Dealing with uncertainties in Physics

1.1 Measurement and Uncertainty

To understand the world around us, we try to make mathematical models to describe the phenomena we observe. Experimentation and quantitative measurement are the means by which these models are explored. However, no measurement is absolutely perfect and there is always some uncertainty. Obviously, the more careful the experiment the more precise the end result.

In order to compare our measurements to the predictions of the model, or to other researchers' measurements, we need some way of indicating the likely uncertainty in our experiment. When comparing different measurements, they can be said to agree if there is any overlap between the uncertainty ranges about the measured values. If the uncertainty bounds are not known, no comparison can be made. The result of any quantitative measurement has two essential components:

- A numerical value (expressed in SI units) which gives the best estimate of the quantity being measured (the *measurand*). This estimate may well be a single measurement or the mean value of a series of measurements.
- A non-negative measure of the uncertainty associated with this estimated value. In experimental physics this may well be the variability or dispersion of a series of similar measurements expressed as a standard uncertainty (standard deviation).

By definition, the term *error* (or measurement error) is the difference between the true value and the measured value. Uncertainty is caused by the interplay of errors which create dispersion around the estimated value of the measurand; the smaller the dispersion, the smaller the uncertainty.
Even if the terms error and uncertainty are used somewhat interchangeably in everyday descriptions, they actually have different meanings. They should not be used as synonyms.
The $\pm$ (plus or minus) symbol that often follows the reported value of a measurand and the numerical quantity that follows this symbol, indicate the uncertainty associated with the particular measurand and not the error.
If repeated measurements are made of the same quantity, statistical procedures can be used to determine the uncertainties in the measurement process. This type of statistical analysis provides uncertainties which are determined from the data themselves without requiring further estimates. The important variables in such analyses are the *mean*, the *standard deviation* and the *standard uncertainty of the mean* (also referred to as the standard deviation of the mean or the standard error of the mean).

Systematic and random error

Experimental errors may be divided into two classes:

- systematic error, often referred to as bias, can be identified as a fixed value for a discrepancy and should be corrected at the earliest practical opportunity in the measurement process
- random error, arising from unpredictable variations which influence the measurement procedure, and are associated with the actual measurement (for example, failure to properly account for temperature fluctuations).

Random errors may be analysed statistically while systematic errors are resistant to statistical analysis. Systematic errors are generally evaluated by nonstatistical procedures.

For this part of the course, we want only a reasonable **estimate** of the errors, rather than a rigorous statistical analysis, which would be too time consuming. In many cases a simple estimate of the **maximum uncertainty** is enough.

Uncertainty in reading a measurement from a scale

It is easy to determine the uncertainty involved in reading the scale of an instrument. A reasonable estimate of this is the smallest division of the scale, or more commonly half of the smallest division.

Example

A ruler with a smallest division of 1 mm is used to measure the length of an object. You might think only one value is involved. In fact there are two, one at each end of the measurement. Hence with an uncertainty of 0.5 mm at each end the total uncertainty is ± 1 mm.
The result might be quoted as 123 ± 1 mm. The ± 1 mm represents the uncertainty in the result, and indicates that the true result lies somewhere between 122 and 124 mm.

The percentage error is simply given by:

$$\frac{1}{123} \times 100 = 0.08\%.$$

Significant Figures

In calculating a result the answer is only as accurate as the least precisely known quantity. Because of this, the number of digits quoted in the final result should not be more than the number of significant figures contained in the least known quantity.

For more detailed definition of significant figures see Wikipedia.

Example

The area of a rectangle of sides 11.3 cm and 6.8 cm is 76.84 cm^2. However, as we only know the lengths of the sides to three and two significant figures respectively, the result must be quoted to **two significant figures only**, which in this case is 77 cm^2 (the eight is rounded up).

NOTE: Additional significant figures should be retained in intermediate calculations, and the final result rounded off.

Reducing Uncertainties

One way of reducing the uncertainty in a measurement is to take a number of readings, typically six, of the same observation and to use the average of these readings as the accepted value of the measured quantity. The standard deviation would then represent the uncertainty in the measurement. Standard deviation, usual symbol σ, is a measure how the numbers spread around the average. The formula for standard deviation is

$$\sigma = \sqrt{\frac{\Sigma_i^N (x_i - x_{mean})^2}{N-1}}$$

For example, in the measurement of the period of a simple pendulum, instead of taking just one measurement for the period and using this result as the accepted value, the period should be determined, say six times, and the average used for the final result.

There are more examples in Wikipedia.

Combining Uncertainties

The tables below give expressions that enable us to calculate the final uncertainty when two quantities, each with their own uncertainties, are combined. Here ΔA and ΔB are the absolute uncertainties associated with the measurements of the quantities A and B respectively, and ΔZ is the uncertainty in the final derived result Z.

To estimate the uncertainty in a derived quantity such as Z, you can make use of the formulas in either of the Tables 1.1 and 1.2 below.

Relationship between Z, A and B	Formula to give the **maximum uncertainties**
$Z = A + B$ or $Z = A - B$	$\Delta Z = \Delta A + \Delta B$
$Z = A \times B$ or $Z = A \div B$	$\frac{\Delta Z}{Z} = \frac{\Delta A}{A} + \frac{\Delta B}{B}$
$Z = A^n$, for all n	$\frac{\Delta Z}{Z} = \lvert n \rvert \frac{\Delta A}{A}$
$Z = \ln A$	$\Delta Z = \frac{\Delta A}{A}$
$Z = e^A$	$\frac{\Delta Z}{Z} = \Delta A$

Table 1.1: These formulae for combining uncertainties give a simple worst case approximation.

Relationship between Z, A and B	Formula to give the **standard error**
$Z = A + B$ or $Z = A - B$	$(\Delta Z)^2 = (\Delta A)^2 + (\Delta B)^2$
$Z = A \times B$ or $Z = A \div B$	$(\frac{\Delta Z}{Z})^2 = (\frac{\Delta A}{A})^2 + (\frac{\Delta B}{B})^2$

Table 1.2: These formulae for combining incoherent (uncorrelated) uncertainties give a better approximation of the measurement uncertainty when the uncertainties are random and not systematically in one direction.

Example of simple worst case (or maximum) uncertainty

A piece of A4 paper is measured on the long side as 29.7 ± 1 mm, and on the short side as 21.0 ± 1 mm.
The area A is therefore $29.7 \times 21.0 = 623.7$ mm^2.
For the uncertainty we use the second formula in Table 1: $\Delta A = A\left(\frac{1}{29.7} + \frac{1}{21}\right) = 50.7$.
We have therefore $A = 623.7$ with uncertainty 50.7 mm^2, and we quote the result (with consideration of significant figures) as:

$$A = (62 \pm 5)\ \text{cm}^2$$

Example of standard uncertainty in random uncorrelated data

A pendulum is timed as it swings back and forth over a small angle yielding a series of measurement of each period such as $T = 1.23 \pm 0.005$ s.
Six such measurements produced results of 1.23, 1.17, 1.21, 1.19, 1.22, 1.21 (all in seconds) with uncertainty ± 0.05 s.
The mean (or average) and standard deviation (SD) are calulated to be:
Mean = 1.205 s, SD = 0.0217 s. The period is then quoted as

$$T = (1.20 \pm 0.02)\ \text{s}$$

Example of combined uncertainties

A cricket ball of mass $m = 0.158 \pm 0.001$ kg is thrown and its trajectory videoed. The speed at the top of its flight is determined from the calibrated video to be $v = 1.76 \pm 0.01$ m s^{-1}.

The momentum at that point is therefore $p = 0.158 \times 1.76 = 0.278$ kg m s^{-1}.

For the uncertainty we have two methods:

- Maximum uncertainty (from Table 1):

$$\Delta p = p\left(\frac{\Delta m}{m} + \frac{\Delta v}{v}\right) = 0.278\left(\frac{0.001}{0.158} + \frac{0.01}{1.76}\right) = 0.00334$$

  Hence we can write: $p = (0.278 \pm 0.003)$ kg m s^{-1}.

- Standard error (from Table 2):

$$\Delta p = p\sqrt{\left(\frac{\Delta m}{m}\right)^2 + \left(\frac{\Delta v}{v}\right)^2} = 0.278\sqrt{\left(\frac{0.001}{0.158}\right)^2 + \left(\frac{0.01}{1.76}\right)^2} = 0.00236$$

  Hence we can write: $p = (0.278 \pm 0.002)$ kg m s^{-1}.

Example of functional relationship

In the Viscosity Lab there is a functional relationship developed from theory which calculates viscosity as a function of the sphere's radius R, the distance the sphere falls d and the time taken t:

$$\eta = a\,\frac{R^2 t}{d}$$

where a is treated as a constant (without uncertainty).
For the maximum uncertainty (Table 1) we have:

$$\frac{\Delta\eta}{\eta} = 2\,\frac{\Delta R}{R} + \frac{\Delta t}{t} + \frac{\Delta d}{d}$$

Note that the minus one (-1) for the dependence on d produces a plus one ($+1$) in the expression for uncertainty.

Precision and Accuracy

Three terms which are often used in association with laboratory errors are:

- accuracy (inaccuracy)
- bias
- precision (imprecision).

Precision is a description of the *random error* in the measurement–how good is the result determined without reference to any true value.

Accuracy referred to how close the measurement is to the "true" value. This only has meaning if there is some indication of the true value. If the measured result is different from the assumed "gold standard", the error is then a *systematic error*. For example, if your measurement of the boiling water temperature is not 100 degrees as expected, in normal pressure conditions. Then there is a chance that there this is a systematic error. The measuring system, that is, your thermometer was not calibrated properly. Fig.1.1 shows a target result as an example of of precision, accuracy and errors.

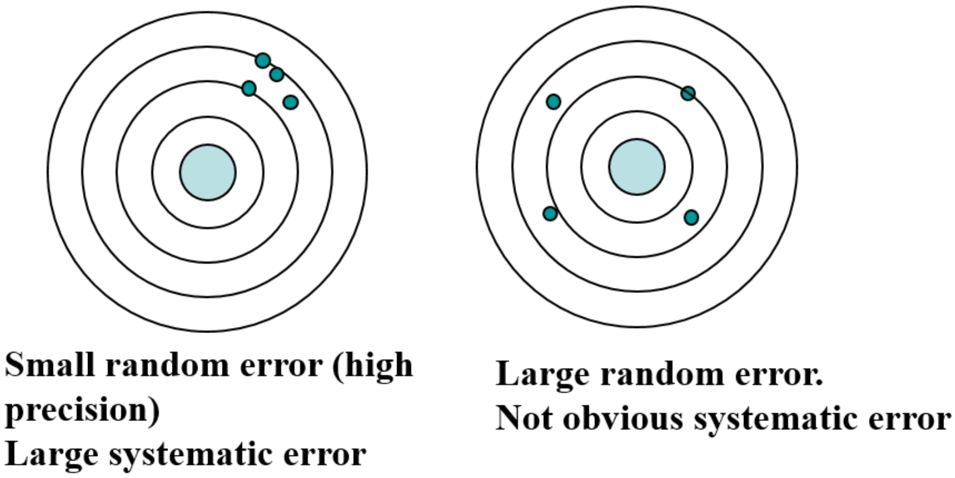

Figure 1.1: Example of two target results. The left one shows the shooter (measurement) has high precision. But there might be something wrong with the sighting of the gun. While the right one shows that the shooter did not do a very good job. The random errors can be estimated from repeated experiments. Systematic errors require a gold standard for comparison.