Q1. $q=\begin{pmatrix} v \\ u \end{pmatrix}$, $f(q)=\begin{pmatrix} u \\ \frac{1}{3}v^3 \end{pmatrix}$, $A(q)=\begin{pmatrix} 0 & 1 \\ v^2 & 0 \end{pmatrix}$

eigen systems: $\lambda^{1,2}=\mp v$, $r^{1,2}=\begin{pmatrix} 1 \\ \mp v \end{pmatrix}$, $\nabla_q \lambda^{1,2}=\begin{pmatrix} \mp 1 \\ 0 \end{pmatrix}$, Thus $\alpha^{1,2}=\frac{1}{(\nabla_q \lambda^{1,2})^T r^{1,2}}=\mp 1$,

where "−" for $p=1$, "+" for $p=2$, here. $q_l=\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $q_r=\begin{pmatrix} 3 \\ \frac{9}{2}-\sqrt{\frac{7}{3}} \end{pmatrix}$.

Consider p-rarefaction wave: $\hat{q}'(\xi)=\alpha^p(\xi)\cdot r^p(\hat{q}(\xi))$, $\xi=\lambda^p(\hat{q}(\xi))$, $\xi=\frac{x}{t}$.

$p=1$: $\begin{cases} \frac{dv}{d\xi}=-1 \\ \frac{du}{d\xi}=v \end{cases}$, ① $\xi_l^1=\lambda^1(q_l)=-v_l=-1$

$v(\xi_l^1)=v_l=1$, $u(\xi_l^1)=u_l=2$, $\Rightarrow v=-\xi$, $u=-\frac{1}{2}\xi^2+\frac{5}{2}$ $\Rightarrow$ 1-rarefaction passes ~~starts from~~ q_l: $u=-\frac{1}{2}v^2+\frac{5}{2}$.

② $\xi_r^1=\lambda^1(q_r)=-v_r=-3$

$v(\xi_r^1)=v_r=3$, $u(\xi_r^1)=u_r=\frac{9}{2}-\sqrt{\frac{7}{3}}$, $\Rightarrow v=-\xi$, $u=-\frac{1}{2}\xi^2+9-\sqrt{\frac{7}{3}}$ $\Rightarrow$ 1-rarefaction passes ~~starts~~ ~~from~~ q_r: $u=-\frac{1}{2}v^2+9-\sqrt{\frac{7}{3}}$.

$\boxed{\xi^2=\lambda^2(q)=v}$

$p=2$: $\begin{cases} \frac{dv}{d\xi}=1 \\ \frac{du}{d\xi}=v \end{cases}$ ① $v(1)=1$, $u(1)=2$, $\xi_l^2=1$ $\Rightarrow v=\xi$, $u=\frac{1}{2}\xi^2+\frac{3}{2}$ $\Rightarrow$ 2-rarefaction passes ~~starts from~~ q_l: $u=\frac{1}{2}v^2+\frac{3}{2}$.

② $v(3)=3$, $u(3)=\frac{9}{2}-\sqrt{\frac{7}{3}}$, $\xi_r^2=3$ $\Rightarrow v=\xi$, $u=\frac{1}{2}\xi^2-\sqrt{\frac{7}{3}}$ $\Rightarrow$ 2-rarefaction passes ~~starts from~~ q_r: $u=\frac{1}{2}v^2-\sqrt{\frac{7}{3}}$.

~~Since $\lambda^1(q_l)=-1$, $\lambda^1(q_r)=-3$, $\lambda^1(q_l)>\lambda^1(q_r)$, also q_l, q_r not even at the same integral curve of any rarefaction wave. So cannot directly connected by rarefaction wave~~

Consider p-shock wave: Utilize ① $f(q)-f(q_l)=s(q-q_l)$, $\begin{pmatrix} u-u_l \\ \frac{1}{3}v^3-\frac{1}{3}v_l^3 \end{pmatrix}=s\begin{pmatrix} v-v_l \\ u-u_l \end{pmatrix}$, $\exists s\in\mathbb{R}$.

then $\mathcal{H}(q_l)=\{(v,u)^T \mid (u-u_l)^2=\frac{1}{3}(v^3-v_l^3)(v-v_l)\}$
$=\{(v,u)^T \mid u=u_l\mp\sqrt{\frac{1}{3}(v^3-v_l^3)(v-v_l)}\}(v>0)$.

where "−" to 1-shock wave pass q_l, "+" to 2-shock wave pass q_l, could be seen from figure.

② $f(q)-f(q_r)=s(q-q_r)$, then $\mathcal{H}(q_r)=\{(v,u)^T \mid u=u_r\mp\sqrt{\frac{1}{3}(v^3-v_r^3)(v-v_r)}\}(v>0)$

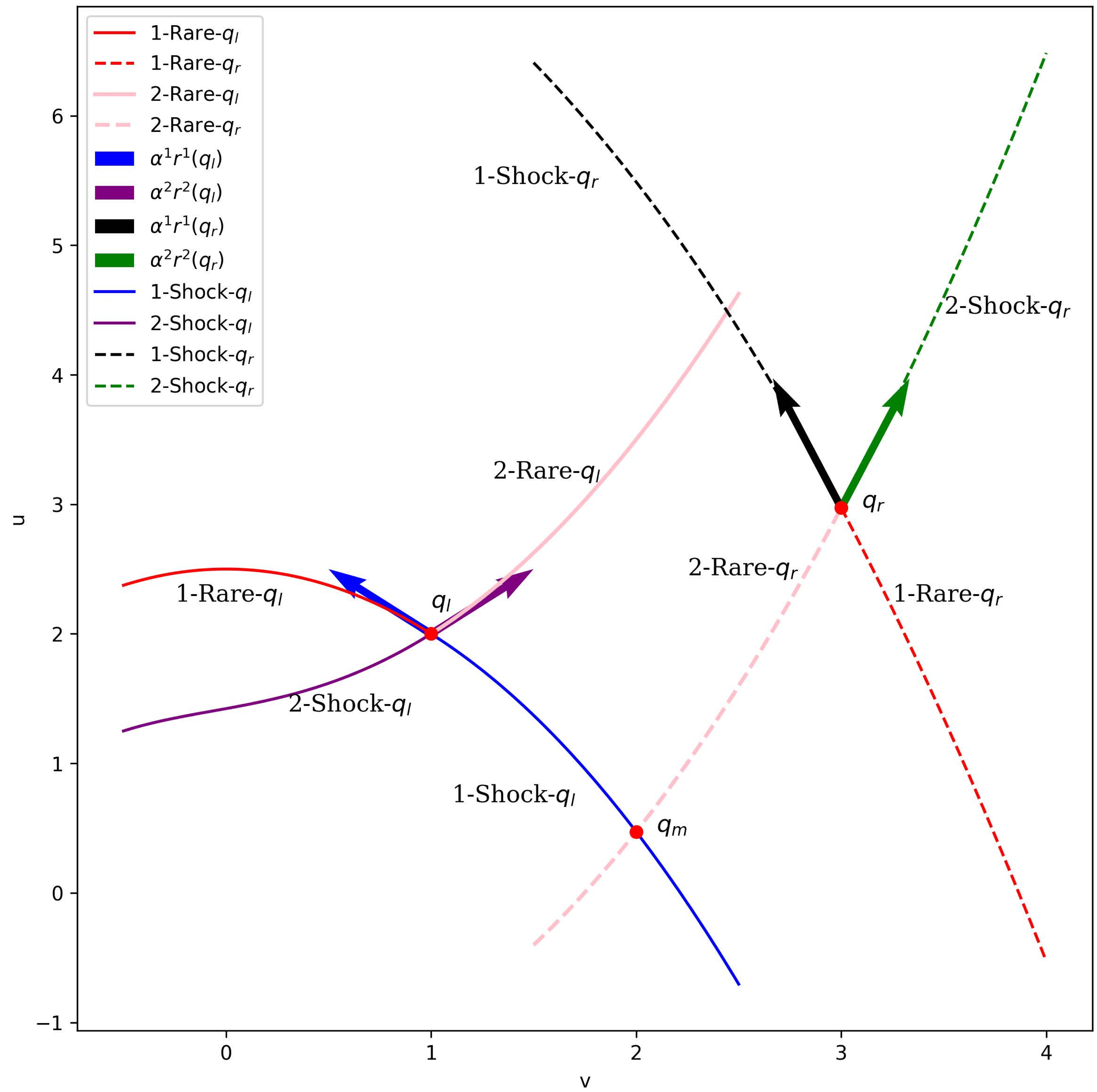
~~Therefore the first path has to be 1-shock wave connected by 1-shock wave that passes q_l but the next connected path is to be determined by calculating possible intersection of 1-shock wave with 2-shock wave and 2-rarefaction wave passing q_r~~

Now we have to determine start from which p-wave, intersecting at which point, end with which p-wave. And the valid part between q_- and q_+ is along q_- to q_+, for rarefaction wave, eigenvalue should be increasing, for shockwave, it should be decreasing.

For Hugoniot locus on $\mathcal{H}(q_l)$: $s_l=\frac{u-u_l}{v-v_l}=\sqrt{\frac{1}{3}}\frac{\mp\sqrt{(v^3-v_l^3)(v-v_l)}}{v-v_l}$, $\lambda^{1,2}(q_l)=\lim\limits_{v\to v_l}=\mp v_l$, similarly $\lambda^{1,2}(q_r)=\mp v_r$.

$\frac{q-q_l}{v-v_l}=\begin{pmatrix} 1 \\ \frac{u-u_l}{v-v_l} \end{pmatrix}$ $\xrightarrow{v\to v_l}$ $r^{1,2}(q_l)=(1,\mp v_l)$. should opposite $\vec{r}$.

~~$q-q_l=\begin{pmatrix} 1 \\ \frac{u-u_l}{v-v_l} \end{pmatrix} \Rightarrow r^{1,2}(q_l)=(1,\mp v_l)$, $r^{1,2}(q_r)=(1,\mp v_r)$~~ ~~opposite $\vec{r}$~~ ~~$\frac{q-q_l}{\|q-q_l\|}=\begin{pmatrix} 1 \\ \frac{u-u_l}{v-v_l} \end{pmatrix}$~~



by the firgue, 1-shock wave starts from q_l intersects with 2-rarefaction wave ends with q_r.

$\frac{1}{2}v^2 - \sqrt{\frac{7}{3}} = 2 - \sqrt{\frac{1}{3}(v^3-1)(v-1)^3}$, $v=2$ is the solution.

~~where~~ assume $q_m = (v_m, u_m)$, $v_m = 2$, $u_m = 2-\sqrt{\frac{7}{3}}$.

therefore $\cancel{\frac{1}{\lambda(q_m)}}$ $f(q_m) - f(q_l) = s(q_m - q_l)$, $s = -\sqrt{\frac{7}{3}}$.

Therefore:

$$q(x,t) = \begin{cases} (1, 2)^T & \frac{x}{t} < -\sqrt{\frac{7}{3}} \\ (2, 2-\sqrt{\frac{7}{3}})^T & \sqrt{\frac{7}{3}} < \frac{x}{t} < \lambda^2(q_m) = v_m = 2 \text{ (2-rarefaction starts from } q_m) \\ (\frac{x}{t}, \frac{1}{2}(\frac{x}{t})^2 - \sqrt{\frac{7}{3}})^T & \lambda^2(q_m) < \frac{x}{t} < \lambda^2(q_r) = v_r = 3 \text{ (2-rarefaction ends with } q_r) \\ (3, \frac{9}{2} - \sqrt{\frac{7}{3}})^T & \frac{x}{t} > \lambda^2(q_r) = v_r = 3 \end{cases}$$

Q2 (a) $f(q) = \begin{pmatrix} \rho u \\ \rho u^2 + \hat{k}\rho^\gamma \end{pmatrix}$ $q = \begin{pmatrix} \rho \\ \rho u \end{pmatrix}$. Let $q_1 = \rho$, $q_2 = \rho u$. $f(q) = \begin{pmatrix} q_2 \\ \frac{q_2^2}{q_1} + \hat{k} q_1^\gamma \end{pmatrix}$

~~Let $z_1 = \sqrt{q_1}$, $z_2 = q_2/\sqrt{q_1}$~~ Let $q_1 = z_1^2$, $q_2 = z_1 z_2$, $\tilde{f}(z) = (z_1 z_2, z_2^2 + \hat{k} z_1^{2\gamma})^T$.

$$\frac{\partial \tilde{f}}{\partial z} = \begin{pmatrix} z_2 & z_1 \\ 2\hat{k}\gamma z_1^{2\gamma-1} & 2z_2 \end{pmatrix}, \quad \frac{\partial \tilde{q}}{\partial z} = \begin{pmatrix} 2z_1 & 0 \\ z_2 & z_1 \end{pmatrix}.$$

Then the integrals are:

$$\int_0^1 \frac{\partial \tilde{f}}{\partial z}(\theta Z_r + (1-\theta) Z_\ell)\, d\theta = \begin{pmatrix} \frac{(Z_2)_\ell + (Z_2)_r}{2} & \frac{(Z_1)_\ell + (Z_1)_r}{2} \\ \hat{k}\frac{[(Z_1)_r]^{2\gamma} - [(Z_1)_\ell]^{2\gamma}}{(Z_1)_r - (Z_1)_\ell} & (Z_2)_\ell + (Z_2)_r \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2\hat{k}\gamma & 0 \end{pmatrix}$$

$$\int_0^1 \frac{\partial \tilde{q}}{\partial z}(\theta Z_r + (1-\theta) Z_\ell)\, d\theta = \begin{pmatrix} (Z_1)_\ell + (Z_1)_r & 0 \\ \frac{(Z_2)_r + (Z_2)_\ell}{2} & \frac{(Z_1)_\ell + (Z_1)_r}{2} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\left(\int_0^1 \frac{\partial \tilde{f}}{\partial z}(\theta Z_r + (1-\theta) Z_\ell)\, d\theta \cdot \left[\int_0^1 \frac{\partial \tilde{q}}{\partial z}(\theta Z_r + (1-\theta) Z_\ell)\, d\theta\right]^{-1}\right) = \begin{pmatrix} 0 & 1 \\ \hat{k}\frac{((Z_1)_r)^{2\gamma} - ((Z_1)_\ell)^{2\gamma}}{((Z_1)_r)^2 - ((Z_1)_\ell)^2} - \left(\frac{(Z_2)_\ell + (Z_2)_r}{(Z_1)_\ell + (Z_1)_r}\right)^2 & \frac{2[(Z_2)_\ell + (Z_2)_r]}{(Z_1)_\ell + (Z_1)_r} \end{pmatrix}$$

$Z_1 = \sqrt{q_1}$, $Z_2 = q_2/\sqrt{q_1}$. $q_\ell = \begin{pmatrix} 1 \\ -a \end{pmatrix}$, $q_r = \begin{pmatrix} 1 \\ a \end{pmatrix}$ $Z_\ell = \begin{pmatrix} 1 \\ -a \end{pmatrix}$, $Z_r = \begin{pmatrix} 1 \\ a \end{pmatrix}$.

therefore $\hat{A} = \begin{pmatrix} 0 & 1 \\ \hat{k}\lim_{x\to 1} \frac{x^{2\gamma}-1}{x^2-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \hat{k}\gamma & 0 \end{pmatrix}$.

(b). linearized $\frac{\partial q}{\partial t} + \hat{A}\frac{\partial q}{\partial x} = 0$. eigen systems of $\hat{A}$: $\lambda^1 = -\sqrt{\hat{k}\gamma}$, $\lambda^2 = \sqrt{\hat{k}\gamma}$, $r^1 = (1, -\sqrt{\hat{k}\gamma})^T$, $r^2 = (1, \sqrt{\hat{k}\gamma})^T$.

$R = (r^1, r^2)$, $R = \begin{pmatrix} 1 & 1 \\ -\sqrt{\hat{k}\gamma} & \sqrt{\hat{k}\gamma} \end{pmatrix}$. $R^{-1} = \frac{1}{|R|}\begin{pmatrix} \sqrt{\hat{k}\gamma} & -1 \\ \sqrt{\hat{k}\gamma} & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -\frac{1}{\sqrt{\hat{k}\gamma}} \\ 1 & \frac{1}{\sqrt{\hat{k}\gamma}} \end{pmatrix}$

Let $w = R^{-1} q$, $q_\ell = \begin{pmatrix} 1 \\ -a \end{pmatrix}$, $q_r = \begin{pmatrix} 1 \\ a \end{pmatrix}$, $w_\ell = R^{-1} q_\ell = [\frac{1}{2}(1 + \frac{a}{\sqrt{\hat{k}\gamma}}), \frac{1}{2}(1 - \frac{a}{\sqrt{\hat{k}\gamma}})]^T$

$w_r = [\frac{1}{2}(1 - \frac{a}{\sqrt{\hat{k}\gamma}}), (1 + \frac{a}{\sqrt{\hat{k}\gamma}})\frac{1}{2}]^T$. Therefore: $w(x,t) = \begin{cases} [\frac{1}{2}(1 + \frac{a}{\sqrt{\hat{k}\gamma}}), \frac{1}{2}(1 - \frac{a}{\sqrt{\hat{k}\gamma}})]^T & \frac{x}{t} < -\sqrt{\hat{k}\gamma} \\ [\frac{1}{2}(1 - \frac{a}{\sqrt{\hat{k}\gamma}}), \frac{1}{2}(1 - \frac{a}{\sqrt{\hat{k}\gamma}})]^T & -\sqrt{\hat{k}\gamma} < \frac{x}{t} < \sqrt{\hat{k}\gamma} \\ [\frac{1}{2}(1 - \frac{a}{\sqrt{\hat{k}\gamma}}), \frac{1}{2}(1 + \frac{a}{\sqrt{\hat{k}\gamma}})]^T & \frac{x}{t} > \sqrt{\hat{k}\gamma}. \end{cases}$

$$q(x,t) = Rw(x,t) = \begin{cases} q_\ell & \frac{x}{t} < -\sqrt{\hat{k}\gamma} \\ \begin{pmatrix} 1 & 1 \\ -\sqrt{\hat{k}\gamma} & \sqrt{\hat{k}\gamma} \end{pmatrix} \frac{1}{2}\begin{pmatrix} 1 - \frac{a}{\sqrt{\hat{k}\gamma}} \\ 1 - \frac{a}{\sqrt{\hat{k}\gamma}} \end{pmatrix} & -\sqrt{\hat{k}\gamma} < \frac{x}{t} < \sqrt{\hat{k}\gamma} \\ q_r & \frac{x}{t} > \sqrt{\hat{k}\gamma} \end{cases}$$

therefore when $\frac{x}{t} \in (-\sqrt{\hat{k}\gamma}, \sqrt{\hat{k}\gamma})$, $q(x,t) = \frac{1}{2}\cdot\begin{pmatrix} 2 - \frac{2a}{\sqrt{\hat{k}\gamma}} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 - \frac{a}{\sqrt{\hat{k}\gamma}} \\ 0 \end{pmatrix}$,

$1 - \frac{a}{\sqrt{\hat{k}\gamma}} \geq 0 \iff a \leq \sqrt{\hat{k}\gamma}$. #.

Q3. Write down the semi-discrete weak form (with only the spatial variable discretized) of the PG method for the heat equation:

$$\frac{\partial q}{\partial t} = \frac{\partial^2 q}{\partial x^2} \quad x\in(0,1). \text{ with periodic boundary conditions.}$$

Show that the numerical solution $Q(x,t)$ and $U(x,t)$ satisfies

$$\frac{d}{dt}\int_0^1 |Q(x,t)|^2\,dx = -\int_0^1 |U(x,t)|^2\,dx \le 0.$$

Pf: To avoid dealing with 2nd-order derivatives, write the heat equation as

$$\frac{\partial q}{\partial t} = \frac{\partial u}{\partial x}$$
$$u = \frac{\partial q}{\partial x}.$$

Numerical solution:

$$Q(x,t)=Q_i(x,t),\quad U(x,t)=U_i(x,t),\quad x\in(x_{i-1/2},x_{i+1/2})$$
$$Q_i(x,t)=\sum_{k=0}^{K} Q_i(x_i^k,t)\,\ell_i^k(x);\quad U_i(x,t)=\sum_{k=0}^{K} U_i(x_i^k,t)\,\ell_i^k(x).$$

Then test functions: $\ell_i^k(x)$: then weak form is as follows:

(A). $$\frac{\partial}{\partial t}\int_{x_{i-1/2}}^{x_{i+1/2}} Q_i(x,t)\,\ell_i^k(x)\,dx = \int_{x_{i-1/2}}^{x_{i+1/2}} \frac{\partial}{\partial x}U_i(x,t)\cdot \ell_i^k(x)\,dx$$

$$\cancel{=\ell_i^k(x)\,U(x,t)}$$

$$= \ell_i^k(x_{i+1/2})\cdot F^Q_{i+1/2}(t) - \ell_i^k(x_{i-1/2})\cdot F^Q_{i-1/2}(t) - \int_{x_{i-1/2}}^{x_{i+1/2}} U_i(x,t)\,\frac{d}{dx}\ell_i^k(x)\,dx$$

(B). $$\int_{x_{i-1/2}}^{x_{i+1/2}} U_i(x,t)\,\ell_i^k(x)\,dx = -\int_{x_{i-1/2}}^{x_{i+1/2}} Q_i(x,t)\,\frac{d\ell_i^k(x)}{dx}\,dx + \ell_i^k(x_{i+1/2})\,F^U_{i+1/2}(t) - \ell_i^k(x_{i-1/2})\,F^U_{i-1/2}(t).$$

$$F^Q_{i+1/2}(t)=\frac{1}{2}\left[U_i(x_{i+1/2},t)+U_{i+1}(x_{i+1/2},t)\right],\quad F^U_{i+1/2}(t)=\frac{1}{2}\left[Q_i(x_{i+1/2},t)+Q_{i+1}(x_{i+1/2},t)\right].$$

Since $$\frac{d}{dt}\int_0^1 |Q(x,t)|^2dx = \sum_{i=1}^{N}\frac{d}{dt}\int_{x_{i-1/2}}^{x_{i+1/2}} |Q_i(x,t)|^2dx = \sum_{i=1}^{N}\frac{d}{dt}\int_{x_{i-1/2}}^{x_{i+1/2}}\sum_{m=0}^{K}\sum_{k=0}^{K} Q_i(x_i^m,t)\,Q_i(x_i^k,t)\,\ell_i^k(x)\,\ell_i^m(x)\,dx$$

$$= \sum_{i=1}^{N}\sum_{m=0}^{K} Q_i(x_i^m,t)\frac{d}{dt}\int_{x_{i-1/2}}^{x_{i+1/2}}\left(\sum_{k=0}^{K} Q_i(x_i^k,t)\,\ell_i^k(x)\right)\cdot \ell_i^m(x)\,dx$$

$$+ \sum_{i=1}^{N}\sum_{k=0}^{K} Q_i(x_i^k,t)\frac{d}{dt}\int_{x_{i-1/2}}^{x_{i+1/2}}\left(\sum_{m=0}^{K} Q_i(x_i^m,t)\,\ell_i^m(x)\right)\cdot \ell_i^k(x)\,dx,$$

where $\sum_{k=0}^{K} Q_i(x_i^k,t)\,\ell_i^k(x) = Q_i(x,t) = \sum_{m=0}^{K} Q_i(x_i^m,t)\,\ell_i^m(x)$

$$\frac{d}{dt}\int_0^1 |Q(x,t)|^2dx = 2\sum_{i=1}^{N}\sum_{k=0}^{K} Q_i(x_i^k,t)\frac{d}{dt}\int_{x_{i-1/2}}^{x_{i+1/2}} Q_i(x,t)\,\ell_i^k(x)\,dx$$

$$\cancel{=2\sum_{i=1}^{N}\sum_{k=0}^{K} Q_i(x_i^k,t)\,\ell_i^k(x_{i+1/2})F^Q_{i+1/2}(t) - Q_i(x_i^k,t)\,\ell_i^k(x_{i-1/2})F^Q_{i-1/2}(t) - \int_{x_{i-1/2}}^{x_{i+1/2}} U_i(x,t)\frac{d}{dx}Q_i(x_i^k,t)\,\ell_i^k(x)\,dx}$$

$$\cancel{=2\sum_{i=1}^{N}\sum_{k=0}^{K} Q_i(x_i^k,t)\,\ell_i^k(x_{i+1/2})F^Q_{i+1/2}(t)\ \ 2\sum_{k=0}^{K} Q_i(x_i^k,t)\ \text{where}\ \sum_{k=0}^{K} -\int_{x_{i-1/2}}^{x_{i+1/2}} U_i(x,t)\frac{d}{dx}Q_i(x_i^k,t)\,\ell_i^k(x)\,dx}$$

$$\cancel{=\sum_{k=0}^{K}\sum_{m=0}^{K} -\int_{x_{i-1/2}}^{x_{i+1/2}} U_i(x_i^m,t)\,Q_i(x_i^k,t)\,\ell_i^m\frac{d}{dx}\ell_i^k(x)\,dx = \sum_{k=0}^{K}\sum_{m=0}^{K} -U_i(x_i^m,t)\int_{x_{i-1/2}}^{x_{i+1/2}}}$$

Essentially next will utilize periodic boundary condition for zero sum and integral by parts to exchange index.

Since $\frac{d}{dt}\int_0^1 |Q(x,t)|^2 dx = \sum_{i=1}^{N} \frac{d}{dt}\int_{x_{i-1/2}}^{x_{i+1/2}} |Q_i(x,t)|^2 dx = \sum_{i=1}^{N} \frac{d}{dt}\int_{x_{i-1/2}}^{x_{i+1/2}} \sum_{m=0}^{k}\sum_{k=0}^{K} Q_i(x_i^m,t)Q_i(x_i^k,t)\cdot \ell_i^k(x)\,\ell_i^m(x)\,dx$

$= \sum_{i=1}^{N}\sum_{m=0}^{K} Q_i(x_i^m,t)\frac{d}{dt}\int_{x_{i-1/2}}^{x_{i+1/2}} \left(\sum_{k=0}^{K} Q_i(x_i^k,t)\ell_i^k(x)\right)\cdot \ell_i^m(x)\,dx$

$+ \sum_{i=1}^{N}\sum_{k=0}^{K} Q_i(x_i^k,t)\frac{d}{dt}\int_{x_{i-1/2}}^{x_{i+1/2}} \left(\sum_{m=0}^{K} Q_i(x_i^m,t)\ell_i^m(x)\right)\cdot \ell_i^k(x)\,dx$

where $\sum_{k=0}^{K}(\cdot)^k = \sum_{m=0}^{K}(\cdot)^m$ part $(\cdot)^k := Q_i(x_i^k,t)\ell_i^k(x)$.

Therefore $\frac{d}{dt}\int_0^1 |Q(x,t)|^2 dx = 2\sum_{i=1}^{N}\sum_{k=0}^{K} Q_i(x_i^k,t)\frac{d}{dt}\int_{x_{i-1/2}}^{x_{i+1/2}} Q_i(x,t)\ell_i^k(x)\,dx$ By (A)

$=2\left[\sum_{i=1}^{N}\sum_{k=0}^{K} Q_i(x_i^k,t)\ell_i^k(x_{i+1/2})F^Q_{i+1/2}(t) - Q_i(x_i^k,t)\ell_i^k(x_{i-1/2})F^Q_{i-1/2}(t) - \int_{x_{i-1/2}}^{x_{i+1/2}} U_i(x,t)\frac{d}{dx}Q_i(x_i^k,t)\ell_i^k(x)\,dx\right].$

Thus the numerical flux term with the periodic boundary condition ($x_{1/2}=0$, $x_{N+\frac12}=1$)

$\sum_{i=1}^{N}\sum_{k=0}^{K} Q_i(x_i^k,t)\ell_i^k(x_{i+1/2})F^Q_{i+1/2}(t) - Q_i(x_i^k,t)\ell_i^k(x_{i-1/2})F^Q_{i-1/2}(t) = \sum_{k=0}^{K}\sum_{i=1}^{N} Q(x^k_{i+1/2},t)F^Q_{i+1/2}(t) - Q(x^k_{i-1/2},t)F^Q_{i-1/2}(t)$

$= \sum_{k=0}^{K}\left[Q(1,t)F^Q_{N+1/2}(t) - Q(0,t)F^Q_{1/2}(t)\right] = 0.$

$-\int_{x_{i-1/2}}^{x_{i+1/2}} U_i(x,t)\frac{d}{dx}Q_i(x_i^k,t)\ell_i^k(x)\,dx = -\sum_{m=0}^{K}\int_{x_{i-1/2}}^{x_{i+1/2}} U_i(x_i^m,t)\ell_i^m(x)Q_i(x_i^k,t)\frac{d}{dx}\ell_i^k(x)\,dx$

Here comes the symmetry by integrals by part, to interchange k,m of $\ell_i(x)$. ~~$\sum_{m=0}^{K} U_i$~~
we need to form $Q_i(x_i^k,t)\ell_i^k(x)$ so that $\int_{x_{i-1/2}}^{x_{i+1/2}} \sum_k Q_i(x_i^k,t)\ell_i^k(x)\frac{d}{dx}\ell_i^m(x)\,dx$

By $\int_{x_{i-1/2}}^{x_{i+1/2}} \ell_i^m(x)\frac{d}{dx}\ell_i^k(x)\,dx = \ell_i^m(x)\ell_i^k(x)\Big|_{x_{i-1/2}}^{x_{i+1/2}} - \int_{x_{i-1/2}}^{x_{i+1/2}} \ell_i^k(x)\frac{d}{dx}\ell_i^m(x)\,dx = \int_{x_{i-1/2}}^{x_{i+1/2}} U_i(x,t)\ell_i^m(x)\,dx.$

we have by (B):

$\sum_{m=0}^{K}\int_{x_{i-1/2}}^{x_{i+1/2}} U_i(x_i^m,t)\ell_i^m(x)Q_i(x_i^k,t)\frac{d}{dx}\ell_i^k(x)\,dx = \sum_{m=0}^{K} U(x^m_{i+1/2},t)Q(x^k_{i+1/2},t) - U(x^m_{i-1/2},t)Q(x^k_{i-1/2},t)$

$- \sum_{m=0}^{K}\int_{x_{i-1/2}}^{x_{i+1/2}} U_i(x_i^m,t)\frac{d}{dx}\ell_i^m(x)\cdot Q_i(x_i^k,t)\ell_i^k(x)\,dx$

Therefore $\sum_{k=0}^{K}\sum_{i=1}^{N} -\int_{x_{i-1/2}}^{x_{i+1/2}} U_i(x,t)\frac{d}{dx}Q_i(x_i^k,t)\ell_i^k(x)\,dx = -\sum_{i=1}^{N}\overbrace{\sum_{k=0}^{K}\sum_{m=0}^{K}\left[U(x^m_{i+1/2},t)Q(x^k_{i+1/2},t) - U(x^m_{i-1/2},t)Q(x^k_{i-1/2},t)\right]}^{\text{①}}$

$+ \sum_{i=1}^{N}\sum_{k=0}^{K}\sum_{m=0}^{K}\int_{x_{i-1/2}}^{x_{i+1/2}} U_i(x_i^m,t)\frac{d}{dx}\ell_i^m(x)\,Q_i(x_i^k,t)\ell_i^k(x)\,dx \xrightarrow[\text{condition, ①}=0]{\text{by peoridic boundary}} \sum_{i=1}^{N}\sum_{m=0}^{K}\int_{x_{i-1/2}}^{x_{i+1/2}} U_i(x_i^m,t)Q_i(x,t)\frac{d}{dx}\ell_i^m(x)\,dx \quad (*)$

By $\int_{x_{i-1/2}}^{x_{i+1/2}} U_i(x,t)\ell_i^m(x)\,dx = -\int_{x_{i-1/2}}^{x_{i+1/2}} Q_i(x,t)\frac{d\ell_i^m(x)}{dx}\,dx + \ell_i^m(x_{i+1/2})F^U_{i+1/2}(t) - \ell_i^m(x_{i-1/2})F^U_{i-1/2}(t)$,
along with peoridic boundary condition, the numerical flux part sum to zearo, leads to only $(*) = \sum_{i=1}^{N}\sum_{m=0}^{K} -\int_{x_{i-1/2}}^{x_{i+1/2}} U_i(x,t)\ell_i^m(x)\cdot U_i(x_i^m,t)\,dx = -\int_0^1 |U(x,t)|^2 dx.$

Therefore $\frac{\partial}{\partial t}\int_0^1 |Q(x,t)|^2 dx = 2\left[-\int_0^1 |U(x,t)|^2 dx\right] \Leftrightarrow \frac{\partial}{\partial t}\frac12\int_0^1 |Q(x,t)|^2 dx = -\int_0^1 |U(x,t)|^2 dx \le 0$. #.