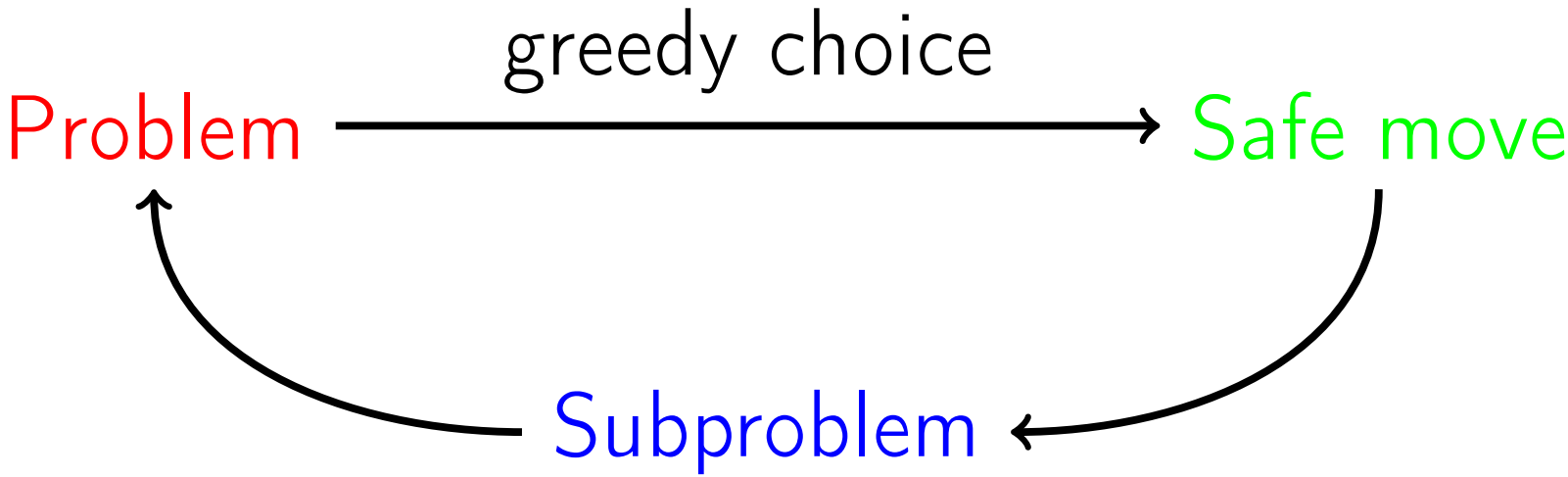
General Strategy



- Make a greedy choice
- Prove that it is a safe move
- Reduce to a subproblem
- Solve the subproblem