Mesh Deformation II

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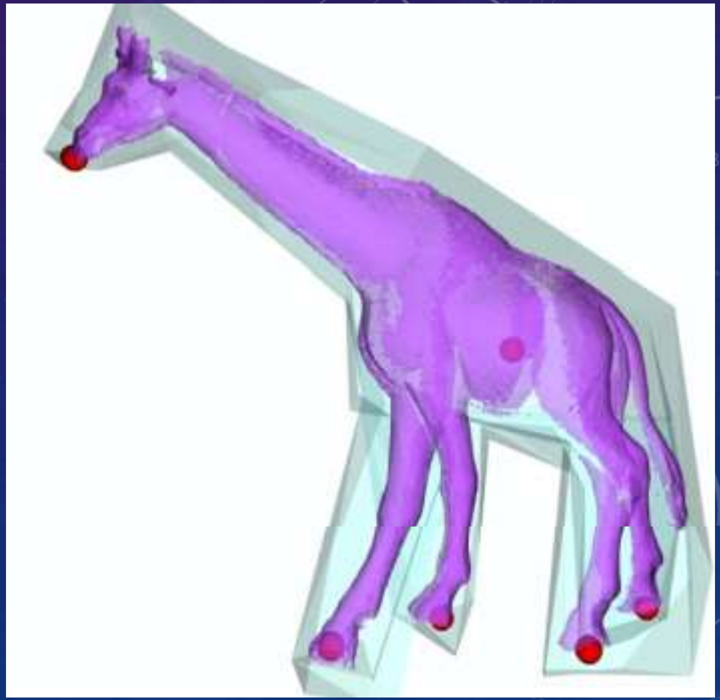
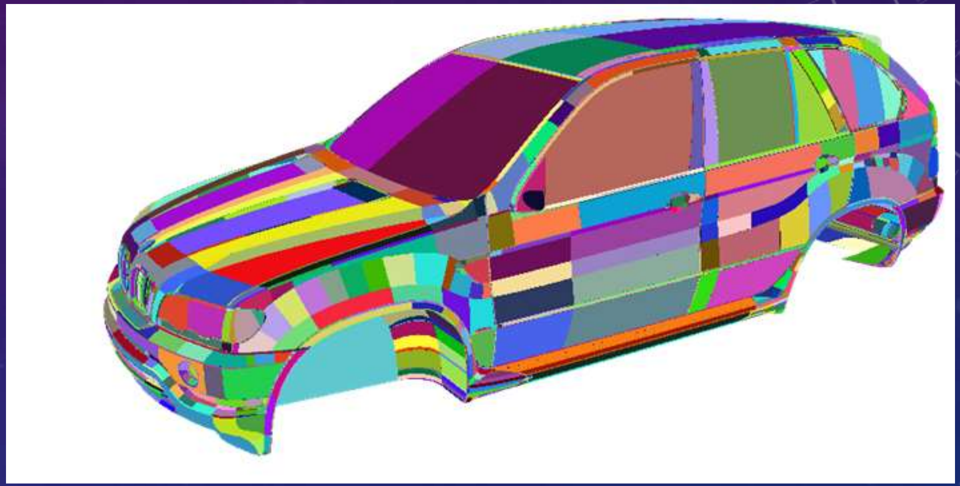
Methods

- Surface-based deformations
- Space deformations
  - Shape is volumetric (planar domain for 2D, polyhedral domain for 3D)
  - Deformation defined in neighborhood of shape
  - Can be applied to any shape representation



Advantages

- Handle arbitrary input
  - Meshes (also non-manifold)
  - Point sets
  - Polygonal soups
  - ...



- 3M triangles
- 10k components
- Not oriented
- Not manifold

Complexity mainly depends on the control object, not the surface

Advantages

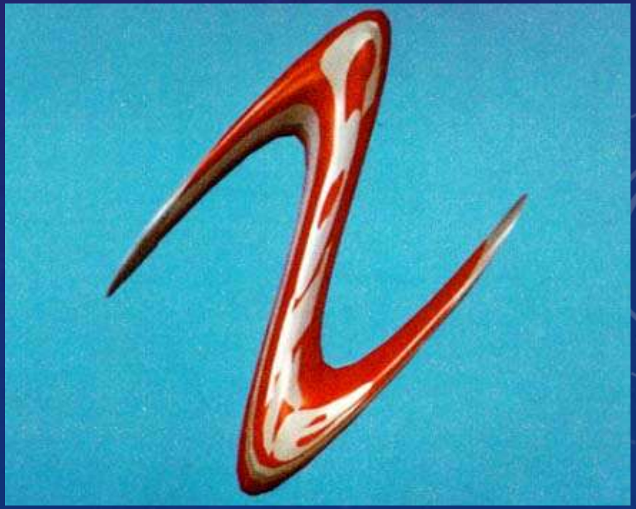
- Easier to analyze: functions on Euclidean domain

$$F: \mathbb{R}^3 \to \mathbb{R}^3$$

Jacobian – local deformation

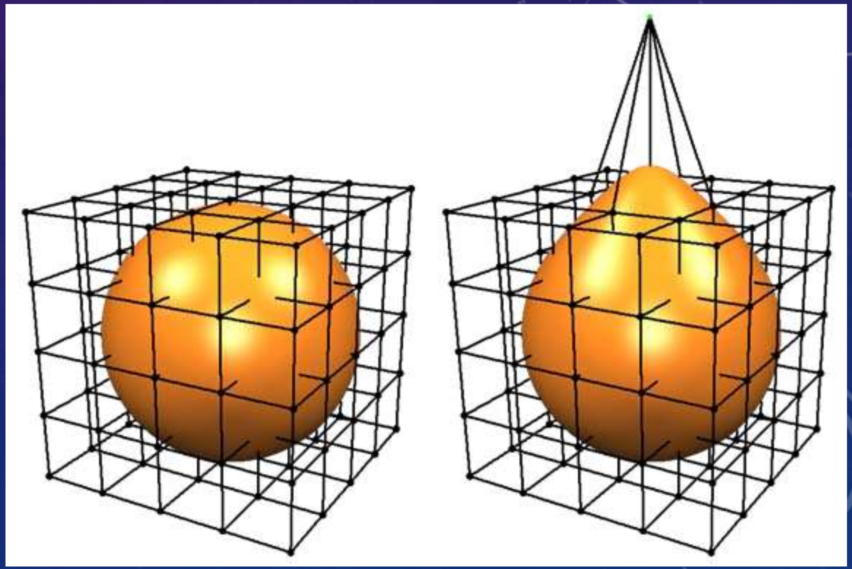
$$J_F = USV^T, S = diag(\sigma_1, \sigma_2, \sigma_3)$$

Distortion : conformal, volume



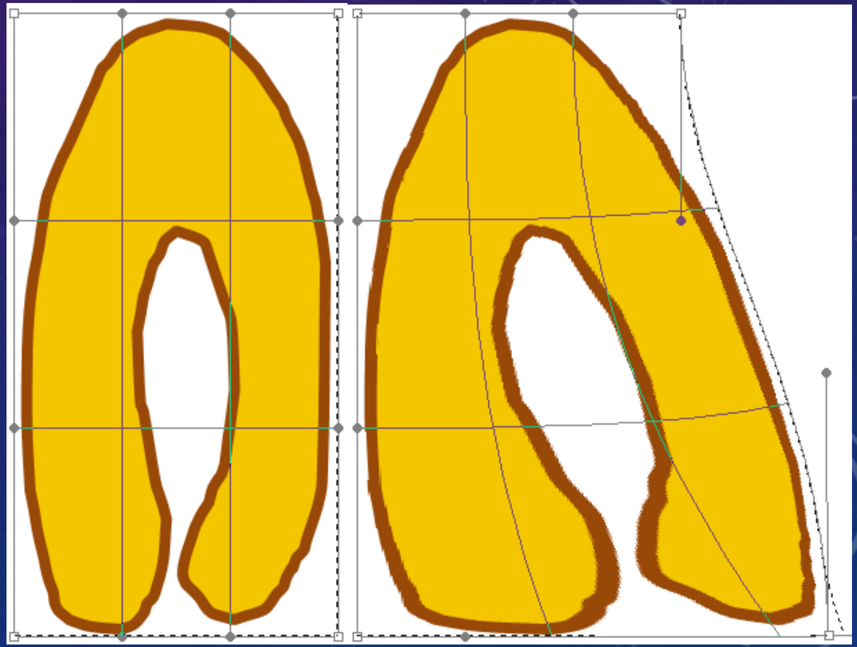
Disadvantages

- The deformation is only loosely aware of the shape that is being edited



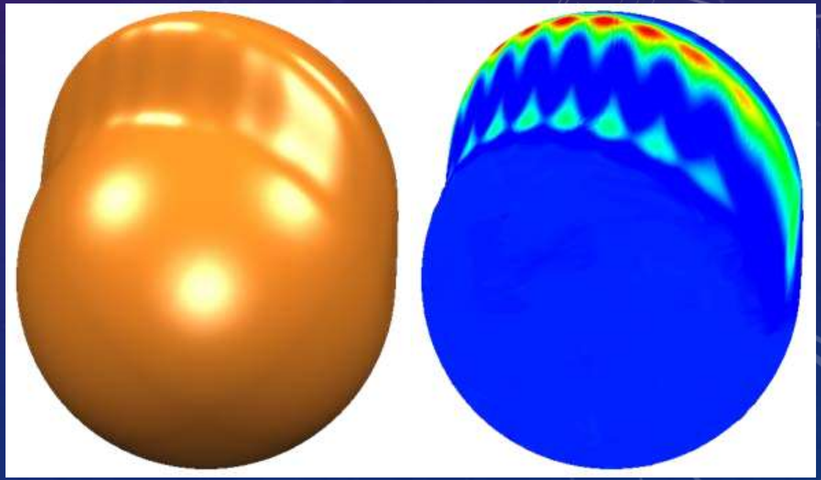
Disadvantages

- The deformation is only loosely aware of the shape that is being edited
- Small Euclidean distance → similar deformation



Disadvantages

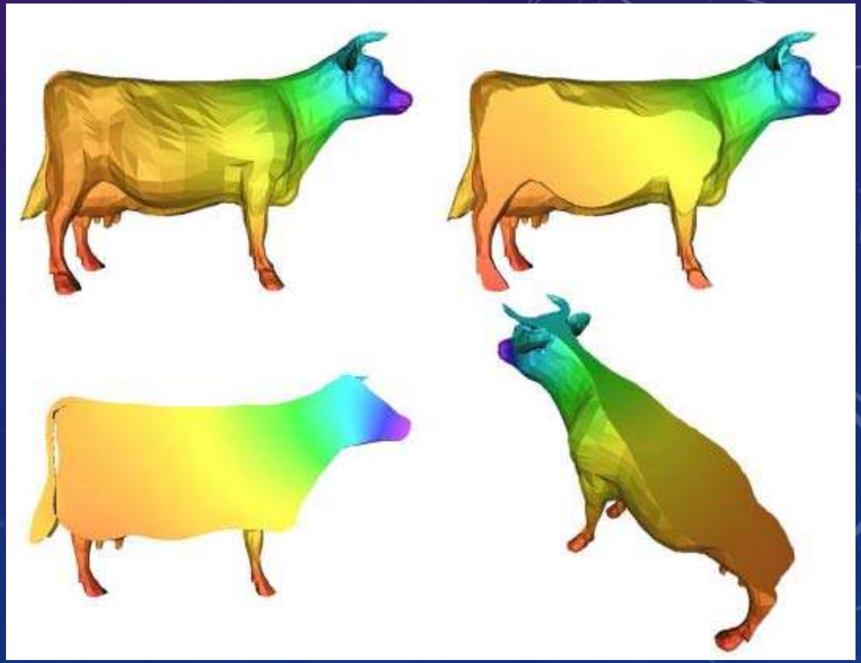
- The deformation is only loosely aware of the shape that is being edited
- Small Euclidean distance → similar deformation
- Local surface detail may be distorted



Space deformations

- User defines displacements $d_i \in \mathbb{R}^3$ for each element of the control object
- Displacements are interpolated to the entire space using basis functions $B_i(x) : \mathbb{R}^3 \to \mathbb{R}$,

$$d(x) = \sum_i d_i B_i(x)$$



- Basis functions should be smooth for aesthetic results

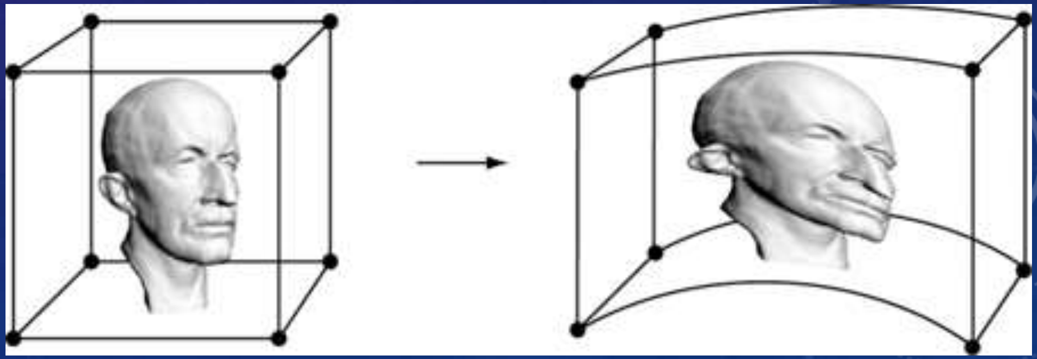
Space deformations

- Trivariate Tensor Product Bases
- Skeleton
- Cage-based deformation

Trivariate Tensor Product Bases

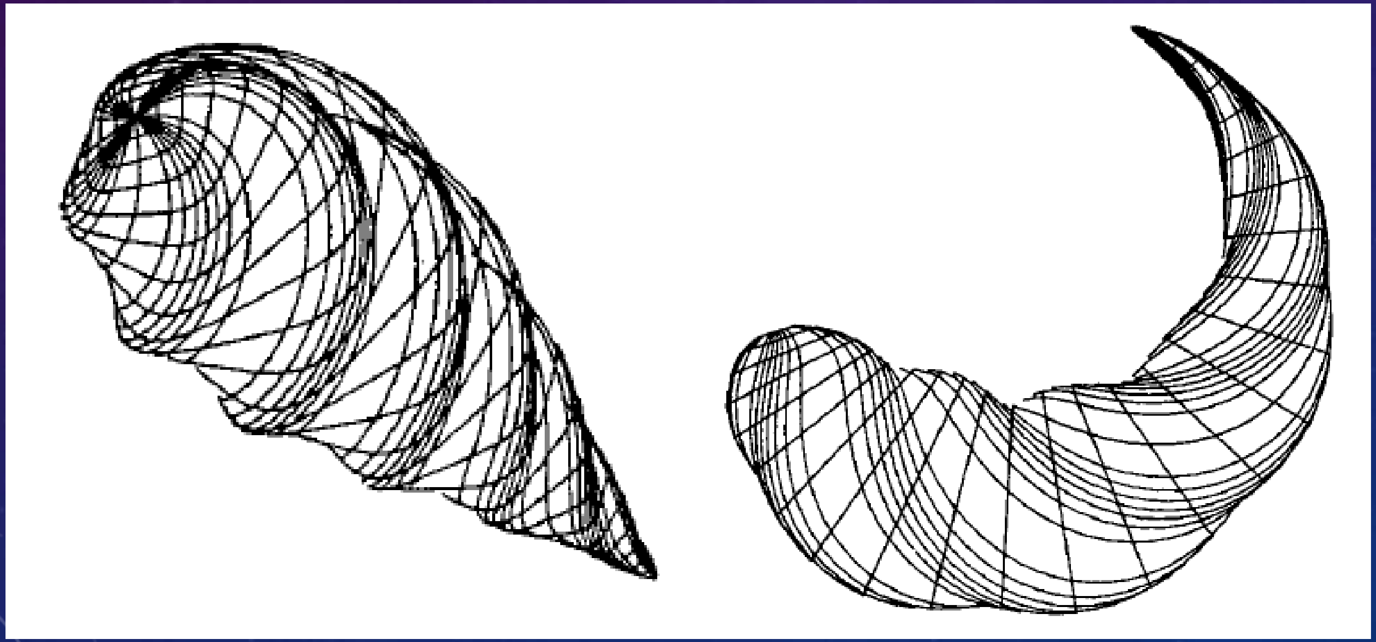
- Control object = lattice
- Basis functions $B_i(x)$ are trivariate tensor-product splines

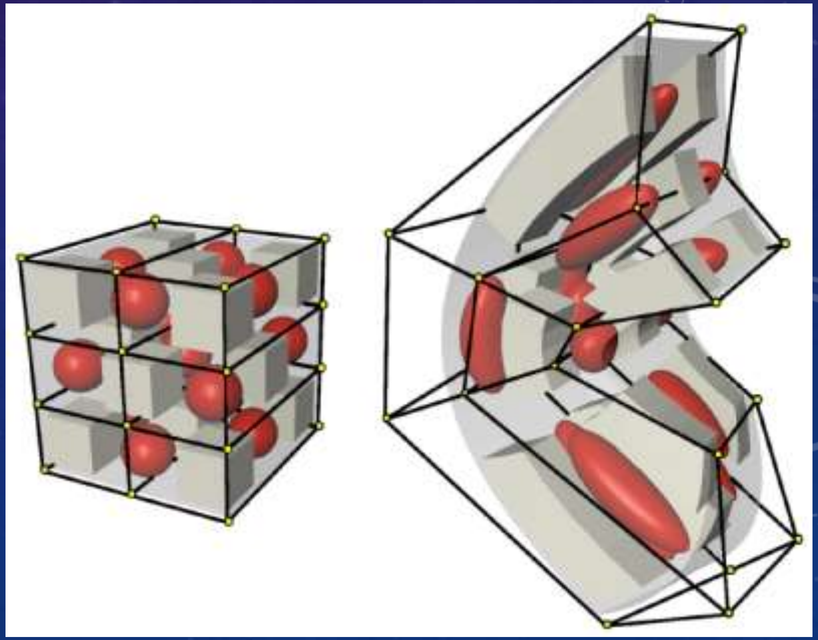
$$d(x) = \sum_{i=0}^{l}\sum_{j=0}^{m}\sum_{k=0}^{n} d_{ijk} N_i(x) N_j(y) N_k(z)$$



Lattice as Control Object

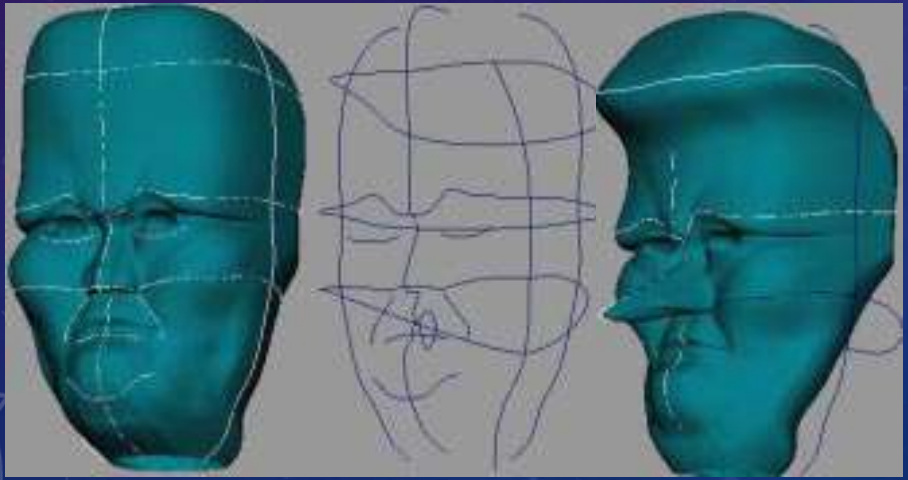
- Interpolate deformation constraints - by least squares

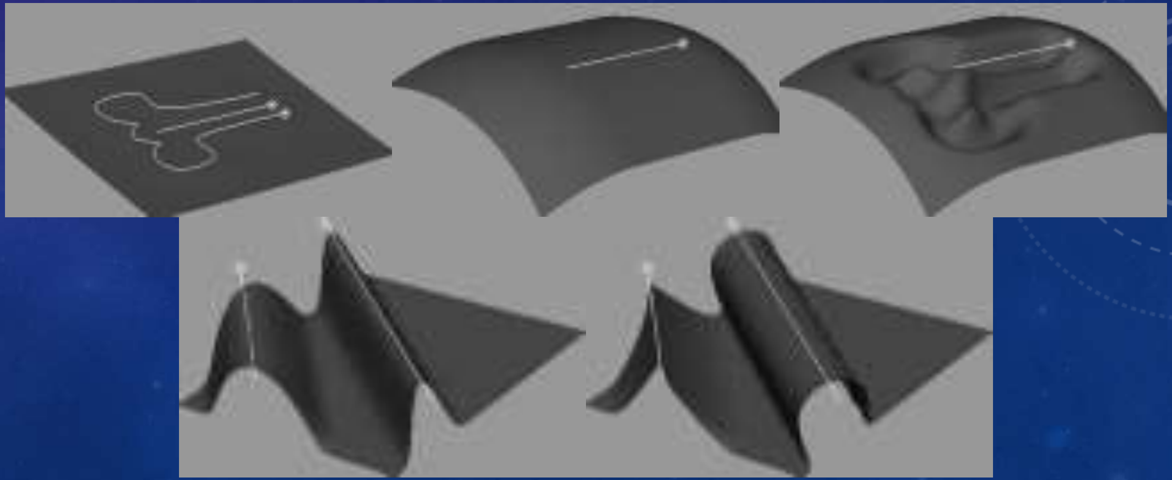




Comparison to wires

- Control objects are arbitrary space curves
- Can place curves along meaningful features of the edited object
- Smooth deformations around the curve with decreasing influence





Comparison to RBF

- Represent deformation by RBFs

$$d(x) = \sum_{i=1}^{l} w_i \, \phi(\|c_i - x\|) + p(x)$$

where w_i weights, $\phi(r) = r^3$ triharmonic basis function, c_i a set of centers and $p(x)$ a polynomial of low degree.

Comparison to RBF

- RBF fitting
  - Interpolate displacement constraints
  - Solve linear system for w_i and p

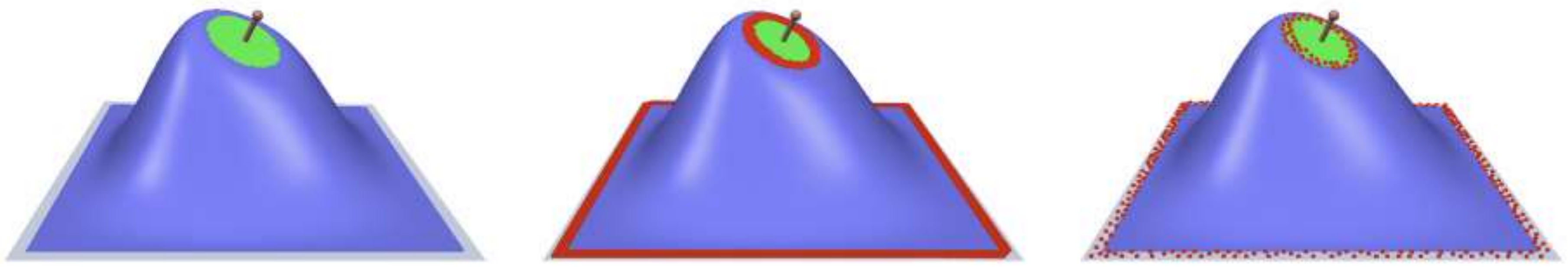


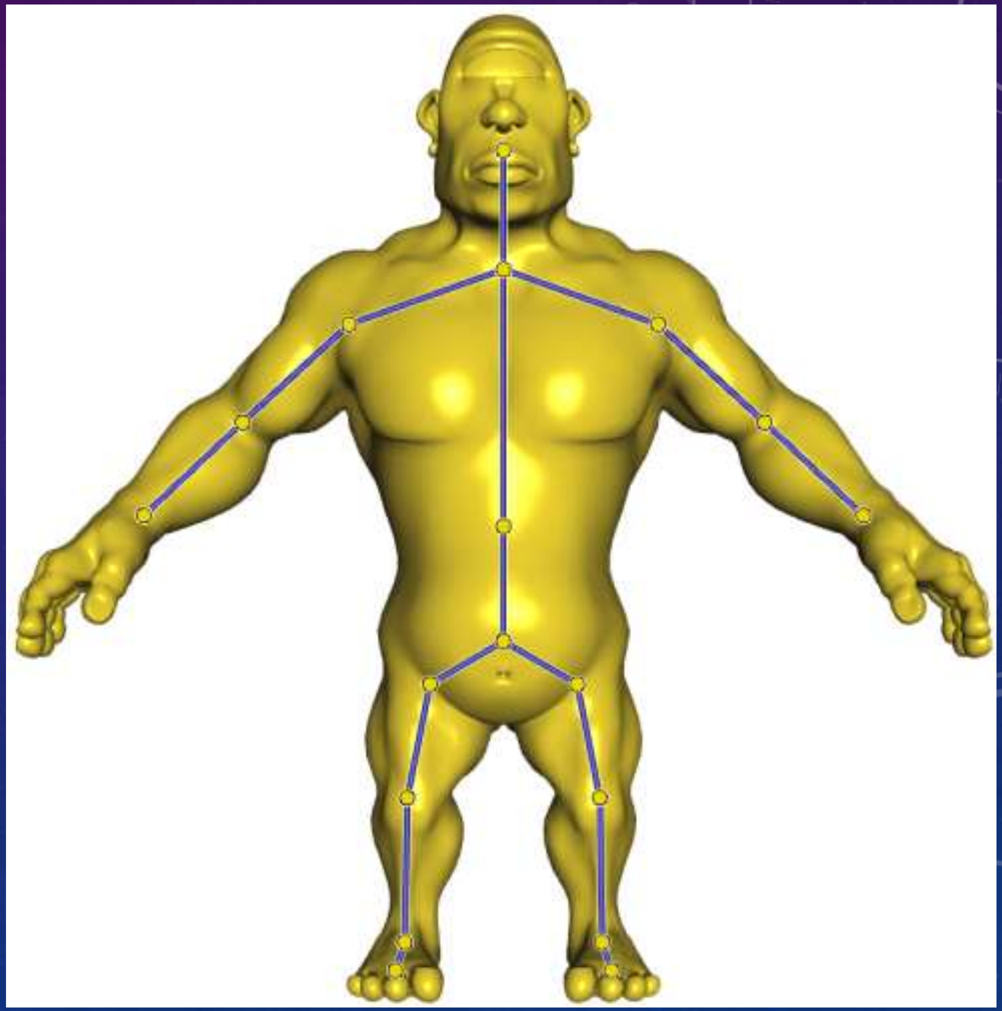
Figure 1: *The blue support region is deformed by smoothly interpolating the affinely transformed green control handle. The fair triharmonic surface-based deformations of [BK04a]* (left) *can be reproduced by the triharmonic space deformation, where the C^2 constraints are defined by the red bands of three points thickness* (center)*. However, the number of centers required for a sufficiently accurate approximation (see Sect. 5) is usually significantly lower, like 20% in this example* (right)*.*

Skeleton

- Place skeleton in shapes
  - Medial Axis Transform (MAT)

Skeleton extraction

- Laplacian shrinking
- Voronoi diagram
- Reeb graph, segmentation …



Skeleton

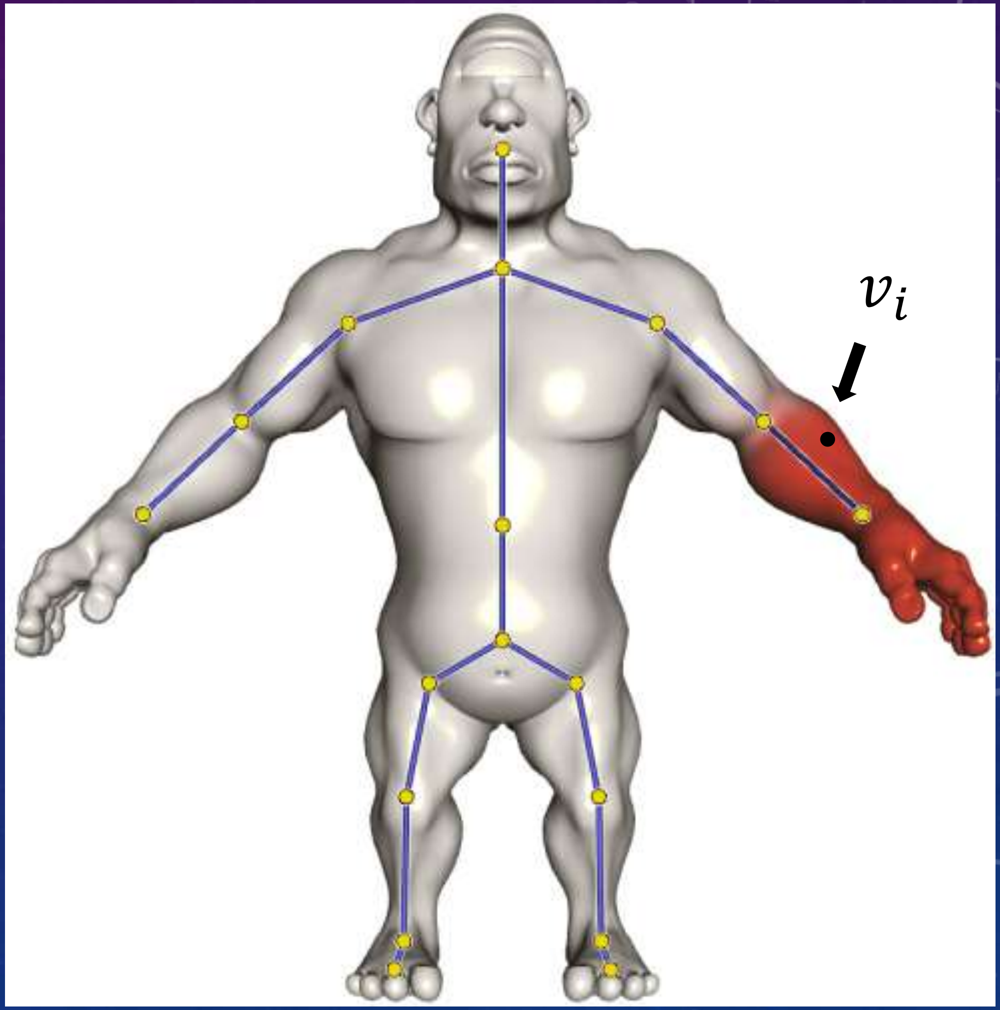
- Compute/paint weights

$$v_i' = \sum_{j=1}^{m} w_{i,j} T_j v_i = \left(\sum_{j=1}^{m} w_{i,j} T_j \right) v_i$$

Skinning:

$w_{i,j} \geq 0$ and $\sum_{j=1}^{m} w_{i,j} = 1$

Sparsity: for each v_i, only a few $w_{i,j} > 0$



Skeleton

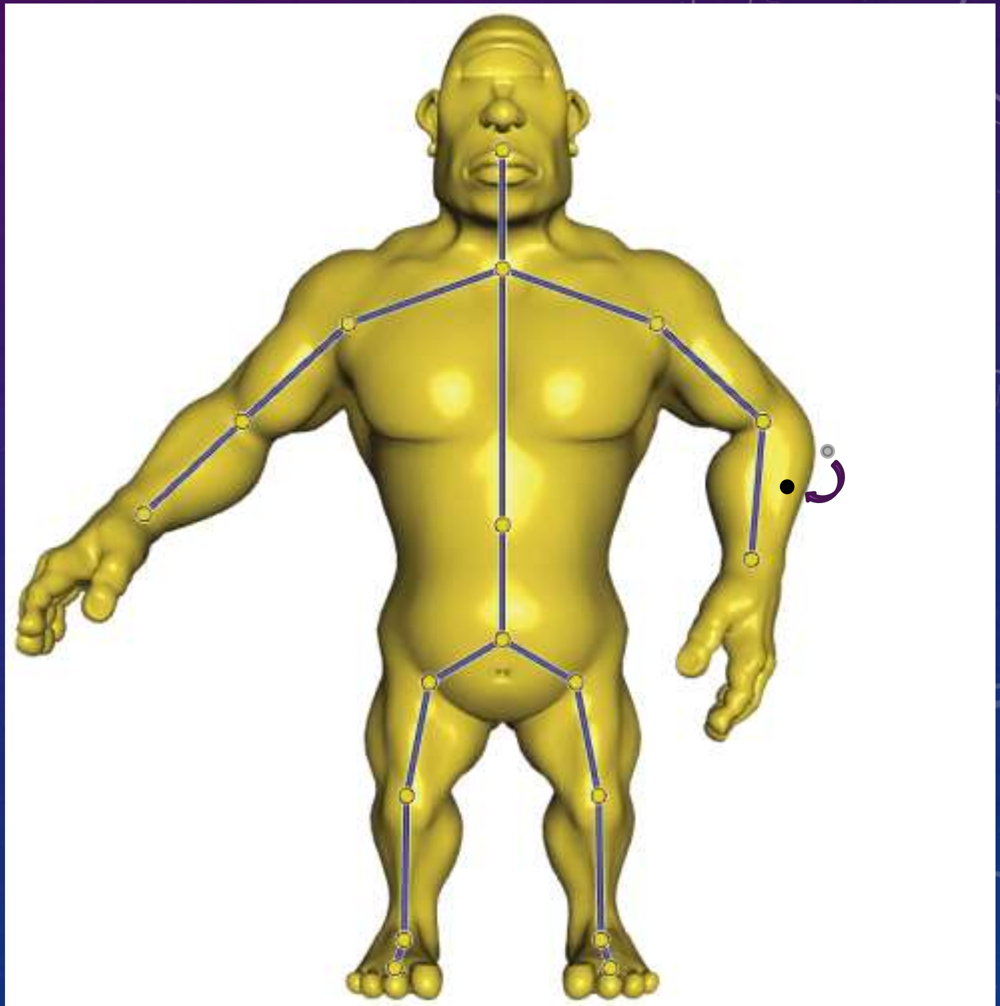
- Deform bones

T_j : rotation + translation

Blending rotations

$$1/2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + 1/2\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

Rotation by 0	Rotation by π	Not a rotation



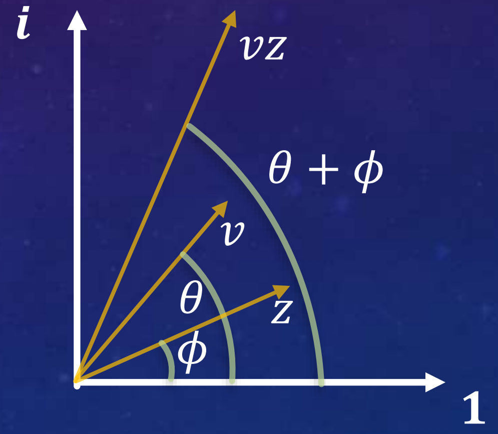
Rotate and scale

- 2D plane : complex number

$$z = x + iy = \exp(\rho + i\theta)$$

where

$$\exp(\rho + i\theta) = \exp\rho\,(\cos\theta + i\sin\theta)$$



Rotate and scale

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$$

- 3D space : quaternions

$\mathbb{H} : q = (q_0, q_1, q_2, q_3) \rightarrow q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k} = (\dot{q}, \vec{q})$

$$pq = (\dot{p}, \vec{p})(\dot{q}, \vec{q}) = (\dot{p}\dot{q} - \vec{p} \cdot \vec{q}, \dot{p}\vec{q} + \dot{q}\vec{p} + \vec{p} \times \vec{q})$$

Conjugate $q^* = (q_0, -q_1, -q_2, -q_3) \rightarrow q_0 - q_1\boldsymbol{i} - q_2\boldsymbol{j} - q_3\boldsymbol{k}$

And $qq^* = q^*q = q_0^2 + q_1^2 + q_2^2 + q_3^2 := |q|^2$

Thus, if $|q| > 0$, we have $q^{-1} = \frac{q^*}{|q|^2}$

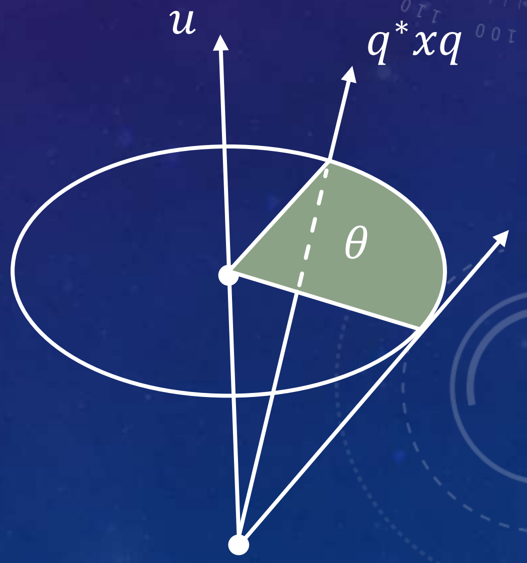
Rotate and scale

- From $\mathbb{R}^3$ to $\mathbb{H}$: $\vec{x} = (x_1, x_2, x_3) \rightarrow x = (0, \vec{x}) = 0 + x_1\boldsymbol{i} + x_2\boldsymbol{j} + x_3\boldsymbol{k}$

Rotation around the axis $\vec{u} = (u_1, u_2, u_3), \|\vec{u}\| = 1$

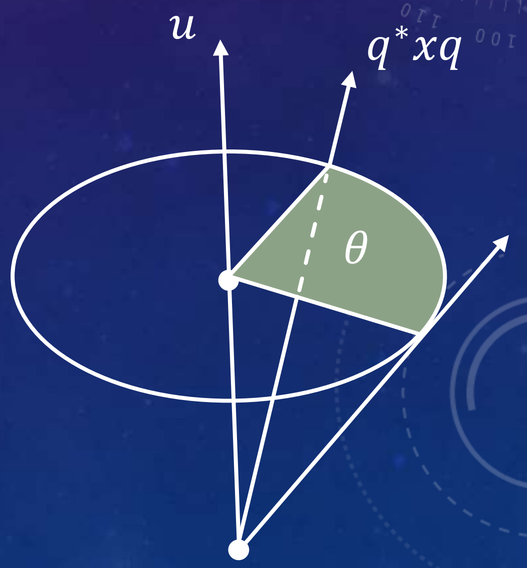
- $q = \left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\vec{u}\right) = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(u_1\boldsymbol{i} + u_2\boldsymbol{j} + u_3\boldsymbol{k})$
- $q^* = \left(\cos\frac{\theta}{2}, -\sin\frac{\theta}{2}\vec{u}\right),\ qq^* = 1 \rightarrow q^{-1} = q^*$
- $y = qxq^* = 0 + y_1 i + y_2 j + y_3 k \rightarrow \vec{y} = (y_1, y_2, y_3)$

Scale : $p = cq, c \in \mathbb{R} \Longrightarrow y' = p^*xp = c^2 y$



Rotate and scale

$$\left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\vec{u}\right)(0,\lambda\vec{u})\left(\cos\frac{\theta}{2}, -\sin\frac{\theta}{2}\vec{u}\right)$$
$$= \left(-\sin\frac{\theta}{2}\lambda, \cos\frac{\theta}{2}\lambda\vec{u}\right)\left(\cos\frac{\theta}{2}, -\sin\frac{\theta}{2}\vec{u}\right)$$
$$= \left(-\sin\frac{\theta}{2}\cos\frac{\theta}{2}\lambda + \sin\frac{\theta}{2}\cos\frac{\theta}{2}\lambda, \cos^2\frac{\theta}{2}\lambda\vec{u} + \sin^2\frac{\theta}{2}\lambda\vec{u}\right)$$
$$= (0,\vec{u})$$



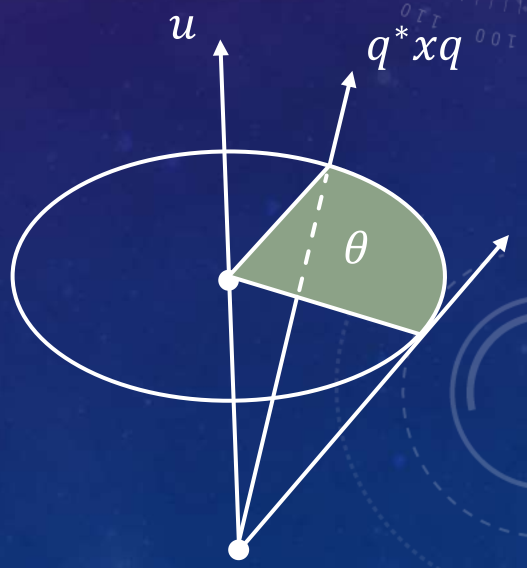
Rotate and scale

For any $\vec{u} \cdot \vec{v} = 0$

$$\left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\vec{u}\right)(0, \vec{v})\left(\cos\frac{\theta}{2}, -\sin\frac{\theta}{2}\vec{u}\right)$$

$$= \left(0, \cos\frac{\theta}{2}\vec{v} + \sin\frac{\theta}{2}\vec{u} \times \vec{v}\right)\left(\cos\frac{\theta}{2}, -\sin\frac{\theta}{2}\vec{u}\right)$$

$$= \left(0, \cos^2\frac{\theta}{2}\vec{v} + \cos\frac{\theta}{2}\sin\frac{\theta}{2}\vec{u} \times \vec{v} - \cos\frac{\theta}{2}\sin\frac{\theta}{2}\vec{v} \times \vec{u} - \sin^2\frac{\theta}{2}\vec{u} \times \vec{v} \times \vec{u}\right) = (0, \cos\theta\,\vec{v} + \sin\theta\,\vec{u} \times \vec{v})$$



Euler's identity

Let unit quaternion $p = (\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\vec{u})$, $q = (\cos\frac{\phi}{2}, \sin\frac{\phi}{2}\vec{u})$

$$pq = (\dot{p}\dot{q} - \vec{p}\cdot\vec{q}, \dot{p}\vec{q} + \dot{q}\vec{p} + \vec{p}\times\vec{q}) = (\cos\frac{\theta+\phi}{2}, \sin\frac{\theta+\phi}{2}\vec{u})$$

Euler's identity

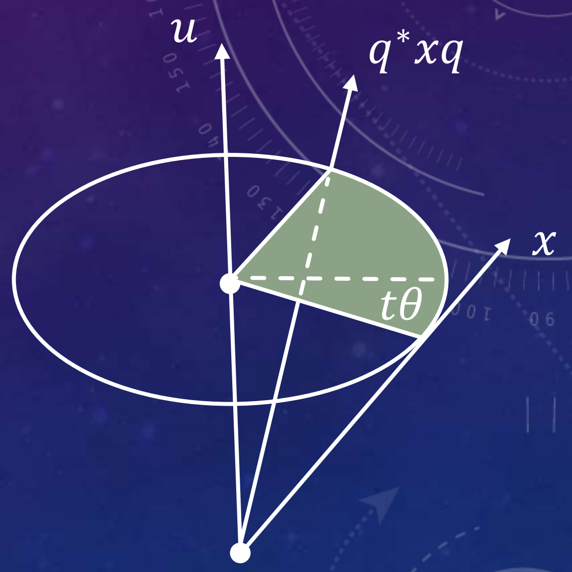
$$\exp\left(\vec{u}\frac{\theta}{2}\right) = \left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\vec{u}\right) \Longrightarrow \log\left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\vec{u}\right) = \vec{u}\frac{\theta}{2}$$

We have $\exp\left(\vec{u}\frac{\theta}{2}\right)^t = \exp\left(\vec{u}t\frac{\theta}{2}\right)$

Blend of quaternion

Two unit quaternion p, q

- Quaternion Linear Blending $QLB(t|p,q) = \frac{(1-t)p+tq}{\|(1-t)p+tq\|}$
- Spherical Linear Interpolation $ScLERP(t|p,q) = p(p^*q)^t$



Let $\sigma = p^*q = \left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\vec{u}\right) = \exp(\frac{\theta}{2}\vec{u})$,

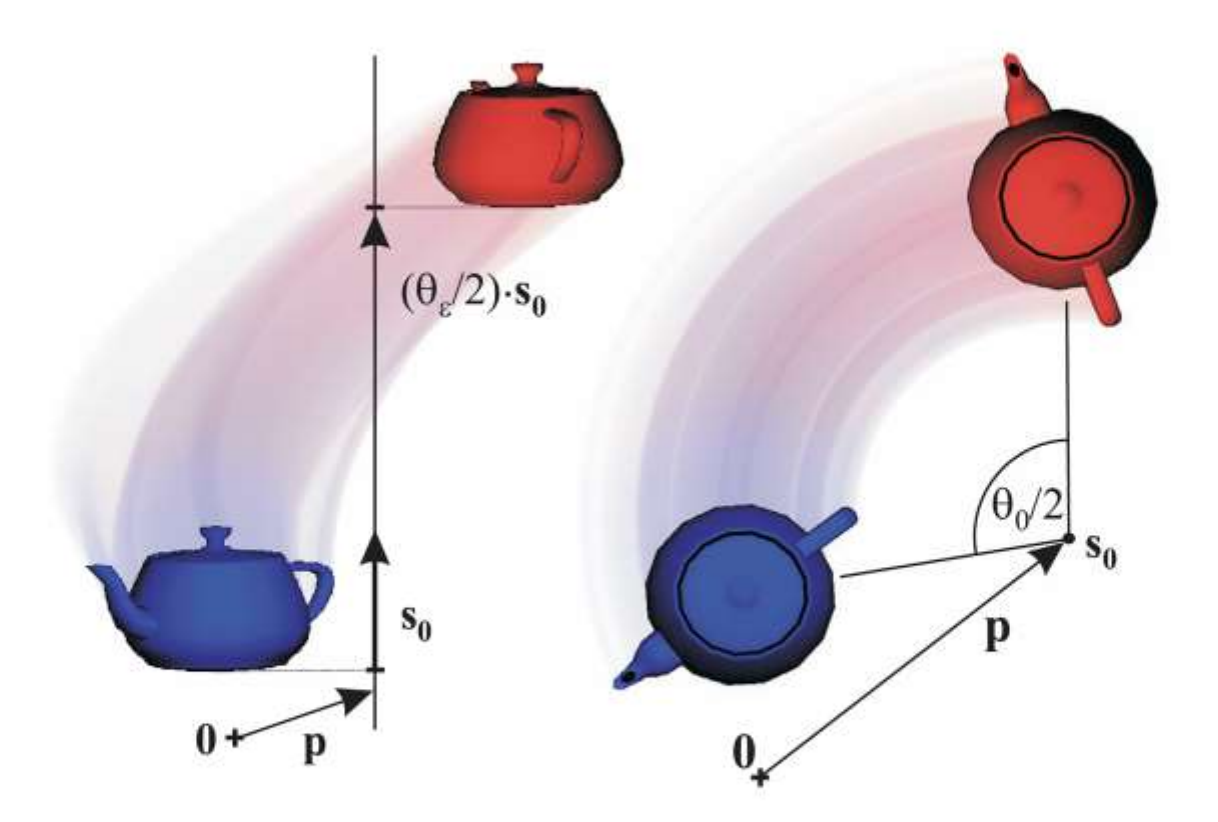
$$QLB(t|p,q) = p\frac{(1-t)+tp^*q}{\|(1-t)p+tq\|} = p\frac{((1-t)+t\cos\frac{\theta}{2}, t\sin\frac{\theta}{2}\vec{u})}{\|(1-t)p+tq\|}$$

$$ScLERP(t|p,q) = p(p^*q)^t = p\exp(t\frac{\theta}{2}\vec{u}) = p(\cos\frac{t\theta}{2}, \sin\frac{t\theta}{2}\vec{u})$$

Rotate and translate

- For example, screw motion
  - Rotation about s_0 by $\frac{\theta_0}{2}$
  - Translation along s_0 with $\frac{\theta_\epsilon}{2}$

s_0 need not pass through origin



Dual quaternion (Clifford algebra)

- Dual quaternion : $\sigma = p + \epsilon q$, where ϵ commute with $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ and $\epsilon^2 = 0$
  - $\sigma_1 + \sigma_2 = (p_1 + p_2) + \epsilon(q_1 + q_2)$
  - $\sigma_1 \otimes \sigma_2 = p_1 p_2 + \epsilon(p_1 q_2 + p_2 q_1)$
- A dual number with $p \neq 0$ has a inverse

$$\sigma^{-1} = p^{-1}(1 - \epsilon q p^{-1})$$

Dual quaternion conjugates

- $\sigma^{\cdot} = p - \epsilon q \Longrightarrow \sigma \otimes \sigma^{\cdot} = (p + \epsilon q)(p - \epsilon q) = pp + \epsilon(qp - pq)$
- $\sigma^{*} = p^{*} + \epsilon q^{*} \Longrightarrow \sigma \otimes \sigma^{*} = (p + \epsilon q)(p^{*} + \epsilon q^{*}) = pp^{*} + \epsilon(qp^{*} + pq^{*})$
- $\sigma^{\diamond} = p^{*} - \epsilon q^{*} \Longrightarrow \sigma \otimes \sigma^{\diamond} = (p + \epsilon q)(p^{*} - \epsilon q^{*}) = pp^{*} + \epsilon(qp^{*} - pq^{*})$

We have $(\sigma_1 \otimes \sigma_2)^{\cdot} = \sigma_1^{\cdot} \otimes \sigma_2^{\cdot},\ (\sigma_1 \otimes \sigma_2)^{*} = \sigma_2^{*} \otimes \sigma_1^{*},\ (\sigma_1 \otimes \sigma_2)^{\diamond} = \sigma_2^{\diamond} \otimes \sigma_1^{\diamond}$

Unit dual quaternion

- Unit : $\sigma \otimes \sigma^* = 1$

$$\begin{cases} pp^* = 1 \\ qp^* + pq^* = 0 \end{cases} \Longrightarrow \begin{cases} p_0^2 + p_1^2 + p_2^2 + p_3^2 = 1 \\ p_0 q_0 + p_1 q_1 + p_2 q_2 + p_3 q_3 = 0 \end{cases}$$

Degrees of freedom (DOFs) – six : rotation + translation

- Rotation quaternion r and translation quaternion $t = (0, \vec{t})$

$$\sigma = r + \epsilon \frac{1}{2} tr \Longrightarrow \sigma \otimes \sigma^* = 1$$

Unit dual quaternion

- For vector $\vec{v}$, corresponding dual quaternion $1 + \epsilon(0, \vec{v}) = 1 + \epsilon v$

$$\sigma \otimes (1 + \epsilon v) \otimes \sigma^{\diamond} = \left(r + \epsilon \frac{1}{2} tr\right) \otimes (1 + \epsilon v) \otimes \left(r^* - \epsilon \frac{1}{2} r^* t^*\right)$$

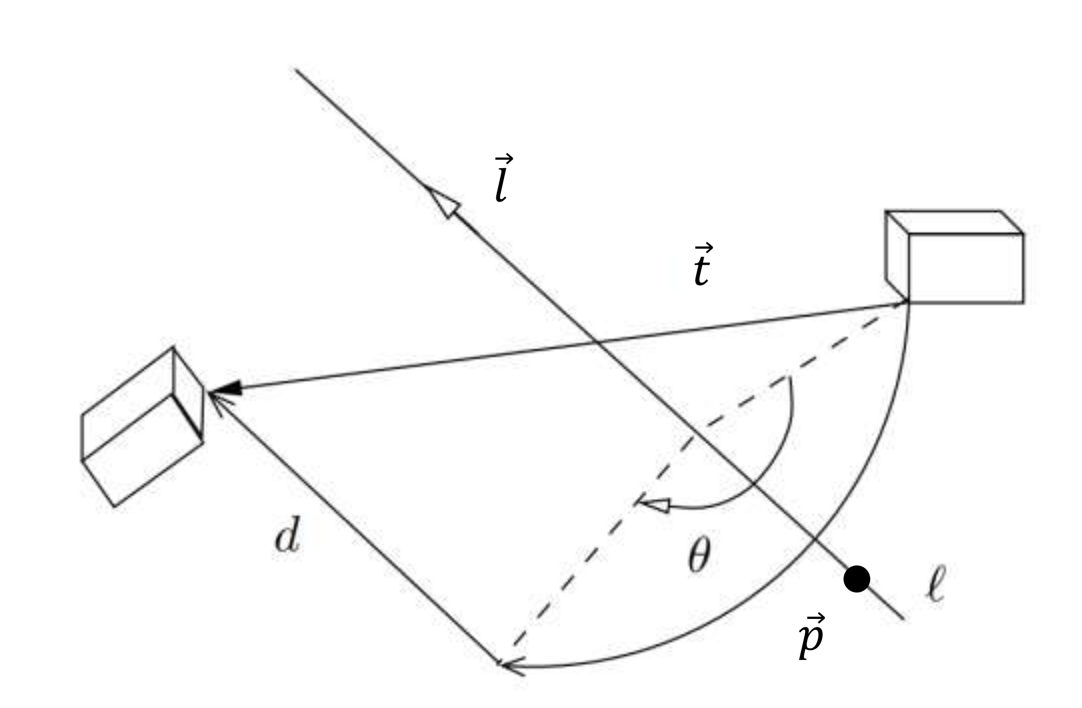
$$= \left(r + \epsilon\left(rv + \frac{1}{2} tr\right)\right) \otimes \left(r^* - \epsilon \frac{1}{2} r^* t^*\right)$$

$$= rr^* + \epsilon\left(rvr^* + \frac{1}{2} trr^* - \frac{1}{2} rr^* t^*\right) = 1 + \epsilon(rvr^* + t)$$

Rigid = Screw

Chasles' theorem.

Any rigid displacement is equivalent to a rotation about some line, called the screw axis, followed by a translation in the direction of the line.



Parameter:

$$d = \vec{t} \cdot \vec{l}, \qquad \vec{m} = \vec{p} \times \vec{l}$$

Rigid = Screw

From $\sigma = p + \epsilon q$, we have

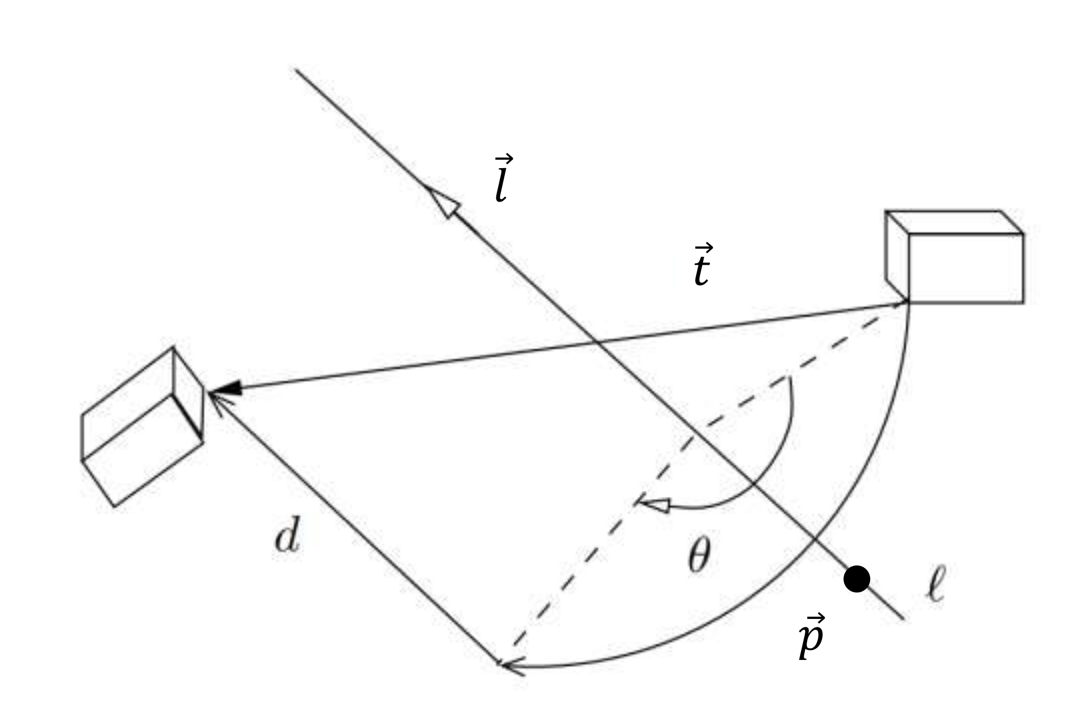
$$p = (\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\vec{l})$$

$$d = \vec{t} \cdot \vec{l} = \mathrm{Im}(2qp^*) \cdot \vec{l}$$

For moment $\boldsymbol{m}$,

$$\vec{m} = \frac{1}{2}\left(\vec{t} \times \vec{l} + \vec{l} \times \left(\vec{t} \times \vec{l}\right) \cot\frac{\theta}{2}\right)$$

$$= \frac{1}{2}\left(\vec{t} \times \vec{l} + \left(\vec{t} - d\vec{l}\right) \cot\frac{\theta}{2}\right)$$



Rigid = Screw

$$\vec{m} = \frac{1}{2}\left(\vec{t} \times \vec{l} + \left(\vec{t} - d\vec{l}\right)\cot\frac{\theta}{2}\right)$$

Multiply $\sin\frac{\theta}{2}$,

$$\vec{m}\sin\frac{\theta}{2} + \vec{l}\frac{d}{2}\cos\frac{\theta}{2}$$
$$= \frac{1}{2}\left(\vec{t} \times \vec{l}\sin\frac{\theta}{2} + \vec{t}\cos\frac{\theta}{2}\right)$$

$$\frac{1}{2}tr = \frac{1}{2}(0, \vec{t})\left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\vec{l}\right)$$
$$= \frac{1}{2}\left(-\vec{t} \cdot \vec{l}\sin\frac{\theta}{2}, \vec{t}\cos\frac{\theta}{2} + \vec{t} \times \vec{l}\sin\frac{\theta}{2}\right)$$
$$= \left(-\frac{d}{2}\sin\frac{\theta}{2}, \vec{m}\sin\frac{\theta}{2} + \vec{l}\frac{d}{2}\cos\frac{\theta}{2}\right)$$

Rigid = Screw

$$\sigma = r + \epsilon\frac{1}{2}tr$$

$$= \left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\vec{l}\right) + \epsilon\left(-\frac{d}{2}\sin\frac{\theta}{2}, \vec{m}\sin\frac{\theta}{2} + \vec{l}\frac{d}{2}\cos\frac{\theta}{2}\right)$$

$$= \left(\cos\frac{\theta}{2} - \epsilon\frac{d}{2}\sin\frac{\theta}{2}, \sin\frac{\theta}{2}\vec{l} + \epsilon(\vec{m}\sin\frac{\theta}{2} + \vec{l}\frac{d}{2}\cos\frac{\theta}{2})\right)$$

$$= (\cos\frac{\theta+\epsilon d}{2}, \sin\frac{\theta+\epsilon d}{2}(\vec{l} + \epsilon\vec{m}))$$

By defining

$$\begin{cases} \cos\dfrac{\theta + \epsilon d}{2} = \cos\dfrac{\theta}{2} - \epsilon\dfrac{d}{2}\sin\dfrac{\theta}{2} \\ \sin\dfrac{\theta + \epsilon d}{2} = \sin\dfrac{\theta}{2} + \epsilon\dfrac{d}{2}\cos\dfrac{\theta}{2} \end{cases}$$

$$\frac{1}{2}tr = \frac{1}{2}(0, \vec{t})\left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\vec{l}\right)$$

$$= \frac{1}{2}\left(-\vec{t}\cdot\vec{l}\sin\frac{\theta}{2}, \vec{t}\cos\frac{\theta}{2} + \vec{t}\times\vec{l}\sin\frac{\theta}{2}\right)$$

$$= (-\frac{d}{2}\sin\frac{\theta}{2}, \vec{m}\sin\frac{\theta}{2} + \vec{l}\frac{d}{2}\cos\frac{\theta}{2})$$

Rigid = Screw

$$\sigma = r + \epsilon \frac{1}{2} tr$$

$$= \left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\vec{l}\right) + \epsilon\left(-\frac{d}{2}\sin\frac{\theta}{2}, \vec{m}\sin\frac{\theta}{2} + \vec{l}\frac{d}{2}\cos\frac{\theta}{2}\right)$$

$$= \left(\cos\frac{\theta}{2} - \epsilon\frac{d}{2}\sin\frac{\theta}{2}, \sin\frac{\theta}{2}\vec{l} + \epsilon(\vec{m}\sin\frac{\theta}{2} + \vec{l}\frac{d}{2}\cos\frac{\theta}{2})\right)$$

$$= (\cos\frac{\theta+\epsilon d}{2}, (\vec{l} + \epsilon\vec{m})\sin\frac{\theta+\epsilon d}{2})$$

By defining

$$\begin{cases} \cos\dfrac{\theta + \epsilon d}{2} = \cos\dfrac{\theta}{2} - \epsilon\dfrac{d}{2}\sin\dfrac{\theta}{2} \\ \sin\dfrac{\theta + \epsilon d}{2} = \sin\dfrac{\theta}{2} + \epsilon\dfrac{d}{2}\cos\dfrac{\theta}{2} \end{cases}$$

Let $\begin{cases} \theta' = \theta + \epsilon d \\ \vec{l}' = \vec{l} + \epsilon\vec{m}' \end{cases}$

$$\sigma = \left(\cos\frac{\theta'}{2}, \vec{l}'\sin\frac{\theta'}{2}\right) = \exp(\vec{l}'\frac{\theta'}{2})$$

$$\exp\left(\vec{l}'\frac{\theta'}{2}\right)^{\mathrm{t}} = \exp(\vec{l}'\frac{t\theta'}{2})$$

Blend of dual quaternion

Two unit dual quaternion σ_1, σ_2

- Quaternion Linear Blending $QLB(t|\sigma_1, \sigma_2) = \frac{(1-t)\sigma_1 + t\sigma_2}{\|(1-t)\sigma_1 + t\sigma_2\|}$
- Spherical Linear Interpolation $ScLERP(t|\sigma_1, \sigma_2) = \sigma_1 \otimes (\sigma_1^* \otimes \sigma_2)^t$

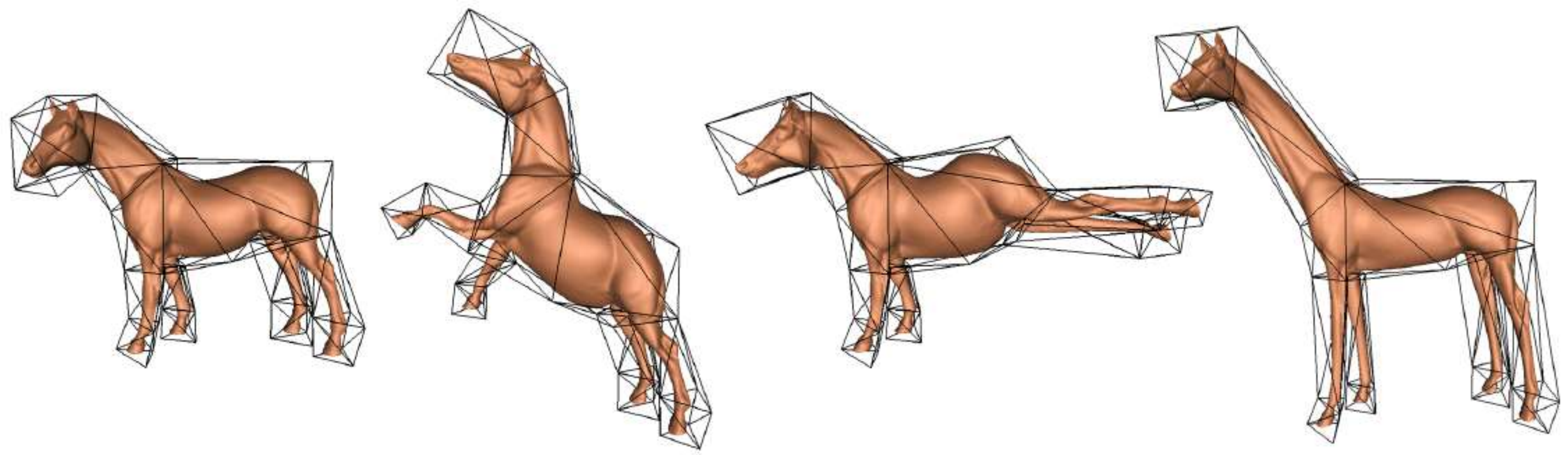
Let $\sigma_1^* \otimes \sigma_2 = \exp(\vec{l}\,\frac{\theta}{2})$

$$QLB(t|\sigma_1, \sigma_2) = \frac{(1-t)\sigma_1 + t\sigma_2}{\|(1-t)\sigma_1 + t\sigma_2\|} = \sigma_1 \otimes \frac{((1-t) + t\cos\frac{\theta}{2}, t\sin\frac{\theta}{2}\vec{l})}{\|(1-t)\sigma_1 + t\sigma_2\|}$$

$$ScLERP(t|p, q) = p(p^* q)^t = p\exp(\vec{l}t\frac{\theta}{2}) = p(\cos\frac{t\theta}{2}, \vec{l}\sin\frac{t\theta}{2})$$

Cage-based deformation

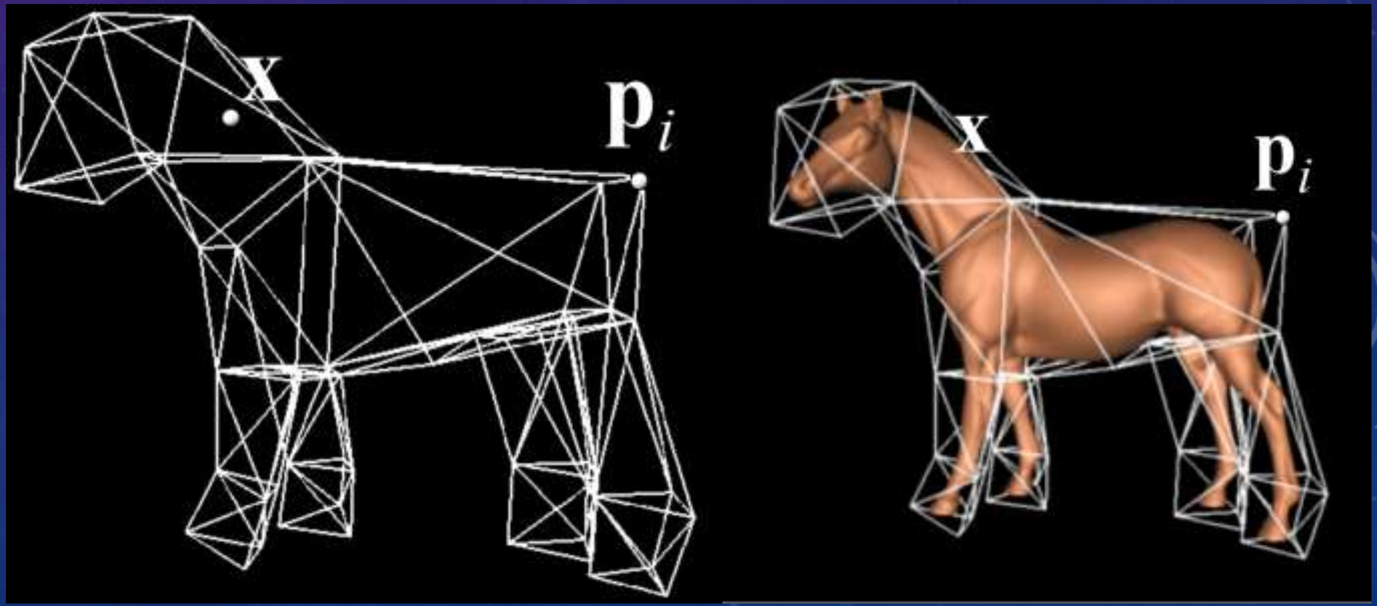
- Cage = crude version of the input shape
- Polytope (not a lattice)



Cage-based deformation

- Each point x in space is represented w.r.t. to the cage elements using coordinate functions

$$x = \sum_{i=1}^{k} w_i(x) p_i$$

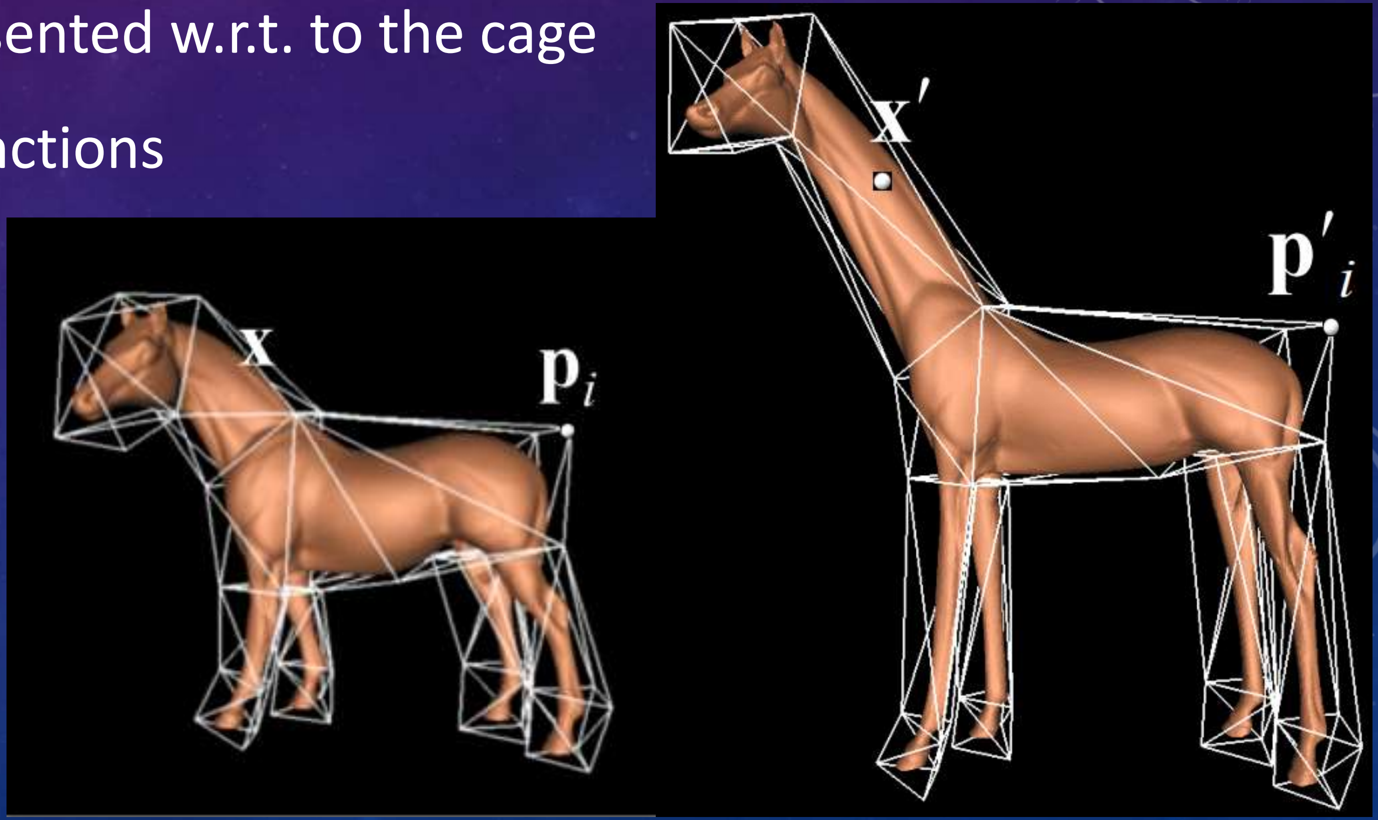


Cage-based deformation

- Each point x in space is represented w.r.t. to the cage elements using coordinate functions

$$x = \sum_{i=1}^{k} w_i(x) p_i$$

$$x' = \sum_{i=1}^{k} w_i(x) p_i'$$



Generalized barycentric coordinates

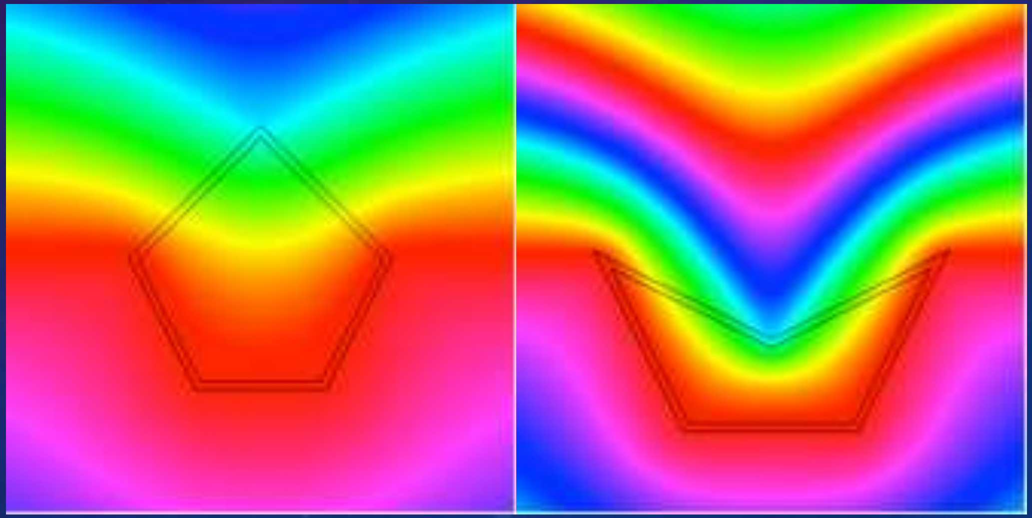
- Lagrange property: $w_i(p_j) = \delta_{ij}$
- Partition of unity: $\forall x, \sum_{i=1}^{k} w_i(x) = 1$
- Reproduction: $\forall x, \sum_{i=1}^{k} w_i(x) p_i = x$

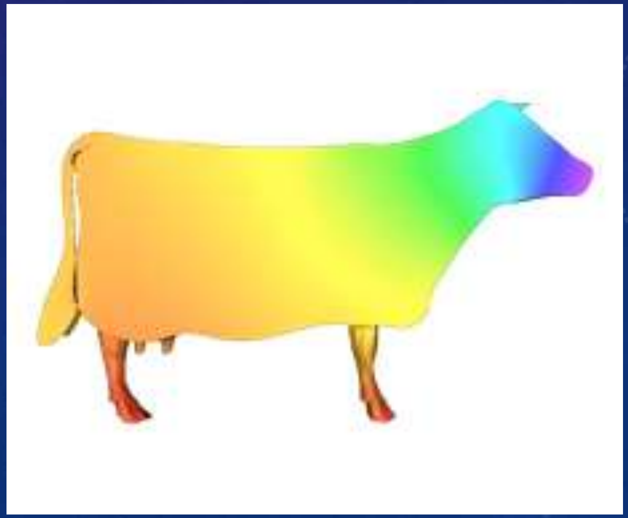
Generalized barycentric coordinates

- Mean-value coordinates
- Harmonic coordinates
- Green coordinates
- Bounded biharmonic weights
- Local barycentric coordinates

Mean-value coordinates

- Mean-value coordinates [Floater 2003, Ju et al. 2005]
  - Generalization of barycentric coordinates
  - Closed-form solution for $w_i(x)$



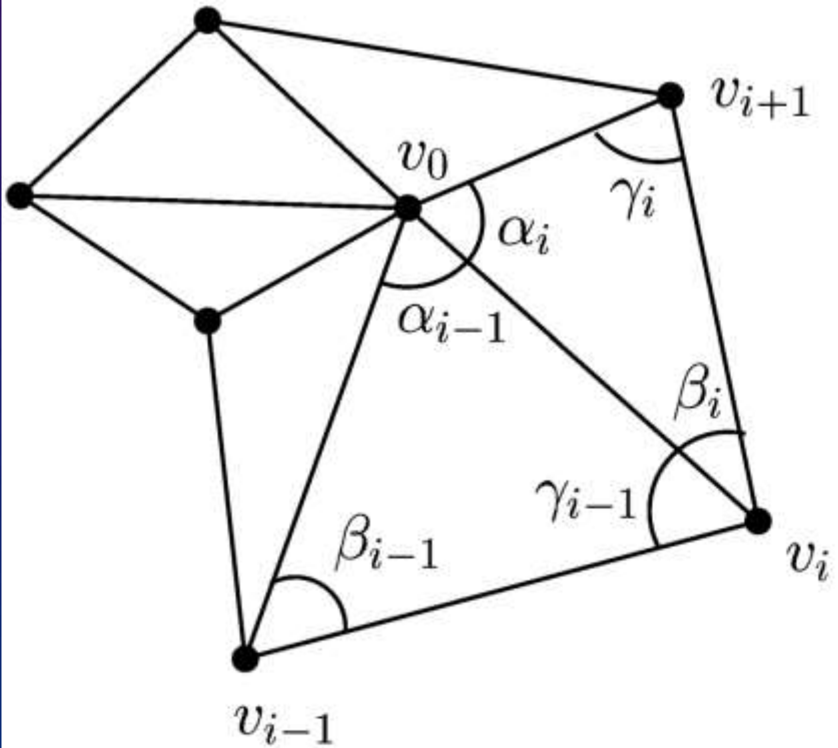


Mean-value coordinates

- Mean-value coordinates

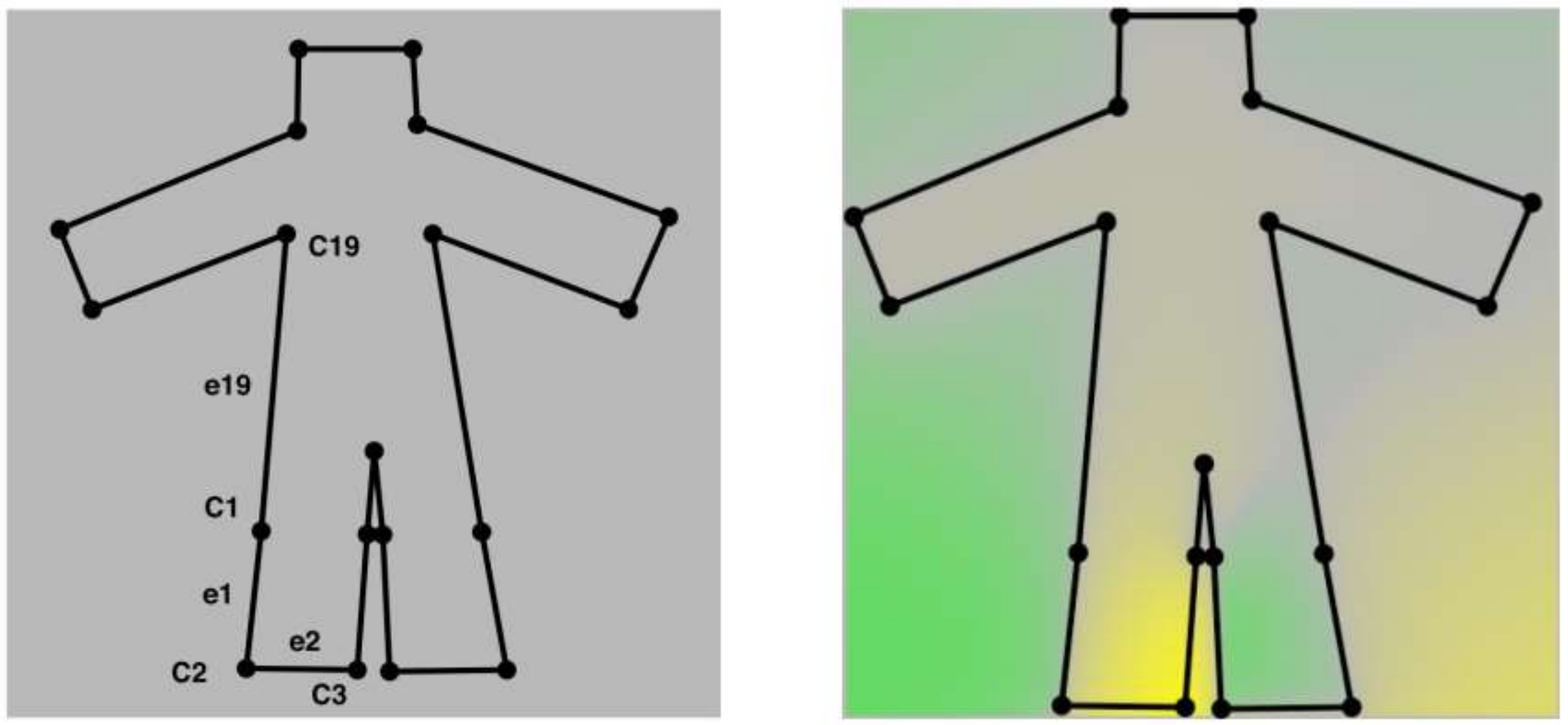
$$\phi_i(v_0) = \frac{\tan\frac{\alpha_{i-1}}{2} + \tan\frac{\alpha_i}{2}}{\|v_i - v_0\|}$$

$$w_i(v_0) = \frac{\phi_i(v_0)}{\sum_{i=1}^{k} \phi_i(v_0)}$$

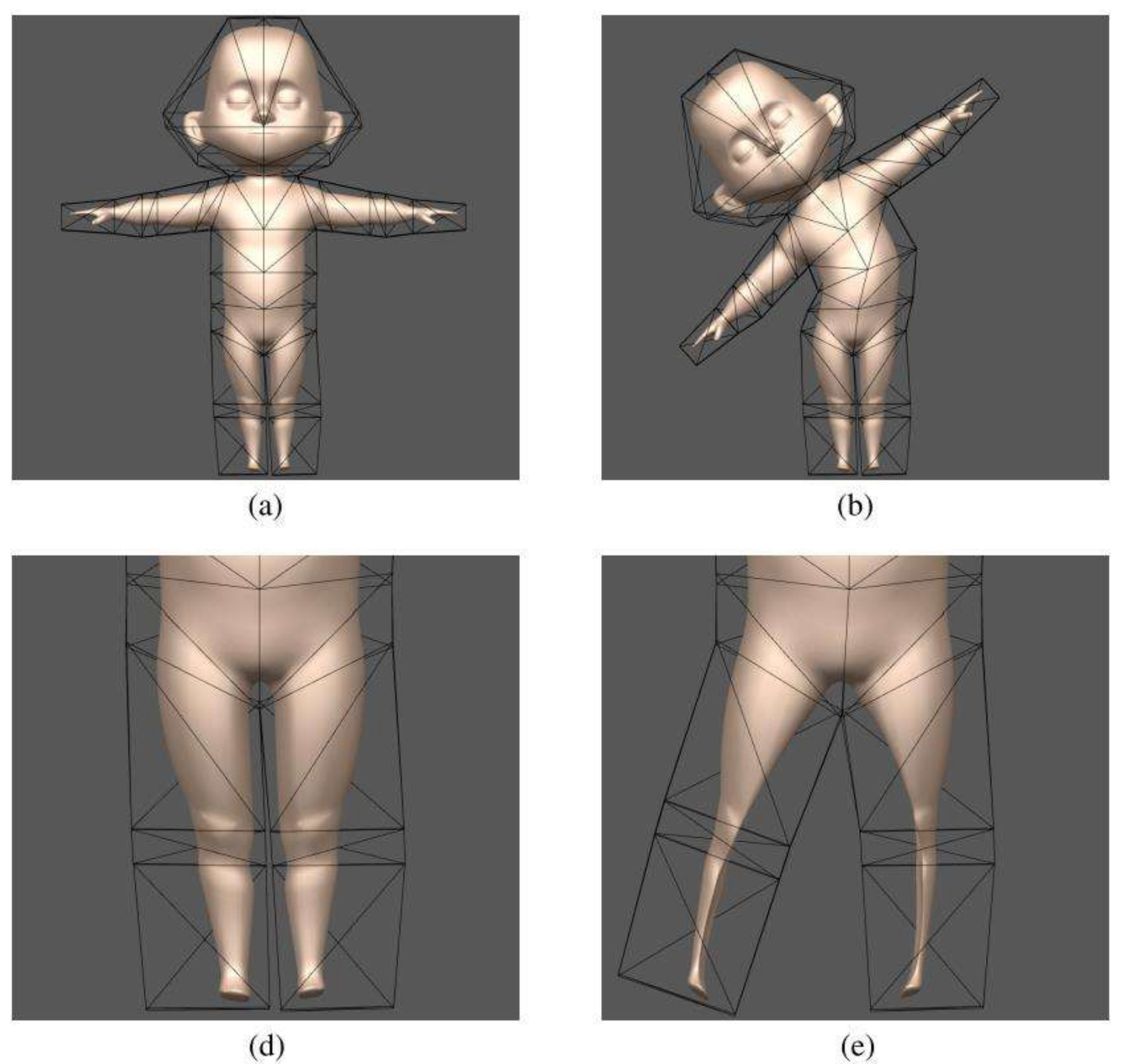


Concave polygon

Yellow indicates positive values, green indicates negative values.



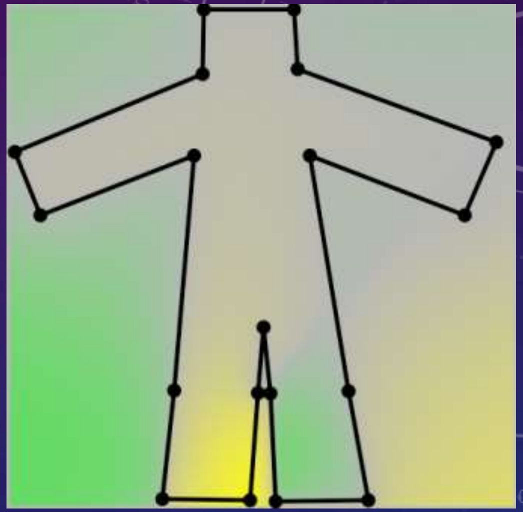
Results



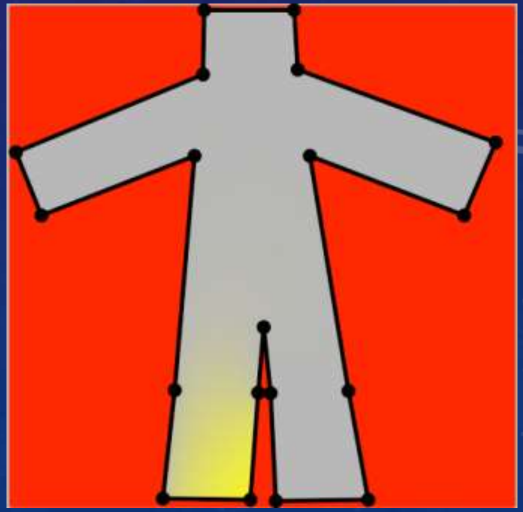
Harmonic coordinates

- Harmonic coordinates [Joshi et al. 2007]
  - Harmonic functions $h_i(x)$ for each cage vertex p_i
  - Solve $\Delta h = 0$

Subject to h_i linear on the boundary s.t. $h_i(p_j) = \delta_{ij}$



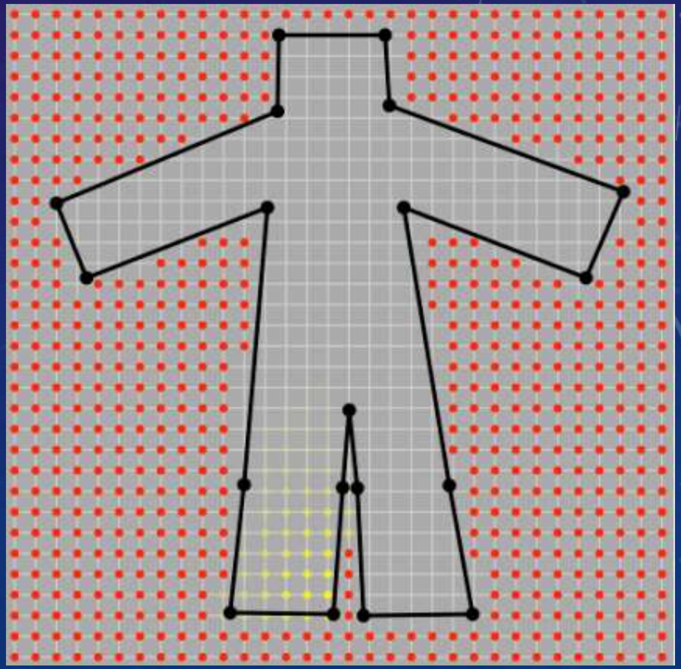
MVC



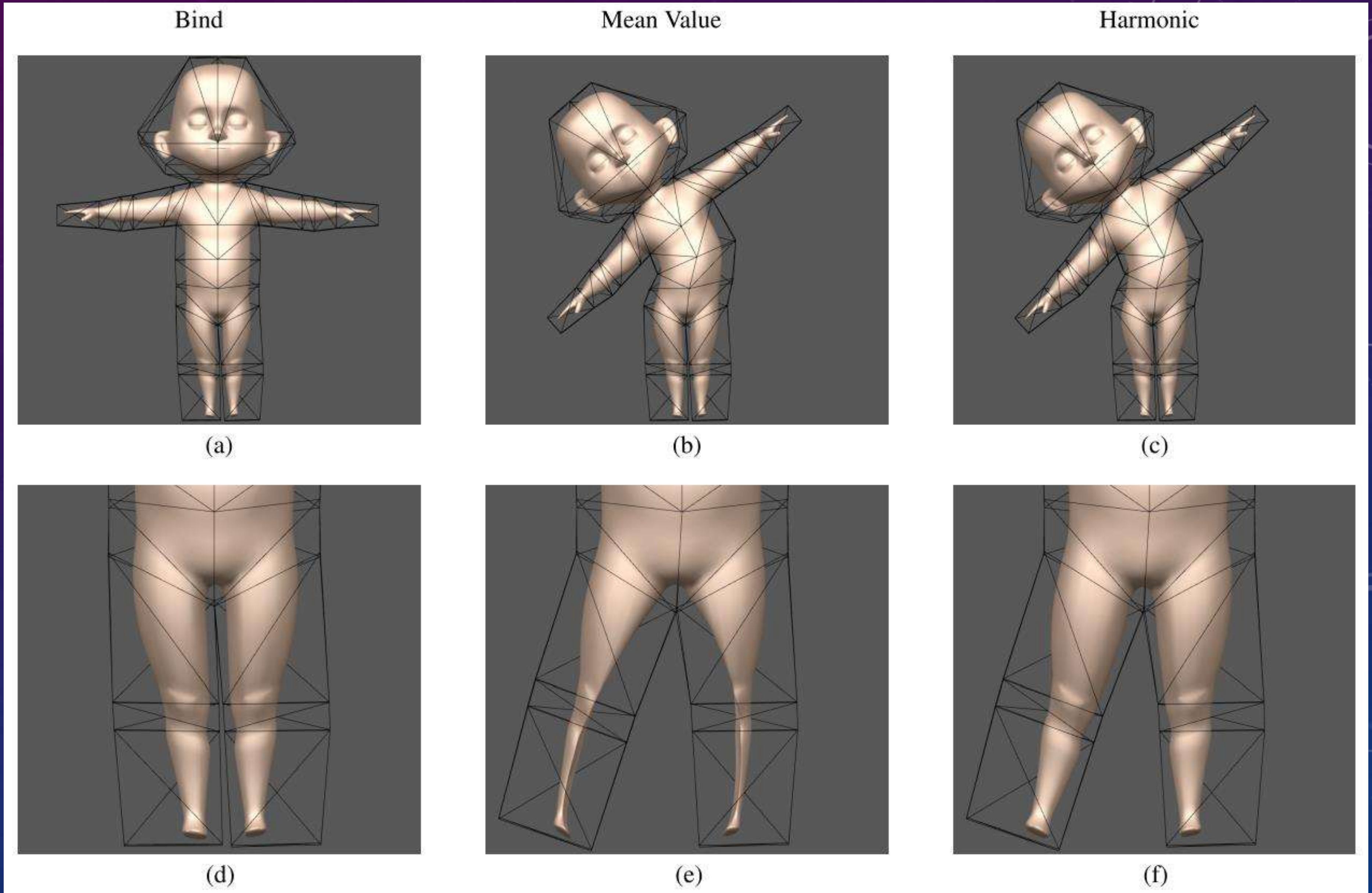
HC

Numerical solution

- Allocate a regular grid of cells that is large enough to enclose the cage
- Volumetric Laplace equation
  - Laplacian smooth: explicit iteration until convergence
  - Hierarchical finite difference solver



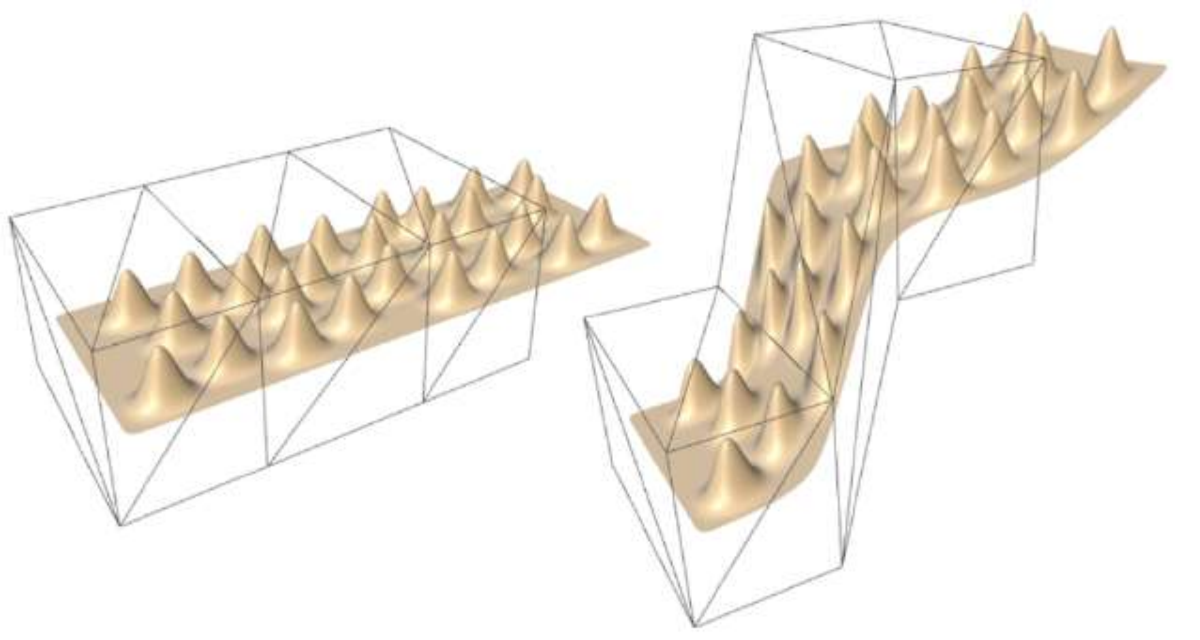
Results



Green coordinates

- Green coordinates [Lipman et al. 2008]
- Observation: previous vertex-based basis functions always lead to affine invariance!

$$x' = \sum_{i=1}^{k} w_i(x) p_i'$$

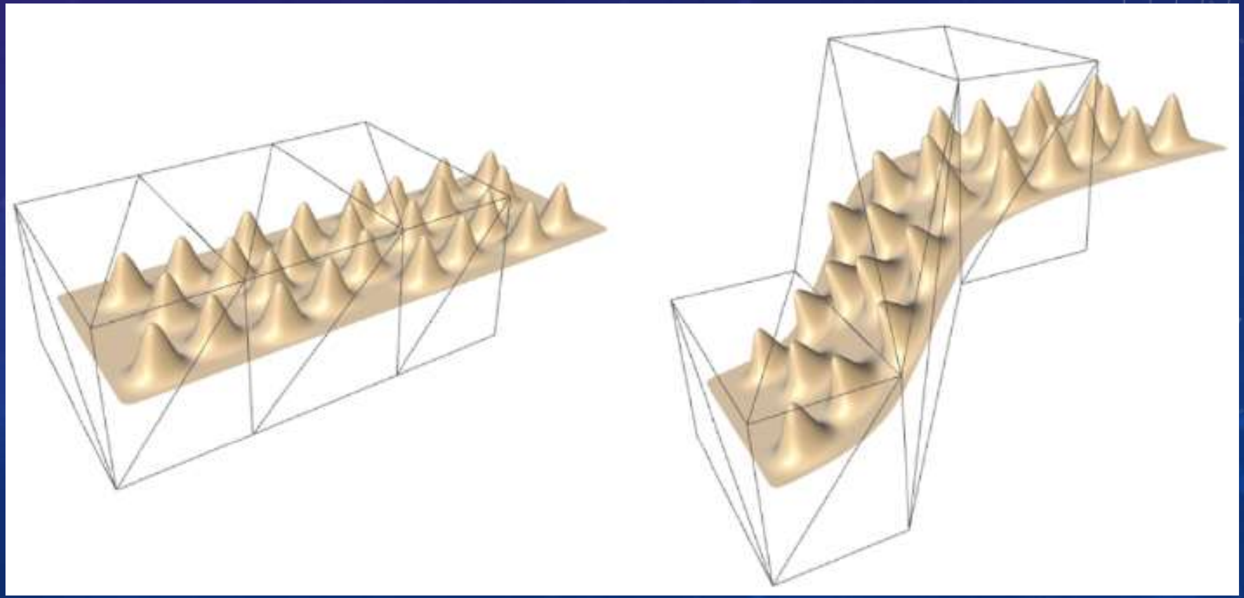


Green coordinates

- Green coordinates [Lipman et al. 2008]
- Correction: Make the coordinates depend on the cage faces as well

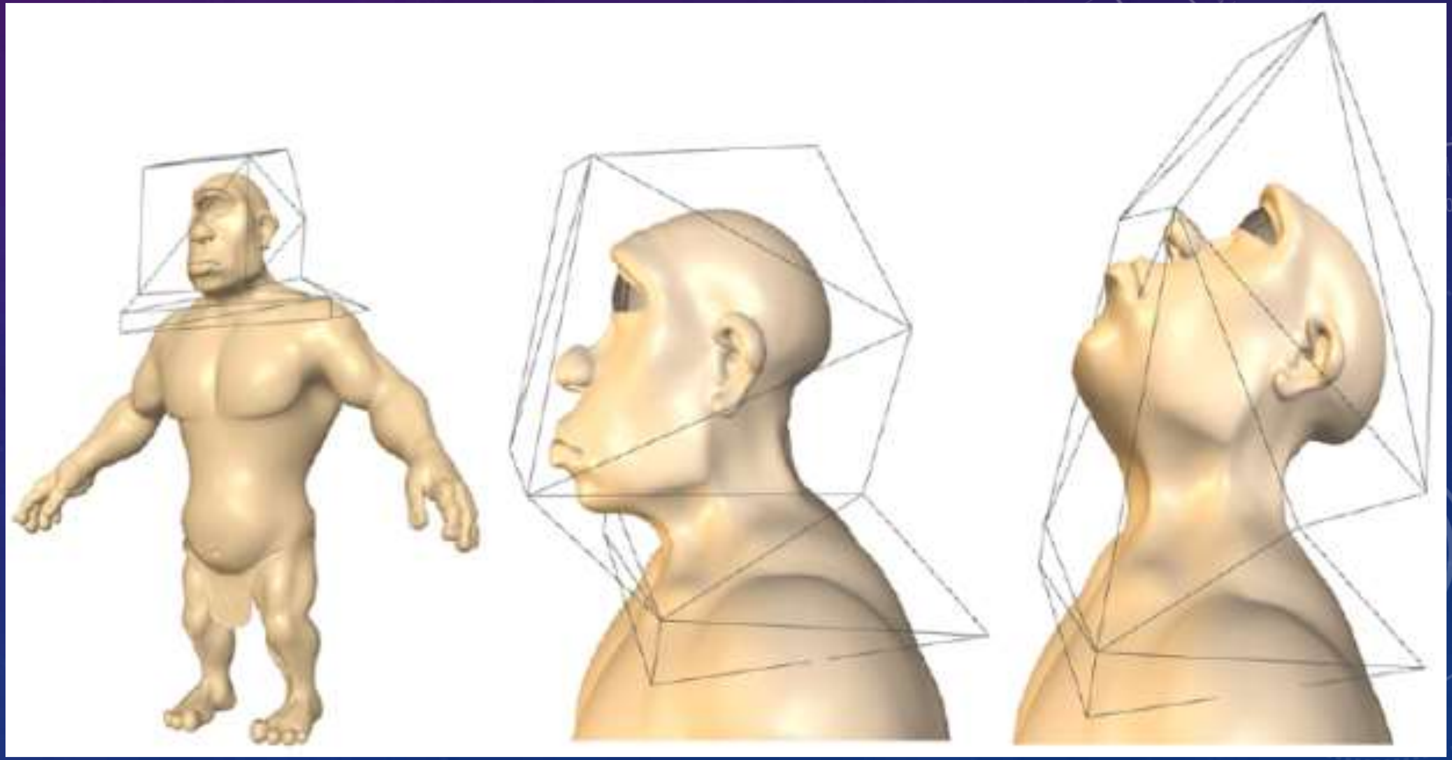
$$x' = \sum_{i=1}^{k} w_i(x) p_i'$$

$$x' = \sum_{i=1}^{k} w_i(x) p_i' + \sum_{i=1}^{k} \phi_i(x) n_i'$$



Green coordinates

- Closed-form solution
  - Conformal in 2D
  - quasi-conformal in 3D
- Hard to control details of embedded surface



Bounded biharmonic weights

$$\underset{w_j,\ j=1,\ldots,m}{\arg\min} \sum_{j=1}^{m} \frac{1}{2} \int_{\Omega} \left\| \Delta w_j \right\|^2 dV \tag{2}$$

$$\text{subject to:} \quad w_j\big|_{H_k} = \delta_{jk} \tag{3}$$

$$w_j\big|_{F} \text{ is linear} \qquad \forall F \in \mathcal{F}_{\mathcal{C}} \tag{4}$$

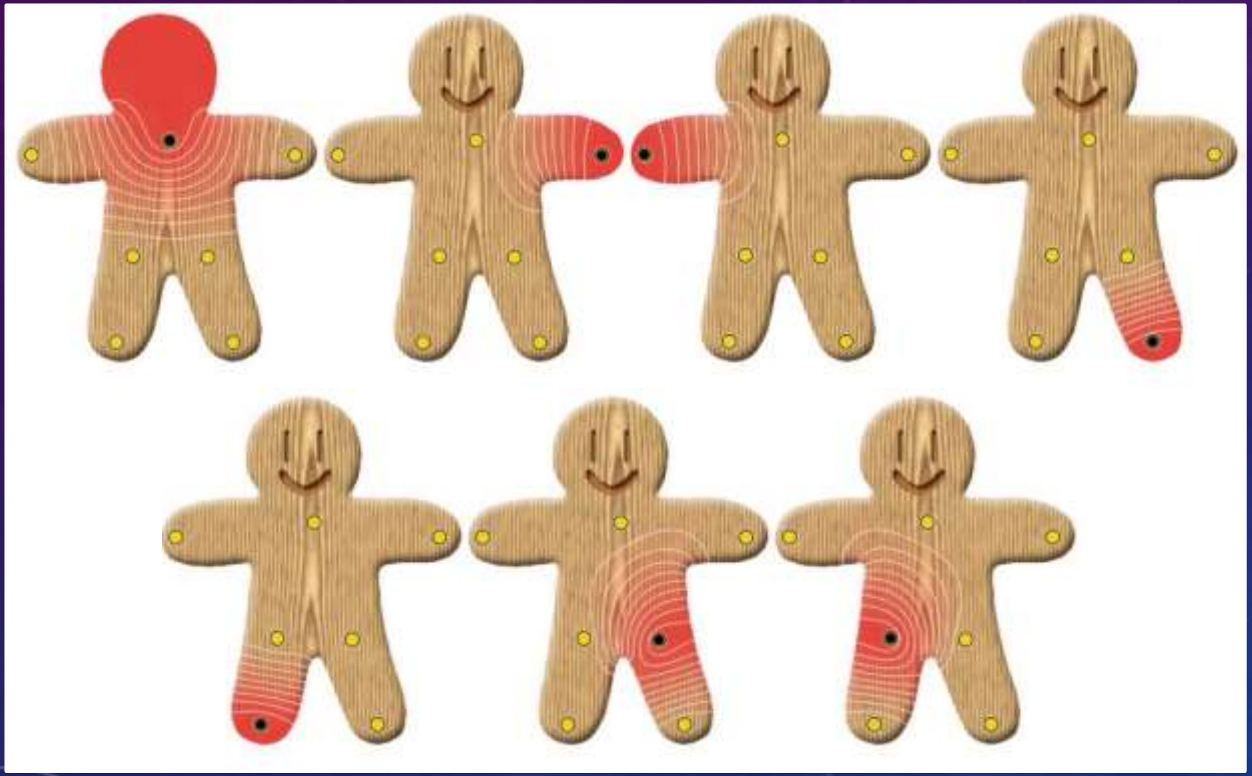
$$\sum_{j=1}^{m} w_j(\mathbf{p}) = 1 \qquad \forall \mathbf{p} \in \Omega \tag{5}$$

$$0 \le w_j(\mathbf{p}) \le 1, \;\; j = 1, \ldots, m, \qquad \forall \mathbf{p} \in \Omega, \tag{6}$$

Properties

- Smoothness ($\Delta^2 w_j = 0$) - C^1 at the handles and C^∞ everywhere else
- Non-negativity
- Shape-awareness: bi-Laplacian operator
- Partition of unity
- Locality and sparsity: just observation
- No local maxima: experimentally observed

Properties



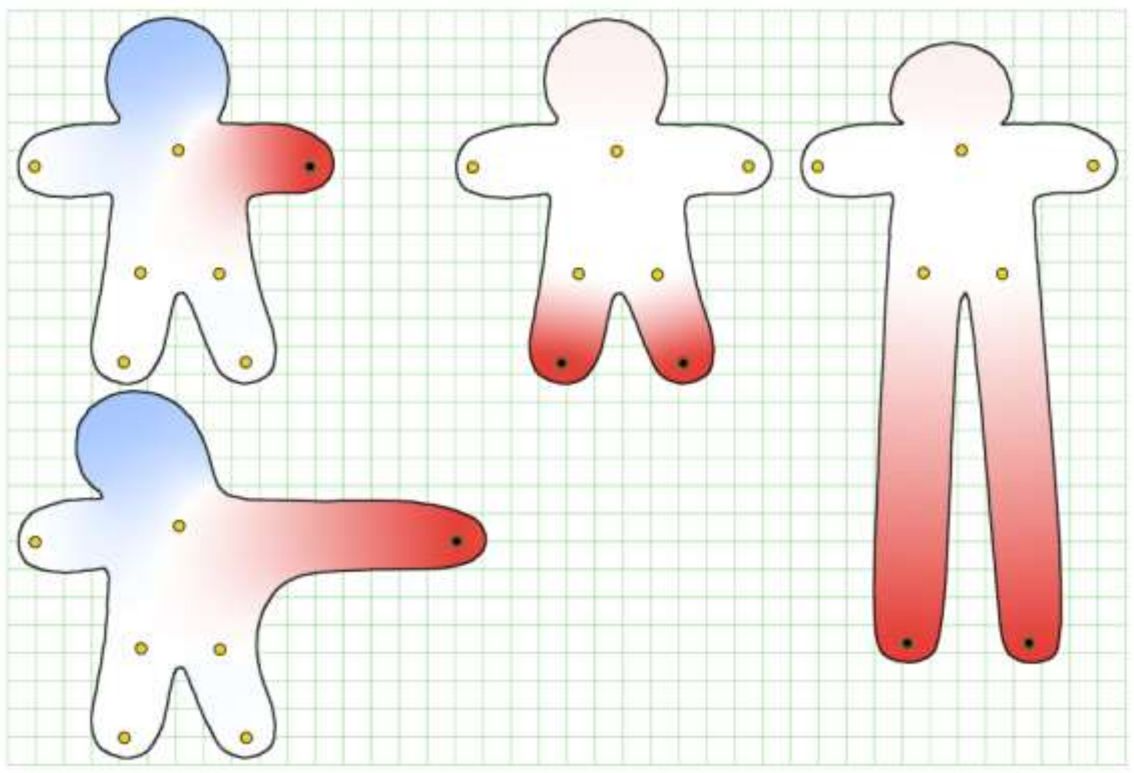


Figure 4: *Weights like unconstrained biharmonic functions that have negative weights (left) and extraneous local maxima (right) lead to undesirable and unintuitive behavior. Notice the shrinking of the head on the right.*

Local barycentric coordinates

- A local change in the value at a single control point will create a global change by propagation into the whole domain
- Global nature
  - The first one is the lack of locality and control over a deformation.
  - The second drawback is scalability. Most practical applications store barycentric coordinates using one scalar value per control point for every vertex of the target domain.

Formulation

$$\min_{w_1,\ldots,w_n} \sum_{i=1}^{n} \int_{\Omega} |\nabla w_i|$$

$$\text{s.t. } \sum_{i=1}^{n} w_i(\mathbf{x})\mathbf{c}_i = \mathbf{x},\ \sum_{i=1}^{n} w_i = 1,\ w_i \geq 0,\ \forall \mathbf{x} \in \Omega,$$

$$w_i(\mathbf{c}_j) = \delta_{ij}\ \forall i, j,$$

$$w_i \text{ is linear on cage edges and faces } \forall i. \tag{5}$$

Locality