Mesh Parameterization II

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Injectivity and Bijectivity

What are 'good' parameterizations?

- Low distortion + injectivity

UV

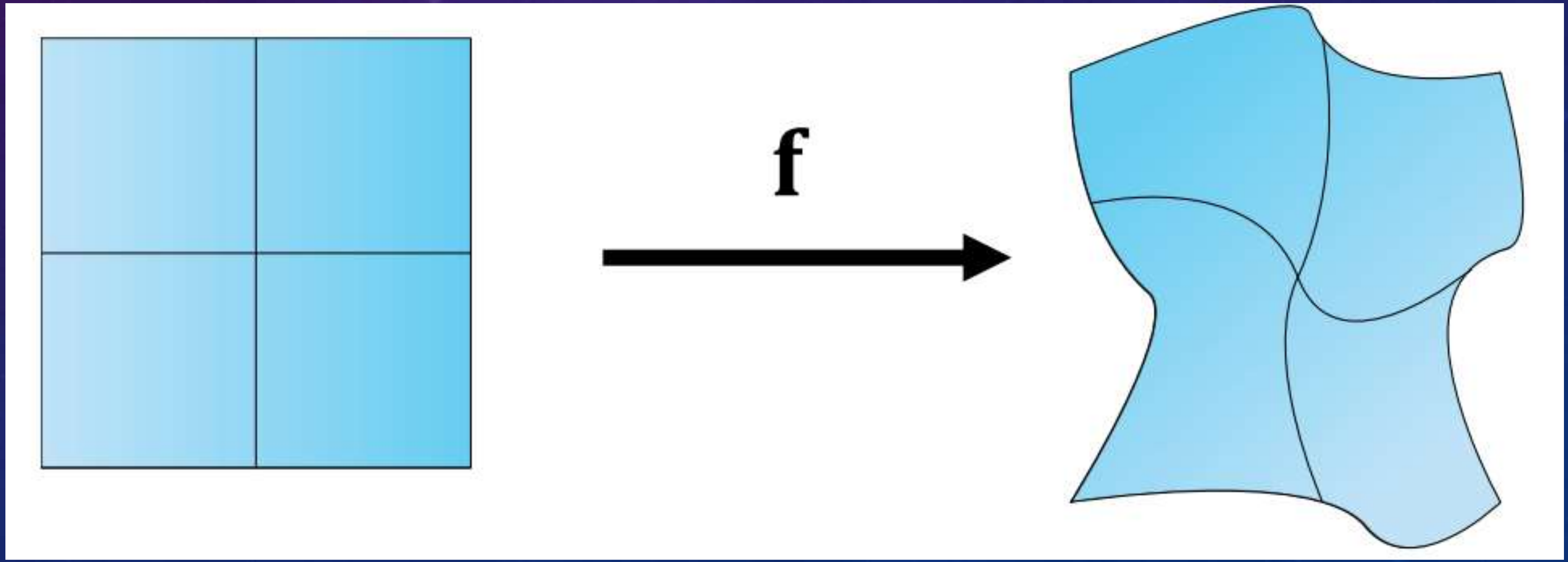
Not injective

Injective

Lower distortion
+ injective

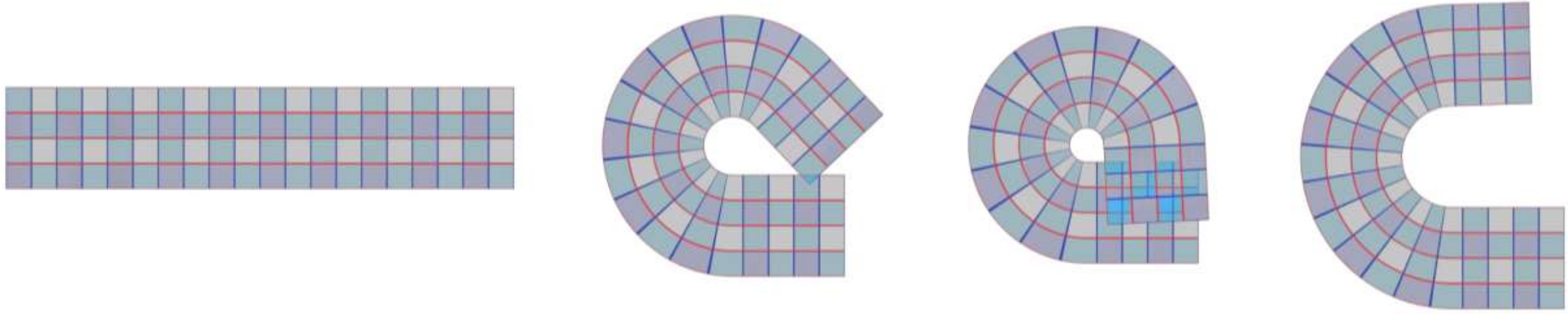
Condition of injectivity

- $\det\left(J_f(x)\right) > 0, \forall\, x$



Bijectivity

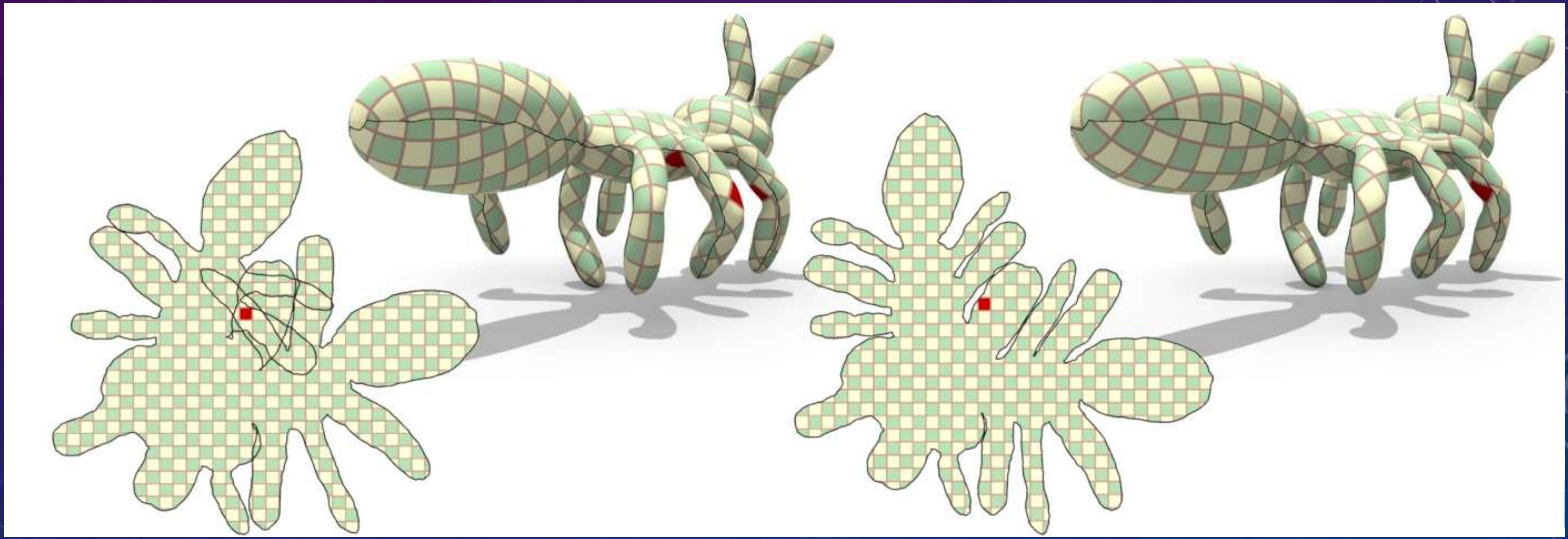
- $\det\left(J_f(x)\right) > 0, \forall\ x$ and $f(\Omega)$ no overlap



Injective, but not bijective

Bijective

Bijectivity



With overlap

Without overlap

Methods for Injective Parameterization

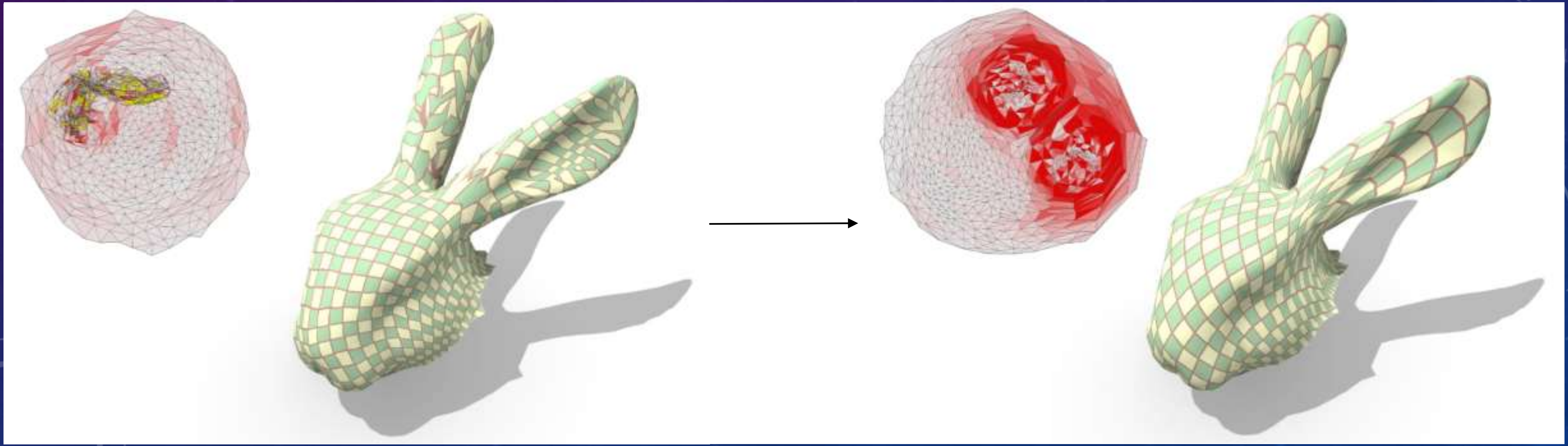
Injective Parameterization

Two cases:

- Flipped initializations → removing flipped elements
- Flip-free initializations → keeping mapping always flip-free

Removing flipped elements

- Convert the flipped elements into flip-free.



Closest point projection

- Signed singular value decomposition: $J_f = USV$
  - U and V are two rotation matrices
  - $S = diag(\sigma_1, \sigma_2), \sigma_1 \geq |\sigma_2|$
- Flip-free constraints: $\sigma_2 > 0$ (constraints)
- Distortion term (energy)

Methods

- Convex subspace
- Local-global optimization
- Penalty function
- Total unsigned area

Convex subspace

- Bounded conformal distortion constraint : $1 \leq \mathcal{T}\left(J_{ijk}\right) = \frac{\sigma}{\tau} \leq K$

$$J_{ijk}^{T} J_{ijk} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{pmatrix}$$

Let $a' = \frac{a+d}{2}, c' = \frac{a-d}{2}, b' = \frac{c-b}{2}, d' = \frac{c+b}{2}$, then

$$\sigma = \sqrt{a'^2 + b'^2} + \sqrt{c'^2 + d'^2}, \qquad \tau = \sqrt{a'^2 + b'^2} - \sqrt{c'^2 + d'^2}$$

Convex subspace

- Bounded conformal distortion constraint : $1 \leq \mathcal{T}\left(J_{ijk}\right) = \frac{\sigma}{\tau} \leq K$

$$\sigma = \sqrt{a'^2 + b'^2} + \sqrt{c'^2 + d'^2}, \qquad \tau = \sqrt{a'^2 + b'^2} - \sqrt{c'^2 + d'^2}$$

$$\tau > 0 \Longrightarrow \sqrt{c'^2 + d'^2} < \sqrt{a'^2 + b'^2}$$

$$\frac{\sigma}{\tau} \leq K \Longrightarrow \sqrt{c'^2 + d'^2} \leq \frac{K-1}{K+1}\sqrt{a'^2 + b'^2}$$

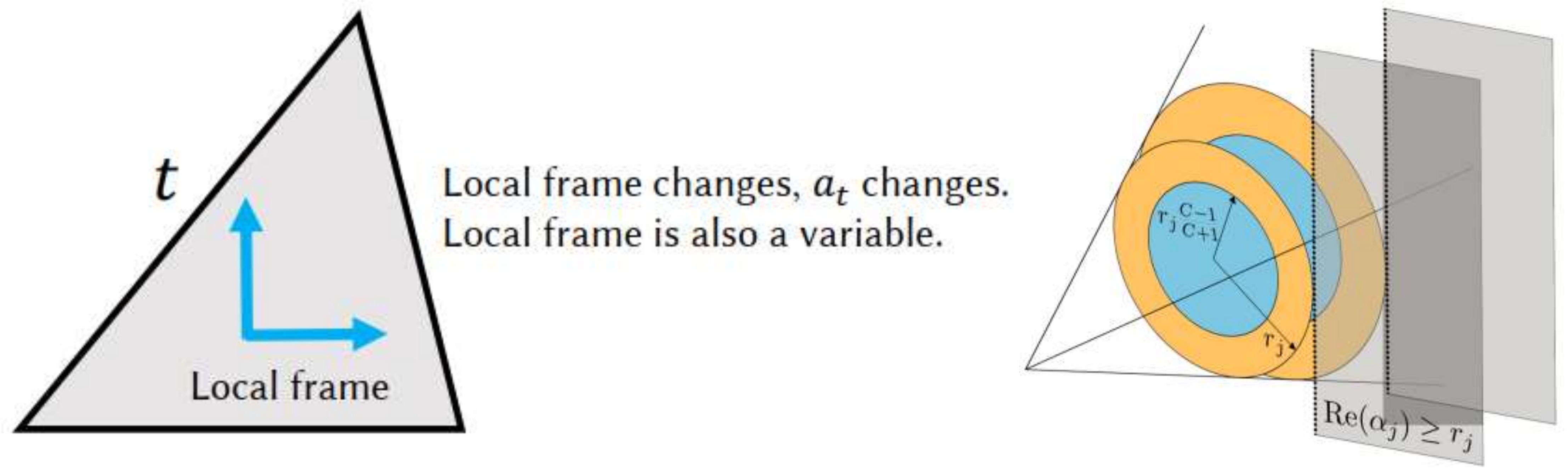
Convex subspace

- Non-convex $\sqrt{c'^2 + d'^2} \le \frac{K-1}{K+1}\sqrt{a'^2 + b'^2}$
- Lipman, Yaron. Bounded distortion mapping spaces for triangular meshes. ACM Transactions on Graphics (TOG) 31.4 (2012): 1-13.

Auxiliary $r > 0$ for each face : $\begin{cases} \sqrt{c'^2 + d'^2} \le \frac{K-1}{K+1} r \text{ (convex)} \\ \sqrt{a'^2 + b'^2} \ge r \text{ (non} - \text{convex)} \end{cases}$

Convex subspace

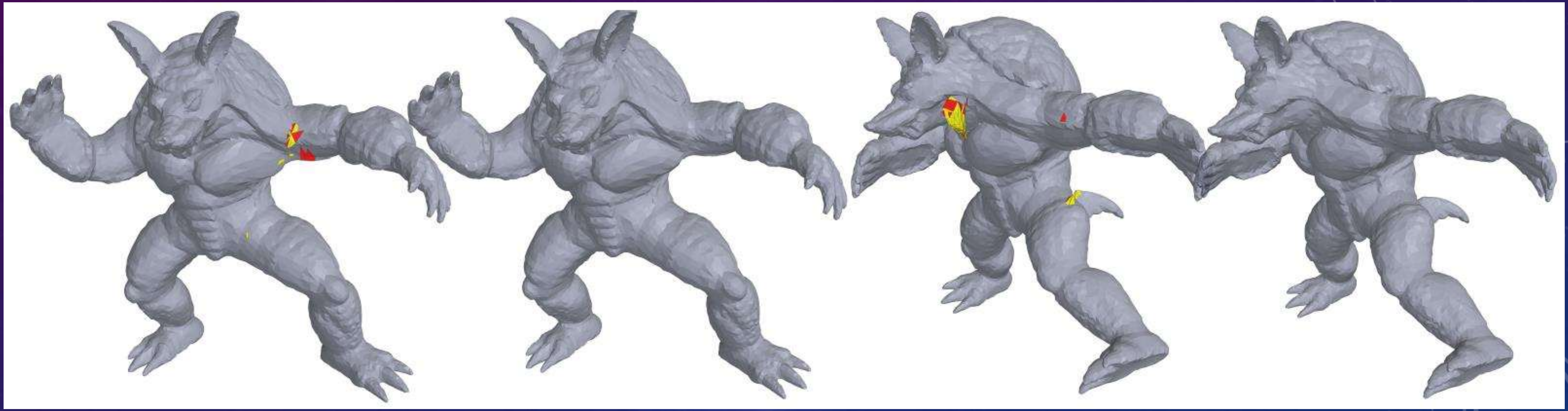
- $a' \geq r \text{ (convex)} \Longrightarrow \sqrt{a'^2 + b'^2} \geq r$



Optimization

- Objective function:
  - LSCM: $E = \sum_{ijk} A_{ijk}(c_{ijk}'^2 + d_{ijk}'^2)$
  - ARAP: $E = \sum_{ijk} A_{ijk}((a_{ijk}' - 1)^2 + b_{ijk}'^2 + c_{ijk}'^2 + d_{ijk}'^2)$
- Optimization:
  - Fix the local frame on each triangle: Second-Order Cone Programming
  - Update local frame to let $b_{ijk}' = 0$

Results



- A small number of iterations $n < 10$
- Quadratic programming is time-consuming
- How to set K?

Local-global optimization

- Auxiliary matrix H_{ijk} for each face

$$\min_{u} \sum_{ijk} \left\| J_{ijk}(u) - H_{ijk} \right\|_F^2 ,$$

s.t. $H_{ijk} \in \mathcal{H} = \{H | 1 \leq \tau\{H\} \leq K\}$ and $Au = b$ (coordinate constraint)

Local-global optimization

- Local step - Signed singular value decomposition

$$\min_{u} \sum_{ijk} \left\| J_{ijk}(u) - H_{ijk} \right\|_F^2 ,$$

s.t. $H_{ijk} \in \mathcal{H} = \{H | 1 \leq \tau\{H\} \leq K\}$

Local-global optimization

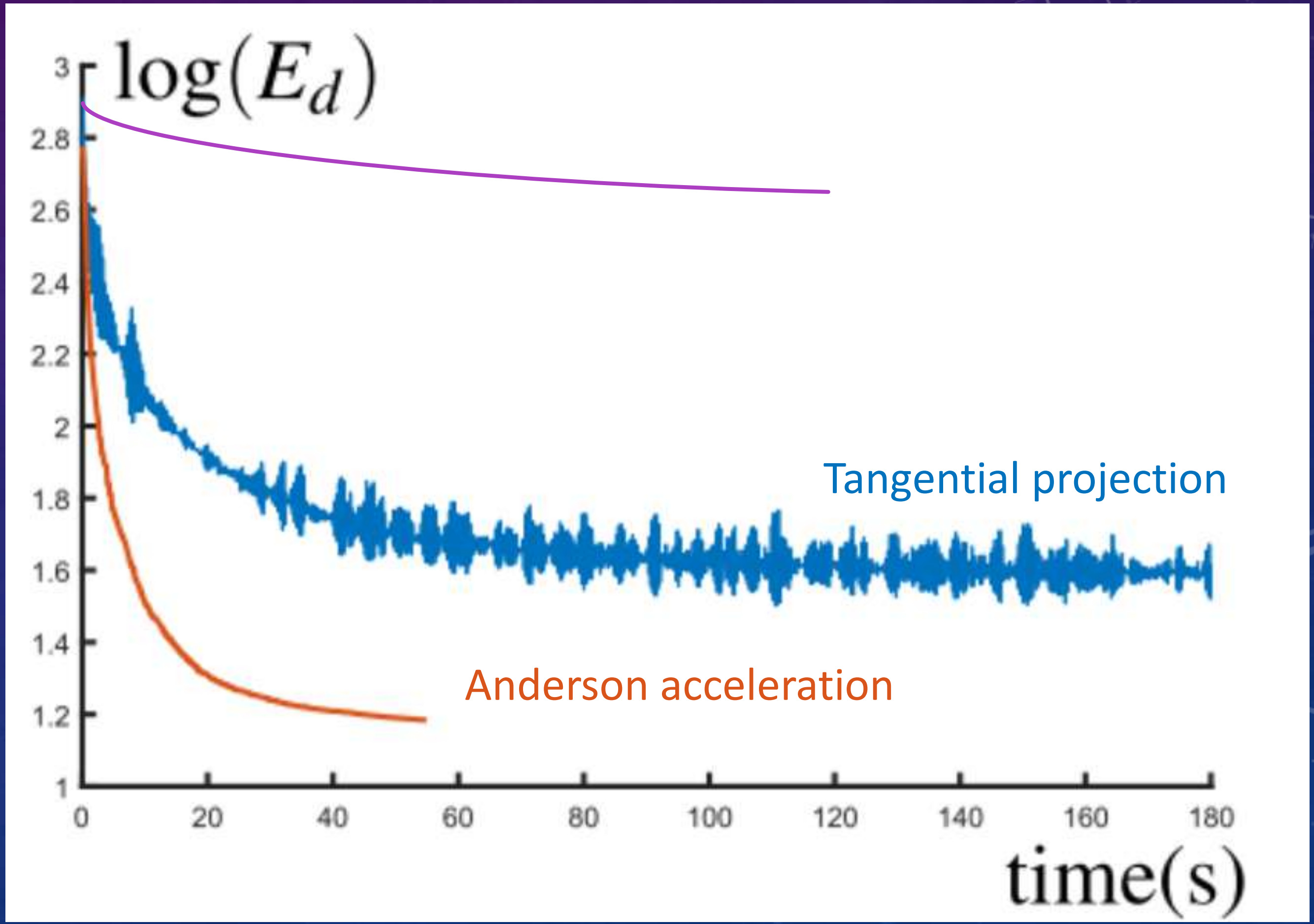
- Global step – constraint least square (KKT condition)

$$\min_{u} \sum_{ijk} \left\| J_{ijk}(u) - H_{ijk} \right\|_F^2 ,$$

s.t. $Au = b$ (coordinate constraint)

Slow convergence

- Acceleration
  - Tangential projection
  - Anderson acceleration



Tangential projection

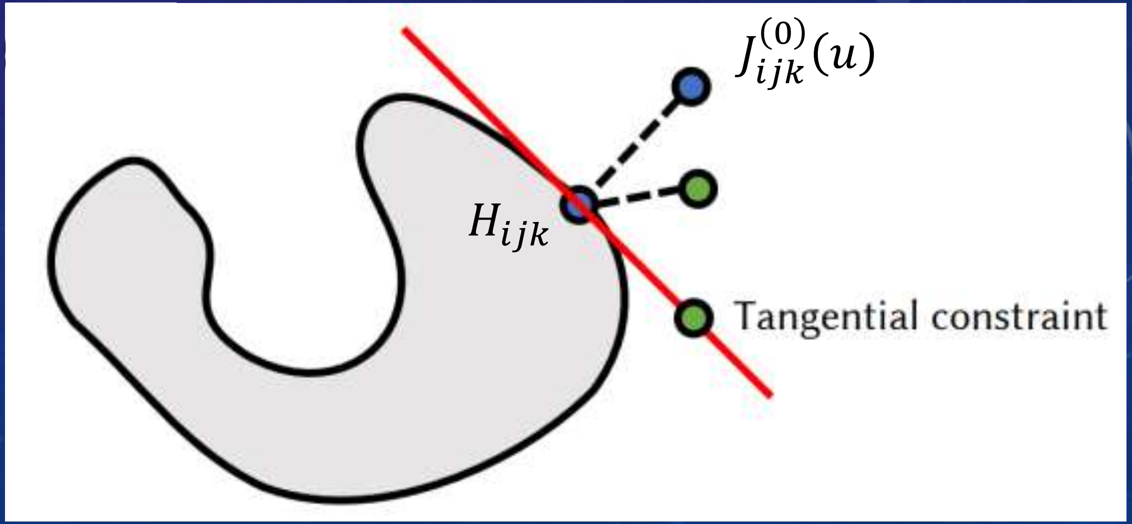
- Kovalsky, Shahar Z., et al. Large-scale bounded distortion mappings. ACM Trans. Graph. 34.6 (2015): 191-1

Global step – constraint least square (KKT condition)

$$\min_{u} \sum_{ijk} \left\| J_{ijk}(u) - H_{ijk} \right\|_F^2,$$

s.t. $Au = b$ (coordinate constraint)

$$\left(J_{ijk}(u) - H_{ijk}\right) \perp \left(J_{ijk}^{(0)}(u) - H_{ijk}\right)$$



Slow convergence

- Acceleration
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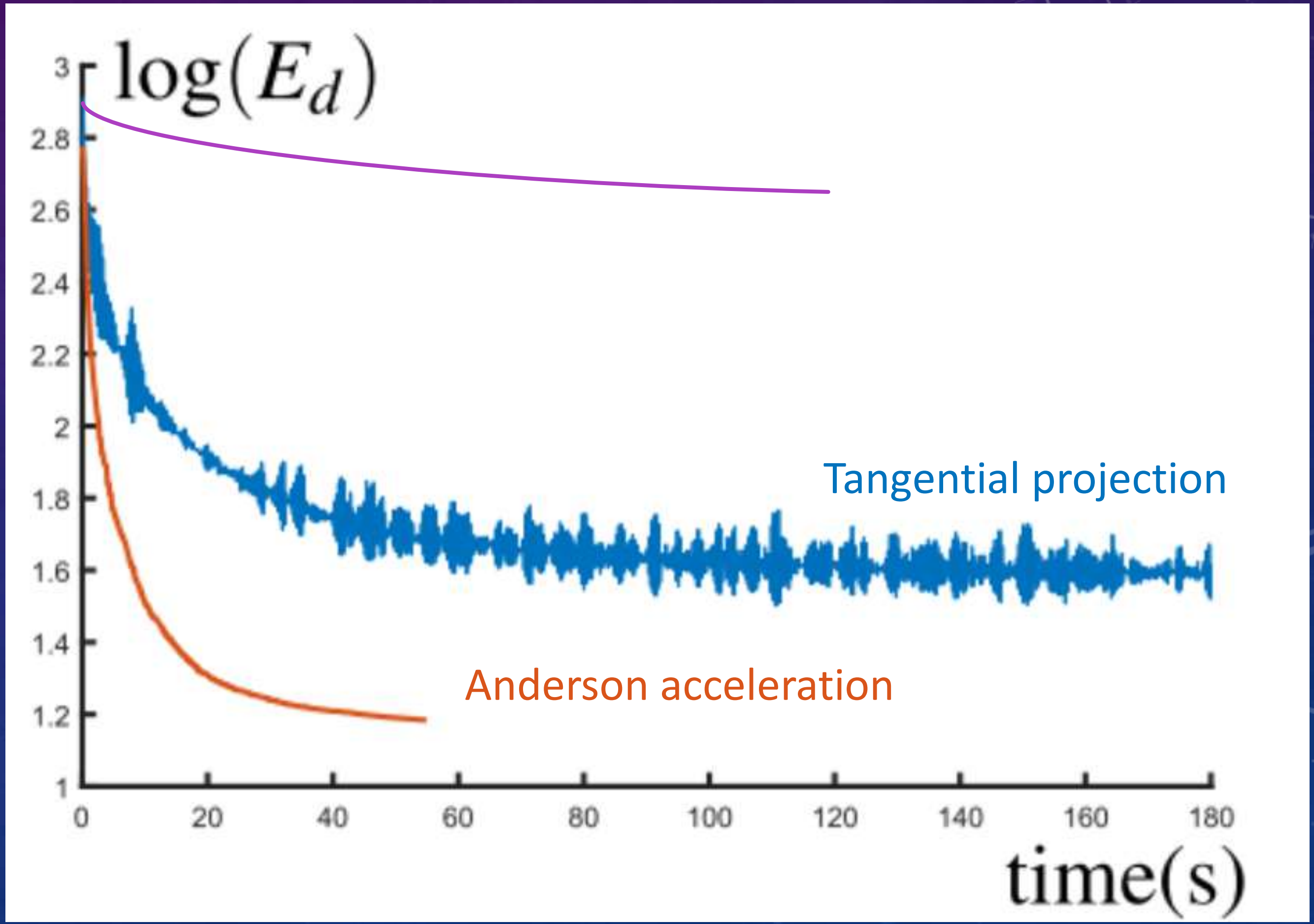


Anderson acceleration

- Su, Jian-Ping, Xiao-Ming Fu, and Ligang Liu. Practical foldover-free volumetric mapping construction. Computer Graphics Forum. Vol. 38. No. 7. 2019
- Peng, Yue, et al. Anderson acceleration for geometry optimization and physics simulation. ACM Transactions on Graphics (TOG) 37.4 (2018): 1-14.

Anderson acceleration

- Fixed-point iteration $u^{k+1} = G(u^k)$. Accelerated iteration

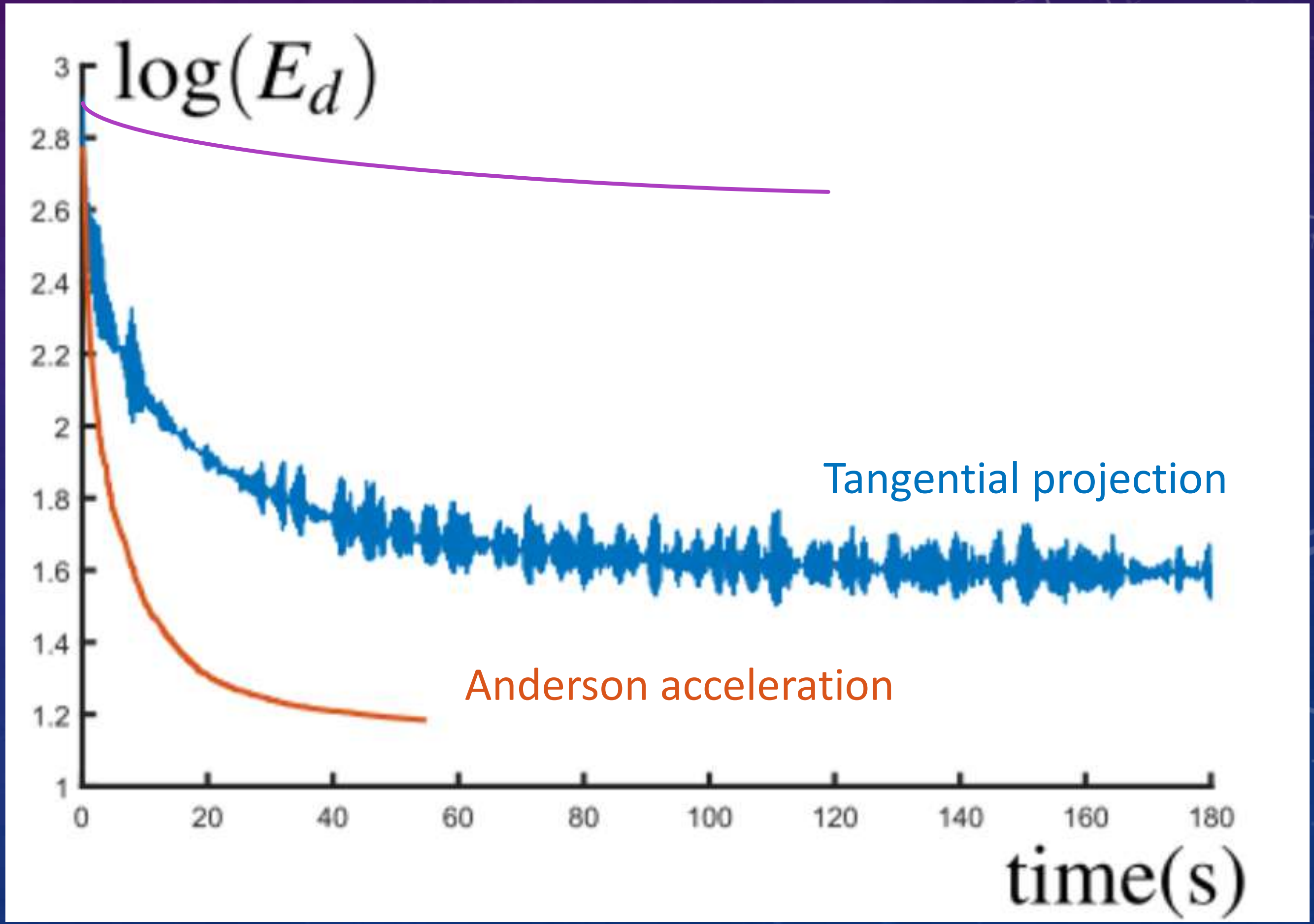
$$u_{AA}^{k+1} = G(u^k) - \sum_{j=1}^{m} \theta_j^* \left(G(u^{k-j+1}) - G(u^{k-j})\right)$$

where $(\theta_1^*, \dots, \theta_m^*)$ is the solution to a linear least-squares problem:

$$\min_{(\theta_1, \dots, \theta_m)} \left\| F^k - \sum_{j=1}^{m} \theta_j (F^{k-j+1} - F^{k-j}) \right\|, F^k = G(u^k) - u^k$$

Slow convergence

- Acceleration
  - Tangential projection
  - Anderson acceleration



Update bound K

$$\min_{u} \sum_{ijk} \left\| J_{ijk}(u) - H_{ijk} \right\|_F^2 ,$$

s.t. $H_{ijk} \in \mathcal{H} = \{H | 1 \leq \tau\{H\} \leq K\}$ and $Au = b$ (coordinate constraint)

Strategy: $K^0 = 4, K^{n+1} = \beta K^n$

Practical Foldover-Free Volumetric Mapping Construction

Submitted to Pacific Graphics 2019

ID:1075

(This video contains no voiceover)

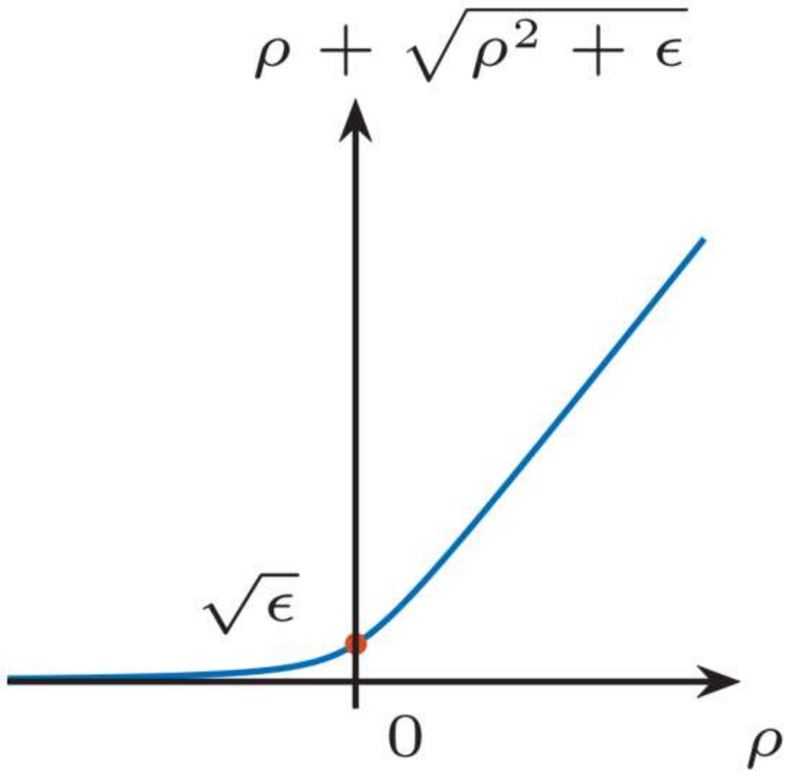
Penalty function

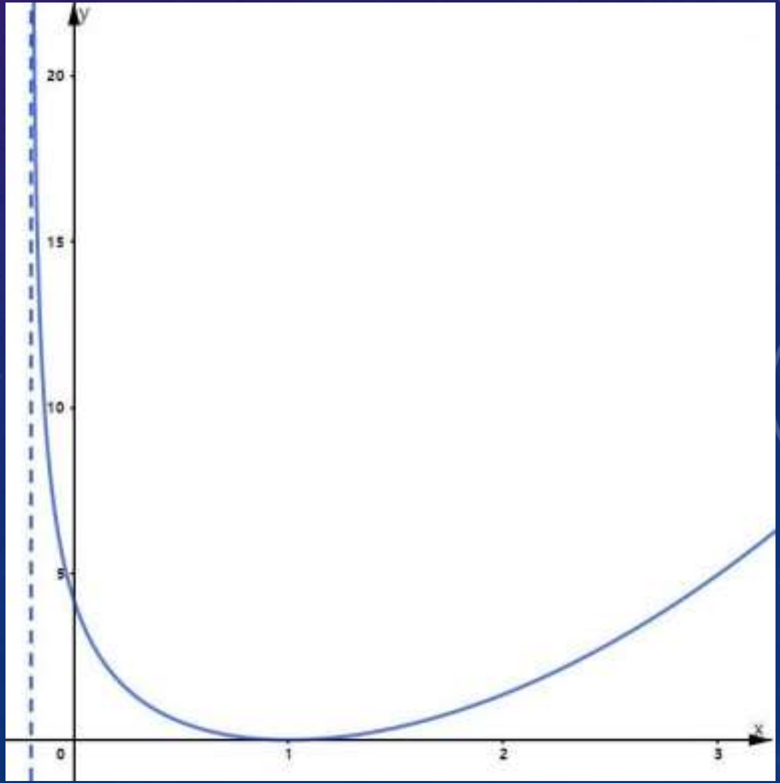
- Key idea: penalize the flipped elements via penalty functions
- Main properties:
  - It is very large to penalize flipped Jacobian matrices;
  - It is very small to accept flip-free Jacobian matrices.
- Solvers are then the challenges
  - Non-linear and non-convex

Penalty function

$$\sum_{ijk} \frac{\|J_{ijk}\|_F^d}{\det(J_{ijk})+\sqrt{\det(J_{ijk})^2+\epsilon}}$$

$$\sum_{ijk}\left(\det(J_{ijk})-1\right)^2+\left(\log\frac{\det(J_{ijk})-\delta}{1-\delta}\right)^2$$





Parameter ϵ

- Garanzha, Vladimir, et al. Foldover-free maps in 50 lines of code. ACM Transactions on Graphics (TOG) 40.4 (2021): 1-16
- Update strategy of ϵ : $\epsilon^0 = 0$

$$\epsilon^{n+1} = \begin{cases} \left(1 - \dfrac{\sigma^n \sqrt{(D_{-}^{n+1})^2 + (\epsilon^n)^2}}{|D_{-}^{n+1}| + \sqrt{(D_{-}^{n+1})^2 + (\epsilon^n)^2}}\right)\epsilon^n, & \text{if } D_{-}^{n+1} < 0 \\ (1 - \sigma^n)\epsilon^n, & \text{if } D_{-}^{n+1} \geq 0 \end{cases}$$

Where $D_{-}^{n+1} = \min\limits_{ijk} \det\left(J_{ijk}^{n+1}\right)$ and $\sigma^n = \max\left(\frac{1}{10}, 1 - \frac{F(U^{n+1},\epsilon^n)}{F(U^n,\epsilon^n)}\right)$

Non-linear solvers

- Block coordinate descent method
- Monotone preconditioned conjugate gradient method
- L-BFGS
- SGD
- Secord-order methods

Total unsigned area (TUA)

- Du, Xingyi, et al. Lifting simplices to find injectivity. ACM Trans. Graph. 39.4 (2020): 120.

Signed area S_t of a triangle t and unsigned area U_t of a triangle t. For any 2D triangulation or 3D tetrahedron $\mathcal{T}$,

- Total signed area $\sum_t S_t = Area(\mathcal{T})$
- Total unsigned area $\sum_t U_t \geq Area(\mathcal{T})$
- $\mathcal{T}$ is flip-free iff $\sum_t U_t = Area(\mathcal{T})$

Methods

- Optimizing total unsigned area for achieving flip-free mappings.
- Challenges:
  - TUA is C^0-continuous as a vertex moves across the supporting line of its opposite edge in a triangle.
  - The triangulation containing degenerate elements is a global minimum of TUA but a non-injective embedding
  - TUA has zero gradients with respect to any vertex surrounded by a ring of consistently oriented triangles

Total lifted content

$$\sum_t \frac{1}{d!} \sqrt{\det(X^T X + \alpha \tilde{X}^T \tilde{X})}$$

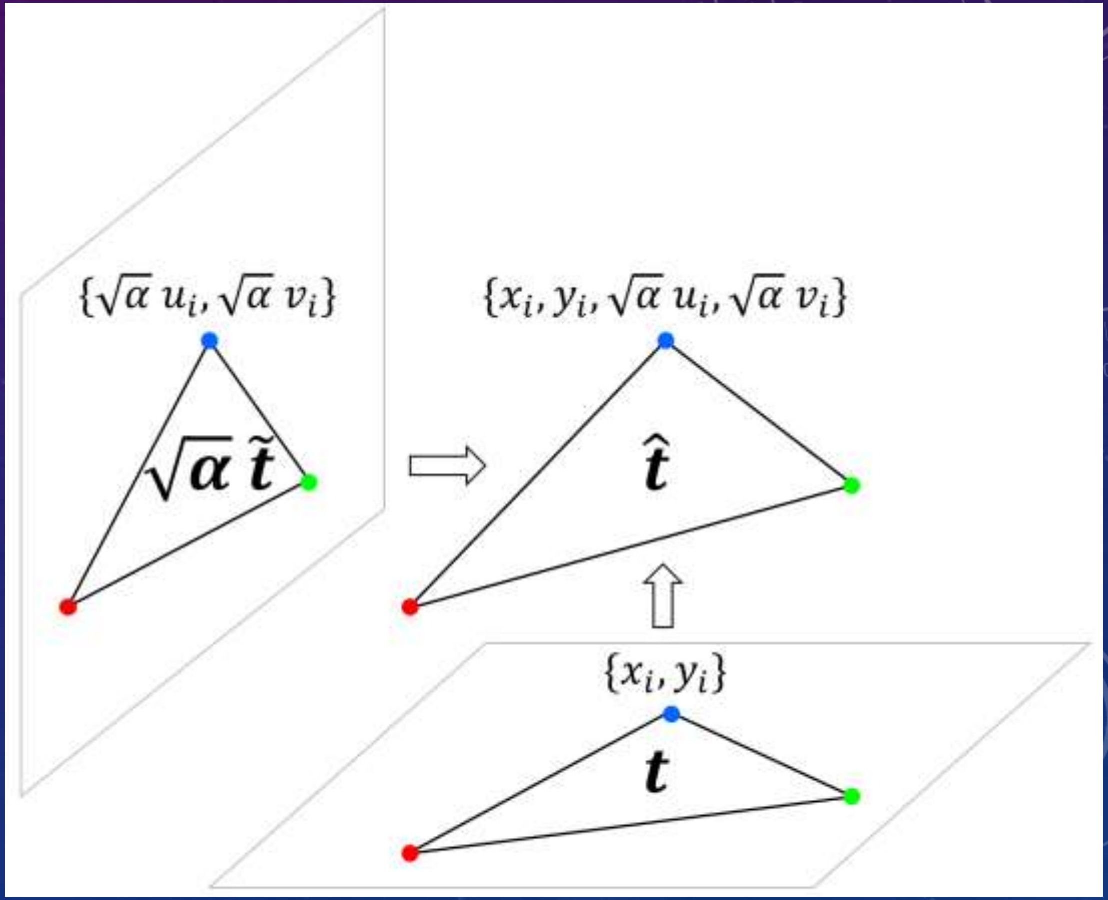
- X: a $d \times d$ matrix whose column vectors are the edge vectors from one vertex to the other d vertices. $U_t = \sqrt{\det(X^T X)}$
- $\tilde{X}$: similar to X, but from an auxiliary simplex, such as an equilateral triangles or tetrahedra of the same size as the input.

Total lifted content

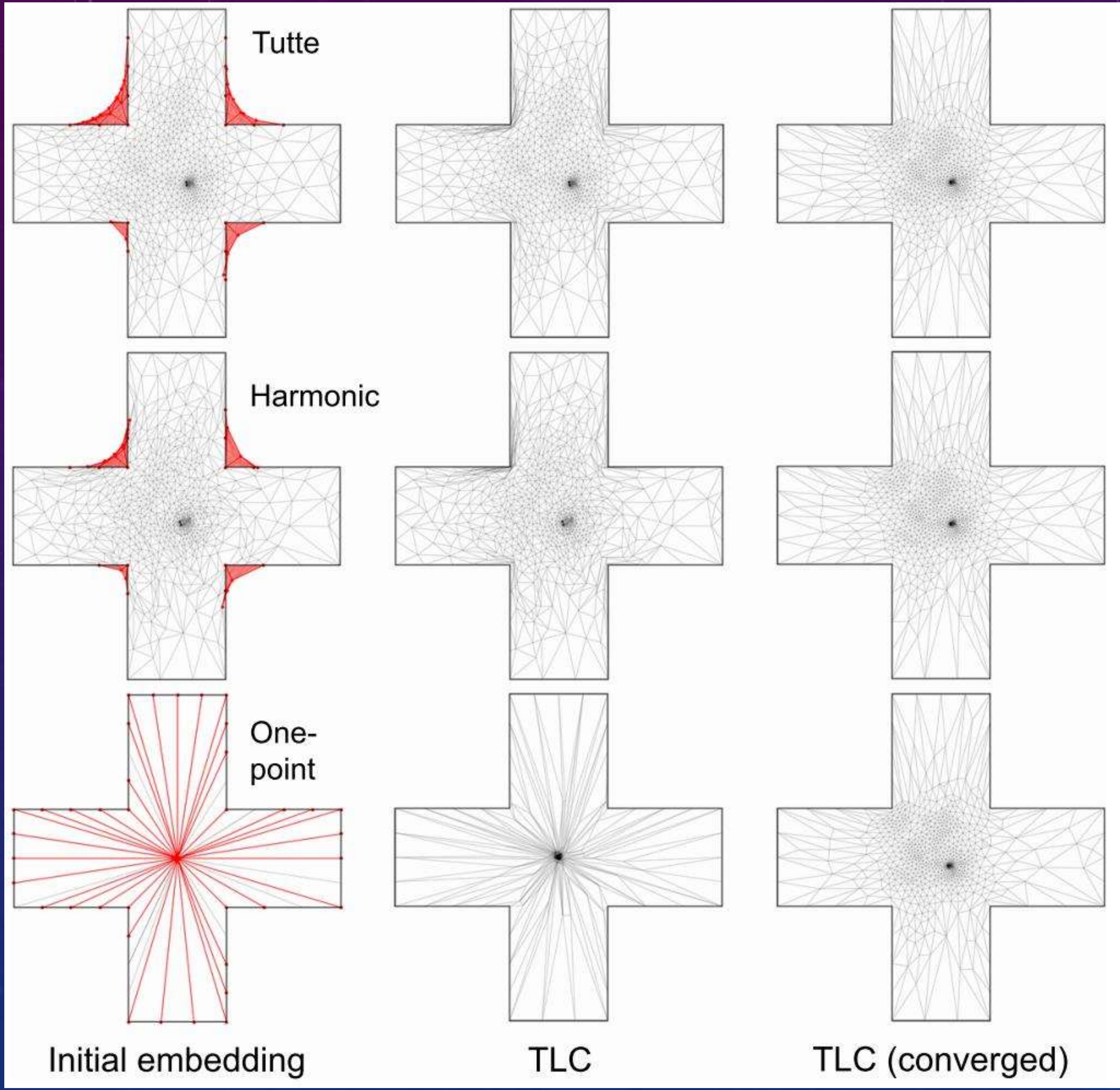
$$\sum_t \frac{1}{d!}\sqrt{\det(X^T X + \alpha \tilde{X}^T \tilde{X})}$$

$$\tilde{X} = \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}$$

- TLC is smooth over the entire space
- TLC has only an injective global minimum for sufficiently small values of α.



Results



Tutte
Harmonic
One-
point
Initial embedding
TLC
TLC (converged)

Injective Parameterization

Two cases:

- Flipped initializations → removing flipped elements
- Flip-free initializations → keeping mapping always flip-free

Flip-free initializations

- Tutte's barycentric mapping

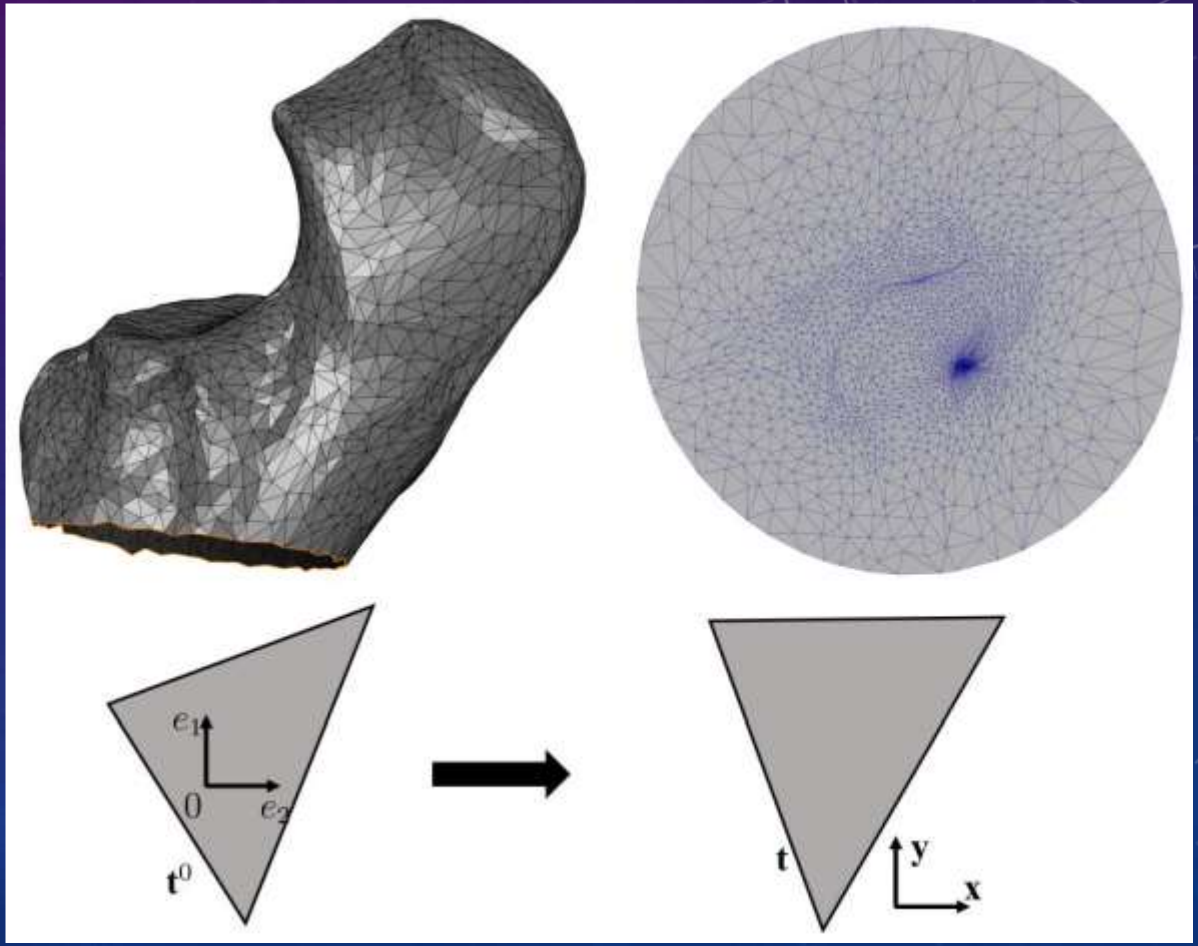
Given a triangulated surface homeomorphic to a disk, if the (u, v) coordinates at the boundary vertices lie on a convex polygon in order, and if the coordinates of the internal vertices are a convex combination of their neighbors, then the (u, v) coordinates form a valid parameterization (without self-intersections, bijective)

Barycentric Mapping

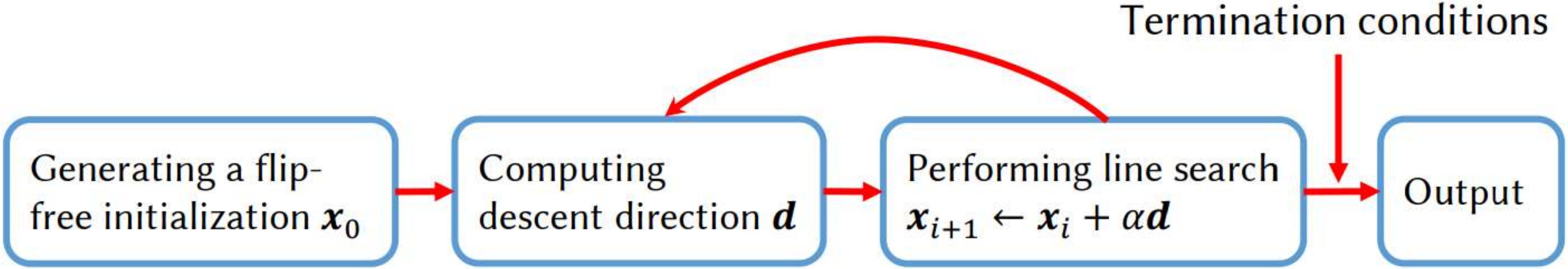
- Homeomorphic to a disk
- A convex polygon - circle, square,……
- A convex combination

$$\binom{u_i}{v_i} = \sum_{j\in\Omega(i)} w_{ij} \binom{u_j}{v_j}, w_{ij} > 0$$

- Solver: linear equation.



Pipeline



$\boldsymbol{d}$: the main difference between methods.

α: the flip-free constraint is always satisfied.

Barrier functions

- Barrier functions diverge to infinity when elements become degenerate, thus inhibiting flips (log of the determinant).
- Distortion metrics explode near degeneracies.

MIPS : $\frac{\sigma_1}{\sigma_2} + \frac{\sigma_2}{\sigma_1} = \frac{\sigma_1^2 + \sigma_2^2}{\sigma_1 \sigma_2} = \frac{\|J\|_F^2}{\det J}$. If $\det J \rightarrow 0$, then MIPS $\rightarrow \infty$

Optimization problem

$$\min_{u,v} \sum_{ijk} A_{ijk}\, \mathcal{D}(J_{ijk}(u,v)) \triangleq \min_{u,v} E(u,v)$$

1. Initial point (u_0, v_0), iter $n = 0$
2. Descent direction $\langle p, \nabla E \rangle < 0$
3. Step size $\min_{\alpha} E((u_n, v_n) + \alpha p)$
4. Update $(u_{n+1}, v_{n+1}) = (u_n, v_n) + \alpha p$, $n = n + 1$

Line search

- Theoretical guarantee - each triangle has a positive area
- Triangle ijk becomes degenerate when its signed area becomes zero:

$$\det\begin{pmatrix}(u_k, v_k) + \alpha p_k - (u_i, v_i) - \alpha p_i \\ (u_j, v_j) + \alpha p_j - (u_i, v_i) - \alpha p_i\end{pmatrix} = 0$$

It is quadratic in α and max step size for this triangle is the smallest positive root.

The max step size for all triangles is the minimum parameter α over all triangles.

Termination conditions

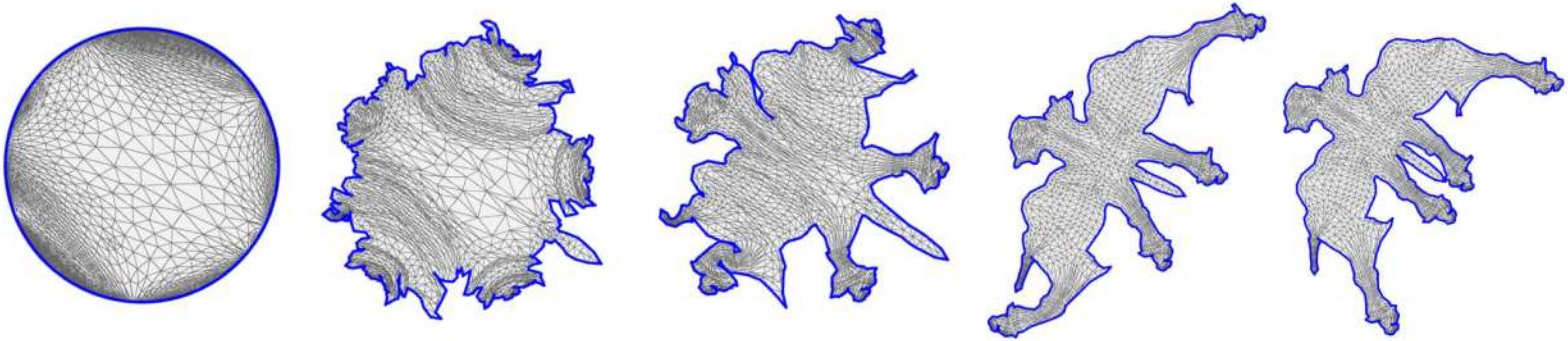
$$\min_{u,v} \sum\nolimits_{ijk} A_{ijk}\, \mathcal{D}(J_{ijk}(u,v)) \triangleq \min_{u,v} E(u,v)$$

1. The gradient is small $||\nabla E|| \leq \epsilon$
2. The absolute or relative error in energy and/or position are small.
3. A fixed number of iterations

Methods for
Bijective Parameterization

Motivation

- From overlap-free initialization → keep overlap free

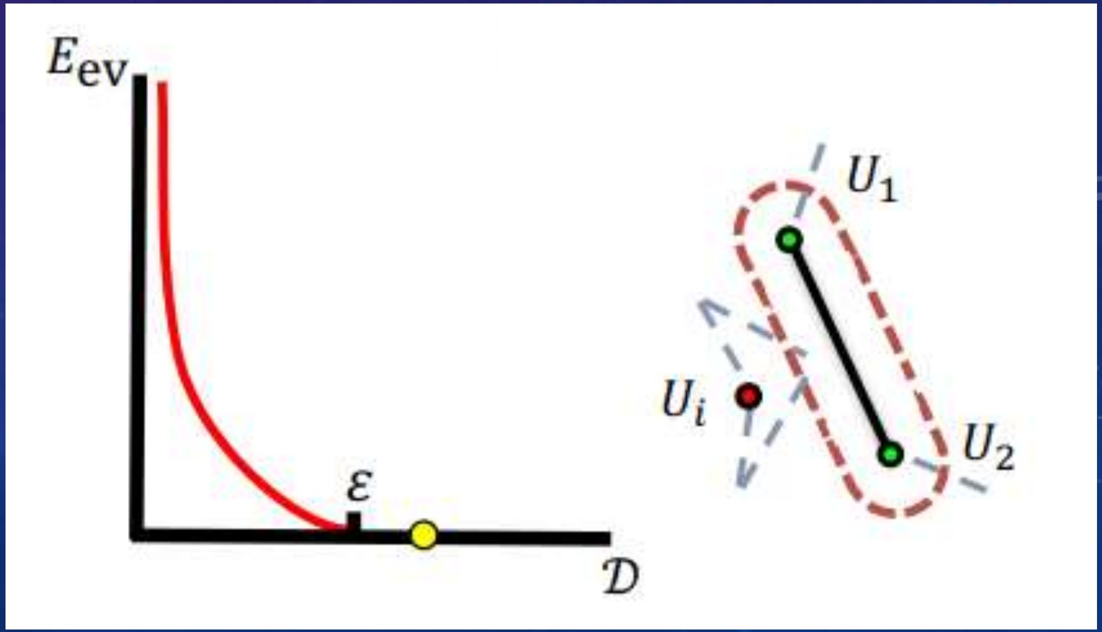


Barrier functions

- Smith J, Schaefer S. Bijective parameterization with free boundaries[J]. ACM Transactions on Graphics (TOG), 2015, 34(4): 1-9.

- Boundary barrier function

$$E_{ev}(b_j, u_i) = \max\left(0, \frac{\epsilon}{\mathcal{D}(b_j, u_i)} - 1\right)^2$$



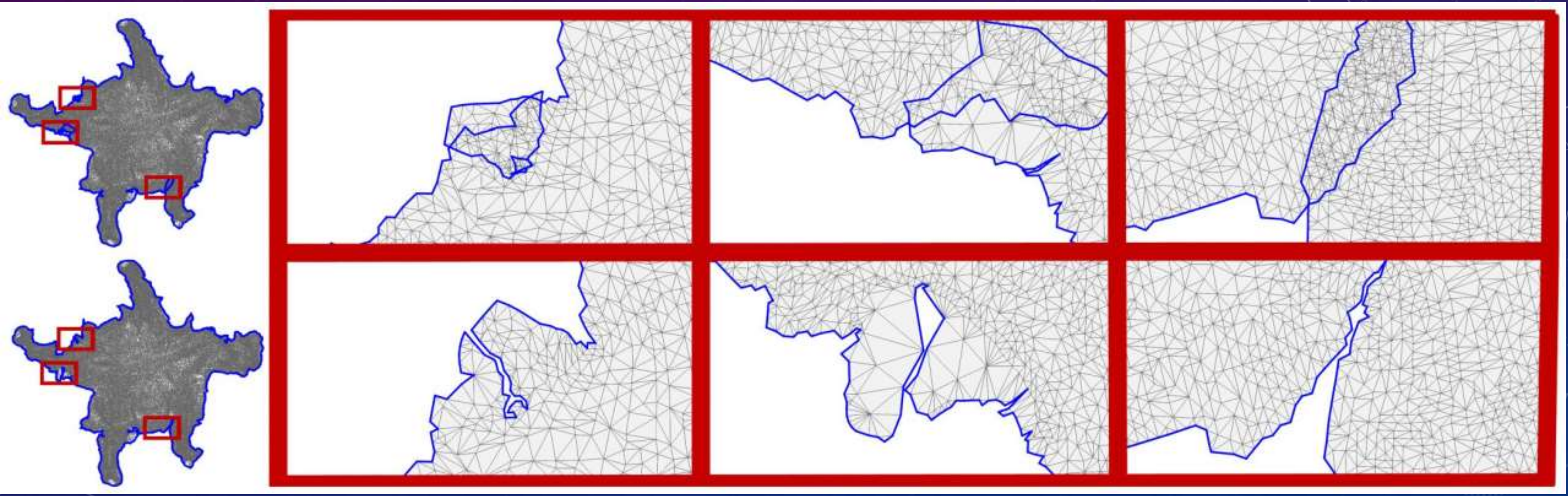
Formulation

$$\min_{u} E_d(u) + \lambda E_b(u)$$

$$E_d(u) = \frac{1}{4}\sum\nolimits_{ijk}\left(\left\|J_{ijk}(u)\right\|_F^2 + \left\|J_{ijk}^{-1}(u)\right\|_F^2\right)$$

$$E_b(u) = \sum_{b_j \in \partial\Omega} \sum_{u_i \in \Omega} E_{ev}(b_j, u_i)$$

Results

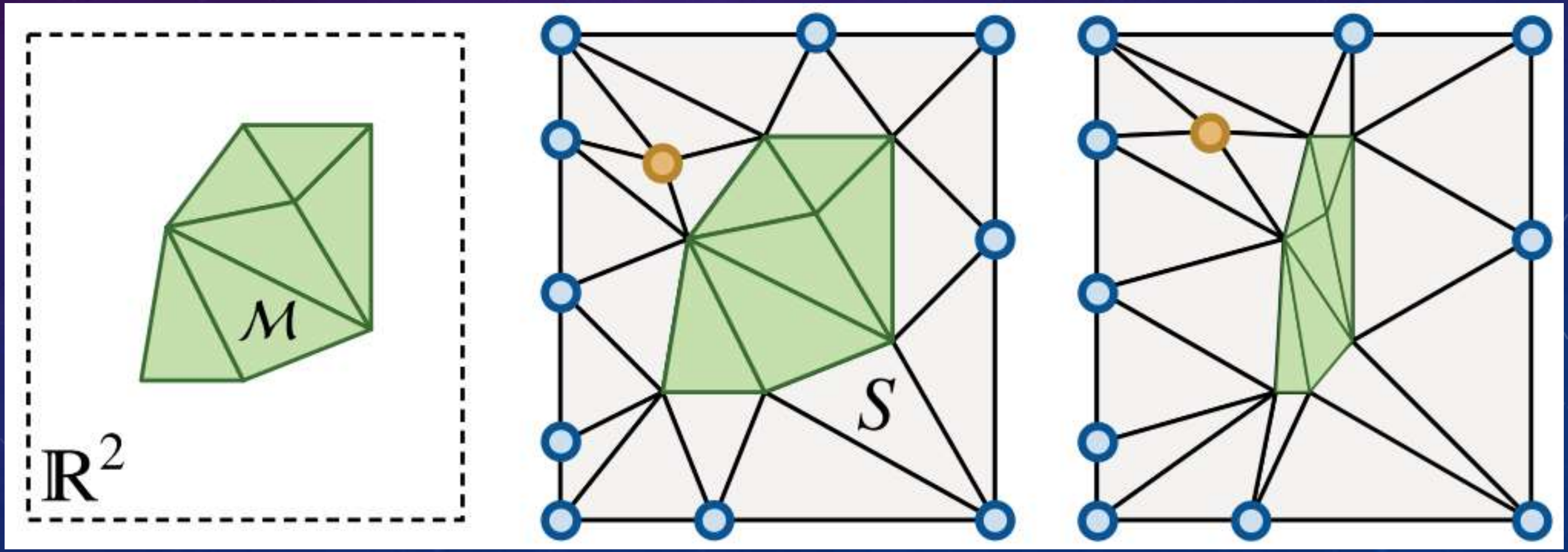


Scaffold

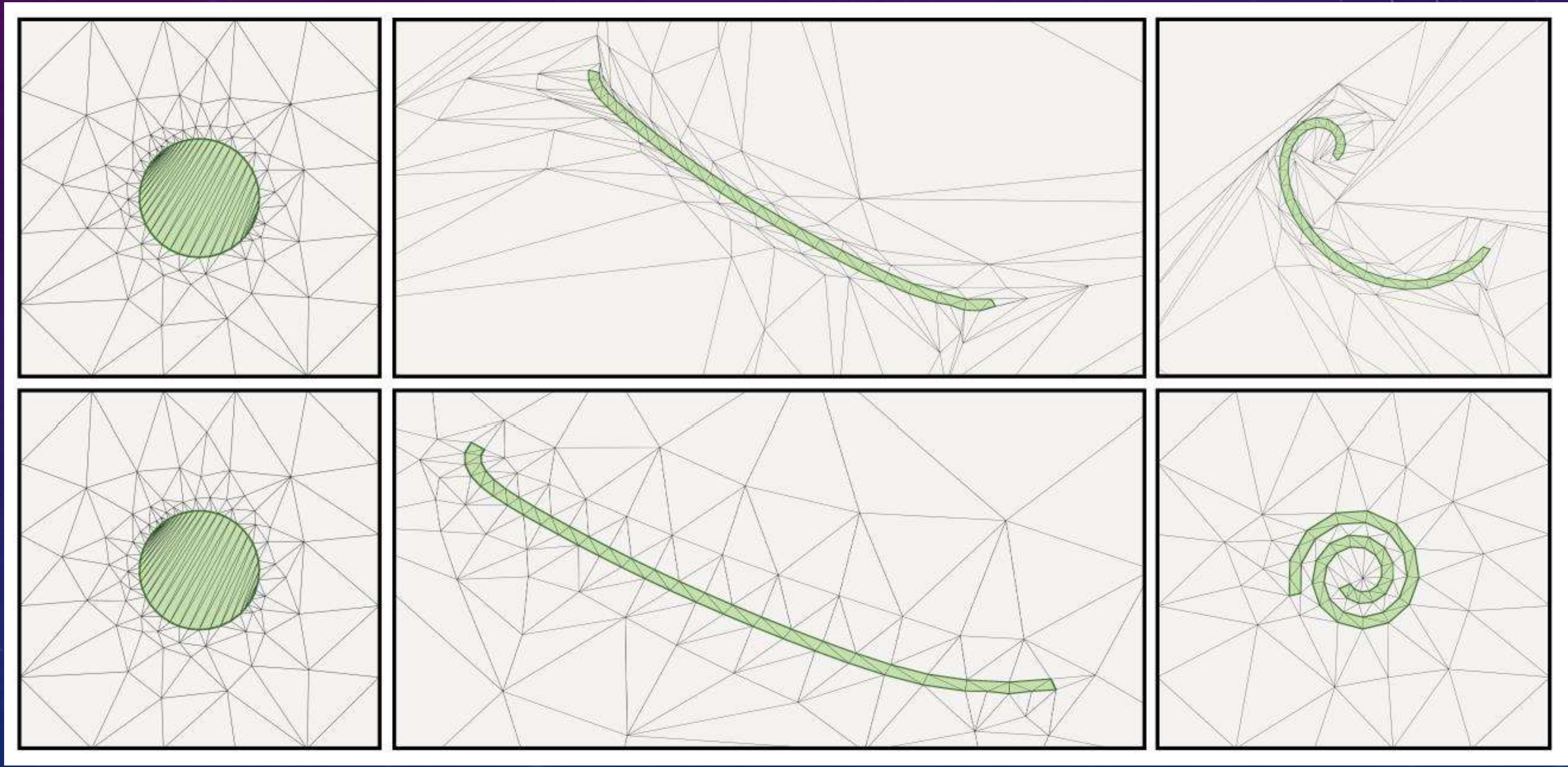
- Jiang Z, Schaefer S, Panozzo D. Simplicial complex augmentation framework for bijective maps[J]. ACM Transactions on Graphics, 2017, 36(6).

Motivation

- Overlap-free $\Longrightarrow$ flip-free



Connectivity updating



Results