考试样题解答（单选题和填空题）

一、单选题

D 1. $-1+\sqrt{3}i = 2\left(\cos\frac{2}{3}\pi + i\sin\frac{2}{3}\pi\right)$

$$\sqrt[3]{-1+\sqrt{3}i} = \sqrt[3]{2}\left(\cos\frac{\frac{2}{3}\pi+2k\pi}{3} + i\sin\frac{\frac{2}{3}\pi+2k\pi}{3}\right) \quad k=0,1,2.$$

$k=0 \quad w_0 = \sqrt[3]{2}\, e^{i\frac{2}{9}\pi}$

$k=1 \quad w_1 = \sqrt[3]{2}\, e^{i\frac{8}{9}\pi}$

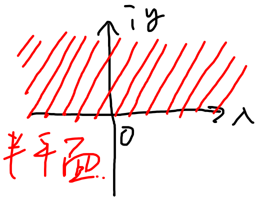
$k=2 \quad w_2 = \sqrt[3]{2}\, e^{i\frac{14}{9}\pi}$

C 2. $\mathrm{Ln}(z)$ 在 $z=0$ 处无定义，在负实半轴上 $(x<0, y=0)$ 不连续.

D 3. $\mathrm{Re}(z^2)=1 \rightarrow \quad z^2=(x+iy)^2 = x^2-y^2+i\cdot 2xy$

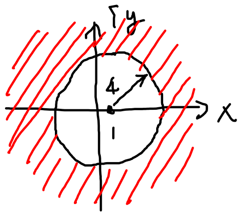
$\mathrm{Re}(z^2) = x^2-y^2=1 \rightarrow$ 双曲

D 4. A. $\mathrm{Im}\, z>0$

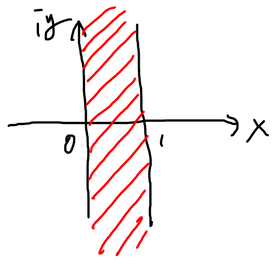


上半平面

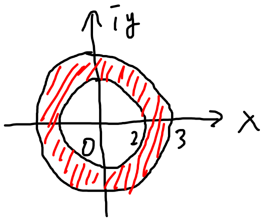
B. $|z-1|>4$



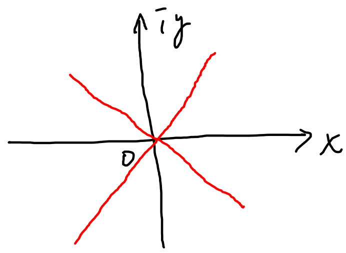
C. $0<\mathrm{Re}(z)<1$



D. $2<|z|<3$



A 5. $f(z)=x^3+iy^3 \quad u=x^3 \quad v=y^3$



$\frac{\partial u}{\partial x}=3x^2 \quad \frac{\partial u}{\partial y}=0$

$\frac{\partial v}{\partial x}=0 \quad \frac{\partial v}{\partial y}=3y^2$

由 C-R 方程 $\begin{cases} \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x} \end{cases} \rightarrow \begin{cases} 3x^2=3y^2 \\ 0=-0 \end{cases} \rightarrow x=\pm y.$

$f(z)$ 在 $y=\pm x$ 上处处可导，全平面无处解析.

D 6、$\mathrm{Arg}(z)=\arg(z)+2k\pi,\ k=0,\pm1,\pm2,\cdots$

主值 唯一介于 $(-\pi,\pi]$ 之间.

B 7、全平面解析函数积分与平面上任何路径无关.

A 8. 数项级数 $\sum_{n=0}^{\infty}a_n$ 绝对收敛 $\longleftrightarrow$ $\sum_{n=0}^{\infty}|a_n|$ 这个正项实级数收敛.

A. $\sum_{n=1}^{\infty}\left|\left(\frac{6+5i}{8}\right)^n\right|=\sum_{n=1}^{\infty}\left(\frac{|6+5i|}{8}\right)^n=\sum_{n=1}^{\infty}\left(\frac{\sqrt{61}}{8}\right)^n$ 收敛.

B. $\sum_{n=1}^{\infty}\left|\frac{i^n}{n}\right|=\sum_{n=1}^{\infty}\frac{1}{n}$ 发散.

C. $\sum_{n=1}^{\infty}\left|\left(\frac{2+i}{2}\right)^n\right|=\sum_{n=1}^{\infty}\left(\frac{\sqrt{5}}{2}\right)^n$ 发散.

D. $\sum_{n=1}^{\infty}\left|\frac{\cos(in)}{2^n}\right|=\sum_{n=1}^{\infty}\left|\frac{\frac{e^{-n}+e^{n}}{2}}{2^n}\right|>\sum_{n=1}^{\infty}\frac{1}{2}\left(\frac{e}{2}\right)^n$ 发散.

C 9、$e^z-1=\left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots\right)-1=z\left(1+\frac{z}{2!}+\frac{z^2}{3!}+\cdots\right)$

$=z\varphi(z)$ $\leftarrow$ $\varphi(0)\neq0.$

$\rightarrow$ $z^2(e^z-1)=z^3\varphi(z)$ 以 $z=0$ 为 3 级零点 (3重根)

$\rightarrow$ $z=0$ 为 $\frac{1}{z^2(e^z-1)}$ 的 3 级极点.

B 10、$C_0=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}f(t)\,dt=\frac{1}{T}\left[\int_{-\frac{T}{2}}^{0}0\,dt+\int_{0}^{\frac{T}{2}}2\,dt\right]=1$ $(n=0)$

$C_n=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}f(t)e^{-in\omega_0t}\,dt=\frac{1}{T}\int_{0}^{\frac{T}{2}}2e^{-i\frac{2n\pi}{T}t}\,dt$ $\omega_0=\frac{2\pi}{T}$ 基频

$=\frac{2}{T}\frac{1}{-\frac{2n\pi i}{T}}e^{-\frac{2n\pi i}{T}t}\Big|_{0}^{\frac{T}{2}}=\frac{i}{n\pi}\left(e^{-n\pi i}-1\right)$ $(n\neq0)$

$\longrightarrow$ $C_1=-\frac{2i}{\pi}$, $C_2=0$, $C_3=-\frac{2i}{3\pi}$

二、填空题

1、 $z=(1+i)^2 = 1^2-1^2+i\cdot 2\cdot 1\cdot 1 = 2i \qquad |z|=2$

或者 $|z| = |(1+i)^2| = |1+i|^2 = \sqrt{2}^2 = 2$

2. $f(z) = x^2-y^2+i\cdot 2xy$, $z_0 = \sqrt{3}-i \rightarrow x=\sqrt{3}$, $y=-1$

$f(z_0) = \sqrt{3}^2-(-1)^2+i\cdot 2\cdot\sqrt{3}\cdot(-1) = 2-2\sqrt{3}i$

3、 $\frac{1}{2}+\frac{\sqrt{3}}{2}i = \cos\frac{1}{3}\pi + i\sin\frac{1}{3}\pi$

$$\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^{999} = \left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^{999} = \cos\left(999\cdot\frac{\pi}{3}\right)+i\sin\left(999\cdot\frac{\pi}{3}\right)$$

$$= \cos 333\pi + i\sin 333\pi = \cos\pi + i\sin\pi = -1$$

4、 $z=\frac{1}{2}i \qquad 2\cos z\sin z = 2\left.\frac{e^{iz}+e^{-iz}}{2}\right|_{z=\frac{1}{2}i} \cdot \left.\frac{e^{iz}-e^{-iz}}{2i}\right|_{z=\frac{1}{2}i}$

$$= \left(e^{-\frac{1}{2}}+e^{\frac{1}{2}}\right)\cdot\frac{-i}{2}\cdot\left(e^{-\frac{1}{2}}-e^{\frac{1}{2}}\right) = \frac{e-e^{-1}}{2}i$$

或者. $2\cos z\sin z = \sin 2z \rightarrow \sin 2z\big|_{z=\frac{1}{2}i} = \sin i = \frac{e^{i\cdot i}-e^{-i\cdot i}}{2i} = \frac{-i}{2}\cdot(e^{-1}-e^{1})$

5、 $f(z) = z\,\mathrm{Re}(z) = (x+iy)\cdot x = x^2+ixy$

$u=x^2 \qquad v=xy$

$$f'(0) = \left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\Big|_{z=0} = (2x+yi)\Big|_{\substack{x=0\\y=0}} = 0$$

6. $\int_C \bar{z}\,dt = \int_0^1 \overline{z(t)}\,dz(t) = \int_0^1 \overline{z(t)}\,z'(t)\,dt$

$$= \int_0^1 \overline{(1-t+it)}\,(1-t+it)'\,dt = \int_0^1 (1-t-it)(-1+i)\,dt$$

$$= (-1+i)\int_0^1(1-t-it)\,dt = (-1+i)\left(t-\frac{t^2}{2}-i\frac{t^2}{2}\right)\Big|_0^1$$

$$= (-1+i)\left(\frac{1}{2}-\frac{1}{2}i\right) = i$$

7. $\oint_{|z|=2} \frac{\sin z}{(z-\frac{\pi}{2})^3} dz = 2\pi i \frac{1}{(3-1)!} (\sin z)'' \Big|_{z=\frac{\pi}{2}} = \pi i (-\sin z)\Big|_{z=\frac{\pi}{2}} = -\pi i$

8、 $\sum_{n=0}^{\infty} (1+i)^n z^n \rightarrow C_n = (1+i)^n$

① 比值法 $\lim_{n\to\infty} \left|\frac{C_{n+1}}{C_n}\right| = \lim_{n\to\infty} \left|\frac{(1+i)^{n+1}}{(1+i)^n}\right| = |1+i| = \sqrt{2} \rightarrow R = \frac{1}{\sqrt{2}}$

② 根值法 $\lim_{n\to\infty} \sqrt[n]{|C_n|} = \lim_{n\to\infty} \sqrt[n]{|1+i|^n} = |1+i| = \sqrt{2} \rightarrow R = \frac{1}{\sqrt{2}}$

9. $f(z) = \frac{1}{z^2(z-i)}$ 圆环 $1 < |z-i| < \infty$

以 i 为中心展开的级数形式 $f(z) = \sum_{n=-\infty}^{+\infty} C_n (z-i)^n$

$\frac{1}{z-i}$ 不用处理，先展开 $\frac{1}{z}$ ：

$\frac{1}{z} = \frac{1}{(z-i)-(-i)} = \frac{1}{1-\frac{-i}{z-i}} \cdot \frac{1}{z-i} = \sum_{n=0}^{\infty} \left(\frac{-i}{z-i}\right)^n \frac{1}{z-i}$

$= \sum_{n=0}^{\infty} \frac{(-i)^n}{(z-i)^{n+1}}$

上式两边同时求导 $\left(\frac{1}{z}\right)' = \frac{-1}{z^2} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(n+1)i^n}{(z-i)^{n+2}}$

$\rightarrow f(z) = \frac{1}{z^2} \cdot \frac{1}{z-i} = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1) i^n}{(z-i)^{n+3}}$

计算 C_{-5} 即求 $\frac{C_{-5}}{(z-i)^5}$ 项系数，上式令 $n=2$，取分子得：-3

10. $F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt = \int_0^{+\infty} e^{-\beta t} e^{-i\omega t} dt$

$= \int_0^{+\infty} e^{-(\beta+i\omega)t} dt = \frac{1}{-(\beta+i\omega)} e^{-(\beta+i\omega)t} \Big|_0^{+\infty}$

这里 $\lim_{t\to+\infty} e^{-(\beta+i\omega)t} = \lim_{t\to+\infty} e^{-\beta t}[\cos(-\omega t) + i\sin(-\omega t)] = 0$

$\rightarrow F(\omega) = \frac{1}{\beta+i\omega} \rightarrow F(2) = \frac{1}{\beta+2i} = \frac{\beta}{\beta^2+4} - i\frac{2}{\beta^2+4}$