Computational Fluid Dynamics

GUO QILONG

February 26, 2024

Contents

0.1 Problem 1
0.2 Numerical Scheme 2
0.2.1 Spatial discretization 2
0.2.2 Boundary conditions: 3
0.3 Results 4
0.3.1 Q1 4
0.3.2 Q2 5

1 Reference **5**

Project 1

0.1 Problem

Let $\Omega = (0,1)\times(0,1)$ and $\gamma = 7/5$. Consider the following two-dimensional Euler equations:

$$\begin{aligned}
&\frac{\partial \tilde{\rho}}{\partial t} + \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} = 0,\\
&\frac{\partial \tilde{u}}{\partial t} + \frac{\partial \tilde{p}}{\partial x} = 0,\\
&\frac{\partial \tilde{v}}{\partial t} + \frac{\partial \tilde{p}}{\partial y} = 0,\\
&\frac{\partial \tilde{p}}{\partial t} + \gamma\frac{\partial \tilde{u}}{\partial x} + \gamma\frac{\partial \tilde{v}}{\partial y} = 0, \qquad (x,y)\in\Omega
\end{aligned}$$

The initial conditions are

$$\begin{aligned}
&\tilde{u}(x,y,0) = \tilde{v}(x,y,0) = 0, \quad (x,y)\in\Omega,\\
&\tilde{\rho}(x,y,0) = \tilde{p}(x,y,0) = \begin{cases} 1, & \text{if } (x,y)\in B,\\ 0, & \text{if } (x,y)\in \Omega\setminus B,\end{cases}
\end{aligned}$$

where $B = \{(x,y)|(x-x_0)^2 + (y-y_0)^2 \leq r^2\}$ and ${}_0 > 0$. The boundary conditions are

$$\tilde{u}(0,y,t) = \tilde{u}(1,y,t) = \tilde{v}(x,0,t) = \tilde{v}(x,1,t) = 0, \qquad \forall x,y \in (0,1), \quad \forall t > 0$$

0.2 Numerical Scheme

The main Numerical Scheme is to utilize Finite Volume Method and Doner-cell upwind method.

0.2.1 Spatial discretization

$$\begin{aligned} 0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N-\frac{1}{2}} < x_{N+\frac{1}{2}} = 1, \\ 0 = y_{\frac{1}{2}} < y_{\frac{3}{2}} < \cdots < y_{N-\frac{1}{2}} < y_{N+\frac{1}{2}} = 1, \end{aligned}$$

where $\Delta x = \Delta y = \frac{1}{N}$.

Finite Volume Method:

$$\begin{aligned} \mathbf{Q}_{i,j}^n &= \frac{1}{\Delta x \Delta y} \iint_{[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]} \mathbf{q}(x,y,0+n\Delta t) dxdy \\ &= \frac{1}{\Delta x \Delta y} \iint_{[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]} \begin{bmatrix} \tilde{\rho}(x,y,0+n\Delta t) \\ \tilde{u}(x,y,0+n\Delta t) \\ \tilde{v}(x,y,0+n\Delta t) \\ \tilde{p}(x,y,0+n\Delta t) \end{bmatrix} dxdy \\ &\approx \begin{bmatrix} \tilde{\rho}(x_i,y_j,0+n\Delta t) \\ \tilde{u}(x_i,y_j,0+n\Delta t) \\ \tilde{v}(x_i,y_j,0+n\Delta t) \\ \tilde{p}(x_i,y_j,0+n\Delta t) \end{bmatrix} := \begin{bmatrix} \tilde{\rho}_{i,j}^n \\ \tilde{u}_{i,j}^n \\ \tilde{v}_{i,j}^n \\ \tilde{p}_{i,j}^n \end{bmatrix}, \end{aligned}$$

where Δt is chosen small to ensure the stability.

Equations:

$$\frac{\partial \mathbf{q}}{\partial t} + \mathbf{A} \frac{\partial \mathbf{q}}{\partial x} + \mathbf{B} \frac{\partial \mathbf{q}}{\partial y} = 0$$

Conservative scheme:

$$\mathbf{Q}_{i,j}^{n+1} = \mathbf{Q}_{i,j}^n - \frac{\Delta t}{\Delta x}(\mathbf{F}_{i+\frac{1}{2},j}^n - \mathbf{F}_{i-\frac{1}{2},j}^n) - \frac{\Delta t}{\Delta y}(\mathbf{G}_{i,j+\frac{1}{2}}^n - \mathbf{G}_{i,j-\frac{1}{2}}^n), \quad \begin{matrix} i = 1 \cdots N, \\ j = 1 \cdots N. \end{matrix}$$

Doner-cell upwind:

$$\begin{aligned} \mathbf{F}_{i+\frac{1}{2},j}^n &= \mathbf{A}_+ \mathbf{Q}_{i,j}^n + \mathbf{A}_- \mathbf{Q}_{i+1,j}^n, \\ \mathbf{G}_{i,j+\frac{1}{2}}^n &= \mathbf{B}_+ \mathbf{Q}_{i,j}^n + \mathbf{B}_- \mathbf{Q}_{i,j+1}^n, \end{aligned}$$

where

$$\mathbf{A} = \mathbf{R}\mathbf{\Lambda}\mathbf{R^{-1}}, \quad \mathbf{A}_+ = \mathbf{R}\mathbf{\Lambda}_+\mathbf{R^{-1}}, \quad \mathbf{A}_- = \mathbf{R}\mathbf{\Lambda}_-\mathbf{R^{-1}},$$

$$\mathbf{B} = \mathbf{S}\mathbf{\Sigma}\mathbf{S^{-1}}, \quad \mathbf{B}_+ = \mathbf{S}\mathbf{\Sigma}_+\mathbf{S^{-1}}, \quad \mathbf{B}_- = \mathbf{R}\mathbf{\Sigma}_-\mathbf{S^{-1}},$$

$\mathbf{\Lambda}$ and $\mathbf{\Sigma}$ are eigenvalues diagonalizable matrice.

0.2.2 Boundary conditions:

Create ghost cell averages by extrapolation

$$\mathbf{Q}^n_{\frac{1}{2},j} = \begin{bmatrix} \tilde{\rho}_{\frac{1}{2},j} \\ \tilde{u}_{\frac{1}{2},j} \\ \tilde{v}_{\frac{1}{2},j} \\ \tilde{p}_{\frac{1}{2},j} \end{bmatrix} = \begin{bmatrix} \frac{3}{2}\tilde{\rho}^n_{1,j} - \frac{1}{2}\tilde{\rho}^n_{2,j} \\ \tilde{u}(0, y_j, n\Delta t) \\ \frac{3}{2}\tilde{v}^n_{1,j} - \frac{1}{2}\tilde{v}^n_{2,j} \\ \frac{3}{2}\tilde{p}^n_{1,j} - \frac{1}{2}\tilde{p}^n_{2,j} \end{bmatrix}, \quad j = 1, \cdots, N, \quad \begin{matrix} \mathbf{Q}^n_{-k,j} = 2\mathbf{Q}^n_{\frac{1}{2},j} - \mathbf{Q}^n_{k+1,j}, \\ k = 0, \cdots, N-1, \end{matrix}$$

$$\mathbf{Q}^n_{N+\frac{1}{2},j} = \begin{bmatrix} \tilde{\rho}_{N+\frac{1}{2},j} \\ \tilde{u}_{N+\frac{1}{2},j} \\ \tilde{v}_{N+\frac{1}{2},j} \\ \tilde{p}_{N+\frac{1}{2},j} \end{bmatrix} = \begin{bmatrix} \frac{3}{2}\tilde{\rho}^n_{N,j} - \frac{1}{2}\tilde{\rho}^n_{N-1,j} \\ \tilde{u}(1, y_j, n\Delta t) \\ \frac{3}{2}\tilde{v}^n_{N,j} - \frac{1}{2}\tilde{v}^n_{N-1,j} \\ \frac{3}{2}\tilde{p}^n_{N,j} - \frac{1}{2}\tilde{p}^n_{N-1,j} \end{bmatrix}, \quad j = 1, \cdots, N, \quad \begin{matrix} \mathbf{Q}^n_{N+k,j} = 2\mathbf{Q}^n_{N+\frac{1}{2},j} - \mathbf{Q}^n_{N-k+1,j}, \\ k = 1, \cdots, N, \end{matrix}$$

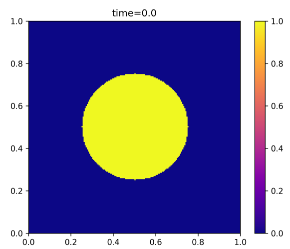
$$\mathbf{Q}^n_{i,\frac{1}{2}} = \begin{bmatrix} \tilde{\rho}_{i,\frac{1}{2}} \\ \tilde{u}_{i,\frac{1}{2}} \\ \tilde{v}_{i,\frac{1}{2}} \\ \tilde{p}_{i,\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} \frac{3}{2}\tilde{\rho}^n_{i,1} - \frac{1}{2}\tilde{\rho}^n_{i,2} \\ \frac{3}{2}\tilde{u}^n_{i,1} - \frac{1}{2}\tilde{u}^n_{i,2} \\ \tilde{v}(x_i, 0, n\Delta t) \\ \frac{3}{2}\tilde{p}^n_{i,1} - \frac{1}{2}\tilde{p}^n_{i,2} \end{bmatrix}, \quad i = 1, \cdots, N, \quad \begin{matrix} \mathbf{Q}^n_{i,-k} = 2\mathbf{Q}^n_{i,\frac{1}{2}} - \mathbf{Q}^n_{i,k+1}, \\ k = 0, \cdots, N-1, \end{matrix}$$

$$\mathbf{Q}^n_{i,N+\frac{1}{2}} = \begin{bmatrix} \tilde{\rho}_{i,N+\frac{1}{2}} \\ \tilde{u}_{i,N+\frac{1}{2}} \\ \tilde{v}_{i,N+\frac{1}{2}} \\ \tilde{p}_{i,N+\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} \frac{3}{2}\tilde{\rho}^n_{i,N} - \frac{1}{2}\tilde{\rho}^n_{i,N-1} \\ \frac{3}{2}\tilde{u}^n_{i,N} - \frac{1}{2}\tilde{u}^n_{i,N-1} \\ \tilde{v}(x_i, 1, n\Delta t) \\ \frac{3}{2}\tilde{p}^n_{i,N} - \frac{1}{2}\tilde{p}^n_{i,N-1} \end{bmatrix}, \quad i = 1, \cdots, N, \quad \begin{matrix} \mathbf{Q}^n_{i,N+k} = 2\mathbf{Q}^n_{i,N+\frac{1}{2}} - \mathbf{Q}^n_{i,N-k+1}, \\ k = 1, \cdots, N, \end{matrix}$$

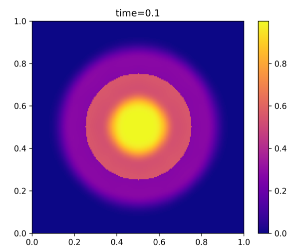
where only $\mathbf{Q}^n_{0,j}, \mathbf{Q}^n_{N+1,j}, \mathbf{Q}^n_{i,0}, \mathbf{Q}^n_{i,N+1}$ are used.

0.3 Results

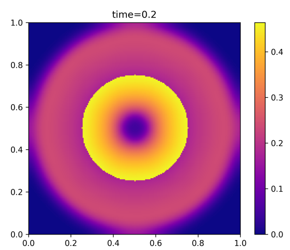
0.3.1 Q1



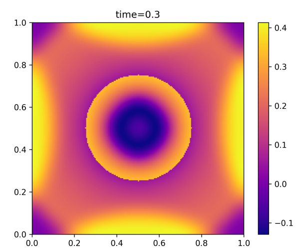
(a) Time=0.00



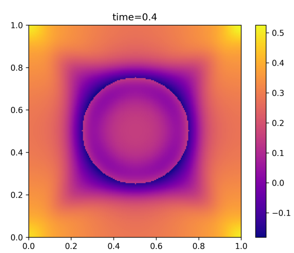
(b) Time=0.10



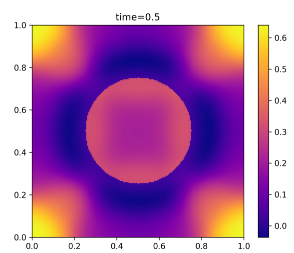
(c) Time=0.20



(d) Time=0.30



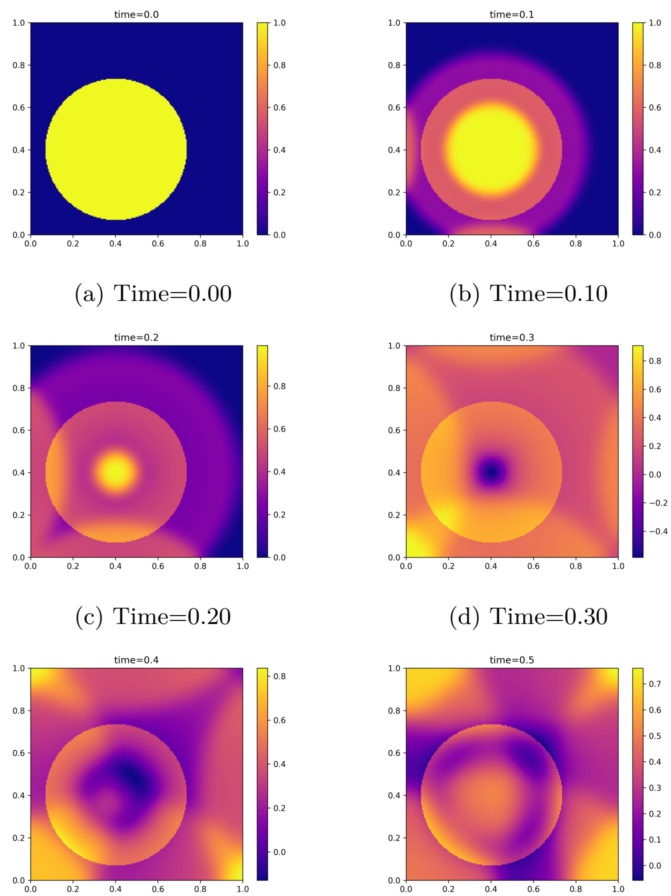
(e) Time=0.40



(f) Time=0.50

Figure 1: $\tilde{\rho}(x, y, t)$, $N = 200$, $\Delta t = 0.0001$, $(x_0, y_0, r) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{4})$

0.3.2 Q2



(a) Time=0.00

(b) Time=0.10

(c) Time=0.20

(d) Time=0.30

(e) Time=0.40

(f) Time=0.50

Figure 2: $\tilde{\rho}(x, y, t)$, $N = 200$, $\Delta t = 0.0001$, $(x_0, y_0, r) = (\frac{2}{5}, \frac{2}{5}, \frac{1}{3})$

1 Reference

1. Cai, Z. (2024, Feb.). Lecture 3,7 [Computational Fluid Dynamics].