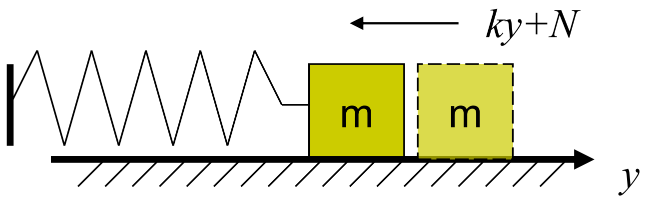
Exercise Lesson

[**Example 1**] Study a physical system comprised of a spring and a mass. Considering the coulomb friction $\pm N$, the motion equation is given by

$$m\ddot{y} + ky \pm N = 0$$

where m is the mass. k is the stiffness coefficient of the spring. y denotes the displacement. Suppose m and k satisfy m=1, k=1. Plot the phase portrait in the phase plane $y - \dot{y}$



Solution:

The motion equation is

$$\ddot{y} + y \pm N = 0$$

Let $x_1 = y \pm N$ and substitute it into the above equation. Then we have

$$\ddot{x}_1 + x_1 = 0 \qquad \textbf{(1)}$$

The state equation of system is given by:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 \end{cases} \qquad \textbf{(2)}$$

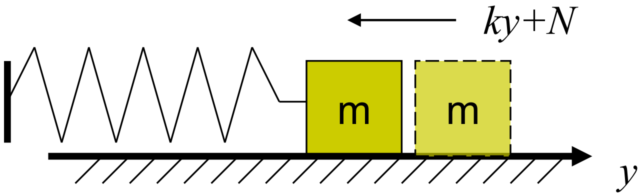
$$\Rightarrow \quad \frac{\mathrm{d}x_2}{\mathrm{d}x_1} = -\frac{x_1}{x_2}$$

Solve this differential equation!

The solution of $\dfrac{\mathrm{d}x_2}{\mathrm{d}x_1} = -\dfrac{x_1}{x_2}$ **is**

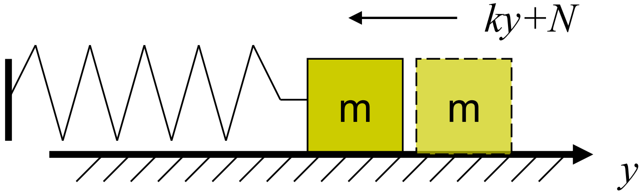
$x_2^2 + x_1^2 = c$ **or** $(\dot{y}^2) + (y \pm N)^2 = c$ **…………(3)**

Eq. (3) is the equation of phase trajectories!



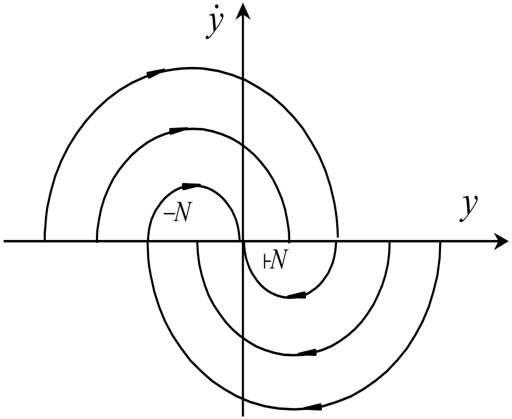
Analysis: If $\dot{y} > 0$ **, the sign of friction is '+'.**

$$m\ddot{y} + ky \pm N = 0$$



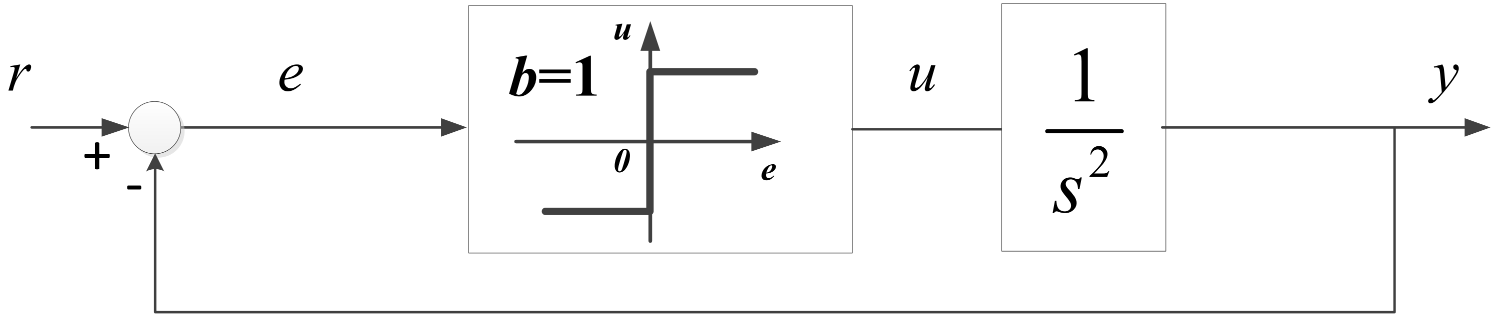
$$(\dot{y}^2) + (y+N)^2 = c, \qquad \text{if } \dot{y} > 0$$

$$(\dot{y}^2) + (y-N)^2 = c, \qquad \text{if } \dot{y} < 0$$



[Example 2] The following nonlinear system is excited with a unit step input signal. If the initial state of the system is

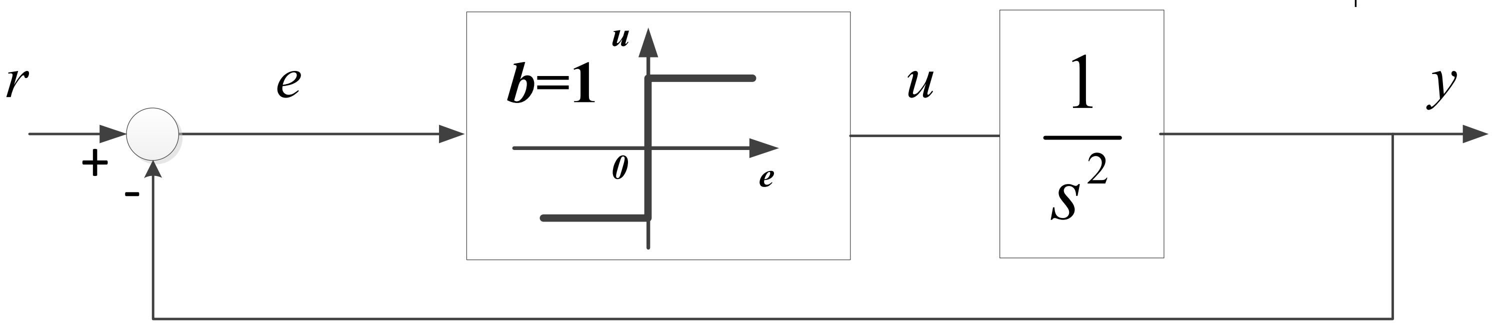
$$y(0) = -8, \quad \dot{y}(0) = 0$$



(1) Plot the phase trajectory of the nonlinear system from the initial point $y(0) = 0, \quad \dot{y}(0) = 0$

(2) Will the system produce the periodic motion? If the motion is existed, please calculate the amplitude A and the period T of the motion.

(1) 分区以及确定坐标轴（坐标轴肯定是y和y的导数）



$$u=\begin{cases}1 & e>0\\ -1 & e<0\end{cases} \qquad r=1(t)$$

$1=e+y$

$e=1-y$

$$u=\ddot{y}=\begin{cases}1 & e>0\\ -1 & e<0\end{cases}$$

$y<1$

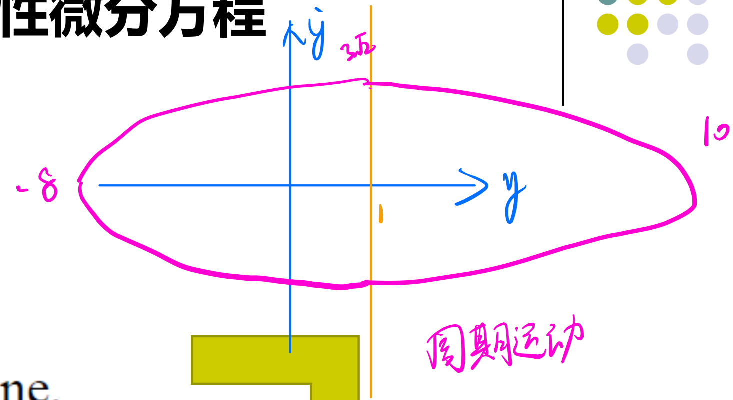
the switch line is y=1 (e=0)

$y>1$

(2) 计算切换线方程

切换线：y=1

(3) 从初始条件开始分区解线性微分方程



$$y(0) = -8,\quad \dot{y}(0) = 0$$

$$e(0) = r(0) - y(0) = 9 > 0$$

The curve is on the left of the switch line.

$$\begin{cases} \ddot{y} = 1 \\ \dot{y} = t + c_1 \\ y = \dfrac{t^2}{2} + c_1 t + c_2 \end{cases} \Rightarrow \begin{cases} c_1 = 0 \\ c_2 = -8 \end{cases}$$

We have $y = \dfrac{\dot{y}^2}{2} - 8$

$$\begin{cases} y = \dfrac{\dot{y}^2}{2} - 8 \\ y = 1 \end{cases} \Rightarrow \begin{cases} y = 1 \\ \dot{y} = 3\sqrt{2} \end{cases} \Rightarrow t_1 = 3\sqrt{2}$$

切换线：y=1

(3) 从初始条件开始分区解线性微分方程

新初始条件： $y(0)=1,\ \ \dot{y}(0)=3\sqrt{2}$

$$\begin{cases} \ddot{y}=-1 \\ \dot{y}=-t+c_1 \\ y=-\dfrac{t^2}{2}+c_1 t+c_2 \end{cases} \Rightarrow \begin{cases} c_1=3\sqrt{2} \\ c_2=1 \end{cases}$$

We have $y=-\dfrac{\dot{y}^2}{2}+10$ and $\begin{cases} \dot{y}=-t+3\sqrt{2} \\ y=-\dfrac{t^2}{2}+3\sqrt{2}t+1 \end{cases}$

$$\begin{cases} y=-\dfrac{\dot{y}^2}{2}+10 \\ y=1 \end{cases} \Rightarrow \begin{cases} y(t_2)=1 \\ \dot{y}(t_2)=\pm 3\sqrt{2} \end{cases} \Rightarrow t_2=6\sqrt{2}$$

切换线：y=1

(3) 从初始条件开始分区解线性微分方程

新初始条件： $y(0)=1,\ \ \dot{y}(0)=-3\sqrt{2}$

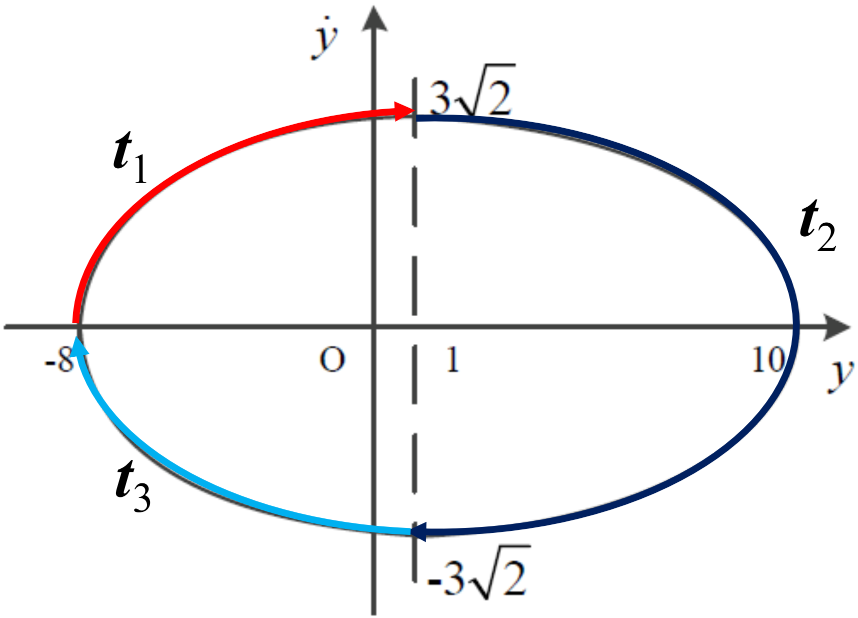
$$\begin{cases} \ddot{y}=1 \\ \dot{y}=t+c_1 \\ y=\dfrac{t^2}{2}+c_1t+c_2 \end{cases} \Rightarrow \begin{cases} c_1=-3\sqrt{2} \\ c_2=1 \end{cases}$$

We have $y=\dfrac{\dot{y}^2}{2}-8$ and $\begin{cases} \dot{y}=t-3\sqrt{2} \\ y=\dfrac{t^2}{2}-3\sqrt{2}t+1 \end{cases}$

And the curve is back to the point $y(t_3)=-8,\ \ \dot{y}(t_3)=0$ at last.

$t_3=3\sqrt{2}$

(4) 连接各区解曲线并画图



$$T = t_1 + t_2 + t_3 = 12\sqrt{2}$$

幅值为9